Research Article

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Chow transformation of coherent sheaves

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Abstract: We define a dual of the Chow transformation of currents on any complex projective manifold. This integral transformation is a factor of a left inverse of the Chow transformation and its composition with the Chow transformation is a right inverse of a linear differential operator, which does not commute with ∂ or $\bar{\partial}$. We obtain a complete intrinsic resolution of the problem of the algebraicity of the cohomology classes. On another hand, in the case of the complex projective space, we give the translation in terms of real-analytic $\mathcal{D}$-modules of the properties of the Chow transformation. Then, the proofs can be simplified by using the conormal currents, which exist for all currents of bidimension (p, p) on the complex projective space, even not closed. This is a consequence of the existence of dual currents, defined on the dual complex projective space. In particular, we obtain a linear differential system of order lower than that of the Gelfand-Gindikin-Graev differential system, characterizing the images by the Chow transformation of smooth differential forms on the complex projective space.

Keywords: approximation by algebraic cycles, complex projective manifold, cycle space, dual current, Penrose transform, Radon transform, real-analytic left differential module

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1 Introduction

Let $X \subset \mathbb{P}_N$ be a complex projective manifold of dimension d_X and let T be a current on X of bidimension (p, p) with $p = d_X - q$. The Chow transform $\hat{C}(T)$ of T is a current of bidegree $(1, 1)$ on the space $C_{d_X-p-1}(X)$ of effective algebraic cycles in X of dimension $d_X - p - 1$. The fact that $\hat{C}(T)$ is of bidegree $(1, 1)$ means that it is of bidegree $(1, 1)$ on each irreducible component M of $C_{d_X-p-1}(X)$ of dimension d_M (see [1,7]). For $[x] \in X$ generic, we set

$$d = \dim\{c \in M, c \ni [x]\},$$

and assuming that the family of cycles $c \in M$ covers X, we have $d + p = d_M - 1$.

We prove that we can choose M such that there is a dual integral transform $\hat{C}^*_M$, defined for $(1, 1)$-currents on M, with values in $\{(p, p)\text{-currents on } X\}$ satisfying the following property (see Theorem 1, subsection 2.3).

Theorem. *There is a linear differential operator*

$$\hat{\mathcal{P}}_M : \{(p, p)\text{-currents on } X\} \to \{(d_X - p, d_X - p)\text{-currents on } X\}$$

with smooth coefficients locally on X such that the inversion formula

$$T = (\hat{\mathcal{P}}_M \hat{C}^*_M \hat{C}_{|M})(T)$$

holds for all $(d_X - p, d_X - p)$-currents T on X.

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This implies that the Chow transformation $\hat{C}$ of currents defined on a complex projective manifold is injective. In this way, we complete the general scheme of integral geometry for the Chow transformation (see [22,23]).

Note that $\hat{\mathcal{P}}_M$ does not commute with ∂ or $\bar{\partial}$ in general, so there is no cohomological consequence. In fact, when S is a $(q-1,q)$-current on X, we have $(\hat{C}^*_M\hat{C}_{|M})(\partial S) = \partial(\hat{C}^*_M\hat{C}_{|M})(S)$, which gives $\partial S = \hat{\mathcal{P}}_M\, \partial(\hat{C}^*_M\hat{C}_{|M})(S)$.
But $S \to (\hat{C}^*_M\hat{C}_{|M})(S)$ is not bijective, since the bidegree has changed.

We use the Chow transformation to solve the problem of the algebraicity of the cohomology classes, by describing explicitly the obstructions.

Theorem. *If T is closed on X and the cohomology class $\{T\}$ is rational, then $\{T\}$ is algebraic.*

This is obtained by solving the equivalent problem of the approximation by the algebraic cycles (see Theorem 4, subsection 4.1 and Theorem 5, subsection 4.2).

Let us mention that our result is consistent with ideas previously suggested by Jean-Pierre Demailly (see [29,30]).

The fact that the Chow transformation $\hat{C}$ of currents defined on a complex projective manifold is injective can also be seen in the following way.

We denote by $j : X \hookrightarrow P(V)$ the embedding of X into $P(V) = \mathbb{P}_N$ and by $\rho : G(p+1, V^*) \to C_{d_X-p-1}(X)$ the induced meromorphic map, which associates $X \cap P(\ker\lambda_0) \cap \ldots \cap P(\ker\lambda_p)$ to a subspace $\mathrm{vect}(\lambda_0, \ldots, \lambda_p) \in G(p+1, V^*)$. With C, the Chow transformation of currents of bidimension (p,p) on the projective space $\mathbb{P}_N$, we have the equality

$$\rho^*\hat{C}(T) = C(j_*T)$$

between $(1,1)$-currents on the Grassmannian $G(p+1, V^*) = G(q, V) = \mathbb{G}_{q-1,N}$. In other words, for the injectivity of $\hat{C}$ at the level of currents it is sufficient to take the irreducible component M of the cycles $c = X \cap P(\ker\lambda_0) \cap \ldots \cap P(\ker\lambda_p)$.

In effect, we know (see [33]) the existence of a dual integral transform C^*, defined for $(1,1)$-currents on $G(p+1, V^*)$, with values in (p,p)-currents on $P(V)$, satisfying the existence of a linear differential operator with smooth coefficients

$$\mathcal{P} : \{(p,p)\text{-currents on } \mathbb{P}_N\} \to \{(q,q)\text{-currents on } \mathbb{P}_N\}$$

such that $T = (\mathcal{P}C^*C)(T)$ for all currents T of bidimension (p,p) on $\mathbb{P}_N$.

The other purpose of this article is also to prove the existence of $\mathcal{P}$ for $\mathbb{P}_N$ using the theory of left $\mathcal{D}$-modules (see [5,8,13,19]). We denote by $\mathcal{A}_{\mathbb{P}_N}$ the sheaf of rings of real-analytic functions with complex values and, $\mathcal{D}_{\mathbb{P}_N}$ the sheaf of rings of real-analytic linear differential operators on $\mathbb{P}_N$. Then, $\bigwedge^{q,q}T\mathbb{P}_N$ is a sheaf of $\mathcal{A}_{\mathbb{P}_N}$-modules and $\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \bigwedge^{q,q}T\mathbb{P}_N$ becomes a sheaf of left $\mathcal{D}_{\mathbb{P}_N}$-modules. The space of smooth differential (q,q)-forms on $\mathbb{P}_N$ can be written as follows:

$$C^\infty_{q,q}(\mathbb{P}_N) = H^0\Big(\mathbb{P}_N, \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}_N}}\Big(\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \bigwedge^{q,q} T\mathbb{P}_N, C^\infty_{\mathbb{P}_N}\Big)\Big),$$

where the left $\mathcal{D}_{\mathbb{P}_N}$-module $C^\infty_{\mathbb{P}_N}$ is the sheaf of smooth functions on $\mathbb{P}_N$.

The transform $C(T)$ is obtained by integrating T on the projective subspaces of $\mathbb{P}_N$ of dimension $q-1$, i.e., $C(T)$ is obtained from T by means of the double fibration

$$\begin{array}{ccc} & \Gamma & \\ & \overset{\varphi}{\swarrow} \quad \overset{\psi}{\searrow} & \\ \mathbb{P}_N & & \mathbb{G}_{q-1,N} \end{array}$$

with $\Gamma \subset \mathbb{P}_N \times \mathbb{G}_{q-1,N}$ the incidence manifold. We denote by Q the universal quotient vector bundle on $\mathbb{G}_{q-1,N} = G(q, V)$ and we replace $\det Q$ by $\mathcal{D}_{\mathbb{G}_{q-1,N}} \otimes_{\mathcal{A}_{\mathbb{G}_{q-1,N}}} \det Q$ to obtain a left $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module. Then, the existence of $\mathcal{P}$ is a consequence of the following result (see Proposition 10, subsection 3.2).

Proposition. *There is a left $\mathcal{D}_{\mathbb{P}_N}$-module $\mathcal{N}$ isomorphic to $\bigwedge^{q,q} T\mathbb{P}_N$ satisfying the following conditions:*

(*i*) $H^0(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}) \otimes H^0(\mathbb{G}_{q-1,N}, \mathcal{A}_{\mathbb{G}_{q-1,N}}) \otimes \mathcal{N}$ *is contained in* $\varphi_* \psi^*(\det Q \otimes \overline{\det Q} \otimes \psi_* \varphi^*(\mathcal{D}_{\mathbb{P}_N} \otimes \bigwedge^{q,q} T\mathbb{P}_N \otimes \mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}))$,

(*ii*) *for every smooth differential (q, q)-form u on $\mathbb{P}_N$, the differential (p, p)-form $C^*C(u)$ is a global solution on $\mathbb{P}_N$ of $\mathcal{N}$ with values in $C^\infty_{\mathbb{P}_N}$.*

Here, $H^0(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}) \otimes H^0(\mathbb{G}_{q-1,N}, \mathcal{A}_{\mathbb{G}_{q-1,N}})$ is a topological tensor product over $\mathbb{C}$.

The proof of existence of $\mathcal{P}$ can also be simplified by using the theory of conormal currents in $\mathbb{P}_N$ (see subsection 3.3). Note that tangent currents at a point of closed positive currents were studied by Dinh–Sibony (see [11]).

The theory of left $\mathcal{D}$-modules also provides the partial differential equations (PDE) of Gelfand et al. (see [14,15]), that characterize the $(1, 1)$-currents Θ on $\mathbb{G}_{q-1,N}$ that are in the image of C.

Denote by $\tau : V^q \to G(q, V)$ the map that associates with $z = (z^0, \dots, z^{q-1})$ in the product the vector subspace $\tau(z) = \mathrm{vect}(z^0, \dots, z^{q-1})$. We use the Plücker coordinates P_I of $z = (z^0, \dots, z^{q-1}) \in V^q$, which are defined by

$$z^0 \wedge \ldots \wedge z^{q-1} = \sum_{|I|=q} P_I e_I \in \bigwedge^q V,$$

where $e_I = e_{i_1} \wedge \ldots \wedge e_{i_q}$ for $I = (i_1, \dots, i_q)$ with $0 \le i_1 < \ldots < i_q \le N$, when $e_0, \dots, e_N$ is an orthonormal basis of V.

A smooth differential $(1, 1)$-form Θ on $G(q, V)$ is a Chow transform when Θ can be written as follows:

$$\tau^*\Theta = \sum_{|I|=|J|=q} C_{I,J} \mathrm{d}P_I \wedge \mathrm{d}\overline{P_J},$$

where the coefficients $C_{I,J}$ are smooth functions of $z \in V^q$ satisfying:

(i) the homogeneity property $C_{I,J}(A.\, z) = |\det A|^{-2} C_{I,J}(z)$ for all $A \in \mathrm{GL}_q(\mathbb{C})$,

(ii) the linear differential equations of order 1

$$\sum_{\substack{0 \le k \le q-1 \\ 0 \le j \le N}} z_j^k \partial_{j'}^k C_{jI',J} + C_{j'I',J} = 0 = \sum_{\substack{0 \le k \le q-1 \\ 0 \le j \le N}} \overline{z}_j^k \overline{\partial}_{j'}^k C_{I,jJ'} + C_{I,j'J'},$$

(iii) the linear differential equations of order 2

$$(\partial_j^k \partial_{j'}^{k'} - \partial_{j'}^k \partial_j^{k'}) C_{I,J} = 0 = (\overline{\partial}_j^k \overline{\partial}_{j'}^{k'} - \overline{\partial}_{j'}^k \overline{\partial}_j^{k'}) C_{I,J},$$

where ∂_j^k is the partial derivative with respect to the homogeneous coordinate z_j^k. Actually, the coefficients $C_{I,J}$ can be expressed as differential operators evaluated in the coefficients Θ_{nl}^{km} of Θ in homogeneous coordinates (see Proposition 7, subsection 3.1).

This can also be translated in the following way (see Theorem 2, subsection 3.1).

Theorem. *There is a left submodule $\mathcal{M} \subset \bigwedge^{1,1} T\mathbb{G}_{q-1,N}$ of finite type satisfying*

(*i*) *the inclusion of* $H^0(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}) \otimes (\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M})$ *into* $\det Q \otimes \overline{\det Q} \otimes \psi_* \varphi^*(\mathcal{D}_{\mathbb{P}_N} \otimes \bigwedge^{q,q} T\mathbb{P}_N \otimes \mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)})$,

(*ii*) *the equality* $\mathrm{Im}\, C = H^0(\mathbb{G}_{q-1,N}, \mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_{q-1,N}}}(\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M}, C^\infty_{\mathbb{G}_{q-1,N}}))$.

