Research Article

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Partial slice regularity and Fueter's theorem in several quaternionic variables

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Abstract: We extend some definitions and give new results about the theory of slice analysis in several quaternionic variables. The sets of slice functions that are slice, slice regular, and circular with respect to given variables are characterized. We introduce new notions of partial spherical value and derivative for functions of several variables that extend those of one variable. We recover some of their properties as circularity, harmonicity, some relations with differential operators, and a Leibniz rule with respect to the slice product as well as studying their behavior in the context of several variables. Then, we prove our main result, which is a generalization of Fueter's theorem for slice regular functions in several variables. This extends the link between slice regular and axially monogenic functions well known in the one variable context.

Keywords: slice-regular functions, functions of a hypercomplex variable, quaternions, clifford algebras

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1 Introduction

Slice regular functions were first introduced in the study by Gentili and Struppa [6] for quaternion-valued functions, defined over Euclidean balls with real center. Exploiting the complex-slice structure of the quaternion algebra $\mathbb{H}$ and following an idea of Cullen [3], they defined slice regular (or Cullen regular) functions as real differentiable functions, which are slice by slice holomorphic. The main purpose of this new hypercomplex theory was to overcome the problem encountered by the theory of quaternionic functions already well established by Fueter [4], in which the class of regular functions does not contain polynomials. On the contrary, the class of slice regular functions contains all the power series with right quaternionic coefficients. The two theories are indeed very skew, since, in general, only constant functions are both Fueter and Cullen regular, even though they present some connections, as Fueter's theorem suggests. We refer the reader to the monograph [5] for a comprehensive treatment of the theory of slice regular functions of one quaternionic variable and to [11,14] for Fueter regular functions.

Interest in this new subject grew rapidly, and a large number of papers were published. The theory was soon generalized to more general domains of definition, the so-called slice domains [1] and extended to octonions [7] and Clifford algebras [2]. A new viewpoint took place after the work of Ghiloni and Perotti [8] with the introduction of stem functions, already used by Fueter to generate axially monogenic functions through Fueter's map [4]. This approach allows to define slice functions, in which no regularity is needed, over any axially symmetric domain and to extend the theory uniformly in any real alternative *-algebra with unity.

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The stem functions' approach suggested the way to construct a several variable analog of the theory in the foundational paper [10], to which the present article contributes to develop some ideas introduced therein. In that article, the importance of partial slice regularity has been pointed out. Indeed, it is possible to interpret the slice regularity of an n variables slice function in terms of the one-variable slice regularity of $2^n - 1$ slice functions [10, Theorem 3.23], obtained as all possible iterations of partial spherical values and derivatives of that function. This result establishes a bridge between the one and several variables theories, which has been frequently exploited, for example, in the study by Perotti [13], where local slice analysis was naturally extended from one to several quaternionic variables. But, the study of partial slice regularity, as well as partial spherical values and derivatives was not developed further, and a more detailed study deserved attention, leading to this work.

We describe the structure of the paper. After briefly recalling the theory of slice regular functions of one and several quaternionic variables, we focus on the study of partial slice properties, i.e., sliceness, slice regularity, or circularity with respect to a specific subset of variables (Section 3). More precisely, given a set of variables $\{x_h\}_{h\in H}$, we characterize (Propositions 3.1, 3.2, and 3.4) the sets $\mathcal{S}_H$, $\mathcal{SR}_H$, and $\mathcal{S}_{c,H}$ of slice functions, which are, respectively, slice, slice regular, and circular with respect to all the variables x_h. The use of stem functions is fundamental as all those characterizations are given through conditions over stem functions. Furthermore, we show that for every choice of $H \in \mathcal{P}(n)$, the set $\mathcal{S}_{c,H}$ forms a subalgebra of the set of slice functions endowed with the slice product $(\mathcal{S}, \odot)$ (Corollary 3.5); $\mathcal{S}_H$ and $\mathcal{SR}_H$ do not share this property.

In Chapter 4, we define partial spherical values and derivatives for functions of several variables, which extend the one-variable analogs. We recover some of their main properties such as harmonicity (Proposition 4.9), representation, and Leibniz formulas (18) and (19), and we find new ones, peculiar of the several variables setting (Proposition 4.4) through characterizations of Chapter 3. Finally, thanks to the harmonicity of the partial spherical derivatives, we prove a generalization of Fueter's theorem for slice regular functions of several quaternionic variables (Theorem 4.10), which extends the link between slice regular and axially monogenic functions in higher dimensions.

2 Preliminaries

We briefly recall the main definitions of the theory of slice regular functions of one and several quaternionic variables. We state here the definitions of [8] and [10], reduced to the quaternionic setting.

2.1 Slice regular functions of one quaternionic variable

Let $\mathbb{H}$ denote the algebra of quaternions with basis elements $\{1, i, j, k\}$. We can embed $\mathbb{R} \subset \mathbb{H}$ as the subalgebra generated by 1, while $\mathrm{Im}(\mathbb{H}) \coloneqq \langle i, j, k\rangle$, whence $\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{H})$. Let $\mathbb{S}_{\mathbb{H}} \coloneqq \{q \in \mathbb{H} | q^2 = -1\} \subset \mathrm{Im}(\mathbb{H})$ be the sphere of square roots of -1, then if $q \in \mathbb{H}\backslash\mathbb{R}$, there exist $\alpha, \beta \in \mathbb{R}$, $J \in \mathbb{S}_{\mathbb{H}}$ such that $q = \alpha + J\beta$. They are unique if we require $\beta > 0$. Every such q generates a sphere we denote with $\mathbb{S}_q = \mathbb{S}_{\alpha,\beta} = \{\alpha + I\beta : I \in \mathbb{S}_{\mathbb{H}}\}$. Given $J \in \mathbb{S}_{\mathbb{H}}$, let $\phi_J : \mathbb{C} \ni \alpha + i\beta \mapsto \alpha + J\beta \in \mathbb{H}$. It is clear [8, (1)] that ϕ_J is a real *-algebras isomorphism onto $\mathbb{C}_J \coloneqq \langle 1, J\rangle_{\mathbb{R}} \subset \mathbb{H}$.

Denote with $\{1, e_1\}$ a basis of $\mathbb{R}^2$. Let $D \subset \mathbb{C}$ be a conjugate invariant domain ($\overline{D} = D$), a function $F : D \to \mathbb{H} \otimes \mathbb{R}^2$ is a stem function if it is complex intrinsic, i.e., $F(\overline{z}) = \overline{F(z)}$, which means that if F has components $F = F_\varnothing + e_1F_1$, they satisfy $F_\varnothing(\overline{z}) = F_\varnothing(z)$ and $F_1(\overline{z}) = -F_1(z)$. Given such a set D, we define its circularization in $\mathbb{H}$ as $\Omega_D \coloneqq \{\alpha + J\beta | \alpha + i\beta \in D, J \in \mathbb{S}_{\mathbb{H}}\} = \bigcup_{\alpha+i\beta\in D}\mathbb{S}_{\alpha,\beta}$. We can associate to every stem function $F = F_\varnothing + e_1F_1 : D \to \mathbb{H} \otimes \mathbb{R}^2$ a unique slice function $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ as follows: if $x = \alpha + J\beta = \phi_J(z)$ for some $z = \alpha + i\beta \in D$ and $J \in \mathbb{S}_{\mathbb{H}}$, we define

$$f(x) = F_\emptyset(z) + JF_1(z).$$

Every slice function can be completely recovered by its value over one slice $\mathbb{C}_J$, with a representation formula [8, Proposition 6]: let $I, J \in \mathbb{S}_\mathbb{H}$, then for every $x = \alpha + I\beta$, it holds

$$f(x) = \frac{1}{2}f((\alpha + J\beta) + f(\alpha - J\beta)) - \frac{IJ}{2}(f(\alpha + J\beta) - f(\alpha + J\beta)). \tag{1}$$

Given a slice function f, we define its spherical value and its spherical derivative as follows:

$$f_s^\circ(x) = \frac{1}{2}(f(x) + f(\bar{x})), \quad f_s'(x) = \frac{1}{2}[\mathrm{Im}(x)]^{-1}(f(x) - f(\bar{x})).$$

Note that the spherical value and the spherical derivative are both slice functions, as $f_s^\circ = \mathcal{I}(F_\emptyset)$ and $f_s' = \mathcal{I}(F_1(z)/\mathrm{Im}(z))$, if $f = \mathcal{I}(F_\emptyset + e_1F_1)$. Moreover, by applying (1) with $I = J$, we obtain

$$f(x) = f_s^\circ(x) + \mathrm{Im}(x)f_s'(x).$$

We can define a product over slice functions. Let F and G be two stem functions with $F = F_\emptyset + e_1F_1$ and $G = G_\emptyset + e_1G_1$, respectively. Define $F \otimes G \coloneqq F_\emptyset G_\emptyset - F_1G_1 + e_1(F_\emptyset G_1 + F_1G_\emptyset)$, which happens to be a stem function. Now, if $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$, define $f \odot g \coloneqq \mathcal{I}(F \otimes G)$. With respect to this product, the spherical derivative satisfies a Lebniz rule:

$$(f \odot g)_s' = f_s' \odot g_s^\circ + f_s^\circ \odot g_s'.$$

Let F be a C^1 stem function. Define

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} - e_1\frac{\partial F}{\partial \beta}\right), \quad \frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} + e_1\frac{\partial F}{\partial \beta}\right).$$

Since both $\partial F/\partial z$ and $\partial F/\partial \bar{z}$ are stem functions, we can define

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \quad \frac{\partial f}{\partial x^c} = \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right).$$

Finally, a slice function $f = \mathcal{I}(F)$ is said to be slice regular if $\partial f/\partial x^c = 0$ or, equivalently, if $\partial F/\partial \bar{z} = 0$. Note that [8, Proposition 8], if $\Omega_D \cap \mathbb{R} \neq \emptyset$, the definition of slice regular function coincide with the one given by Gentili and Struppa [6], namely that, for every $J \in \mathbb{S}_\mathbb{H}$, the restriction of f, $f_J : \Omega_D \cap \mathbb{C}_J \to \mathbb{H}$ is holomorphic with respect to the complex structure defined by multiplication by J.