Thus, if u is a smooth differential (q, q)-form on $\mathbb{P}_N$, then $C(u)$ is a global solution on $\mathbb{G}_{q-1,N}$ of the left $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module $\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M}$ with values in $C^\infty_{\mathbb{G}_{q-1,N}}$.

2 Chow transformation of holomorphic vector bundles on $\mathbb{P}_N$

We denote by $P(V) = \mathbb{P}_N$ the projective space of lines in a complex vector space V of dimension $N + 1$ and $[x]$ the point of $P(V)$ associated to a nonzero vector x in V. For $1 \le q \le N$, let $G(q, V) = \mathbb{G}_{q-1,N}$ be the Grassmannian of vector subspaces of V of dimension q, and let $P(s)$ be the projective subspace of $P(V)$ associated to $s \in G(q, V)$.

Let Γ be the incidence manifold, i.e., the submanifold in $P(V) \times G(q, V)$ of $([x], s)$ satisfying $x \in s$. Then, $\varphi : \Gamma \to P(V)$ and $\psi : \Gamma \to G(q, V)$ are the restrictions to Γ of the canonical projections.

Let $E \to \mathbb{P}_N$ be a holomorphic vector bundle over $\mathbb{P}_N$. The Penrose transform of E (see [4,17,27]) is the coherent sheaf

$$\mathcal{F} = \psi_* \varphi^* E$$

over $\mathbb{G}_{q-1,N}$. In the general case, $\mathcal{F}$ is not necessarily locally free. But there is $k_0 \in \mathbb{N}$ such that $\psi_* \varphi^*(E \otimes O_{\mathbb{P}_N}(k))$ is a vector bundle for every $k \ge k_0$. So we will implicitly replace E by $E \otimes O_{\mathbb{P}_N}(k) = E(k)$.

2.1 Inversion in the derived category

For $[x] \in P(V)$, the fiber $\varphi^{-1}([x])$ is identified with $G(q-1, V/\mathbb{C}x)$, thus its dimension is $d = (q-1)(N+1-q)$.

We denote by $\omega_\varphi = \bigwedge^d \Omega^1_\varphi$ the determinant bundle of Ω^1_φ, where Ω^1_φ is obtained in the following manner. Thanks to the morphism $\Gamma \xrightarrow{\varphi} \mathbb{P}_N$, we have the exact sequence $T\Gamma \xrightarrow{d\varphi} \varphi^* T\mathbb{P}_N \to 0$, then by duality we obtain the exact sequence

$$0 \to \varphi^* \Omega^1_{\mathbb{P}_N} \to \Omega^1_\Gamma \to \Omega^1_\varphi \to 0$$

with $\Omega^1_\varphi = T^*\Gamma / \varphi^* T^* \mathbb{P}_N$. If we take $\mathcal{L} = \omega_\varphi^{-1}$, we have

$$\begin{aligned}\mathcal{L}_{[x],s} &= \bigwedge^d ((s/\mathbb{C}x)^* \otimes (V/s)) \\ &= (\det (s/\mathbb{C}x)^*)^{N+1-q} \otimes (\det (V/s))^{q-1} \\ &= (\det (V/\mathbb{C}x)^*)^{N+1-q} \otimes (\det Q_s)^N,\end{aligned}$$

where Q is the universal quotient vector bundle on $G(q, V)$.

We can also take the holomorphic vector bundle $\mathcal{L}$ of rank 1 over Γ defined by $\mathcal{L}_{[x],s} = \bigwedge^{q-1}(s/\mathbb{C}x)^*$. Then, the exact sequence

$$0 \to s/\mathbb{C}x \to V/\mathbb{C}x \to Q_s \to 0$$

allows us to write

$$\mathcal{L}_{[x],s} = \det (V/\mathbb{C}x)^* \otimes \det Q_s.$$

However, since we will be integrating on the fibers of φ, we will only use the holomorphic line bundle

$$\mathcal{L} = \psi^* \det Q \tag{1}$$

on the incidence manifold Γ, with $\det Q > 0$ on the Grassmannian $G(q, V)$.

Proposition 1. *The correspondence $E \to \mathcal{F}$ is injective, and we can retrieve E from the coherent sheaf*

$$E' = \varphi_*(\mathcal{L} \otimes \psi^* \mathcal{F})$$

calculated in the derived category on $\mathbb{P}_N$.

Proof. We begin by calculating $\psi^* \mathcal{F}$ using the base change (see [21])

$$\begin{array}{ccc} \Gamma' & \overset{p_2}{\to} & \Gamma \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \psi} \\ \Gamma & \overset{\psi}{\to} & G(q, V), \end{array}$$

where $\Gamma' = \{(a, b) \in \Gamma^2, \psi(a) = \psi(b)\}$ and $p_1 : \Gamma' \to \Gamma$ and $p_2 : \Gamma' \to \Gamma$ are the restrictions of the canonical projections. In the derived category, we have $\psi^*\mathcal{F} = p_{1*}p_2^*\varphi^*E$ and then the projection formula

$$\varphi_*(\mathcal{L} \otimes \psi^*\psi_*\varphi^*E) = \varphi_*(\mathcal{L} \otimes p_{1*}p_2^*\varphi^*E) = (\varphi \circ p_1)_*(p_1^*\mathcal{L} \otimes (\varphi \circ p_2)^*E).$$

With

$$q : \Gamma' \to P(V) \times P(V)$$

defined by $q([x], [x'], s) = ([x], [x'])$, using the commutation formula $\varphi \circ p_i = \mathrm{pr}_i \circ q$, we arrive at

$$(\mathrm{pr}_1 \circ q)_*(p_1^*\mathcal{L} \otimes (\mathrm{pr}_2 \circ q)^*E) = \mathrm{pr}_{1*}(q_*p_1^*\mathcal{L} \otimes \mathrm{pr}_2^*E).$$

Now we have to determine the coherent sheaf $q_*p_1^*\mathcal{L}$ on $\mathbb{P}_N \times \mathbb{P}_N$, which is a generic holomorphic vector bundle on $\mathbb{P}_N \times \mathbb{P}_N$.

In effect, for $[x] \neq [x']$, the fiber $q^{-1}([x], [x']) = \{s \supset \mathrm{vect}(x, x')\} \subset \varphi^{-1}([x])$ is identified to the Grassmannian $G(q-2, V/(\mathbb{C}x \oplus \mathbb{C}x'))$ and

$$(q_*p_1^*\mathcal{L})_{[x],[x']} = H^0(q^{-1}([x], [x']), \mathcal{L}) = H^0(q^{-1}([x], [x']), \psi^*\det Q).$$

Thanks to the exact sequence

$$0 \to s/(\mathbb{C}x \oplus \mathbb{C}x') \to V/(\mathbb{C}x \oplus \mathbb{C}x') \to Q_s \to 0 \tag{2}$$

on $q^{-1}([x], [x'])$, we obtain $(q_*p_1^*\mathcal{L})_{[x],[x']} = \bigwedge^{N+1-q}(V/(\mathbb{C}x \oplus \mathbb{C}x'))$ because $H^0(G(q, V), \det Q) = \bigwedge^{N+1-q}V$ by the Bott theorem (see [6]).

This generic holomorphic vector bundle can be extended by a coherent sheaf in $\mathbb{P}_N \times \mathbb{P}_N$ in the following way. When $[x] \in \mathbb{P}_N$ is fixed, we consider $\mathcal{N}_{[x]}$ the image sheaf of the morphism $\mathbb{C}x' \to V/\mathbb{C}x$ for all $[x'] \in \mathbb{P}_N$. Then, $\mathcal{N}_{[x]}$ extends $(\mathbb{C}x \oplus \mathbb{C}x')/\mathbb{C}x$ and $\mathcal{A} = (V/\mathbb{C}x)/\mathcal{N}_{[x]}$ extends $V/(\mathbb{C}x \oplus \mathbb{C}x')$ when $[x']$ varies in $\mathbb{P}_N$. In such a way,

$$\bigwedge^{N+1-q}\mathcal{A} = \bigwedge^{N+1-q}((V/\mathbb{C}x)/\mathcal{N}_{[x]})$$

extends $\bigwedge^{N+1-q}(V/(\mathbb{C}x \oplus \mathbb{C}x'))$. Note that $\bigwedge^{N+1-q}\mathcal{A}$ is also a quotient of $\bigwedge^{N+1-q}(V/\mathbb{C}x)$ since the canonical map $V/\mathbb{C}x \to (V/\mathbb{C}x)/\mathcal{N}_{[x]}$ induces a surjection $\bigwedge^{N+1-q}(V/\mathbb{C}x) \to \bigwedge^{N+1-q}\mathcal{A}$.

But, for all fixed $[x] \in \mathbb{P}_N$, with $i_{[x]}([x']) = ([x], [x']) \in \mathbb{P}_N \times \mathbb{P}_N$, we have

$$E'_{[x]} = H^0(P(V), i_{[x]}^*(q_*p_1^*\mathcal{L}) \otimes E).$$

By taking the limit when $[x'] \overset{\neq}{\to} [x]$, we conclude that

$$i_{[x]}^*(q_*p_1^*\mathcal{L}) = \bigwedge^{N+1-q}\mathcal{A}.$$

Since $(\mathcal{N}_{[x]})_{[x]} \otimes_{\mathcal{O}_{\mathbb{P}_N,[x]}} \mathbb{C} = \{0\}$, the evaluation at $[x]$ gives a morphism

$$H^0\left(P(V), \left(\bigwedge^{N+1-q}\mathcal{A}\right) \otimes E\right) \to \bigwedge^{N+1-q}(V/\mathbb{C}x) \otimes E_{[x]}. \tag{3}$$

In effect, the fiber $(\bigwedge^{N+1-q}\mathcal{A})_{[x]} \otimes_{\mathcal{O}_{\mathbb{P}_N,[x]}} \mathbb{C}$ is equal to

$$\bigwedge^{N+1-q}\left(\mathcal{A}_{[x]} \otimes_{\mathcal{O}_{\mathbb{P}_N,[x]}} \mathbb{C}\right) = \bigwedge^{N+1-q}(\{(V/\mathbb{C}x) \otimes \mathbb{C}\}/\left\{(\mathcal{N}_{[x]})_{[x]} \otimes_{\mathcal{O}_{\mathbb{P}_N,[x]}} \mathbb{C}\right\}) = \bigwedge^{N+1-q}(V/\mathbb{C}x).$$

Moreover, by replacing E by $E(k)$ with k large enough, we can assume the morphism (3) to be surjective.

So there is a surjective morphism

$$E'_{[x]} \to \bigwedge^{N+1-q}(V/\mathbb{C}x) \otimes E_{[x]},$$

i.e. we recover $\bigwedge^{N+1-q}(V/O_{\mathbb{P}_N}(-1)) \otimes E$ from E'. By cancellation, we can recover E from E' too. □

Note that the coherent sheaf $q_* p_1^* \mathcal{L}$ involved in the proof of Proposition 1 can also be obtained using the blow up $\tilde{\Gamma}' \xrightarrow{\mu'} \Gamma'$ of Γ' along $q^{-1}(D_{\mathbb{P}_N})$, where $D_{\mathbb{P}_N}$ is the diagonal in $\mathbb{P}_N \times \mathbb{P}_N$. We can use a commutative diagram

$$\begin{array}{ccc} \tilde{\Gamma}' & \xrightarrow{\sigma} & M \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Gamma' & \xrightarrow{q} & \mathbb{P}_N \times \mathbb{P}_N, \end{array}$$

where μ is the blow up of $\mathbb{P}_N \times \mathbb{P}_N$ along $D_{\mathbb{P}_N}$ and σ is equidimensional. Then, $q_* p_1^* \mathcal{L} = \mu_* \sigma_* \mu'^* p_1^* \mathcal{L}$ is the direct image by μ of a holomorphic vector bundle defined on M also obtained by extension of the holomorphic vector bundle $\bigwedge^{N+1-q}(V/(\mathbb{C}x \oplus \mathbb{C}x'))$ defined for $([x],[x']) \in \mathbb{P}_N \times \mathbb{P}_N \backslash D_{\mathbb{P}_N}$. In effect, with $v([x],[x'])$ the natural morphism $\mathbb{C}x' \to V/\mathbb{C}x$, we have

$$M = \{([x],[x'],[u]),\ u \in \mathrm{Hom}(\mathbb{C}x', V/\mathbb{C}x)\backslash\{0\} \text{ and } u \text{ colinear to } v([x],[x'])\}$$

and the inverse image in M of $V/(\mathbb{C}x \oplus \mathbb{C}x')$ is nothing than the quotient vector bundle $(V/\mathbb{C}x)/\mathrm{Im}\, u$.

Corollary 1. *The correspondence that associates $\mathcal{F} = \psi_* \varphi^* \mathcal{E}$ to any coherent sheaf $\mathcal{E}$ on $\mathbb{P}_N$ is injective and we can retrieve $\mathcal{E}$ from*

$$\mathcal{E}' = \varphi_*(\mathcal{L} \otimes \psi^* \mathcal{F})$$

calculated in the derived category on $\mathbb{P}_N$.

Proof. We use a projective resolution

$$0 \longrightarrow E_N \longrightarrow E_{N-1} \longrightarrow \dots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

with holomorphic vector bundles E_i defined in $\mathbb{P}_N$. If k is large enough, the complex

$$0 \longrightarrow \psi_*\varphi^*(E_N(k)) \longrightarrow \psi_*\varphi^*(E_{N-1}(k)) \longrightarrow \dots \longrightarrow \psi_*\varphi^*(E_1(k)) \longrightarrow \psi_*\varphi^*(E_0(k)) \longrightarrow \psi_*\varphi^*(\mathcal{E}(k)) \longrightarrow 0$$

is exact too (see [27]), and we retrieve $E_i(k)$ from $\psi_*\varphi^*(E_i(k))$ by Proposition 1. □

For $\mathcal{B}$, a coherent sheaf of O_Γ-modules, the projection formula in the derived category

$$\psi_* \mathrm{Hom}(\mathcal{B}, \psi^* O_{G(q,V)}) = \mathrm{Hom}(\psi_* \mathcal{B}, O_{G(q,V)})$$

can be written as $\psi_*(\mathcal{B}^\vee) = (\psi_* \mathcal{B})^\vee$. So in Corollary 1, we can replace $\mathcal{L}$ by $\mathcal{L}^*$.

Proposition 2. *Let Z be an effective algebraic cycle in $P(V) = \mathbb{P}_N$ of codimension q and let $\mathcal{I}_Z$ be the ideal sheaves of Z. If $\mathcal{E} = (O_{\mathbb{P}_N}/\mathcal{I}_Z)(k)$, then*

$$\mathcal{F} = \psi_* \varphi^* \mathcal{E} = \psi_* \varphi^* (O_{\mathbb{P}_N}/\mathcal{I}_Z \otimes O_{\mathbb{P}_N}(k))$$

is a coherent $O_{G(q,V)}$-module, and for each $s \in G(q,V)$, we have

$$\mathcal{F}_s \otimes_{O_{G(q,V),s}} \mathbb{C} = H^0(P(s), O_{P(s)}(k)/(\mathcal{I}_{Z \cap P(s)} \cdot O_{P(s)}(k))).$$

Proof. We first express

$$\mathcal{F} = \psi_*(O_\Gamma/\mathcal{I}_{\varphi^{-1}(Z)} \otimes \varphi^* O_{\mathbb{P}_N}(k)) = \psi_*(j_* O_{\varphi^{-1}(Z)} \otimes \varphi^* O_{\mathbb{P}_N}(k))$$

with $\varphi^{-1}(Z) \xrightarrow{j} \Gamma$ the natural injection. Since $\varphi^* O_{\mathbb{P}_N}(k)$ is locally free, it becomes

$$\mathcal{F} = \psi_* j_* (O_{\varphi^{-1}(Z)} \otimes j^* \varphi^* O_{\mathbb{P}_N}(k)) = i_* \nu_* (\varphi^* O_{\mathbb{P}_N}(k)_{|\varphi^{-1}(Z)})$$

with $\Sigma \xrightarrow{i} G(q, V)$ the natural injection and ν the modification $\varphi^{-1}(Z) \xrightarrow{\psi} \Sigma$. For $s \in \Sigma$ generic, the intersection $Z \cap P(s)$ is a point $[x_s]$ and the fiber over s of the line bundle $\nu_*(\varphi^* O_{\mathbb{P}_N}(k)_{|\varphi^{-1}(Z)})$ appears as being

$$(\mathbb{C}x_s)^{-k} = H^0(P(s), O_{P(s)}(k)) / H^0(P(s), \mathcal{I}_{[x_s]} \cdot O_{P(s)}(k)).$$

Moreover, the short exact sequence

$$0 \to \mathcal{I}_{[x]} \cdot O_{P(s)}(k) \to O_{P(s)}(k) \to O_{P(s)}(k) / (\mathcal{I}_{[x]} \cdot O_{P(s)}(k)) \to 0$$

implies in cohomology the exact sequence as follows:

$$0 \to H^0(P(s), \mathcal{I}_{[x]} \cdot O_{P(s)}(k)) \to H^0(P(s), O_{P(s)}(k)) \to H^0(P(s), O_{P(s)}(k) / (\mathcal{I}_{[x]} \cdot O_{P(s)}(k))) \to 0$$

in cohomology because $H^1(P(s), \mathcal{I}_{[x]} \cdot O_{P(s)}(k)) = \{0\}$ by ampleness. We conclude that

$$H^0(P(s), O_{P(s)}(k) / (\mathcal{I}_{[x_s]} \cdot O_{P(s)}(k))) = \mathcal{F}_s \otimes_{O_{G(q,V),s}} \mathbb{C}. \qquad \square$$

2.2 Yang-Mills connections

We equip the holomorphic vector bundle E on $P(V) = \mathbb{P}_N$ with a Hermitian metric h and the line bundle $O_{\mathbb{P}_N}(k)$ with the Hermitian metric induced by the Hermitian scalar product on $V = \mathbb{C}^{N+1}$.

The induced Hermitian metric on $\psi_* \varphi^*(E \otimes O_{\mathbb{P}_N}(k))$ is defined in the following way. For $\tilde{u}$, $\tilde{v} \in H^0(P(s), (E \otimes O_{\mathbb{P}_N}(k))_{|P(s)})$, we denote by u, v the associated vectors in $\psi_* \varphi^* (E \otimes O_{\mathbb{P}_N}(k))_s$ and we set

$$\langle u, v \rangle = \int_{[x] \in P(s)} \langle \tilde{u}([x]), \tilde{v}([x]) \rangle \omega_{[x]}^{q-1}, \tag{4}$$

where ω is the Fubini-Study form on $P(V) = \mathbb{P}_N$.

We can express the above $\int_{P(s)}$ by writing a point in $P(s)$ as being $[x]$ with $x = t \cdot z = t_1 z^0 + \ldots + t_q z^{q-1}$ where $t \in \mathbb{S}^{2q-1}$ and $s = \operatorname{vect}(z^0, \ldots, z^{q-1})$. We obtain

$$\langle u, v \rangle = \alpha_q \Delta(z) \int_{t \in \mathbb{S}^{2q-1}} \langle \tilde{u}(t \cdot z), \tilde{v}(t \cdot z) \rangle \Phi(t),$$

where $\alpha_q = \frac{(q-1)!\, i^{q^2-1}}{(2\pi)^{q-1}}$ and $\Delta(z)$ is the Gram determinant of z, while

$$\Phi(t) = \frac{i}{4\pi} \sum_{1 \le k \le q} (-1)^{k-1} (t_k \mathrm{d}t_1 \wedge \ldots \wedge \widehat{\mathrm{d}t_k} \wedge \ldots \wedge \mathrm{d}t_q \wedge \mathrm{d}\bar{t}_1 \wedge \ldots \wedge \mathrm{d}\bar{t}_q + (-1)^q \bar{t}_k \mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_q \wedge \mathrm{d}\bar{t}_1 \wedge \ldots \wedge \widehat{\mathrm{d}\bar{t}_k} \wedge \ldots \wedge \mathrm{d}\bar{t}_q).$$

Let us now express the Chern connection D on $F = \psi_* \varphi^* E$, where we implicitly replace E by $E(k) = E \otimes O_{\mathbb{P}_N}(k)$.

When W is an open subset of $G(q, V)$, to define a section $u \in H^0(W, F)$ that is to say a continuous section on W of the sheaf $O(F)$ is equivalent to define $\tilde{u}([x], s) \in E_{[x]}$ for every $([x], s) \in \psi^{-1}(W)$ satisfying $\tilde{u}([x], s)$ is holomorphic on $\psi^{-1}(W)$.

The $(1, 0)$-part of D can be calculated by means of the relation

$$(\partial \langle u, v \rangle)(\sigma) = \langle D'_\sigma u, v \rangle + \langle u, d''_{\bar{\sigma}} v \rangle$$

for $u, v \in O(F)(W)$ and $\sigma \in H^0(W, TG(q, V))$. In W, we have

$$\langle u, v\rangle = \left(\psi_{|\psi^{-1}(W)}\right)_*(f\varphi^*\omega^{q-1}),$$

where $f([x], s) = \langle\tilde{u}([x], s), \tilde{v}([x], s)\rangle$ for $([x], s) \in \psi^{-1}(W)$. So $\partial\langle u, v\rangle = (\psi_{|\psi^{-1}(W)})_*(\partial f \wedge \varphi^*\omega^{q-1})$, which gives

$$(\partial\langle u, v\rangle)_s(\sigma_s) = \int_{[x]\in P(s)} (\partial f \wedge \varphi^*\omega^{q-1})_{([x],s)}((\xi, \sigma_s), \dots),$$

where the lifting $\xi \in T_{[x]}P(V)$ is such that $(\xi, \sigma_s) \in T_{([x],s)}\Gamma$ for $[x] \in P(s)$. The above contraction is equal to

$$(\partial f_{([x],s)}(\xi, \sigma_s))\omega_{[x]}^{q-1} - \partial f_{([x],s)} \wedge \omega_{[x]}^{q-1}(\xi, \dots)$$

and its restriction to $T_{[x]}P(s)$ is equal to $A([x], s)\omega^{q-1}_{[x]|T_{[x]}P(s)}$ for some function $A([x], s)$.

To calculate $A([x], s)$, we choose $e_0, \dots, e_N$ an orthonormal basis of $V = \mathbb{C}^{N+1}$, and we assume that $\mathrm{vect}(e_0, \dots, e_{q-1}) = s$, and $e_0 = \frac{x}{\|x\|}$.

For $s \in W$, we set $i_s : P(s) = \psi^{-1}(s) \hookrightarrow \psi^{-1}(W) \subset \Gamma$ defined by $i_s([x]) = ([x], s)$. We use that $i_s^*\partial f = \partial i_s^* f = \partial f(\cdot, s)$ and by identifying $(e_j \bmod \mathbb{C}e_0) \otimes e_0^*$ with e_j, we obtain

$$\omega_{[x]}^{q-1} = \left(\frac{i}{2\pi}\sum_{1\le j\le N} e_j^* \wedge \bar{e}_j^*\right)^{q-1} = C\sum_{|I|=q-1} e_I^* \wedge \bar{e}_I^*$$

$$\omega_{[x]}^{q-1}(\xi, \dots) = C\sum_{1\le j\le N}\sum_{|J|=q-2} \xi_j e_J^* \wedge \bar{e}_{jJ}^*,$$

where $\xi = \sum_{1\le j\le N}\xi_j(e_j \bmod \mathbb{C}e_0) \otimes e_0^*$. Here, $C = \frac{i^{(q-1)^2}(q-1)!}{(2\pi)^{q-1}}$ and $I = (i_1, \dots, i_{q-1})$ with $1 \le i_1 < \dots < i_{q-1} \le N$, while $J = (j_1, \dots, j_{q-2})$ with $1 \le j_1 < \dots < j_{q-2} \le N$. In restriction to $T_{[x]}P(s)$, we have to take $jJ = \{1, \dots, q-1\}$ and conclude that

$$(\partial\langle u, v\rangle)_s(\sigma_s) = \int_{[x]\in P(s)} \partial f_{([x],s)}(\xi - (\mathrm{d}p_s)_{[x]}(\xi), \sigma_s)\omega_{[x]}^{q-1},$$

where $p_s : P(V) \longrightarrow P(V)$ is the projective map associated to the orthogonal projection of V onto s.