2.2 Slice regular functions of several quaternionic variables

Let n be a positive integer, and let $\mathcal{P}(n)$ denote all possible subsets of $\{1, \dots, n\}$. Given an ordered set $K = \{k_1, \dots, k_p\} \in \mathcal{P}(n)$, with $k_1 < \dots < k_p$ and an associated p-tuple $(q_{k_1}, \dots, q_{k_p}) \in \mathbb{H}^p$, we define $q_K \coloneqq q_{k_1} \cdot \ldots \cdot q_{k_p}$ (with $q_\emptyset \coloneqq 1$) and for any $\tilde{q} \in \mathbb{H}$, $[q_K, \tilde{q}] \coloneqq q_K \cdot \tilde{q}$.

Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, set $\bar{z}^h \coloneqq (z_1, \dots, z_{h-1}, \bar{z}_h, z_{h+1}, \dots, z_n)$, $\forall h = 1, \dots, n$. A set $D \subset \mathbb{C}^n$ is called invariant with respect to complex conjugation whenever $z \in D$ if and only if $\bar{z}^h \in D$ for every $h \in \{1, \dots, n\}$. We define its circularization $\Omega_D \subset \mathbb{H}^n$ as follows:

$$\Omega_D \coloneqq \{(\alpha_1 + J_1\beta_1, \dots, \alpha_n + J_n\beta_n) | (\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n) \in D, J_1, \dots, J_n \in \mathbb{S}_\mathbb{H}\},$$

and we call circular those sets Ω such that $\Omega = \Omega_D$ for some $D \subset \mathbb{C}^n$, invariant with respect to complex conjugation. From now on, we will always assume D an invariant subset of $\mathbb{C}^n$ with respect to complex conjugation and Ω_D a circular set of $\mathbb{H}^n$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of $\mathbb{R}^n$ and denote with $\{e_K\}_{K\in\mathcal{P}(n)}$ a basis of $\mathbb{R}^{2^n}$. Consider a function $F : D \to \mathbb{H} \otimes \mathbb{R}^{2^n}$, $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$, in which its components $\{F_K\}_{K\in\mathcal{P}(n)}$ are $\mathbb{H}$-valued functions. We call F a stem function if $\forall K \in \mathcal{P}(n)$, $\forall h = 1, \dots, n$

$$F_K(\bar{z}^h) = (-1)^{|K\cap\{h\}|} F_K(z). \tag{2}$$

Write Stem(D) for the set of all stem functions from D to $\mathbb{H} \otimes \mathbb{R}^{2^n}$.

A map $f : \Omega_D \subset \mathbb{H}^n \to \mathbb{H}$ is called slice function if there exists a stem function $F : D \to \mathbb{H} \otimes \mathbb{R}^{2^n}$, $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$, such that

$$f(x) = \sum_{K\in\mathcal{P}(n)} [J_K, F_K(z)], \quad \forall x \in \Omega_D,$$

where $x = (x_1, \dots, x_n)$, with $x_i = \alpha_i + J_i\beta_i$, for some $\alpha_i, \beta_i \in \mathbb{R}$, $J_i \in \mathbb{S}_{\mathbb{H}}$ and $z = (z_1, \dots, z_n) \in D$, $z_i = \alpha_i + i\beta_i$, for $i = 1, \dots, n$. Note that (2) is necessary to make slice functions well defined. We say that f is induced by F. $\mathcal{S}(\Omega_D)$ will denote the set of all slice functions from Ω_D to $\mathbb{H}$ and $\mathcal{I} : \text{Stem}(D) \to \mathcal{S}(\Omega_D)$ will be the map sending a stem function to its induced slice function. From [10, Proposition 2.12], every slice function is induced by a unique stem function, so $\mathcal{I}$ is an injective map.

We can define slice functions through a commutative diagram too: for any $J_1, \dots, J_n \in \mathbb{S}_{\mathbb{H}}$, we define

$$\phi_{J_1} \times \cdots \times \phi_{J_n} : \mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (\phi_{J_1}(z_1), \dots, \phi_{J_n}(z_n)) \in \mathbb{H}^n$$

and

$$\Phi_{J_1,\dots,J_n} : \mathbb{H} \otimes \mathbb{R}^{2^n} \ni \sum_{K\in\mathcal{P}(n)} e_K a_K \mapsto \sum_{K\in\mathcal{P}(n)} [J_K, a_K] \in \mathbb{H}.$$

Given $F \in \text{Stem}(D)$, we can define its induced slice function $f = \mathcal{I}(F)$ as the unique slice function that makes the following diagram commutative for any $J_1, \dots, J_n \in \mathbb{S}_{\mathbb{H}}$:

$$\begin{tikzcd} D \arrow[r, "F"] \arrow[d, "\phi_{J_1}\times\dots\times\phi_{J_n}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{H}\otimes\mathbb{R}^{2^n} \arrow[d, "\Phi_{J_1,\dots,J_n}"] \\ \Omega_D \arrow[r, "f"'] & \mathbb{H} \end{tikzcd}$$

As described in [10, Definition 2.31, Lemma 2.32], equip $\mathbb{R}^{2^n}$ with a Δ-product $\otimes : \mathbb{R}^{2^n} \times \mathbb{R}^{2^n} \to \mathbb{R}^{2^n}$, is defined on each basis element as follows:

$$e_H \otimes e_K \coloneqq (-1)^{|H\cap K|} e_{H\Delta K},$$

where $H\Delta K = (H \cup K)\setminus(H \cap K)$ and extended by linearity to all $\mathbb{R}^{2^n}$. This product induces a product on $\mathbb{H} \otimes \mathbb{R}^{2^n}$: given $a, b \in \mathbb{H} \otimes \mathbb{R}^{2^n}$, $a = \sum_{H\in\mathcal{P}(n)} e_H a_H$, and $b = \sum_{K\in\mathcal{P}(n)} e_K b_K$, with $a_H, b_K \in \mathbb{H}$, define

$$a \otimes b \coloneqq \sum_{H,K\in\mathcal{P}(n)} (e_H \otimes e_K)(a_H b_K) = \sum_{H,K\in\mathcal{P}(n)} (-1)^{|H\cap K|} e_{H\Delta K} a_H b_K,$$

where $a_H b_K$ is just the usual product of quaternions. Furthermore, we can define a product between stem functions as the pointwise product induced by $\otimes$: let $F, G \in \text{Stem}(D)$, define $(F \otimes G)(z) \coloneqq F(z) \otimes G(z)$. More precisely, if $F = \sum_{H\in\mathcal{P}(n)} e_H F_H$ and $G = \sum_{K\in\mathcal{P}(n)} e_K G_K$,

$$(F \otimes G)(z) \coloneqq \sum_{H,K\in\mathcal{P}(n)} (-1)^{|H\cap K|} e_{H\Delta K} F_H(z) G_K(z).$$

The advantage of this definition is that the product of two stem functions is again a stem function [10, Lemma 2.34], and this allows to define a product on slice functions, too. Let $f, g \in \mathcal{S}(\Omega_D)$, with $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$, and then define the slice tensor product $f \odot g$ between f and g as follows:

$$f \odot g := \mathcal{I}(F \otimes G).$$

Equip $\mathbb{R}^{2^n}$ with the family of commutative complex structures $\mathcal{J} = \{\mathcal{J}_h : \mathbb{R}^{2^n} \to \mathbb{R}^{2^n}\}_{h=1}^n$, where each $\mathcal{J}_h$ is defined over any basis element e_K of $\mathbb{R}^{2^n}$ as

$$\mathcal{J}_h(e_K) := (-1)^{|K\cap\{h\}|} e_{K\Delta\{h\}} = \begin{cases} e_{K\cup\{h\}} & \text{if } h \notin K \\ -e_{K\setminus\{h\}} & \text{if } h \in K, \end{cases}$$

and extended by linearity to all $\mathbb{R}^{2^n}$. $\mathcal{J}$ induces a family of commutative complex structure on $\mathbb{H} \otimes \mathbb{R}^{2^n}$ (by abuse of notation, we use the same symbol) $\mathcal{J} = \{\mathcal{J}_h : \mathbb{H} \otimes \mathbb{R}^{2^n} \to \mathbb{H} \otimes \mathbb{R}^{2^n}\}_{h=1}^n$ according to the following formula:

$$\mathcal{J}_h(q \otimes a) := q \otimes \mathcal{J}_h(a) \quad \forall q \in \mathbb{H}, \quad \forall a \in \mathbb{R}^{2^n}.$$

We can associate two Cauchy-Riemann operators to each complex structure $\mathcal{J}_h$. Given $F \in \mathrm{Stem}(D) \cap C^1(D)$, we define

$$\partial_h F := \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} - \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right), \quad \bar{\partial}_h F := \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} + \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right).$$

Note that, if F is a stem function, so are $\partial_h F$ and $\bar{\partial}_h F$ [10, Lemma 3.9]. Thus, if $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D) := \mathcal{I}(\mathrm{Stem}(D) \cap C^1(\Omega_D))$, we can define the partial derivatives for every $h = 1, \dots, n$

$$\frac{\partial f}{\partial x_h} := \mathcal{I}(\partial_h F), \quad \frac{\partial f}{\partial x_h^c} := \mathcal{I}(\bar{\partial}_h F).$$

A C^1 stem function $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$ is called h-holomorphic with respect to $\mathcal{J}$ if $\bar{\partial}_h F \equiv 0$ or equivalently [10, Lemma 3.12], if its components satisfies a system of Cauchy-Riemann equations:

$$\frac{\partial F_K}{\partial \alpha_h} = \frac{\partial F_{K\cup\{h\}}}{\partial \beta_h}, \quad \frac{\partial F_K}{\partial \beta_h} = -\frac{\partial F_{K\cup\{h\}}}{\partial \alpha_h}, \quad \forall K \in \mathcal{P}(n), h \notin K, \tag{3}$$

and it is called holomorphic if it is h-holomorphic for every $h = 1, \dots, n$. Finally, given a holomorphic stem function F, the induced slice function $\mathcal{I}(F)$ will be called the slice regular function. The set of all slice regular functions from Ω_D to $\mathbb{H}$ will be denoted by $\mathcal{SR}(\Omega_D)$. By [10, Proposition 3.13], $f \in \mathcal{SR}(\Omega_D)$ if and only if $\frac{\partial f}{\partial x_h^c} = 0$ for every $h = 1, \dots, n$.

We recall two other operators on $\mathbb{H}$, known as Cauchy-Riemann-Fueter operators:

$$\partial_{\mathrm{CRF}} := \frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta} - j\frac{\partial}{\partial \gamma} - k\frac{\partial}{\partial \delta}, \quad \bar{\partial}_{\mathrm{CRF}} := \frac{\partial}{\partial \alpha} + i\frac{\partial}{\partial \beta} + j\frac{\partial}{\partial \gamma} + k\frac{\partial}{\partial \delta},$$

where α, β, γ, and δ denote the four real components of a quaternion $x = \alpha + i\beta + j\gamma + k\delta$. Functions in the kernel of $\bar{\partial}_{\mathrm{CRF}}$ are usually called Fueter regular (or monogenic in the context of Clifford algebras). The importance of these operators is evident as they factorize the Laplacian, indeed

$$\partial_{\mathrm{CRF}}\bar{\partial}_{\mathrm{CRF}} = \bar{\partial}_{\mathrm{CRF}}\partial_{\mathrm{CRF}} = \Delta.$$

Thus, monogenic functions are in particular harmonic. We can extend these operators to $\mathbb{H}^n$: for a slice function $f : \Omega_D \to \mathbb{H}$, we define, for any $h = 1, \dots, n$, ∂_{x_h} and $\bar{\partial}_{x_h}$ as the Cauchy-Riemann-Fueter operators with respect to $x_h := \alpha_h + i\beta_h + j\gamma_h + k\delta_h$:

$$\partial_{x_h} := \frac{\partial}{\partial \alpha_h} - i\frac{\partial}{\partial \beta_h} - j\frac{\partial}{\partial \gamma_h} - k\frac{\partial}{\partial \delta_h}, \quad \bar{\partial}_{x_h} := \frac{\partial}{\partial \alpha_h} + i\frac{\partial}{\partial \beta_h} + j\frac{\partial}{\partial \gamma_h} + k\frac{\partial}{\partial \delta_h}.$$

For every $h = 1, \dots, n$, it holds

$$\partial_{x_h}\bar{\partial}_{x_h} = \bar{\partial}_{x_h}\partial_{x_h} = \Delta_h,$$

where $\Delta_h = \frac{\partial^2}{\partial \alpha_h^2} + \frac{\partial^2}{\partial \beta_h^2} + \frac{\partial^2}{\partial \gamma_h^2} + \frac{\partial^2}{\partial \delta_h^2}$. Finally, denote by $\mathcal{M}_h(\Omega) \coloneqq \{f : \Omega \to \mathbb{H} : \bar{\partial}_{x_h} f = 0\}$ the set of monogenic functions with respect to x_h and let $\mathcal{AM}_h(\Omega_D) \coloneqq \mathcal{M}_h(\Omega_D) \cap \mathcal{S}^1(\Omega_D)$ be the set of axially monogenic functions with respect to x_h, i.e., the set of slice functions which are monogenic with respect to x_h.

3 Characterization of $\mathcal{S}_H$, $\mathcal{SR}_H$, and $\mathcal{S}_{c,H}$

Let $f : \Omega_D \subset \mathbb{H}^n \to \mathbb{H}$ and $h = 1, \dots, n$. For any $y = (y_1, \dots, y_n) \in \Omega_D$, let

$$\Omega_{D,h}(y) \coloneqq \{x \in \mathbb{H} \,|\, (y_1, \dots, y_{h-1}, x, y_{h+1}, \dots, y_n) \in \Omega_D\} \subset \mathbb{H}.$$

It is easy to see [10, Section 2] that $\Omega_{D,h}(y)$ is a circular set of $\mathbb{H}$, more precisely $\Omega_{D,h}(y) = \Omega_{D_h(z)}$, where

$$D_h(z) \coloneqq \{w \in \mathbb{C} \,|\, (z_1, \dots, z_{h-1}, w, z_{h+1}, \dots, z_n) \in D\},$$

and $z = (z_1, \dots, z_n)$ is such that $y \in \Omega_{\{z\}}$.

Definition 3.1. We say that a slice function $f \in \mathcal{S}(\Omega_D)$ is *slice* (resp. *slice regular*) *with respect to* x_h if, $\forall y \in \Omega_D$, its restriction

$$f_h^y : \Omega_{D,h}(y) \to \mathbb{H}, \quad f_h^y(x) \coloneqq f(y_1, \dots, y_{h-1}, x, y_{h+1}, \dots, y_n)$$

is a one variable slice (resp. slice regular) function, as defined in §2.1. We denote by $\mathcal{S}_h(\Omega_D)$ (resp. $\mathcal{SR}_h(\Omega_D)$) the set of slice functions from Ω_D to $\mathbb{H}$ that are slice (resp. slice regular) with respect to x_h. For $H \in \mathcal{P}(n)$, define $\mathcal{S}_H(\Omega_D) \coloneqq \bigcap_{h \in H} \mathcal{S}_h(\Omega_D)$, $\mathcal{SR}_H(\Omega_D) \coloneqq \bigcap_{h \in H} \mathcal{SR}_h(\Omega_D)$. Note that, by definition, $\mathcal{SR}_H(\Omega_D) \subset \mathcal{S}_H(\Omega_D) \subset \mathcal{S}(\Omega_D)$.

We say that f is *circular with respect to* x_h if $\forall y = (y_1, \dots, y_n) \in \Omega_D$, f_h^y is constant on $\mathbb{S}_{y_h} \subset \mathbb{H}$. The set of slice functions that are circular with respect to x_h will be denoted by $\mathcal{S}_{c,h}(\Omega_D) \subset \mathcal{S}(\Omega_D)$. Note that f is circular with respect to x_h if and only if for every orthogonal transformation $T : \mathbb{H} \to \mathbb{H}$ that fixes 1, it holds $f(x_1, \dots, x_{h-1}, T(x_h), x_{h+1}, \dots, x_n) = f(x_1, \dots, x_n)$, for any $(x_1, \dots, x_n) \in \Omega_D$. In this case, if $x_h = \alpha_h + J_h \beta_h$, f does not depend on J_h. Finally, if $H \in \mathcal{P}(n)$, set $\mathcal{S}_{c,H}(\Omega_D) \coloneqq \bigcap_{h \in H} \mathcal{S}_{c,h}(\Omega_D)$.

Every slice function is, in particular, slice with respect to the first variable [10, Proposition 2.23], i.e., $\mathcal{S}_1(\Omega_D) = \mathcal{S}(\Omega_D)$, but in general, $\mathcal{S}_h(\Omega_D) \subsetneq \mathcal{S}(\Omega_D)$. The next proposition characterizes the set $\mathcal{S}_H(\Omega_D)$ for any $H \in \mathcal{P}(n)$ in terms of stem functions.

Proposition 3.1. *For every $H \in \mathcal{P}(n)$, it holds*

$$\mathcal{S}_H(\Omega_D) = \left\{ \mathcal{I}(F) : F \in \mathrm{Stem}(D), \quad F = \sum_{K \in H^c} e_H F_K + \sum_{h \in H} e_{\{h\}} \sum_{Q \subset \{h+1, \dots, n\} \setminus H} e_Q F_{\{h\} \cup Q} \right\}. \tag{4}$$

In particular, for any $h \in \{1, \dots, n\}$,

$$\mathcal{S}_h(\Omega_D) = \left\{ \mathcal{I}(F) : F \in \mathrm{Stem}(D), \quad F = \sum_{K \in \mathcal{P}(n), h \notin K} e_H F_K + e_{\{h\}} \sum_{Q \subset \{h+1, \dots, n\}} e_Q F_{\{h\} \cup Q} \right\}. \tag{5}$$

Equivalently, $f = \mathcal{I}(F) \in \mathcal{S}_H(\Omega_D)$ if and only if $F_{P \cup \{h\} \cup Q} = 0$, $\forall h \in H$, $\forall Q \subset \{h+1, \dots, n\}$, $\forall P \in \mathcal{P}(h-1)$ with $P \neq \emptyset$.

Proof. Since $\mathcal{S}_H(\Omega_D) \coloneqq \bigcap_{h \in H} \mathcal{S}_h(\Omega_D)$, it is sufficient to assume $H = \{h\}$ for some $h = 1, \dots, n$.