We set

$$X([x], s) = (\xi - (\mathrm{d}p_s)_{[x]}(\xi), \sigma_s) \in T_{([x],s)}\Gamma \tag{5}$$

since $T_{([x],s)}\psi^{-1}(s) \subset T_{([x],s)}\Gamma$, and we have

$$(\partial\langle u, v\rangle)_s(\sigma_s) = \int_{[x]\in P(s)} \left\{\langle\left(D'_{\varphi^*E}\right)_X\tilde{u}, \tilde{v}\rangle + \langle\tilde{u}, d''_X\tilde{v}\rangle\right\}\omega_{[x]}^{q-1} = \int_{[x]\in P(s)} \langle\left(D'_{\varphi^*E}\right)_X\tilde{u}, \tilde{v}\rangle\omega_{[x]}^{q-1}$$

when we take $\tilde{v}$ holomorphic on $\psi^{-1}(W)$.

We define

$$\Pi_E : C^\infty(\psi^{-1}(W), \varphi^*E) \to C^\infty(\psi^{-1}(W), \varphi^*E)$$

the linear operator corresponding for each $s \in W$ to the orthogonal projection

$$C^\infty(P(s), E) \to H^0(P(s), E) \subset C^\infty(P(s), E)$$

where we use the Hermitian scalar product (4) on $C^\infty(P(s), E)$. The above formula becomes

$$(\partial\langle u, v\rangle)_s(\sigma_s) = \int_{[x]\in P(s)} \langle(\Pi_E \circ \left(D'_{\varphi^*E}\right)_X)\tilde{u}, \tilde{v}\rangle\omega_{[x]}^{q-1}.$$

Proposition 3. *For all $u \in O(F)(W)$, we have*

$$(D'_\sigma u)^\sim = (\Pi_E \circ \big(D'_{\varphi^*E}\big)_X)\tilde{u} = (\Pi_E \circ (\varphi^* D'_E)_X)\tilde{u}$$

on $\psi^{-1}(W)$, where $\tilde{u} \in H^0(\psi^{-1}(W), \varphi^* E)$ corresponds to u.

Recall that we have a morphism $\psi^*\mathcal{F} = \psi^*\psi_*\varphi^*E \to \varphi^*E$. When $\mathcal{F} = O(F)$ is locally free, this morphism is nothing more than the evaluation morphism

$$F_s = H^0(P(s), E) \to E_{[x]}$$

for $x \in s$. We can assume that it is surjective, and so we obtain an injection $\varphi^*E^* \hookrightarrow \psi^*F^*$.

Corollary 2. *Let D_{F^*} be the Chern connection induced on F^*. Then, for every section $h \in C^\infty(\Gamma, \varphi^*E^*)$, we have $(\psi^* D_{F^*})(h) \in C^\infty_{1,0}(\Gamma, \varphi^*E^*)$.*

We calculate now the curvature Θ_F of the Chern connection $D = D_F$ by the formula

$$\Theta_F(\sigma_1, \overline{\sigma}_2)u = D_{\sigma_1} D_{\overline{\sigma}_2} u - D_{\overline{\sigma}_2} D_{\sigma_1} u - D_{[\sigma_1, \overline{\sigma}_2]} u$$

by assuming that $[\sigma_1, \overline{\sigma}_2] = 0$ and u is a local holomorphic section of F. Thus, we have

$$\Theta_F(\sigma_1, \overline{\sigma}_2)u = D'_{\sigma_1} d''_{\overline{\sigma}_2} u - d''_{\overline{\sigma}_2} D'_{\sigma_1} u = -d''_{\overline{\sigma}_2} D'_{\sigma_1} u.$$

Let $X_i \in T_{([x],s)}\Gamma / T_{([x],s)}\psi^{-1}(s)$ be such that $\mathrm{d}\psi_{([x],s)}(X_i) = \sigma_i$ and the equality (5) holds, then

$$\begin{aligned}(\Theta_F(\sigma_1, \overline{\sigma}_2)u)^\sim &= -d''_{\overline{X}_2}(\Pi_E(\varphi^* D_E)_{X_1}\tilde{u})\\ &= (\Pi_E \circ \Theta_{\varphi^*E}(X_1, \overline{X}_2))\tilde{u} + (\Pi_E \circ (\varphi^* D_E)_{[X_1, \overline{X}_2]})\tilde{u} + \big(\Pi_E d''_{\overline{X}_2} - d''_{\overline{X}_2}\Pi_E\big)(\varphi^* D_E)_{X_1}\tilde{u}.\end{aligned}$$

Since $\Pi_E^* \tilde{v} = \Pi_E \tilde{v} = \tilde{v}$, we obtain

$$\langle \Theta_F(\sigma_1, \overline{\sigma}_2)u, v\rangle = \int_{P(s)} \langle \Theta_{\varphi^*E}(X_1, \overline{X}_2)\tilde{u} + (\varphi^* D'_E)_{[X_1, \overline{X}_2]}\tilde{u} + \big(\Pi_E d''_{\overline{X}_2} - d''_{\overline{X}_2}\Pi_E\big)(\varphi^* D'_E)_{X_1}\tilde{u}, \tilde{v}\rangle \omega^{q-1},$$

where $\mathrm{d}\varphi_{([x],s)}(X_i) = \xi_i - (\mathrm{d}p_s)_{[x]}(\xi_i)$ with

$$\xi_i = -x^{-1} \otimes (\sigma_{i,s}(x) \bmod \mathbb{C}x) \in T_{[x]}P(V) = \mathrm{Hom}(\mathbb{C}x, V/\mathbb{C}x)$$

modulo $T_{[x]}P(s)$.

We consider the term involving the Lie bracket $[X_1, \overline{X}_2]$. We use homogeneous coordinates on $G(q, V)$, i.e., we write $s = \mathrm{vect}(z^0, \dots, z^{q-1})$, then, for $0 \le k, m \le q-1$, and $0 \le j, l \le N$, we choose

$$\begin{aligned}\sigma_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vect}(z^0, \dots, z^{k-1}, z^k + te_j, z^{k+1}, \dots, z^{q-1})_{|t=0} = \partial^k_j,\\ \overline{\sigma}_2 &= \overline{\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vect}(z^0, \dots, z^{m-1}, z^m + te_l, z^{m+1}, \dots, z^{q-1})_{|t=0}} = \overline{\partial}^m_l\end{aligned}$$

which gives

$$\begin{aligned}\xi_1 - (\mathrm{d}p_s)_{[x]}(\xi_1) &= (\text{component of } x \text{ on } z^k)((\mathrm{d}p_s)_{[x]} - \mathrm{id})(x^{-1} \otimes (e_j \bmod \mathbb{C}x)),\\ \overline{\xi_2 - (\mathrm{d}p_s)_{[x]}(\xi_2)} &= \overline{(\text{component of } x \text{ on } z^m)((\mathrm{d}p_s)_{[x]} - \mathrm{id})(x^{-1} \otimes (e_l \bmod \mathbb{C}x))}.\end{aligned}$$

Since σ_1 and σ_2 are holomorphic vector fields, we have $[(\xi_1, \sigma_1), \overline{(\xi_2, \sigma_2)}] = 0$. Then, $((\mathrm{d}p_s)_{[x]}(\xi_i), 0) \in T_{([x],s)}\psi^{-1}(s)$ and the $\psi^{-1}(s)$ is a C^∞-foliation of Γ. By involutivity, we have

$$[((\mathrm{d}p_s)_{[x]}(\xi_1), 0), \overline{((\mathrm{d}p_s)_{[x]}(\xi_2), 0)}] \in T_{([x],s)}\psi^{-1}(s) \oplus \overline{T_{([x],s)}\psi^{-1}(s)}.$$

Finally, the Lie brackets $[(\xi_1, \sigma_1), \overline{((\mathrm{d}p_s)_{[x]}(\xi_2), 0)}]$ and $[((\mathrm{d}p_s)_{[x]}(\xi_1), 0), \overline{(\xi_2, \sigma_2)}]$ belong to the space $T_{([x],s)}\psi^{-1}(s) \oplus \overline{T_{([x],s)}\psi^{-1}(s)}$, since each $(\mathrm{d}p_s)_{[x]}(\xi_i)$ decomposes on holomorphic sections of $T_{([x],s)}\psi^{-1}(s)$. So we conclude that

$$[X_1, \overline{X}_2] \in T_{([x],s)}\psi^{-1}(s) \oplus \overline{T_{([x],s)}\psi^{-1}(s)}.$$

On another hand, we set $Y_i([x], s) = \mathrm{d}\varphi_{([x],s)}(X_i([x], s))$ and use that

$$Y_i([x], s) \in T_{[x]}P(\mathbb{C}x \oplus s^\perp) \subset (\varphi^* TP(V))_{([x],s)}.$$

We have $(\mathrm{d}\varphi)([X_1, \overline{X}_2]) = [Y_1, \overline{Y}_2]$. We fix $W \in G(p+2, V)$ and set $\mathcal{F}_W = \{([x], s) \in \Gamma, \mathbb{C}x \oplus s^\perp = W\}$. For all $([x], s) \in \mathcal{F}_W$, we have $Y_i([x], s) \in T_{[x]}P(W)$ and the $\mathcal{F}_W$ is a foliation of Γ, so $[Y_1, \overline{Y}_2]_{([x],s)} \in T_{[x]}P(W) \oplus \overline{T_{[x]}P(W)}$ for all $([x], s) \in \mathcal{F}_W$. In other words, we obtain

$$[X_1, \overline{X}_2] \in T_{[x]}P(W) \oplus \overline{T_{[x]}P(W)},$$

which implies that $[X_1, \overline{X}_2] = 0$.

Finally, a transposition in the last term yields that the curvature $\Theta_F \in C^\infty_{1,1}(G(q, V), \operatorname{End} F)$ has the following expression:

$$\langle \Theta_F(\sigma_1, \overline{\sigma}_2)u, v\rangle = \int_{P(s)} \left\{ \langle \Theta_{\varphi^* E}(X_1, \overline{X}_2)\tilde{u}, \tilde{v}\rangle - \langle (\varphi^* D'_E)_{X_1}\tilde{u}, (I - \Pi_E)(\varphi^* D'_E)_{X_2}\tilde{v}\rangle \right\} \omega^{q-1}. \tag{6}$$

In Section 2.3, we calculate, in particular, $\Theta_{L_E} \in C^\infty_{1,1}(G(q, V))$ for some holomorphic line bundle $L_E \subset \psi_* \varphi^*(\bigwedge^{r-q+1} E)$ over $G(q, V)$, when E is a Hermitian holomorphic vector bundle over $P(V)$ of rank r.

2.3 Chow transform of a Chern class

Let $X \subset \mathbb{P}_N$ be a complex projective manifold of dimension d_X and let T be a smooth differential form on X of bidimension (p, p) with $p = d_X - q$. The Chow transform $\hat{C}(T)$ of T is a current of bidegree $(1, 1)$ on the space $C_{q-1}(X)$ of effective algebraic cycles in X of dimension $q - 1$, obtained in the following way. Let $\hat{\Gamma}$ be the incidence variety, i.e., the subvariety in $X \times C_{q-1}(X)$ of (x, c) satisfying $c \ni x$. With $\hat{\varphi} : \hat{\Gamma} \to X$ and $\hat{\psi} : \hat{\Gamma} \to C_{q-1}(X)$, the restrictions to $\hat{\Gamma}$ of the canonical projections, we set

$$\hat{C}(T) = \hat{\psi}_* \hat{\varphi}^* T.$$

The fact that $\hat{C}(T)$ is of bidegree $(1, 1)$ means that it is of bidegree $(1, 1)$ on each irreducible component M of $C_{q-1}(X)$, which covers X.

When E is a Hermitian holomorphic vector bundle over X of rank $r_E = r$, we denote by

$$\Theta_E \in C^\infty_{1,1}(X, \operatorname{End} E)$$

the differential $(1, 1)$-form of curvature of E. With $c(E)$, the total Chern class of E, whose component of bidegree (q, q) is $c_q(E)$, we calculate the Chow transform $\hat{\psi}_* \hat{\varphi}^* c_q(\Theta_E)$ in terms of the Penrose transform $\hat{\psi}_* \hat{\varphi}^*(\bigwedge^{r-q+1} E)$, which is a holomorphic vector bundle over $C_{q-1}(X)$.