($\Rightarrow$) $f \in \mathcal{S}_h(\Omega_D)$ means that $\forall y \in \Omega_D$, the one-variable function f_h^y is slice; thus, it must satisfy representation formula (1): namely, if $x = a + Ib \in \Omega_{D,h}(y)$ and $J \in \mathbb{S}_{\mathbb{H}}$, it holds

$$f_h^y(x) = \frac{1}{2}(f_h^y(a+Jb) + f_h^y(a-Jb)) - \frac{IJ}{2}(f_h^y(a+Jb) - f_h^y(a-Jb)). \tag{6}$$

Set $z = (z_1, \ldots, z_n)$, $z' = (z_1, \ldots, z_{h-1})$, $z'' = (z_{h+1}, \ldots, z_n)$, $y = (\phi_{J_1} \times \ldots \times \phi_{J_n})(z)$, for some $J_1, \ldots, J_n \in \mathbb{S}_{\mathbb{H}}$, $w = a + ib$, $x = \phi_I(w)$, $L_s = M_s = J_s$ for $s \neq h$, $L_h \coloneqq I$ and $M_h \coloneqq J$. Then we have

$$f_h^y(x) = \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] + \sum_{K\in\mathcal{P}(n), h\notin K} [L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')], \tag{7}$$

$$f_h^y(a+Jb) = \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] + \sum_{K\in\mathcal{P}(n), h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')],$$

and

$$\begin{aligned} f_h^y(a-Jb) &= \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', \overline{w}, z'')] + \sum_{K\in\mathcal{P}(n), h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', \overline{w}, z'')] \\ &= \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] - \sum_{K\in\mathcal{P}(n), h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')], \end{aligned}$$

where we have used (2). Thus, the right-hand side of (6) becomes

$$\begin{aligned} &\frac{1}{2}(f_h^y(a+Jb) + f_h^y(a-Jb)) - \frac{I}{2}[J(f_h^y(a+Jb) - f_h^y(a-Jb))] \\ &= \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] - IJ \sum_{K\in\mathcal{P}(n), h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]. \end{aligned} \tag{8}$$

Comparing (7) and (8), (6) is satisfied if and only if

$$\sum_{K\in\mathcal{P}(n), h\notin K} [L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')] = -IJ \sum_{K\in\mathcal{P}(n), h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]. \tag{9}$$

Since (6) is assumed to be true for every $I, J, J_1, \ldots, J_n \in \mathbb{S}_{\mathbb{H}}$ and every z', w, z'', (9) holds if and only if $\forall K \subset \{1, \ldots, n\}\setminus\{h\}$

$$[L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')] = -IJ[M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]. \tag{10}$$

Indeed, if (10) were not true, there would be a $K \subset \mathcal{P}(\{1, \ldots, n\}\setminus\{h\})$ such that

$$[L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')] \neq -IJ[M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')],$$

but for $J_1 = \cdots = J_n = J = I$, we would have

$$(-1)^{|K\cup\{h\}|} F_{K\cup\{h\}}(z', w, z'') \neq (-1)^{|K\cup\{h\}|} F_{K\cup\{h\}}(z', w, z''),$$

which is false. Let us represent $\{K \in \mathcal{P}(n) | h \notin K\} = \{P \sqcup Q | P \in \mathcal{P}(h-1), \quad Q \subset \{h+1, \ldots, n\}\}$. Suppose $P \neq \varnothing$, then $\forall Q \subset \{h+1, \ldots, n\}$, (10) becomes

$$[L_{(P\cup\{h\}\cup Q)}, F_{P\cup\{h\}\cup Q}(z', w, z'')] = -IJ[M_{(P\cup\{h\}\cup Q)}, F_{P\cup\{h\}\cup Q}(z', w, z'')],$$

and this implies that $F_{P\cup\{h\}\cup Q} \equiv 0$. Indeed, if $F_{P\cup\{h\}\cup Q} \neq 0$, the previous equation would reduce to $J_P I = -IJJ_P J$, which does not hold for every choice of I, J, J_P.

($\Leftarrow$) *Vice versa*, suppose F takes the form

$$F = \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_K + e_h \sum_{Q\subset\{h+1,\ldots,n\}} e_Q F_{\{h\}\cup Q}.$$

Following the notation mentioned earlier, it holds

$$f_h^y(x) = \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] + I \sum_{Q\subset\{h+1,\ldots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')].$$

Thus, consider the function $G_h^y = G_{1,h}^y + iG_{2,h}^y$, with

$$G_{1,h}^{y}(w) \coloneqq \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')], \quad G_{2,h}^{y}(w) \coloneqq \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')].$$

G_h^y is a one-variable stem function, indeed,

$$\begin{aligned} G_h^y(\overline{w}) &= \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', \overline{w}, z'')] + i \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', \overline{w}, z'')] \\ &= \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')] - i \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')] = \overline{G_h^y(w)}, \end{aligned}$$

and $f_h^y = \mathcal{I}(G_h^y)$, by construction, so $f \in \mathcal{S}_h(\Omega_D)$. □

Remark 1. By the previous proof, we can better understand the set $\mathcal{S}_H(\Omega_D)$: let $f = \mathcal{I}(F) \in \mathcal{S}_H(\Omega_D)$, then for any $x \in \Omega_D$ with $x = (\phi_{J_1} \times \cdots \times \phi_{J_n})(z)$, $f(x)$ takes the form

$$f(x) = \sum_{K\in H^c} [J_K, F_K(z)] + \sum_{h\in H} J_h \sum_{Q\subset\{h+1,\dots,n\}\setminus H} [J_Q, F_{\{h\}\cup Q}(z)].$$

Moreover, for any $h \in H$ and any $y = (y_1, \dots, y_n)$, f_h^y is a one-variable slice function, induced by the stem function G_h^y, with components

$$G_{1,h}^{y}(w) \coloneqq \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')], \quad G_{2,h}^{y}(w) \coloneqq \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')], \tag{11}$$

where $z = (z', z_h, z'')$ and $y = (\phi_{J_1} \times \cdots \times \phi_{J_n})(z)$.

Now, we deal with partial slice regularity.

Proposition 3.2. *For every $H \in \mathcal{P}(n)$ it holds*

$$\mathcal{SR}_H(\Omega_D) = \mathcal{S}_H(\Omega_D) \bigcap_{h\in H} \ker(\partial/\partial x_h^c).$$

Proof. Since $\mathcal{SR}_H(\Omega_D) \coloneqq \bigcap_{h\in H}\mathcal{SR}_h(\Omega_D)$, it is sufficient to assume $H = \{h\}$ for some $h = 1, \dots, n$.

($\subset$) By definition, $\mathcal{SR}_h(\Omega_D) \subset \mathcal{S}_h(\Omega_D)$, so let $f = \mathcal{I}(F)$, with

$$F = \sum_{K\in\mathcal{P}(n),h\notin K} e_K F_K + e_h \sum_{Q\subset\{h+1,\dots,n\}} e_Q F_{\{h\}\cup Q}, \tag{12}$$

thanks to (5). For any $y \in \Omega_D$, f_h^y is induced by the stem function $G_h^y = G_{1,h}^y + iG_{2,h}^y$, with

$$G_{1,h}^{y}(w) \coloneqq \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')], \quad G_{2,h}^{y}(w) \coloneqq \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')].$$

By definition, $f \in \mathcal{SR}_h(\Omega_D)$ means that $\forall y \in \Omega_D$, the stem function G_h^y is holomorphic, i.e., recalling (11), it must hold that for every $z = (z', z_h, z'') \in D$, $w \in D_h(z)$ and $\forall J_j \in \mathbb{S}_{\mathbb{H}}$ that

$$\begin{cases} \sum_{P,Q} [J_{P\cup Q}, \partial_{\alpha_h} F_{P\cup Q}(z', w, z'')] = \sum_Q [J_Q, \partial_{\beta_h} F_{\{h\}\cup Q}(z', w, z'')] \\ \sum_{P,Q} [J_{P\cup Q}, \partial_{\beta_h} F_{P\cup Q}(z', w, z'')] = -\sum_Q [J_Q, \partial_{\alpha_h} F_{\{h\}\cup Q}(z', w, z'')], \end{cases}$$

where in the aforementioned sums $P \in \mathcal{P}(h-1)$ and $Q \subset \{h+1, \dots, n\}$. Now, since that system is true for every choice of imaginary unit J_j, proceeding as in the proof of Proposition 3.1, we can deduce that an equivalence between each term of the sum holds. Let any $Q \subset \{h+1, \dots, n\}$: if $P \neq \varnothing$, equality can hold only if $\partial_{\alpha_h} F_{P\cup Q} = \partial_{\beta_h} F_{P\cup Q} = 0$, and this trivially proves that the components $F_{P\cup Q}$ satisfy (3), since $F_{P\cup\{h\}\cup Q} = 0$, by (5). Otherwise, let $P = \varnothing$, then the previous system becomes

$$\begin{cases} \partial_{\alpha_h} F_Q = \partial_{\beta_h} F_{\{h\}\cup Q} \\ \partial_{\beta_h} F_Q = -\partial_{\alpha_h} F_{\{h\}\cup Q} \end{cases}$$

and (3) are satisfied too. This proves that F is h-holomorphic, which means that $f \in \ker(\partial/\partial x_h^c)$.

($\supset$) Suppose $f \in \mathcal{S}_h(\Omega_D) \cap \ker(\partial/\partial x_h^c)$, then F satisfies (12) and (3). As in the proof of Proposition 3.1, $K = P \sqcup Q$, with $P \in \mathcal{P}(h-1)$ and $Q \subset \{h+1, \dots, n\}$. Since, by (12), $F_{P\cup\{h\}\cup Q} \equiv 0$, $\forall P \in \mathcal{P}(h-1)\setminus\{\emptyset\}$, $\forall Q \subset \{h+1, \dots, n\}$ the h-holomorphicity of F reduces to the following conditions:

$$\begin{cases} \partial_{\alpha_h} F_{P\cup Q} = \partial_{\beta_h} F_{P\cup Q} = 0 \\ \partial_{\alpha_h} F_Q = \partial_{\beta_h} F_{\{h\}\cup Q} \\ \partial_{\beta_h} F_Q = \partial_{\alpha_h} F_{\{h\}\cup Q}. \end{cases} \tag{13}$$

On the other hand, $f \in \mathcal{SR}_h(\Omega_D)$ if and only if G_h^y is a slice regular function $\forall y \in \Omega_D$, which means that $\partial_\alpha G_{1,h}^y = \partial_\beta G_{2,h}^y$ and $\partial_\beta G_{1,h}^y = -\partial_\alpha G_{2,h}^y$, which, by definition of G_h^y, is equivalent to

$$\begin{cases} \partial_{\alpha_h} \sum\limits_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z)] = \partial_{\beta_h} \sum\limits_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z)] \\ \partial_{\beta_h} \sum\limits_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z)] = -\partial_{\alpha_h} \sum\limits_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z)], \end{cases}$$

where $y = (\phi_{J_1} \times \dots \times \phi_{J_n})(z)$, $z = (z_1, \dots, z_n)$, $z_j = \alpha_j + i\beta_j$. Let us prove the first row of the system. By using the first two equation of (13) and splitting $K = P \sqcup Q$, we can write the left-hand side as follows:

$$\begin{aligned} &\partial_{\alpha_h} \sum_{P\in\mathcal{P}(h-1), Q\subset\{h+1,\dots,n\}} [J_{P\cup Q}, F_{P\cup Q}(z', w, z'')] \\ &= \sum_{P\in\mathcal{P}(h-1), Q\subset\{h+1,\dots,n\}} [J_{P\cup Q}, \partial_{\alpha_h} F_{P\cup Q}(z', w, z'')] = \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, \partial_{\alpha_h} F_Q(z', w, z'')] \\ &= \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, \partial_{\beta_h} F_{\{h\}\cup Q}(z', w, z'')] = \partial_{\beta_h} \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')]. \end{aligned}$$

The second equation is proved in the same way. □

Corollary 3.3. *Let $f \in \mathcal{SR}(\Omega_D)$ and $H \in \mathcal{P}(n)$. Then $f \in \mathcal{S}_H(\Omega_D)$ if and only if $f \in \mathcal{SR}_H(\Omega_D)$.*

Proof. The "if" part is trivial. *Vice versa*, note that from [10, Proposition 3.13], $f \in \mathcal{SR}(\Omega_D)$ implies $\partial f/\partial x_h^c = 0$, $\forall h = 1, \dots, n$, and hence, $\mathcal{S}_H(\Omega_D) \cap \mathcal{SR}(\Omega_D) \subset \mathcal{S}_H(\Omega_D) \bigcap_{h\in H} \ker(\partial/\partial x_h^c) = \mathcal{SR}_H(\Omega_D)$, by Proposition 3.2. □

Finally, we characterize circularity.

Proposition 3.4. *For every $H \in \mathcal{P}(n)$, it holds*

$$\mathcal{S}_{c,H}(\Omega_D) = \left\{ \mathcal{I}(F) : F \in \mathrm{Stem}(D), F = \sum_{K\subset H^c} e_K F_K \right\}. \tag{14}$$

In particular, $\mathcal{S}_{c,H}(\Omega_D) \subset \mathcal{S}_H(\Omega_D)$.

Proof. Since $\mathcal{S}_{c,H}(\Omega_D) = \bigcap_{h\in H} \mathcal{S}_{c,h}(\Omega_D)$, it is sufficient to assume $H = \{h\}$ for some $h = 1, \dots, n$. Let any $y = (y_1, \dots, y_n) \in \Omega_D$, with $y_j \coloneqq \alpha_j + J_j\beta_j$, $z_j \coloneqq \alpha_j + i\beta_j$, set $z' = (z_1, \dots, z_{h-1})$ and $z'' = (z_{h+1}, \dots, z_n)$. $f \in \mathcal{S}_{c,h}(\Omega_D)$ if for every $x = a + Ib$, $f_h^y(x)$ does not depend on I. Let $w \coloneqq a + ib$, $M_p \coloneqq J_p$ if $p \neq h$ and $M_h = I$, then

$$f_h^y(x) = \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] + \sum_{K\in\mathcal{P}(n), h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')].$$

It is clear that $f_h^y(a+Ib)$ does not depend on I if and only if $F_{K\cup\{h\}}=0$ for every $K\in\mathcal{P}(n)$. Finally, by comparing (4) and (14), we see that $\mathcal{S}_{c,H}(\Omega_D)\subset\mathcal{S}_H(\Omega_D)$. □

Note that functions of form (14) were introduced in [10] as H^c-reduced slice functions, and hence, we can say that $f\in\mathcal{S}_{c,H}(\Omega_D)$ if and only if it is H^c reduced. It is easy now to prove the following property.

Corollary 3.5. *For every $H\in\mathcal{P}(n)$, the set $\mathcal{S}_{c,H}(\Omega_D)$ is a real subalgebra of $(\mathcal{S}(\Omega_D),\odot)$.*

Proof. We need to show that if $f,g\in\mathcal{S}_{c,H}(\Omega_D)$, then $f\odot g\in\mathcal{S}_{c,H}(\Omega_D)$. Let $f=\mathcal{I}(F)$ and $g\in\mathcal{I}(G)$, with $F=\sum_{K\subset H^c}e_KF_K$ and $G=\sum_{T\subset H^c}e_TG_T$, by (14). Then

$$F\otimes G=\sum_{K,T\subset H^c}(-1)^{|K\cap T|}e_{K\Delta T}F_KG_T,$$

with $K\Delta T=(K\cup T)\backslash(K\cap T)\subset K\cup T\subset H^c$. Then, again (14) implies $f\odot g\in\mathcal{S}_{c,H}(\Omega_D)$. □

Note that the previous result does not apply to $\mathcal{S}_H(\Omega_D)$, nor $\mathcal{SR}_H(\Omega_D)$, unless for $\mathcal{S}_1(\Omega_D)=\mathcal{S}(\Omega_D)$ and $\mathcal{SR}_1(\Omega_D)$. Indeed, for example, $x_1,x_2\in\mathcal{SR}_2(\Omega_D)$, while $x_1\odot x_2\notin\mathcal{S}_2(\Omega_D)$.

Slice regularity and circularity are hardly compatible.

Proposition 3.6. *Let $f\in\mathcal{S}_{c,h}(\Omega_D)\cap\mathcal{SR}_h(\Omega_D)$. Then f is locally constant with respect to x_h.*

Proof. Let $x_h=a_h+J_hb_h$ and $f=\mathcal{I}(F)$. Since $f\in\mathcal{S}_{c,h}(\Omega_D)$, f does not depend on J_h and $F_{K\cup\{h\}}=0$ for any $K\in\mathcal{P}(n)$. Moreover, $f\in\mathcal{SR}_h(\Omega_D)\subset\ker(\partial/\partial x_h^c)$, by Proposition 3.2, so by (3),

$$\frac{\partial F_K}{\partial\alpha_h}=\frac{\partial F_{K\cup\{h\}}}{\partial\beta_h}=0=\frac{\partial F_{K\cup\{h\}}}{\partial\alpha_h}=-\frac{\partial F_K}{\partial\beta_h}.$$

Thus, f does not depend neither on α_h nor β_h and so it is locally constant with respect to x_h. □

Example 1. Consider the following polynomial function $f:\mathbb{H}^3\to\mathbb{H}$, $f(x_1,x_2,x_3)\coloneqq x_1x_3+x_2x_3^2k$, which happens to be a slice regular function, [10, Proposition 3.14]. We claim that $f\in\mathcal{SR}_2(\Omega_D)$. Let us explicit the components of the stem function inducing f: let $z=(z_1,z_2,z_3)\in\mathbb{C}^3$, with $z_j\coloneqq\alpha_j+i\beta_j$, then $f=\mathcal{I}(F)$, with $F=\sum_{K\in\mathcal{P}(3)}e_KF_K$, where

$$\begin{gathered}F_\varnothing(z)=\alpha_1\alpha_3+\alpha_2(\alpha_3^2-\beta_3^2)k,\quad F_{\{1\}}(z)=\beta_1\alpha_3,\quad F_{\{2\}}(z)=\beta_2(\alpha_3^2-\beta_3^2)k,\\ F_{\{3\}}(z)=\alpha_1\beta_3+2\alpha_2\alpha_3\beta_3k,\quad F_{\{1,2\}}(z)=0,\quad F_{\{1,3\}}(z)=\beta_1\beta_3,\quad F_{\{2,3\}}(z)=2\beta_2\alpha_3\beta_3k,\quad F_{\{1,2,3\}}(z)=0.\end{gathered}$$

Thus, F has the structure required by (5) for $h=2$, so $f\in\mathcal{S}_2(\Omega_D)$. Moreover, for $K=\varnothing,\{1\},\{3\},\{1,3\}$, it holds

$$\frac{\partial F_K}{\partial\alpha_2}=\frac{\partial F_{K\cup\{2\}}}{\partial\beta_2},\quad\frac{\partial F_K}{\partial\beta_2}=-\frac{\partial F_{K\cup\{2\}}}{\partial\alpha_2},$$

so $f\in\ker(\partial/\partial x_2^c)$ and so $f\in\mathcal{SR}_2(\Omega_D)=\mathcal{S}_2(\Omega_D)\cap\ker(\partial/\partial x_2^c)$.

We could have proven the claim by definition, through Remark 1, which explicitly gives us the stem function that induces the corresponding one variable slice function, for every choice of y. Fix any $y=(y_1,y_2,y_3)\in\mathbb{H}^3$, then f_2^y is a slice regular function, induced by the holomorphic stem function $G_2^y=G_{1,2}^y+iG_{2,2}^y$, with

$$G_{1,2}^y(\alpha+i\beta)=y_1y_3+\alpha y_3^2k,\quad G_{2,2}^y(\alpha+i\beta)=\beta y_3^2k.$$

4 Partial spherical derivatives

For $h \in \{1, \dots, n\}$, define $\mathbb{R}_h := \{(x_1, \dots, x_n) | x_h \in \mathbb{R}\}$ and for $H \in \mathcal{P}(n)$, $\mathbb{R}_H := \bigcup_{h \in H} \mathbb{R}_h$.