Proposition 4. *For E, a Hermitian holomorphic vector bundle over X, we have $\hat{\psi}_* \hat{\varphi}^* c_q(\Theta_E) = c_1(\Theta_{L_E})$ with some singular Hermitian metric on a holomorphic line bundle $L_E \subset \hat{\psi}_* \hat{\varphi}^*(\bigwedge^{r_E - q+1} E)$ over the cycle space $C_{q-1}(X)$.*

Proof. We set $L = O_{\mathbb{P}_N}(1)_{|X}$, then by replacing E by $E \otimes L^k$ with k large enough, we can assume that there are sections $f_0, \dots, f_{r-q}$ in $H^0(X, E)$ such that $c_q(E) = \{Z\}$, where

$$Z = \{[x] \in X, f_0([x]), \dots, f_{r-q}([x]) \text{ linearly dependent}\}$$

is the degeneracy locus. So Z is the set of zeroes of the section

$$f_0 \wedge \ldots \wedge f_{r-q} \in H^0\left(X, \bigwedge^{r-q+1} E\right),$$

which induces a surjective morphism $\bigwedge^{r-q+1} E^* \to \mathcal{I}_Z$ equivalent to an injective morphism $\mathcal{I}_Z^\vee \to \bigwedge^{r-q+1} E$. Furthermore, we obtain an injective morphism

$$\hat{\psi}_* \hat{\varphi}^*(\mathcal{I}_Z^\vee) \to \hat{\psi}_* \hat{\varphi}^*\left(\bigwedge^{r-q+1} E\right),$$

which is equivalent to a surjective morphism $(\hat{\psi}_* \hat{\varphi}^*(\bigwedge^{r-q+1} E))^* \to (\hat{\psi}_* \hat{\varphi}^*(\mathcal{I}_Z^\vee))^\vee$.

We denote by $D_Z = \hat{C}(Z) = \hat{\psi}_* \hat{\varphi}^* Z \subset C_{q-1}(X)$ the Chow divisor of Z, in such a way that $\hat{\psi}_* \hat{\varphi}^* \mathcal{I}_Z = \mathcal{I}_{D_Z}$. In the derived category, we have (see [16])

$$(\hat{\psi}_* \hat{\varphi}^*(\mathcal{I}_Z^\vee))^\vee = \hat{\psi}_* \hat{\varphi}^*(\mathcal{I}_Z) = \mathcal{I}_{D_Z}.$$

Thus, there is a section g of $\hat{\psi}_* \hat{\varphi}^*(\bigwedge^{r-q+1} E)$, whose set of zeroes is equal to D_Z. But the section $g \in H^0(C_{q-1}(X), \mathcal{O}(D_Z))$, therefore $\mathcal{O}(D_Z) \subset \hat{\psi}_* \hat{\varphi}^*(\bigwedge^{r-q+1} E)$. Since $\mathcal{O}(D_Z)$ is continuous with respect to Z, it is independent of Z, so $\mathcal{O}(D_Z) = L_E$.

By [9,10], we can approximate the closed differential (q, q)-form $c_q(\Theta_E)$ by rational algebraic cycles Z_j of codimension q in X cohomologous to $c_q(E)$. Then, the divisor $D_{Z_j} = \hat{\psi}_* \hat{\varphi}^* Z_j$ in $C_{q-1}(X)$ is the set of zeroes of a holomorphic section g_j of $\mathcal{O}(D_j)$, thus

$$\left[D_{Z_j}\right] = c_1\left(\Theta_{\mathcal{O}(D_j)}\right)$$

for some singular Hermitian metric on $\mathcal{O}(D_j)$ depending on g_j. By taking the limit when $j \to \infty$, we obtain $\hat{\psi}_* \hat{\varphi}^* c_q(\Theta_E) = c_1(\Theta_{L_E})$ for some singular Hermitian metric on L_E. □

We now prove the injectivity of the Chow transformation $\hat{C} = \hat{\psi}_* \hat{\varphi}^*$ acting on the (q, q)-currents on X. Let d_M be the dimension of an irreducible component M of $C_{q-1}(X)$ and for $[x] \in X$ generic, let

$$d = \dim\{c \in M, c \ni [x]\}$$

be the dimension of the fiber $M \cap \hat{\varphi}^{-1}([x])$. Assume that the family of cycles $c \in M$ covers X, then $d + d_X = d_M + q - 1$. On M, we define first $\Omega = \hat{C}(\omega_{|X}^q) = \hat{\psi}_* \hat{\varphi}^*(\omega_{|X}^q)$, where $\omega_{|X} = c_1(\Theta_L) = c_1\left(\Theta_{\mathcal{O}_{\mathbb{P}_N}(1)}\right)_{|X}$. Then, for Θ a $(1, 1)$-current on M, we define the dual integral transform:

$$\hat{C}_M^*(\Theta) = \hat{\varphi}_* \hat{\psi}^*\left(\Theta \wedge \Omega_{|M}^{d_M - 2}\right),$$

which is a current of bidegree $(d_X - q, d_X - q)$ on X.

Theorem 1. *There is an irreducible component M of $C_{q-1}(X)$ such that the transform*

$$\hat{C}_M^* \hat{C}_{|M} : \{(q, q)\text{-}currents\ on\ X\} \to \{(d_X - q, d_X - q)\text{-}currents\ on\ X\}$$

is invariant by transposition and satisfies

$$\hat{C}_M^*(\hat{C}(T)_{|M}) = 0 \Rightarrow T = 0$$

for all (q, q)-currents T on X. Then, a left inverse of $\hat{C}_M^* \hat{C}_{|M}$ is a linear differential operator $\hat{\mathcal{P}}_M$, which does not commute to ∂ nor $\bar{\partial}$ in general.

Proof. We assume that $(\hat{C}_M^* \hat{C}_{|M})(T) = 0$. When u is a smooth differential (q, q)-form on X, we have first

$$\langle (\hat{C}_M^* \hat{C}_{|M})(T), u \rangle = \int_M \hat{C}(T) \wedge \hat{C}(u) \wedge \Omega^{d_M - 2} = \langle T, (\hat{C}_M^* \hat{C}_{|M})(u) \rangle. \tag{7}$$

For an algebraic subset $Z \subset X$ of pure dimension $d_X - q$ and a $f \in C^\infty(Z)$, we take further $u = f[Z]$ and we use that

$$\hat{C}(f[Z]) = \hat{\psi}_*\hat{\varphi}^*(f[Z]) = \hat{\psi}_*((f \circ \hat{\varphi})[\hat{\varphi}^{-1}Z]) = (f \circ \hat{\varphi} \circ \hat{\nu}^{-1})[D_Z],$$

where $\hat{\nu} : \hat{\varphi}^{-1}Z \xrightarrow{\hat{\psi}} D_Z$ is a modification and $Z \cap c$ is a point $\hat{\pi}(c) = (\hat{\varphi} \circ \hat{\nu}^{-1})(c)$ for $c \in D_Z$ generic. In other words, $\hat{C}(f[Z]) = (f \circ \hat{\pi})[D_Z]$, which gives

$$\langle (\hat{C}^*_M \hat{C}_{|M})(T), u\rangle = \int_Z f\Big(\hat{\pi}_{|D_Z \cap M}\Big)_*\big(\big(\hat{C}(T) \wedge \Omega^{d_M - 2}\big)_{|D_Z\cap M}\big).$$

We obtain the condition

$$\Big(\hat{\pi}_{|D_Z \cap M}\Big)_*\big(\big(\hat{C}(T) \wedge \Omega^{d_M-2}\big)_{|D_Z \cap M}\big) = 0 \quad \forall Z, \tag{8}$$

which implies, for all Z, the condition

$$(\hat{C}(T) \wedge \Omega^{d_M-2} \wedge [D_Z])_{|M} = 0.$$

We arrive at the conclusion that $(\hat{C}(T) \wedge \Omega^{d_M-2})_{|M} = 0$. Thanks to the Lefschetz isomorphism, this implies that $\hat{C}(T) = 0$ on M, and thus $T = 0$ if M is such that $\hat{C}_{|M}$ is injective.

In fact, we take $T = \sum_k f_k[Z_k]$, where $f_k \in C^\infty(Z_k)$. At the same time, we take $Z = Z_l$ with a fixed index l. With $g_k = f_k \circ \hat{\pi}_k$, we have $\hat{C}(T) = \sum_k g_k[D_{Z_k}]$ and the condition (8) becomes

$$\Big(\hat{\pi}_{l|D_{Z_l}\cap M}\Big)_*\left(\left(\sum_k g_k\big[D_{Z_k}\big] \wedge \Omega^{d_M-2}\right)_{|D_{Z_l}\cap M}\right) = 0.$$

The restriction $[D_{Z_l}]_{|D_{Z_l}}$ can be assumed smooth, in the cohomology class of $\{D_{Z_l}\}_{|D_{Z_l}} = \hat{C}(\{Z_l\})_{|D_{Z_l}}$. On another hand, $[D_{Z_k}]_{|D_{Z_l}} = [D_{Z_k} \cap D_{Z_l}]$ has a singular support for $k \neq l$. We obtain $f_l = 0$ when $(\hat{C}^*_M\hat{C}_{|M})(T) = 0$, so $T = 0$. In other words, $(\hat{C}^*_M\hat{C}_{|M})$ is injective, in restriction to the $\sum_k f_k[Z_k]$.

Furthermore, we have the injectivity in the space of all currents of bidimension $(d_X - q, d_X - q)$ on X, thanks to an argument of density. □

So we can choose M such that the relation $\mathrm{id} = \hat{\mathcal{P}}_M\hat{C}^*_M\hat{C}_{|M}$ is satisfied. Then, $\hat{C}^*_M\hat{C}_{|M}$ is injective thus bijective, since this transformation is invariant by transposition by (7).

But there is no corollary in cohomology, since $\hat{\mathcal{P}}_M$ does not commute to ∂ or $\bar{\partial}$ in general. We have only the following property, with respect to the algebraic cohomology. Denote by $H^{p,p}_{\mathrm{alg}}(X) \subset H^{p,p}(X)$ the subspace of the cohomology classes of algebraic cycles of X of codimension p with complex coefficients (see [12,20,28]).

Proposition 5. *The transformation* $\hat{C} = \hat{\psi}_*\hat{\varphi}^*$ *is injective from* $H^{p,p}_{\mathrm{alg}}(X)^*$ *to* $\oplus_i H^{d_{M_i}-1,d_{M_i}-1}_{\mathrm{alg}}(M_i)^*$, *where* M_i *is the family of the irreducible components of* $C_{q-1}(X)$ *with* $p = d_X - q$.

Proof. We prove that the transformation

$$\hat{\varphi}_*\hat{\psi}^* : \bigoplus_i H^{d_{M_i}-1,d_{M_i}-1}_{\mathrm{alg}}(M_i) \to H^{p,p}_{\mathrm{alg}}(X)$$

is surjective. Let $Y \subset X$ be an irreducible algebraic subset of dimension $q = d_X - p$. We write $Y = \bigcup_H (Y \cap H)$ where H belongs to the set of all algebraic hypersurfaces of X. With $c = Y \cap H \in C_{q-1}(X)$, the irreducible component M of c in $C_{q-1}(X)$ does not depend on H when H varies continuously. Then $Y = \bigcup_{c\in C} c$ where C is an algebraic curve in M, in other words $Y = \hat{\varphi}(\hat{\psi}^{-1}(C)) \Leftrightarrow [Y] = \hat{\varphi}_*\hat{\psi}^*([C])$. □

3 Chow transformation of currents on $\mathbb{P}_N$ and sheaves of left modules over the ring of real-analytic linear differential operators

Given T a current on $\mathbb{P}_N = P(V)$ of bidegree (q, q), we set

$$C(T) = \psi_* \varphi^* T = \mathrm{pr}_{2*}([\Gamma] \wedge \mathrm{pr}_1^* T)$$

as being the Chow transform of T. Then, $C(T)$ is a current of bidegree $(1, 1)$ on $\mathbb{G}_{q-1,N} = G(q, V)$.

3.1 Characterization by PDE of the image of the Chow transformation of currents

We obtain, by slicing and projecting, the linear differential equations characterizing the images by the Chow transformation of the (q, q)-currents on $\mathbb{P}_N$. In this way, we obtain a linear differential system of order lower than that of the Gelfand-Gindikin-Graev differential system.

If $\Theta = C(u)$ is a smooth differential $(1, 1)$-form on $G(q, V)$, which is in im C, then a first condition is that for all $W \in G(q + 1, V)$, the restriction $\Theta_{|P(W^*)}$ of Θ to $P(W^*) = G(q, W) \subset G(q, V)$ is closed, i.e.,

$$(\mathrm{d}\Theta)_{|P(W^*)} = 0. \tag{9}$$

This follows from the slicing formula

$$C(u)_{|P(W^*)} = C_W(u_{|P(W)}) \tag{10}$$

for all $W \in G(q + 1, V)$, with C_W, the Chow transformation on $P(W)$. In effect, since $u_{|P(W)}$ is of bidegree (q, q) on $P(W)$, thus closed, $\Theta_{|P(W^*)}$ is closed too.