Definition 4.1. Let $F : D \subset \mathbb{C}^n \to \mathbb{H} \otimes \mathbb{R}^{2^n}$ be a stem function. Define for $h = 1, \dots, n$ and for $H = \{h_1, \dots, h_p\} \in \mathcal{P}(n)$

$$F_h^{\circ}(z) := \sum_{K \in \mathcal{P}(n), h \notin K} e_K F_K(z),$$
$$F_H^{\circ}(z) := \sum_{K \subset H^c} e_K F_K(z) = (\dots(F_{h_1}^{\circ})_{h_2}^{\circ} \dots)_{h_p}^{\circ}(z)$$

and

$$F_h'(z) := \beta_h^{-1} \sum_{K \in \mathcal{P}(n), h \notin K} e_K F_{K \cup \{h\}}(z), \quad \text{if } z \in D \setminus \mathbb{R}_h \tag{15}$$

$$F_H'(z) := \beta_H^{-1} \sum_{K \in H^c} e_K F_{K \cup H}(z) = (\dots(F_{h_1}')_{h_2}' \dots)_{h_p}'(z), \quad \text{if } z \in D \setminus \mathbb{R}_H, \tag{16}$$

where $z = (z_1, \dots, z_n)$ with $z_j = \alpha_j + i\beta_j$ and $\beta_H = \prod_{h \in H} \beta_h$.

Lemma 4.1. *For every $H \in \mathcal{P}(n)$, F_H°, and F_H' are well-defined stem functions on D and $D \setminus \mathbb{R}_H$, respectively.*

Proof. First, let us prove that F_H° and F_H' are well defined, i.e., their definition does not depend on the order of H. Indeed, for any $i, j = 1, \dots, n$, it holds

$$(F_i')_j'(z) = \sum_{K \in \mathcal{P}(n), i,j \notin K} e_K \beta_j^{-1} \beta_i^{-1} F_{K \cup \{i,j\}}(z) = (F_j')_i'(z)$$

and analogously for $(F_i^{\circ})_j^{\circ}$. Without loss of generality, assume $H = \{h\}$, for some $h = 1, \dots, n$. F_h° is trivially a stem function because its non zero components are the same of F. Let us explicit $F_h' = \sum_{K \in \mathcal{P}(n)} e_K G_K$, with

$$G_K(z) = \begin{cases} \beta_h^{-1} F_{K \cup \{h\}} & \text{if } h \notin K \\ 0 & \text{if } h \in K, \end{cases}$$

we will show that every component of F_h' satisfies (2). Let us consider only the components G_K, with $h \notin K$, otherwise (2) is trivial. For any $m \neq h$, we have

$$G_K(\overline{z}^m) = \beta_h^{-1} F_{K \cup \{h\}}(\overline{z}^m) = \beta_h^{-1} (-1)^{|K \cap \{m\}|} F_{K \cup \{h\}}(z) = (-1)^{|K \cap \{m\}|} G_K(z),$$

while, for $m = h$,

$$G_K(\overline{z}^h) = (-\beta_h^{-1}) F_{K \cup \{h\}}(\overline{z}^h) = (-\beta_h^{-1})(-F_{K \cup \{h\}}(z)) = \beta_h^{-1} F_{K \cup \{h\}}(z) = G_K(z). \qquad \square$$

The previous lemma allows to make the following:

Definition 4.2. Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. For $h \in \{1, \dots, n\}$, we define its *spherical x_h-value and x_h-derivative* respectively as follows:

$$f_{s,h}^{\circ} := \mathcal{I}(F_h^{\circ}), \quad f_{s,h}' := \mathcal{I}(F_h').$$

Analogously, for $H \in \mathcal{P}(n)$, define

$$f_{s,H}^{\circ} := \mathcal{I}(F_H^{\circ}), \quad f_{s,H}' := \mathcal{I}(F_H').$$

Note that $f_{s,H}^{\circ} \in \mathcal{S}(\Omega_D)$, while $f_{s,H}' \in \mathcal{S}(\Omega_{D_H})$, where $\Omega_{D_H} := \Omega_D \setminus \mathbb{R}_H$.

We stress that the terms spherical value and spherical derivatives have been already used in [10, Section 2.3] in the context of slice functions of several quaternionic variables, but they refer to different objects. With respect to our definition, spherical values and derivatives are more related to the truncated spherical derivatives $\mathcal{D}_\varepsilon(f)$ [10, Definition 2.24], where for $h \in \{1, \dots, n\}$ and $\varepsilon : \{1, \dots, h\} \to \{0, 1\}$, $\mathcal{D}_\varepsilon(f) \coloneqq \mathcal{D}_{x_h}^{\varepsilon(h)} \cdots \mathcal{D}_{x_1}^{\varepsilon(1)}(f)$, with $\mathcal{D}_{x_l}^1(f) = f'_{s,l}$ and $\mathcal{D}_{x_l}^0(f) = f'_{s,l}$. Indeed, it holds $\mathcal{D}_\varepsilon(f) = (f'_{s,H})^\circ_{s,K}$, with $H = \varepsilon^{-1}(1)$ and $K = \varepsilon^{-1}(0)$.

The following proposition justifies the names given to $f^\circ_{s,h}$ and $f'_{s,h}$, comparing them to their one-variable analogs (§2.1). Note that we have to assume $f \in \mathcal{S}_h(\Omega_D)$, in order for the spherical derivative to agree with it.

Proposition 4.2. *Let $f \in \mathcal{S}(\Omega_D)$ and $h = 1, \dots, n$. Then it holds*

(1) $\forall x = (x_1, \dots, x_n) \in \Omega_D$

$$f^\circ_{s,h}(x) = \frac{1}{2}(f(x) + f(\overline{x}^h)) = (f^x_h)^\circ_s(x_h);$$

(2) *if* $f \in \mathcal{S}_h(\Omega_D)$, $\forall x \in \Omega_D \setminus \mathbb{R}_h$

$$f'_{s,h}(x) = [2\mathrm{Im}(x_h)]^{-1}(f(x) - f(\overline{x}^h)) = (f^x_h)'_s(x_h). \tag{17}$$

In particular, if we assume $f \in \mathcal{S}^1(\Omega_D)$, then we can extend the definition of $f'_{s,h}$ to all Ω_D, thanks to [8, Proposition 7, (2)].

Proof. Let $f = \mathcal{I}(F)$, with $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$. Then for any $z \in D$ and $x = (\phi_{J_1} \times \cdots \times \phi_{J_n})(z)$, we obtain

$$\begin{aligned} f(x) + f(\overline{x}^h) &= \sum_{K \in \mathcal{P}(n)} ([J_K, F_K(z)] + [J_K, F_K(\overline{z}^h)]) \\ &= \sum_{K \in \mathcal{P}(n)} ([J_K, F_K(z)] + (-1)^{|K \cap \{h\}|}[J_K, F_K(z)]) \\ &= \sum_{K \in \mathcal{P}(n), h \notin K} (2[J_K, F_K(z)]) = 2 f^\circ_{s,h}(x). \end{aligned}$$

Now, assume $f \in \mathcal{S}_h(\Omega_D)$, then by (5),

$$f(x) = \sum_{h \notin K} [J_K, F_K(z)] + J_h \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, F_{\{h\} \cup Q}(z)],$$

and so

$$f'_{s,h}(x) = \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, \beta_h^{-1} F_{\{h\} \cup Q}(z)].$$

On the other hand, let $x = (\phi_{J_1} \times \cdots \times \phi_{J_n})(z)$, then by (2), we have

$$\begin{aligned} f(x) - f(\overline{x}^h) &= \sum_{K \in \mathcal{P}(n), h \notin K} [J_K, F_K(z)] + J_h \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, F_{\{h\} \cup Q}(z)] + \\ &\quad - \sum_{K \in \mathcal{P}(n), h \notin K} [J_K, F_K(\overline{z}^h)] - J_h \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, F_{\{h\} \cup Q}(\overline{z}^h)] \\ &= 2J_h \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, F_{\{h\} \cup Q}(z)], \end{aligned}$$

from which

$$\begin{aligned} [2\mathrm{Im}(x_h)]^{-1} (f(x) - f(\overline{x}^h)) &= [2J_h \beta_h]^{-1} \left(2J_h \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, F_{\{h\} \cup Q}(z)] \right) \\ &= \sum_{Q \subset \{h+1, \dots, n\}} [J_Q, \beta_h^{-1} F_{\{h\} \cup Q}(z)] = f'_{s,h}(x). \end{aligned}$$

□

We extend from [12] properties of the spherical derivative of one-variable slice regular functions to several variables.

Lemma 4.3. *If* $f \in \mathcal{SR}_h(\Omega_D)$, *the following hold*:
(1) $\bar{\partial}_{x_h} f = -2f'_{s,h}$;
(2) $\Delta_h f = -4\frac{\partial f'_{s,h}}{\partial x_h} = -2\partial_{x_h}(f'_{s,h})$.

Proof.
(1) Note that $\forall y = (y_1, \dots, y_n) \in \Omega_D$, $f_h^y \in \mathcal{SR}(\Omega_{D,h}(y))$, then we can apply (17) and [12, Corollary 6.2, (a)] to obtain

$$\bar{\partial}_{x_h} f(y) = \bar{\partial}_{\mathrm{CRF}}(f_h^y)(y_h) = -2(f_h^y)'_s(y_h) = -2f'_{s,h}(y).$$

(2) By (17), [12, Corollary 6.2, (c), Theorem 6.3, (c)], and [9, Theorem 2.2 (ii)], we have

$$\Delta_h f(y) = \Delta(f_h^y)(y_h) = -4\frac{\partial (f_h^y)'_s}{\partial x}(y_h) = -4\theta(f_h^y)'_s(y_h) = -2\partial_{\mathrm{CRF}}(f_h^y)'_s(y_h) = -2\partial_{x_h} f'_{s,h}(y),$$

where $(\theta f)(x) = \frac{1}{2}\left(\frac{\partial f}{\partial \alpha}(x) + \frac{\operatorname{Im}(x)}{|\operatorname{Im}(x)|^2}(\beta\frac{\partial f}{\partial \beta}(x) + \gamma\frac{\partial f}{\partial \gamma}(x) + \delta\frac{\partial f}{\partial \delta}(x))\right)$ satisfies $\theta f = \frac{\partial f}{\partial x}$ and $2\theta f'_s = \partial_{\mathrm{CRF}} f'_s$ for any slice function f.[1]

The next proposition presents some properties of partial spherical values and derivatives peculiar of the several variables setting.