Conversely, if this condition is satisfied, then $\Theta_{|P(W^*)} \in \mathrm{im}\, C_W$. Then, a second condition is that $C_W^{-1}(\Theta_{|P(W^*)})$ is the restriction to W of a smooth differential (q, q)-form independent of $W \in G(q + 1, V)$.

Third, a projection argument allows us to complete the characterization of smooth differential $(1, 1)$-forms Θ on $G(q, V)$, which are in im C.

In effect, for W' a vectorial subspace of V of dimension $q - 2$, we set

$$G_{W'} = \{s \in G(q, V), W' \subset s\} \simeq G(2, V / W')$$

and $\tilde{C}_{W'}$ denotes the Chow transformation acting on currents in $P(V / W')$. We denote by $\pi_{W'} : V \longrightarrow V / W'$ the vectorial projection and by $\tilde{\pi}_{W'} : P(V) \longrightarrow P(V / W')$ the induced projective map.

When u is a differential form of bidegree (q, q) of class C^∞ in $P(V)$, we have the equality

$$C(u)_{|G_{W'}} = \tilde{C}_{W'}(\tilde{\pi}_{W'*} u).$$

The direct image $\tilde{\pi}_{W'*} u$ is calculated by integrating u along the fibers of $\tilde{\pi}_{W'}$ and is a smooth differential $(2, 2)$-form in $P(V / W')$.

Proposition 6. *A smooth differential* $(1, 1)$*-form* Θ *on* $G(q, V)$ *is the Chow transform of a smooth differential* (q, q)*-form on* $P(V)$*, when*
- (*i*) *the condition* $\mathrm{d}\Theta_{|G(q,W)} = \mathrm{d}\Theta_{|P(W^*)} = 0$ *is satisfied for all* $W \in G(q + 1, V)$,
- (*ii*) *there is a smooth differential* (q, q)*-form* u *on* $P(V)$ *such that* $\Theta_{|P(W^*)} = C_W(u_{|P(W)})$ *for all* $W \in G(q + 1, V)$,
- (*iii*) *the restriction* $\Theta_{|G_{W'}} \in \mathrm{im}\, \tilde{C}_{W'}$ *for all* $W' \in G(q - 2, V)$.

Proof. We have $(\Theta - C(u))_{|P(W^*)} = 0$ and $(\Theta - C(u))_{|G_{W'}} \in \mathrm{im}\, \tilde{C}_{W'}$. By replacing Θ by $\Theta - C(u)$, it remains to prove that

$$\Theta_{|P(W^*)} = 0 \quad \text{and} \quad \Theta_{|G_{W'}} \in \mathrm{im}\, \tilde{C}_{W'} \Rightarrow \Theta = 0.$$

So assume $\Theta_{|G_{W'}} = \tilde{C}_{W'}(u_{W'})$ with $u_{W'}$ of bidegree $(2, 2)$ in $P(V / W') = \mathbb{P}_{N-q+2}$. For $W' \subset W$, we take the restriction to

$$\{s \in G(q,V),\, W' \subset s \subset W\} = G(2, W/W') = G(2,\mathbb{C}^3) = \mathbb{P}_2^*,$$

and we obtain

$$\tilde{C}_{W'}(u_{W'})_{|\mathbb{P}_2^*} = (\Theta_{|P(W^*)})_{|\mathbb{P}_2^*} = 0.$$

Therefore, $u_{W'|P(W/W')} = 0$ and $u_{W'} = 0$. We conclude that $\Theta_{|G_{W'}} = 0$. For $s = \mathrm{vect}(z^0,\dots,z^{q-1})$ and $0 \le k$, $m \le q-1$, we choose $W' = \mathrm{vect}(z^0,\dots,\widehat{z^k},\dots,\widehat{z^m},\dots,z^{q-1})$ and conclude that $\Theta_s(\sigma_1,\overline{\sigma}_2) = 0$ with

$$\sigma_1 = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vect}(z^0,\dots,z^{k-1}, z^k + te_j, z^{k+1},\dots,z^{q-1})_{|t=0} = \partial_j^k,$$
$$\overline{\sigma}_2 = \overline{\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vect}(z^0,\dots,z^{m-1}, z^m + te_l, z^{m+1},\dots,z^{q-1})_{|t=0}} = \overline{\partial}_l^m$$

where $e_0,\dots,e_N$ is a basis of V. In other words, $\Theta_s = 0$ in homogeneous coordinates. □

Denote by $\tau : V^q \longrightarrow G(q,V)$ the map that associates with $z = (z^0,\dots,z^{q-1})$ in the product the vector subspace $\tau(z) = \mathrm{vect}(z^0,\dots,z^{q-1})$. Assume that the smooth differential (1, 1)-form Θ on $G(q,V)$ is given in homogeneous coordinates as follows:

$$\tau^*\Theta = \sum_{\substack{0\le k,m\le q-1\\ 0\le n,l\le N}} \Theta_{nl}^{km}\,\mathrm{d}z_n^k \wedge \mathrm{d}\overline{z}_l^m.$$

Recall the linear differential equations in the coefficients of Θ that are equivalent to condition (i) of Proposition 6 and are given in [14,15]. These are

$$\partial_{j'}^{k'}\Theta_{jl}^{km} - \partial_j^k\Theta_{j'l}^{k'm} = \partial_{j'}^k\Theta_{jl}^{k'm} - \partial_j^{k'}\Theta_{j'l}^{km}$$

and

$$\overline{\partial}_{l'}^{m'}\Theta_{jl}^{km} - \overline{\partial}_l^m\Theta_{jl'}^{km'} = \overline{\partial}_{l'}^m\Theta_{jl}^{km'} - \overline{\partial}_l^{m'}\Theta_{jl'}^{km}$$

for $0 \le k, k', m, m' \le q-1$, and $0 \le j, j', l, l' \le N$.

In effect, we have

$$\tau^*\mathrm{d}\Theta = \sum \partial_{j'}^{k'}\Theta_{jl}^{km}\mathrm{d}z_{j'}^{k'} \wedge \mathrm{d}z_j^k \wedge \mathrm{d}\overline{z}_l^m + \sum \overline{\partial}_{l'}^{m'}\Theta_{jl}^{km}\mathrm{d}z_j^k \wedge \mathrm{d}\overline{z}_l^m \wedge \mathrm{d}\overline{z}_{l'}^{m'}.$$

We take $W = \mathrm{vect}(a, z^0,\dots,z^{q-1})$ with $a \in V\backslash\{0\}$ in such a way that an hyperplane of W can be written as $\mathrm{vect}(t_k a + z^k)_{0\le k\le q-1}$ with $(t_k) \in \mathbb{C}^q$. In the inverse image in $\mathbb{C}^q$ of $\tau^*\mathrm{d}\Theta$ at the point 0, the coefficient of $\mathrm{d}t_{k'} \wedge \mathrm{d}t_k \wedge \mathrm{d}\overline{t}_m$ is

$$\sum_{j',j,l} (\partial_{j'}^{k'}\Theta_{jl}^{km} - \partial_{j'}^k\Theta_{jl}^{k'm})a_{j'}a_j\overline{a}_l,$$

while the coefficient of $\mathrm{d}t_k \wedge \mathrm{d}\overline{t}_m \wedge \mathrm{d}\overline{t}_{m'}$ is

$$\sum_{j,l,l'} (\overline{\partial}_{l'}^{m'}\Theta_{jl}^{km} - \overline{\partial}_{l'}^m\Theta_{jl}^{km'})a_j\overline{a}_l\overline{a}_{l'}.$$

When the smooth differential (1, 1)-form Θ on $G(q,V)$ satisfies the property (i) of the Proposition 6, then the function

$$C_{jI,lJ} = \frac{1}{((q-1)!)^2}\sum_{0\le k,m\le q-1}(-1)^{k+m}\partial_I^{\hat{k}}\overline{\partial}_J^{\hat{m}}\Theta_{jl}^{km} \tag{11}$$

depends only on jI and lJ. We have denoted here by $\partial_I^{\hat{k}}$ the linear differential operator

$$\det\left(\frac{\partial}{\partial z_j^{k'}}\right)_{\substack{k'\neq k\\ j\in I}}$$

for $|I| = |J| = q - 1$. For the proof, we write

$$\sum_{0\le k\le q-1} (-1)^k \partial_I^{\hat{k}} \Theta_{jl}^{km} = (-1)^{q-1} \sum_{0\le\sigma_1,\dots,\sigma_{q-1},k\le q-1} \varepsilon(\sigma_1,\dots,\sigma_{q-1},k)\partial_{i_1}^{\sigma_1}\dots\partial_{i_{q-1}}^{\sigma_{q-1}}\Theta_{jl}^{km}$$

and we use

$$\partial_{i_\alpha}^{\sigma_\alpha}\Theta_{jl}^{km} - \partial_{i_\alpha}^{k}\Theta_{jl}^{\sigma_\alpha m} = \partial_j^k \Theta_{i_\alpha l}^{\sigma_\alpha m} - \partial_j^{\sigma_\alpha}\Theta_{i_\alpha l}^{km}$$

for $1 \le \alpha \le q - 1$.

The Plücker coordinates P_I of $z = (z^0, \dots, z^{q-1}) \in V^q$ are defined by

$$z^0 \wedge \dots \wedge z^{q-1} = \sum_{|I|=q} P_I e_I \in \bigwedge^q V,$$

where $e_I = e_{i_1} \wedge \dots \wedge e_{i_q}$ for $I = (i_1, \dots, i_q)$ with $0 \le i_1 < \dots < i_q \le N$.

Lemma 1. *If* Θ *satisfies the property* (i) *of the Proposition* 6, *then we have the decomposition*

$$\tau^*\Theta = \sum_{|I|=|J|=q} C_{I,J} \mathrm{d}P_I \wedge \mathrm{d}\overline{P}_J,$$

where the coefficients $C_{I,J}$ *are smooth functions of* z *satisfying the differential equations*

$$\sum_{\substack{0\le k\le q-1\\0\le j\le N}} z_j^k \partial_{j'}^k C_{jI',J} + C_{j'I',J} = 0 = \sum_{\substack{0\le k\le q-1\\0\le j\le N}} \overline{z}_j^k \overline{\partial}_{j'}^k C_{I,jJ'} + C_{I,j'J'}$$

and the homogeneity property $C_{I,J}(A.\,z) = |\det A|^{-2} C_{I,J}(z)$ *for all* $A \in \mathrm{GL}_q(\mathbb{C})$.

Proof. See [30]. □

For u, a smooth differential (q, q)-form on $\mathbb{P}_N$, the inverse image of u by the canonical map $\pi : V \longrightarrow P(V)$ is written as follows:

$$\pi^* u = \sum_{|I|=|J|=q} u_{I,J} \mathrm{d}x_I \wedge \mathrm{d}\overline{x}_J,$$

with x_j the coordinates of $x \in V$ with respect to an orthonormal basis $e_0, \dots, e_N$ of V, and with smooth functions $u_{I,J}$ on $V\backslash\{0\}$ satisfying the property of homogeneity $u_{I,J}(\lambda x) = |\lambda|^{-2q} u_{I,J}(x)$ for $\lambda \in \mathbb{C}^*$ and satisfying for $|I'| = |J'| = q - 1$, $|I| = |J| = q$ the relations

$$\sum_{0\le j\le N} x_j u_{jI',J}(x) = 0 = \sum_{0\le j\le N} \overline{x}_j u_{I,jJ'}(x)$$

that are equivalent to the fact that the radial contractions of $\pi^* u$ are 0. By differentiating, these relations are also equivalent to

$$\sum_{0\le j\le N} x_j \frac{\partial}{\partial x_{j'}} u_{jI',J}(x) + u_{j'I',J}(x) = 0 = \sum_{0\le j\le N} \overline{x}_j \frac{\partial}{\partial \overline{x}_{j'}} u_{I,jJ'}(x) + u_{I,j'J'}(x) \tag{12}$$

for $0 \le j' \le N$.