Proposition 4.4. *Let* $f \in \mathcal{S}(\Omega_D)$, $h \in \{1, \dots, n\}$, *and* $H \in \mathcal{P}(n)$, *with* $p = \min H^c$ if $H \neq \{1, \dots, n\}$. *Then*
(1) $f^\circ_{s,H} \in \mathcal{S}_{c,H}(\Omega_D) \cap \mathcal{S}_p(\Omega_D)$ *and* $f'_{s,H} \in \mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})$;
(2) *if* $f \in \mathcal{S}_h(\Omega_D)$, $f'_{s,h} \in \mathcal{S}_{h+1}(\Omega_{D_H}) \cap \mathcal{S}_{c,\{1,\dots,h\}}(\Omega_{D_H})$;
(3) *if* $f \in \mathcal{S}_{c,h}(\Omega_D)$, $f^\circ_{s,h} = f$, *and* $f'_{s,h} = 0$;
(4) *if* $h \in H$, $H \cap \{1, \dots, h-1\} \neq \emptyset$, *and* $f \in \mathcal{S}_h(\Omega_D)$, *then* $f'_{s,H} = 0$;
(5) $(f^\circ_{s,h})^\circ_{s,h} = f^\circ_{s,h}$ *and* $(f'_{s,h})'_{s,h} = 0$.

Proof.
(1) If $f = \mathcal{I}(F)$, by definition $f^\circ_{s,h} = \sum_{K\subset H^c}[J_K, F_K]$, hence by Proposition 3.4, $f^\circ_{s,H} \in \mathcal{S}_{c,H}(\Omega_D)$. Moreover, we can write it as follows:

$$f^\circ_{s,h} = \sum_{K\subset(H\cup p)^c}[J_K, F_K] + J_p \sum_{K\subset(H\cup p)^c}[J_K, F_{K\cup p}],$$

so $f^\circ_{s,h} \in \mathcal{S}_p(\Omega_D)$. In the same way, one can prove that $f'_{s,H} \in \mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})$.
(2) By Proposition 3.1, F takes the form

$$F = \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_K + e_{\{h\}} \sum_{Q\subset\{h+1,\dots,n\}} e_Q F_{\{h\}\cup Q},$$

hence,

$$F'_h = \beta_h^{-1} \sum_{Q\subset\{h+1,\dots,n\}} e_Q F_{\{h\}\cup Q}.$$

This shows that $f'_{s,h} \in \mathcal{S}_{c,\{1,\dots,h\}}(\Omega_{D_h})$, by Proposition 3.4. Finally, by Proposition 3.1, $f'_{s,h} \in \mathcal{S}_{h+1}(\Omega_{D_h})$.
(3) By Proposition 3.4, $F = \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_K$, so $F'_h = 0$ and $F^\circ_h = F$.

1 In [12] the factor 1/2 is omitted in the definition of θ.

(4) Let $i \in H \cap \{1, \dots, h-1\} \neq \varnothing$, since $f \in \mathcal{S}_h(\Omega_D)$, by (2) $f'_{s,h} \in \mathcal{S}_{c,i}(\Omega_{D_i})$ and by (3) $(f'_{s,h})'_{s,i} = 0$. In particular, $f'_{s,H} = 0$.

(5) It follows from (1) and (3). □

Partial spherical derivatives do not affect regularity in other variables.

Proposition 4.5. *Let $f \in \mathcal{S}^1(\Omega_D)$. Suppose that $f \in \ker(\partial/\partial x_t^c)$ for some $t = 1, \dots, n$, then $f'_{s,h} \in \ker(\partial/\partial x_t^c)$, $\forall h \neq t$.*

Proof. Let $f = \mathcal{I}(F)$, with $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$, so $f'_{s,h} = \mathcal{I}(F'_h)$, with $F'_h = \sum_{K \in \mathcal{P}(n)} e_K G_K$, $G_K = 0$, if $h \in K$ and $G_K = \beta_h^{-1} F_{K \cup \{h\}}$, if $h \notin K$. Let $K \in \mathcal{P}(n)$, with $h, t \notin K$, then by the regularity of F, it holds

$$\begin{cases} \dfrac{\partial G_K}{\partial \alpha_t} = \dfrac{\partial \beta_h^{-1} F_{K\cup\{h\}}}{\partial \alpha_t} = \beta_h^{-1} \dfrac{\partial F_{K\cup\{h\}}}{\partial \alpha_t} = \beta_h^{-1} \dfrac{\partial F_{K\cup\{h\}\cup\{t\}}}{\partial \beta_t} = \dfrac{\partial G_{K\cup\{t\}}}{\partial \beta_t} \\[2ex] \dfrac{\partial G_K}{\partial \beta_t} = \dfrac{\partial \beta_h^{-1} F_{K\cup\{h\}}}{\partial \beta_t} = \beta_h^{-1} \dfrac{\partial F_{K\cup\{h\}}}{\partial \beta_t} = -\beta_h^{-1} \dfrac{\partial F_{K\cup\{h\}\cup\{t\}}}{\partial \alpha_t} = -\dfrac{\partial G_{K\cup\{t\}}}{\partial \alpha_t}. \end{cases}$$

This proves that F'_h is t-holomorphic, and hence, $f'_{s,h} \in \ker(\partial/\partial x_t^c)$. □

As recalled in Section 2.1, every one variable slice function f can be decomposed as $f(x) = f_s^\circ(x) + \operatorname{Im}(x) f'_s(x)$. We now give a similar decomposition for every variable, through the slice product.

Proposition 4.6. *Let $f \in \mathcal{S}(\Omega_D)$, then for any $h = 1, \dots, n$, we can decompose*

$$f = f_{s,h}^\circ + \operatorname{Im}(x_h) \odot f'_{s,h}. \tag{18}$$

Proof. Let $f = \mathcal{I}(F)$, with $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$. Suppose first $x \in \mathbb{R}_h$, i.e., $\operatorname{Im}(x_h)(x) = 0$, then by (2), with the usual notation, we have

$$f(x) = \sum_{K \in \mathcal{P}(n)} [J_K, F_K(z)] = \sum_{K \in \mathcal{P}(n), h \notin K} [J_K, F_K(z)] = f_{s,h}^\circ(x).$$

Now, suppose $x \in \Omega_D \setminus \mathbb{R}_h$ and define $\operatorname{Im}(Z_h)(z_1, \dots, z_n) := e_h \beta_h$, where $z_j := \alpha_j + i\beta_j$. $\operatorname{Im}(Z_h) \in \operatorname{Stem}(D)$ and $\mathcal{I}(\operatorname{Im}(Z_h)) = \operatorname{Im}(x_h)$. Then

$$\begin{aligned} F_h^\circ + \operatorname{Im}(Z_h) \otimes F'_h &= \sum_{K \in \mathcal{P}(n), h \notin K} e_K F_K + (e_h \beta_h) \otimes \left(\sum_{K \in \mathcal{P}(n), h \notin K} e_K \beta_h^{-1} F_{K \cup \{h\}} \right) \\ &= \sum_{K \in \mathcal{P}(n), h \notin K} e_K F_K + \sum_{K \in \mathcal{P}(n), h \notin K} e_{K \cup \{h\}} F_{K \cup \{h\}} = F. \end{aligned}$$

Finally, $f = \mathcal{I}(F) = \mathcal{I}(F_h^\circ + \operatorname{Im}(Z_h) \otimes F'_h) = f_{s,h}^\circ + \operatorname{Im}(x_h) \odot f'_{s,h}$. □

Next proposition shows that the partial spherical derivatives satisfies a Leibniz-type formula, analog to the one-dimensional case.

Proposition 4.7. (Leibniz rule) *Let $f, g \in \mathcal{S}(\Omega_D)$. It holds*

$$(f \odot g)'_{s,h} = f'_{s,h} \odot g_{s,h}^\circ + f_{s,h}^\circ \odot g'_{s,h}. \tag{19}$$

Proof. Let $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$. We have to show that $(F \otimes G)'_h = F'_h \otimes G_h^\circ + F_h^\circ \otimes G'_h$. By [10, Lemma 2.34], we have $F'_h \otimes G_h^\circ = \sum_{K \in \mathcal{P}(n), h \notin K} e_K (F'_h \otimes G_h^\circ)_K$, where

$$(F_h' \otimes G_h^\circ)_K = \sum_{K_1,K_2,K_3\in\mathcal{D}(K)} (-1)^{|K_3|}(F_h')_{K_1\cup K_3}(G_h^\circ)_{K_2\cup K_3},$$

and $\mathcal{D}(K) \coloneqq \{(K_1,K_2,K_3)\in\mathcal{P}(n)^3 | K = K_1 \sqcup K_2, K_3\cap K = \varnothing\}$. By definition of F_h' and G_h°, the previous equation reduces to

$$(F_h' \otimes G_h^\circ)_K = \sum_{K_1,K_2,K_3\in\mathcal{D}_h'(K)} (-1)^{|K_3|}F_{K_1\cup K_3\cup\{h\}}G_{K_2\cup K_3},$$

with $\mathcal{D}_h'(K) \coloneqq \{(K_1,K_2,K_3)\in\mathcal{P}(n)^3 | K = K_1 \sqcup K_2, K_3\cap (K\cup\{h\}) = \varnothing\}$. In the very same way, we obtain

$$(F_h^\circ \otimes G_h')_K = \sum_{K_1,K_2,K_3\in\mathcal{D}_h'(K)} (-1)^{|K_3|}F_{K_1\cup K_3}G_{K_2\cup K_3\cup\{h\}},$$

and hence,

$$F_h'\otimes G_h^\circ + F_h^\circ\otimes G_h' = \sum_{K\in\mathcal{P}(n),h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}_h'(K)} (-1)^{|K_3|}(F_{K_1\cup K_3\cup\{h\}}G_{K_2\cup K_3} + F_{K_1\cup K_3}G_{K_2\cup K_3\cup\{h\}}).$$