The inverse image by τ of the Chow transform $C(u)$ is then written as follows:

$$\tau^* C(u) = \sum_{\substack{0\le k,m\le q-1\\0\le n,l\le N}} C_{nl}^{km}(u) \mathrm{d}z_n^k \wedge \mathrm{d}\overline{z}_l^m.$$

The coefficients $C_{nl}^{km}(u)$ can be expressed (see [30]) by means of the $u_{I,J}$ in the following way:

$$C_{nl}^{km}(u)(z)=(-1)^{k+m}\sum_{|I|=|J|=q-1}\tilde{u}_{nI,lJ}(z)z_I^{\hat{k}}\overline{z}_J^{\hat{m}}$$

with $z_I^{\hat{k}}=\det\left(z_{i_\alpha}^{k'}\right)_{\substack{1\le\alpha\le q-1\\0\le k'\le q-1,k'\ne k}}$ and the transform

$$\tilde{u}_{nI,lJ}(z)=\int_{t\in\mathbb{S}^{2q-1}}u_{nI,lJ}(t\cdot z)\Phi(t),\tag{13}$$

where $t\cdot z=t_1z^0+\ldots+t_qz^{q-1}$ and

$$\Phi(t)=\frac{i}{4\pi}\sum_{1\le k\le q}(-1)^{k-1}(t_k\mathrm{d}t_1\wedge\ldots\wedge\widehat{\mathrm{d}t_k}\wedge\ldots\wedge\mathrm{d}t_q\wedge\mathrm{d}\bar{t}_1\wedge\ldots\wedge\mathrm{d}\bar{t}_q+(-1)^q\bar{t}_k\mathrm{d}t_1\wedge\ldots\wedge\mathrm{d}t_q\wedge\mathrm{d}\bar{t}_1\wedge\ldots\wedge\widehat{\mathrm{d}\bar{t}_k}\wedge\ldots\wedge\mathrm{d}\bar{t}_q).$$

In such a way, it becomes

$$\tau^*C(u)=\sum_{|I|=|J|=q}\tilde{u}_{I,J}\mathrm{d}P_I\wedge\mathrm{d}\overline{P_J}.$$

Set $f_{I,J}([x])=\|x\|^{2q}u_{I,J}(x)$ and let $\mathcal{R}_{q-1}(f_{I,J})$ be the Radon transform of $f_{I,J}$ obtained by integration of that function in the projective subspaces of $P(V)$ of dimension $q-1$. In addition (see [24–26]),

$$\mathcal{R}_{q-1}(f_{I,J})_{\tau(z)}=\alpha_q\tilde{u}_{I,J}(z)\Delta(z),$$

where $\alpha_q=\frac{(q-1)!\,i^{q^2-1}}{(2\pi)^{q-1}}$ and $\Delta(z)$ is the Gram determinant of z. The transforms $\tilde{u}_{I,J}$ satisfy the following properties.

With $A=(a_{k,m})_{0\le k,m\le q-1}\in GL_q(\mathbb{C})$, set

$$A.\,(z^0,\ldots,z^{q-1})=\left(\sum_{0\le k\le q-1}a_{k,0}z^k,\ldots,\sum_{0\le k\le q-1}a_{k,q-1}z^k\right).$$

Then, $\tilde{u}_{I,J}(A.\,z)=|\det A|^{-2}\tilde{u}_{I,J}(z)$.

On another hand, the relations (12) can be expressed as follows:

$$\sum_{\substack{0\le k\le q-1\\0\le j\le N}}z_j^k\partial_{j'}^k\tilde{u}_{jI',J}(z)+\tilde{u}_{j'I',J}(z)=0=\sum_{\substack{0\le k\le q-1\\0\le j\le N}}\overline{z}_j^k\overline{\partial}_{j'}^k\tilde{u}_{I,jJ'}(z)+\tilde{u}_{I,j'J'}(z)$$

for $0\le j'\le N$.

Finally, the $\tilde{u}_{I,J}$ satisfy the John linear differential equations

$$(\partial_j^k\partial_{j'}^{k'}-\partial_{j'}^k\partial_j^{k'})\tilde{u}_{I,J}=0=(\overline{\partial}_j^k\overline{\partial}_{j'}^{k'}-\overline{\partial}_{j'}^k\overline{\partial}_j^{k'})\tilde{u}_{I,J}.$$

Proposition 7. *A smooth differential* $(1,1)$*-form* Θ *on* $G(q,V)$ *is a Chow transform when* Θ *satisfies the property* (i) *of the Proposition 6 and the coefficients* $C_{I,J}$ *of the Lemma 1 satisfy the differential equations*

$$(\partial_j^k\partial_{j'}^{k'}-\partial_{j'}^k\partial_j^{k'})C_{I,J}=0=(\overline{\partial}_j^k\overline{\partial}_{j'}^{k'}-\overline{\partial}_{j'}^k\overline{\partial}_j^{k'})C_{I,J}.$$

Proof. If $\Theta=C(u)$, then $C_{I,J}=\tilde{u}_{I,J}$ by unicity. □

Recall now the linear differential equations equivalent to property (iii) of Proposition 6 that are given in [30]. To this hand, in Proposition 6, we take

$$W'=\mathrm{vect}(z^0,\ldots,\widehat{z^k},\ldots,\widehat{z^{k'}},\ldots,z^{q-1})$$

for $0\le k,k'\le q-1$ fixed and we use that the inverse image in V^2 of $\Theta_{|G_{W'}}$ is

$$\sum_{0\le j,l\le N}(\Theta_{jl}^{kk}dz_j^k\wedge d\bar z_l^k+\Theta_{jl}^{kk'}dz_j^k\wedge d\bar z_l^{k'}+\Theta_{jl}^{k'k}dz_j^{k'}\wedge d\bar z_l^k+\Theta_{jl}^{k'k'}dz_j^{k'}\wedge d\bar z_l^{k'})$$

so the expressions

$$\partial_i^{k'}\bar\partial_j^{k'}\Theta_{j''l''}^{kk}-\partial_i^{k'}\bar\partial_j^{k}\Theta_{j''l''}^{kk'}-\partial_i^{k}\bar\partial_j^{k'}\Theta_{j''l''}^{k'k}+\partial_i^{k}\bar\partial_j^{k}\Theta_{j''l''}^{k'k'}$$

should be annihilated by $\partial_l^k\partial_{l'}^{k'}-\partial_{l'}^k\partial_l^{k'}$ and $\bar\partial_l^k\bar\partial_{l'}^{k'}-\bar\partial_{l'}^k\bar\partial_l^{k'}$ by applying Proposition 7 for $q=2$.

Finally, mention that the linear differential equations equivalent to property (ii) of Proposition 6 are given explicitly in Proposition 4 of [33].

Our purpose for the rest of this subsection is to obtain the linear differential system characterizing the Chow transforms of the (q,q)-currents on $\mathbb{P}_N$ by using the theory of sheaves of left modules over the rings of real-analytic linear differential operators.

We denote by $\mathcal{A}_{\mathbb{P}_N}$ the sheaf of rings of real-analytic functions with complex values and $\mathcal{D}_{\mathbb{P}_N}$ the sheaf of rings of real-analytic linear differential operators on $\mathbb{P}_N$. Then, $\bigwedge^{q,q}T\mathbb{P}_N$ is a sheaf of $\mathcal{A}_{\mathbb{P}_N}$-modules and $\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\bigwedge^{q,q}T\mathbb{P}_N$ becomes a sheaf of left $\mathcal{D}_{\mathbb{P}_N}$-modules. The space of smooth differential (q,q)-forms on $\mathbb{P}_N$ can be written as follows:

$$C_{q,q}^\infty(\mathbb{P}_N)=H^0\Big(\mathbb{P}_N,\mathrm{Hom}_{\mathcal{D}_{\mathbb{P}_N}}\Big(\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\bigwedge^{q,q}T\mathbb{P}_N,C_{\mathbb{P}_N}^\infty\Big)\Big),$$

where the left $\mathcal{D}_{\mathbb{P}_N}$-module $C_{\mathbb{P}_N}^\infty$ is the sheaf of smooth functions on $\mathbb{P}_N$.

We retrieve the conditions on $\tilde u_{I,J}$ by transforming the left $\mathcal{D}_{\mathbb{P}_N}$-module $\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\bigwedge^{q,q}T\mathbb{P}_N$ on $\mathbb{P}_N$ into the left $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module

$$\psi_*\varphi^*\Big(\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\bigwedge^{q,q}T\mathbb{P}_N\otimes_{\mathcal{A}_{\mathbb{P}_N}}\mathcal{O}_{\mathbb{P}_N}(-1)\otimes_{\mathcal{A}_{\mathbb{P}_N}}\overline{\mathcal{O}_{\mathbb{P}_N}(-1)}\Big)$$

on $\mathbb{G}_{q-1,N}$. In order to calculate this transform, we use the inclusion

$$\lambda:\bigwedge^{q,q}T^*\mathbb{P}_N\hookrightarrow\bigoplus_{|I|=|J|=q}(\mathcal{O}_{\mathbb{P}_N}(-q)\otimes\overline{\mathcal{O}_{\mathbb{P}_N}(-q)})\otimes(e_I^*\wedge\bar e_J^*),$$

which associates to a differential (q,q)-form u the previous coefficients $(u_{I,J})$. It follows that $\bigwedge^{q,q}T\mathbb{P}_N$ is the quotient of $\oplus_{|I|=|J|=q}(\mathcal{O}_{\mathbb{P}_N}(q)\otimes\overline{\mathcal{O}_{\mathbb{P}_N}(q)})\otimes(e_I\wedge\bar e_J)$ by $\mathrm{Ker}\,{}^t\lambda$, where

$$(v_{I,J})\in\mathrm{Ker}\,{}^t\lambda\Leftrightarrow v_{I,J}=\sum_{iI'=I}x_ig_{I',J}+\sum_{jJ'=J}\bar x_jh_{I,J'}$$

for systems $(g_{I',J})$ and $(h_{I,J'})$ of functions. Furthermore, $\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\bigwedge^{q,q}T\mathbb{P}_N$ becomes the quotient of

$$\bigoplus_{|I|=|J|=q}(\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\mathcal{O}_{\mathbb{P}_N}(q)\otimes_{\mathcal{A}_{\mathbb{P}_N}}\overline{\mathcal{O}_{\mathbb{P}_N}(q)})\otimes(e_I\wedge\bar e_J)$$

by the submodule of $v_{I,J}$ such that $v_{I,J}=\sum_{iI'=I}g_{I',J}x_i+\sum_{jJ'=J}h_{I,J'}\bar x_j$ for systems $(g_{I',J})$ and $(h_{I,J'})$ of linear differential operators with real-analytic coefficients.

We set $f_{I,J}([x])=\|x\|^{2q}u_{I,J}(x)$ and associate to the differential (q,q)-form u the coefficients $(f_{I,J})$ instead of the $(u_{I,J})$. In addition, we have that $\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\bigwedge^{q,q}T\mathbb{P}_N$ is the quotient by a left submodule $\mathcal{S}$ of

$$\bigoplus_{|I|=|J|=q}\mathcal{D}_{\mathbb{P}_N}\otimes(e_I\wedge\bar e_J).$$

In such a way, we calculate first the left $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module

$$\psi_*\varphi^*(\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\mathcal{O}_{\mathbb{P}_N}(-1)\otimes_{\mathcal{A}_{\mathbb{P}_N}}\overline{\mathcal{O}_{\mathbb{P}_N}(-1)})=\mathrm{pr}_{2*}\Big\{i_*\mathcal{O}_\Gamma\otimes\mathrm{pr}_1^*(\mathcal{D}_{\mathbb{P}_N}\otimes_{\mathcal{A}_{\mathbb{P}_N}}\mathcal{O}_{\mathbb{P}_N}(-1)\otimes_{\mathcal{A}_{\mathbb{P}_N}}\overline{\mathcal{O}_{\mathbb{P}_N}(-1)})\Big\},$$

where $i:\Gamma\hookrightarrow\mathbb{P}_N\times\mathbb{G}_{q-1,N}$ is the inclusion and pr_{2*} is calculated in the sense of $\mathcal{D}$-modules. So we determine

$$\mathrm{pr}_{2*}\Big\{i_*\mathcal{O}_\Gamma \otimes \mathrm{pr}_1^*\mathcal{D}_{\mathbb{P}_N} \otimes \mathrm{pr}_2^*\mathcal{D}_{\mathbb{G}_{q-1,N}} \otimes \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)})\Big\}$$
$$= \mathrm{pr}_{2*}\Big\{i_*\mathcal{O}_\Gamma \otimes \mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}} \otimes \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)})\Big\},$$

where the left $\mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}$-module $i_*\mathcal{O}_\Gamma$ is defined in the following manner.