On the other hand, $F\otimes G = \sum_{K\in\mathcal{P}(n)} e_K(F\otimes G)_K$, where

$$(F\otimes G)_K = \sum_{K_1,K_2,K_3\in\mathcal{D}(K)} (-1)^{|K_3|}F_{K_1\cup K_3}G_{K_2\cup K_3}.$$

Thus,

$$\begin{aligned}(F\otimes G)_h' &= \sum_{K\in\mathcal{P}(n),h\notin K} e_K\beta_h^{-1}(F\otimes G)_{K\cup\{h\}}\\ &= \sum_{K\in\mathcal{P}(n),h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}(K\cup\{h\})} (-1)^{|K_3|}F_{K_1\cup K_3}G_{K_2\cup K_3}.\end{aligned}$$

Note that

$$\begin{aligned}\mathcal{D}(K\cup\{h\}) &= \{(K_1,K_2,K_3)\in\mathcal{P}(n)^3 | K\cup\{h\} = K_1\sqcup K_2, K_3\cap(K\cup\{h\}) = \varnothing\}\\ &= \{(K_1\cup\{h\},K_2,K_3),(K_1,K_2\cup\{h\},K_3) | (K_1,K_2,K_3)\in\mathcal{D}_h'(K)\},\end{aligned}$$

so

$$\begin{aligned}(F\otimes G)_h' &= \sum_{K\in\mathcal{P}(n),h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}(K\cup\{h\})} (-1)^{|K_3|}F_{K_1\cup K_3}G_{K_2\cup K_3}\\ &= \sum_{K\in\mathcal{P}(n),h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}_h'(K)} (-1)^{|K_3|}(F_{K_1\cup\{h\}\cup K_3}G_{K_2\cup K_3} + F_{K_1\cup K_3}G_{K_2\cup\{h\}\cup K_3})\\ &= F_h'\otimes G_h^\circ + F_h^\circ\otimes G_h'.\end{aligned}$$

□

Corollary 4.8. *Let $f\in\mathcal{S}(\Omega_D)$ and $g\in\mathcal{S}_{c,H}(\Omega_D)$ for some $H\in\mathcal{P}(n)$, then $(f\odot g)_{s,H}' = f_{s,H}'\odot g$.*

Proof. We proceed by induction over $|H|$. Suppose first $|H| = 1$, then it follows from Propositions 4.7 and 4.4 (3). Now, suppose by induction that $(f\odot g)_{s,H}' = f_{s,H}'\odot g$ and let $h\notin H$, and then in the same way, we have

$$(f\odot g)_{s,H\cup\{h\}}' = (f_{s,h}'\odot g_{s,h}^\circ + f_{s,h}^\circ\odot g_{s,h}')_{s,H}' = (f_{s,h}'\odot g)_{s,H}' = f_{s,H\cup\{h\}}'\odot g.$$

□

The next result highlights a fundamental property of partial spherical derivatives, i.e., harmonicity. The only requirement is regularity in such variable. This extends the result for one-variable slice regular functions [12, Theorem 6.3, (c)].

Proposition 4.9. *Let $f\in\mathcal{S}^1(\Omega_D)$. Suppose that $f\in\ker(\partial/\partial x_h^c)$, for some $h = 1,\dots,n$. Then*

$$\Delta_h f_{s,h}' = 0.$$

Proof. Let us introduce a slightly different notation: let $x = (x_1,\dots,x_n)\in\Omega_D$, with $x_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l = \alpha_l + J_l b_l$, where

$$J_l \coloneqq \frac{i\beta_l + j\gamma_l + k\delta_l}{\sqrt{\beta_l^2 + \gamma_l^2 + \delta_l^2}}, \quad b_l(\beta_l, \gamma_l, \delta_l) \coloneqq \sqrt{\beta_l^2 + \gamma_l^2 + \delta_l^2}.$$

Let $f = \mathcal{I}(F)$, with $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$, then, by definition, $f'_{s,h}(x) = \sum_{K\in\mathcal{P}(n), h\notin K}[J_K, b_h^{-1}F_{K\cup\{h\}}(z)]$ and so

$$\Delta_h f'_{s,h} = \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, \Delta_h(b_h^{-1}F_{K\cup\{h\}})].$$

Thus, it is enough to prove that

$$\Delta_h(b_h^{-1}F_{K\cup\{h\}}) = (\partial^2_{\alpha_h} + \partial^2_{\beta_h} + \partial^2_{\gamma_h} + \partial^2_{\delta_h})(b_h^{-1}F_{K\cup\{h\}}(z', \alpha_h + ib_h(\beta_h, \gamma_h, \delta_h), z'')) = 0.$$

Immediately, we obtain $\partial^2_{\alpha_h}(b_h^{-1}F_{K\cup\{h\}}) = b_h^{-1}\partial^2_{\alpha_h}F_{K\cup\{h\}}$. Moreover, by

$$\partial_{\beta_h} b_h = \beta_h / b_h, \quad \partial_{\beta_h} F_{K\cup\{h\}} = \frac{\beta_h}{b_h}\partial_{b_h}F_{K\cup\{h\}},$$

we find

$$\partial_{\beta_h}(b_h^{-1}F_{K\cup\{h\}}) = -\frac{\beta_h}{b_h^3}F_{K\cup\{h\}} + \frac{\beta_h}{b_h^2}\partial_{b_h}F_{K\cup\{h\}}$$

and

$$\begin{aligned}
\partial^2_{\beta_h}(b_h^{-1}F_{K\cup\{h\}}) &= \partial_{\beta_h}\left(-\frac{\beta_h}{b_h^3}F_{K\cup\{h\}} + \frac{\beta_h}{b_h^2}\partial_{b_h}F_{K\cup\{h\}}\right) \\
&= \frac{3\beta_h^2 - b_h^2}{b_h^5}F_{K\cup\{h\}} - \frac{\beta_h^2}{b_h^4}\partial_{b_h}F_{K\cup\{h\}} + \frac{b_h^2 - 2\beta_h^2}{b_h^4}\partial_{b_h}F_{K\cup\{h\}} + \frac{\beta_h^2}{b_h^3}\partial^2_{b_h}F_{K\cup\{h\}} \\
&= \frac{3\beta_h^2 - b_h^2}{b_h^5}F_{K\cup\{h\}} + \frac{b_h^2 - 3\beta_h^2}{b_h^4}\partial_{b_h}F_{K\cup\{h\}} + \frac{\beta_h^2}{b_h^3}\partial^2_{b_h}F_{K\cup\{h\}}.
\end{aligned}$$

Analogously for γ_h and δ_h:

$$\partial^2_{\gamma_h}(b_h^{-1}F_{K\cup\{h\}}) = \frac{3\gamma_h^2 - b_h^2}{b_h^5}F_{K\cup\{h\}} + \frac{b_h^2 - 3\gamma_h^2}{b_h^4}\partial_{b_h}F_{K\cup\{h\}} + \frac{\gamma_h^2}{b_h^3}\partial^2_{b_h}F_{K\cup\{h\}},$$

$$\partial^2_{\delta_h}(b_h^{-1}F_{K\cup\{h\}}) = \frac{3\delta_h^2 - b_h^2}{b_h^5}F_{K\cup\{h\}} + \frac{b_h^2 - 3\delta_h^2}{b_h^4}\partial_{b_h}F_{K\cup\{h\}} + \frac{\delta_h^2}{b_h^3}\partial^2_{b_h}F_{K\cup\{h\}}.$$

So

$$(\partial^2_{\beta_h} + \partial^2_{\gamma_h} + \partial^2_{\delta_h})(b_h^{-1}F_{K\cup\{h\}}) = b_h^{-1}\partial^2_{b_h}F_{K\cup\{h\}}$$

and finally,

$$\Delta_h(b_h^{-1}F_{K\cup\{h\}}) = b_h^{-1}(\partial^2_{\alpha_h} + \partial^2_{b_h})F_{K\cup\{h\}} = 0,$$

since $f \in \ker(\partial/\partial x_h^c)$, which implies the h-holomorphicity of every F_K. □

Our last application is a generalization to several variables of Fueter's theorem, which is a fundamental result in hypercomplex analysis. In modern language, it states that, given a slice regular function $f : \Omega_D \subset \mathbb{H} \to \mathbb{H}$, its Laplacian generates an axially monogenic function, i.e.,

$$\overline{\partial}_{\mathrm{CRF}}\Delta f = 0.$$

Theorem 4.10. *Let $\Omega_D \subset \mathbb{H}^n$ be a circular set and let $f \in \mathcal{SR}_h(\Omega_D)$ be a slice function, which is slice regular with respect to x_h, for some $h = 1, \ldots, n$. Then $\Delta_h f$ is an axially monogenic function with respect to x_h, i.e.,*

$$\Delta_h f \in \ker(\overline{\partial}_{x_h}).$$

In other words, the Fueter map extends to

$$\Delta_h : \mathcal{SR}_h(\Omega_D) \to \mathcal{AM}_h(\Omega_D).$$

Proof. Since $f \in \mathcal{SR}_h(\Omega_D)$, we can apply Lemma 4.3 1. and Proposition 4.9

$$\overline{\partial}_{x_h}\Delta_h f = \Delta_h \overline{\partial}_{x_h} f = -2\Delta_h f'_{s,h} = 0. \qquad \square$$

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