We identify $\mathbb{G}_{q-1,N} = G(q, V) = G(p+1, V^*)$, i.e., we define $s = \mathrm{vect}(z^0, \dots, z^{q-1}) \in \mathbb{G}_{q-1,N}$ by $0 = \xi^0 = \dots = \xi^p$ with $\mathrm{vect}(\xi^0, \dots, \xi^p) \subset V^*$ the polar subspace of s. We denote by

$$\mathcal{I} \subset \mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}$$

the left ideal of $\mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}$ generated, on the one hand, by the local holomorphic functions that define $\Gamma \subset \mathbb{P}_N \times \mathbb{G}_{q-1,N}$ and their conjugates, and on the other hand, by the local holomorphic tangent vectors to Γ and their conjugates. Since the inverse image of Γ in $V \times (V^*)^{p+1}$ is

$$\{(x, (\xi^0, \dots, \xi^p)) \in V \times (V^*)^{p+1}, \langle \xi^k, x\rangle = 0\},$$

the inverse image of $\mathcal{I}$ in $\mathcal{D}_V \otimes \mathcal{D}_{(V^*)^{p+1}}$ is

$$\tilde{\mathcal{I}} = \langle f, \overline{f}, X, \overline{X}\rangle,$$

where $f = \langle \xi^k, x\rangle$ and $X = \frac{\partial_j^k}{x_j} - \frac{\partial_{j'}^k}{x_{j'}}$ or $X = \frac{\partial_j^k}{x_j} - \frac{1}{\xi_{j'}^k}\frac{\partial}{\partial x_{j'}}$ or $X = \frac{1}{\xi_j^k}\frac{\partial}{\partial x_j} - \frac{1}{\xi_{j'}^k}\frac{\partial}{\partial x_{j'}}$ for $0 \le k \le p$ and $0 \le j, j' \le N$. Here, we have setted $\partial_j^k = \frac{\partial}{\partial \xi_j^k}$ when $\xi^k = \sum_{0\le j\le N}\xi_j^k e_j^* \in V^*$. Then, by definition

$$i_*\mathcal{O}_\Gamma = \mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}/\mathcal{I},$$

and when $W \subset \mathbb{G}_{q-1,N}$ is open, it remains to express

$$H^0(\mathbb{P}_N \times W, \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}/\mathcal{I}).$$

Lemma 2. *For every linear differential operator $P \in \mathcal{D}_{\mathbb{P}_N} \otimes \mathcal{D}_{\mathbb{G}_{q-1,N}}$ on $\mathbb{P}_N \times W$, there are real-analytic functions h_α on $\mathbb{P}_N$ and linear differential operators $Q_\alpha \in \mathcal{D}_{\mathbb{G}_{q-1,N}}(W)$, for α in an infinite set, such that $P - \sum_\alpha h_\alpha Q_\alpha \in \mathcal{I}(W)$.*

Proof. With $X = \frac{\partial_j^k}{x_j} - \frac{1}{\xi_{j'}^k}\frac{\partial}{\partial x_{j'}} \in \tilde{\mathcal{I}}$, we have $x_j\frac{\partial}{\partial x_{j'}} = \xi_{j'}^k\partial_j^k - x_j\xi_{j'}^k X \in \mathcal{D}_{\mathbb{G}_{q-1,N}} + \tilde{\mathcal{I}}$. Therefore, when $|\alpha| = |\beta|$, we have the same property

$$x_0^{\alpha_0}\dots x_N^{\alpha_N}\frac{\partial^{|\beta|}}{\partial x_0^{\beta_0}\dots\partial x_N^{\beta_N}} \in \mathcal{D}_{\mathbb{G}_{q-1,N}} + \tilde{\mathcal{I}}.$$ □

Note that on the open subset of $\mathbb{P}_N \times \mathbb{G}_{q-1,N}$ defined by $\langle \xi^k, x\rangle \ne 0$, the relation

$$\partial_j^k(\langle \xi^k, x\rangle u) = x_j u + \langle \xi^k, x\rangle\partial_j^k u = x_j u + \frac{\langle \xi^k, x\rangle}{\overline{\langle \xi^k, x\rangle}}\partial_j^k(\overline{\langle \xi^k, x\rangle}u)$$

provides $x_j \in \tilde{\mathcal{I}} + \langle \xi^k, x\rangle(\overline{\langle \xi^k, x\rangle})^{-1}\tilde{\mathcal{I}}$, but the function $\langle \xi^k, x\rangle(\overline{\langle \xi^k, x\rangle})^{-1}$ is only bounded.

We use the morphism

$$H^0\big(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}\big) \otimes \mathcal{D}_{\mathbb{G}_{q-1,N}}(W) \to H^0(\mathbb{P}_N \times W, \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}/\mathcal{I}),$$

which is obtained by associating to $Q \in \mathcal{D}_{\mathbb{G}_{q-1,N}}(W)$, the section of

$$\mathrm{Hom}(\mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(1)}), \mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}})$$

on $\mathbb{P}_N \times W$ defined by $\lambda \otimes \overline{\mu} \to \lambda\overline{\mu}Q$ when λ and μ are linear forms on $V = \mathbb{C}^{N+1}$.

Let $E_{jj'}^{kk'}$ be the real-analytic linear differential operators on $\mathbb{G}_{q-1,N}$ such that

$$\Delta(z)(\partial_j^k\partial_{j'}^{k'} - \partial_{j'}^k\partial_j^{k'})(\Delta(z)^{-1}F(\tau(z))) \Leftrightarrow E_{jj'}^{kk'}(F)(\tau(z)) = 0$$

for all $z \in V^q$, when $F \in C^\infty(\mathbb{G}_{q-1,N})$. We consider the left ideal of $\mathcal{D}_{\mathbb{G}_{q-1,N}}$ generated by the $E = E_{jj'}^{kk'}$ and their conjugates $\overline{E}$. By the natural morphism,

$$\mathcal{D}_{\mathbb{G}_{q-1,N}}(W) \to H^0(W, \det Q \otimes \overline{\det Q}) \otimes H^0(\mathbb{P}_N \times W, \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} / \mathcal{I}) \tag{14}$$

the linear differential operator $E_{jj'}^{kk'}$ is send to $\Delta(z) \otimes (\partial_j^k \partial_{j'}^{k'} - \partial_{j'}^k \partial_j^{k'})$ and thus to 0.

In effect, if u is any smooth section of $(\det Q \otimes \overline{\det Q})^{-1}$, we use the formula

$$\partial_{j'}^{k'} \partial_j^k (\langle \xi^k, x\rangle u) = x_j \partial_{j'}^{k'} u + \langle \xi^k, x\rangle \partial_{j'}^{k'} \partial_j^k u + \delta_{k'}^k x_{j'} \partial_j^k u,$$

which gives $\langle \xi^k, x\rangle (\partial_j^k \partial_{j'}^{k'} - \partial_{j'}^k \partial_j^{k'}) \in \langle f, X\rangle$.

Therefore, the morphism of

$$H^0\big(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}\big) \otimes \big(\mathcal{D}_{\mathbb{G}_{q-1,N}} / \langle E, \overline{E}\rangle\big)(W)$$

into

$$H^0(W, \det Q \otimes \overline{\det Q}) \otimes H^0(\mathbb{P}_N \times W, \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} / \mathcal{I})$$

is injective, and we have the following result.

Proposition 8. *The $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module $H^0(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}) \otimes (\mathcal{D}_{\mathbb{G}_{q-1,N}} / \langle E, \overline{E}\rangle)$ is contained in*

$$\det Q \otimes \overline{\det Q} \otimes \psi_* \varphi^* (\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \mathcal{O}_{\mathbb{P}_N}(-1) \otimes_{\mathcal{A}_{\mathbb{P}_N}} \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}).$$

When $W \subset \mathbb{G}_{q-1,N}$ is open, it remains to calculate the space of continuous sections on W of

$$\mathrm{pr}_{2*}\left\{\mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \big(\mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} / \mathcal{I}\big) \otimes \left(\bigoplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} \otimes (e_I \wedge \overline{e}_J)\right) / \mathrm{pr}_1^* \mathcal{S}\right\}$$

that is to say the space of continuous sections on $\mathbb{P}_N \times W$ of

$$\mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \left\{\frac{\oplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} \otimes (e_I \wedge \overline{e}_J)}{\mathcal{I}(\oplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} \otimes (e_I \wedge \overline{e}_J)) + \mathrm{pr}_1^* \mathcal{S}}\right\}.$$

For all fixed $0 \le j' \le N$ and $|K'| = q-1, |L| = q$, we define the element $Q = Q_{j',K',L} \in \oplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} \otimes (e_I \wedge \overline{e}_J)$ by

$$Q(\tilde{u}_{I,J}) = \sum_{\substack{0 \le k \le q-1 \\ 0 \le j \le N}} \xi_{j'}^k \partial_j^k \tilde{u}_{jK',L}(\xi)$$

for all $(\tilde{u}_{I,J})$. For $0 \le j' \le N$ and $|K| = q, |L'| = q-1$, we define the element $\overline{R} = \overline{R}_{j',K,L'} \in \oplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} \otimes (e_I \wedge \overline{e}_J)$ by

$$\overline{R}(\tilde{u}_{I,J}) = \sum_{\substack{0 \le k \le q-1 \\ 0 \le j \le N}} \overline{\xi}_{j'}^k \overline{\partial}_j^k \tilde{u}_{K,jL'}(\xi)$$

for all $(\tilde{u}_{I,J})$. We use the formula

$$\sum_{\substack{0 \le k \le q-1 \\ 0 \le j \le N}} \xi_{j'}^k \partial_j^k (\langle \xi^l, x\rangle \tilde{u}_{jK',L}(\xi)) = \langle \xi^l, x\rangle Q(\tilde{u}_{I,J}) + \sum_{0 \le j \le N} \xi_{j'}^l x_j \tilde{u}_{jK',L}(\xi), \tag{15}$$

which gives

$$Q(\langle \xi^l, x\rangle \tilde{u}_{I,J}) - \langle \xi^l, x\rangle Q(\tilde{u}_{I,J}) = \sum_{I,J} \left(\sum_{mI'=I} g_{I'J} x_m\right) \tilde{u}_{I,J}$$

with $g_{I'J} = 0$ if $I' \neq K'$ and $g_{K'J} = 0$ if $J \neq L$, while $g_{K'L} = \xi^l_{j'}$. So Q is a continuous section of

$$\mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}) \otimes \left\{\mathcal{I}\left(\bigoplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{P}_N \times \mathbb{G}_{q-1,N}} \otimes (e_I \wedge \bar{e}_J)\right) + \mathrm{pr}_1^*\mathcal{S}\right\}.$$

For $|K| = |L| = q$, we define the linear differential operator E_{KL} by

$$E_{KL}(\tilde{u}_{I,J}) = \Delta(\partial^k_j \partial^{k'}_{j'} - \partial^k_{j'} \partial^{k'}_j)\tilde{u}_{K,L},$$

and we obtain an injective morphism from

$$H^0\big(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}\big) \otimes \left\{\left(\bigoplus_{|I|=|J|=q} \mathcal{D}_{\mathbb{G}_{q-1,N}} \otimes \left(\frac{\partial}{\partial P_I} \wedge \frac{\partial}{\partial \overline{P_J}}\right)\right) / (\langle \Delta Q, \Delta \overline{R}\rangle + \langle E_{KL}, \overline{E}_{KL}\rangle)\right\}$$

into $\det Q \otimes \overline{\det Q} \otimes \psi_* \varphi^*(\mathcal{D}_{\mathbb{P}_N} \otimes \bigwedge_{q,q} T\mathbb{P}_N \otimes \mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)})$.

We denote here by $\mathrm{Im}\, C$ the image of the Chow transformation C acting on the smooth differential (q, q)-forms u on $\mathbb{P}_N$. By applying the Proposition 7, we arrive at the following result.

Theorem 2. *There is a left submodule $\mathcal{M} \subset \bigwedge^{1,1} T\mathbb{G}_{q-1,N}$ of finite type satisfying*
(*i*) *the inclusion of* $H^0(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}) \otimes (\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M})$ *into* $\det Q \otimes \overline{\det Q} \otimes \psi_* \varphi^*(\mathcal{D}_{\mathbb{P}_N} \otimes \bigwedge^{q,q} T\mathbb{P}_N \otimes \mathcal{O}_{\mathbb{P}_N}(-1) \otimes \overline{\mathcal{O}_{\mathbb{P}_N}(-1)})$,
(*ii*) *the equality* $\mathrm{Im}\, C = H^0(\mathbb{G}_{q-1,N}, \mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_{q-1,N}}}(\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M}, C^\infty_{\mathbb{G}_{q-1,N}}))$.

Thus, if u is a smooth differential (q, q)-form on $\mathbb{P}_N$, then $C(u)$ is a global solution on $\mathbb{G}_{q-1,N}$ of the left $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module $\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M}$ with values in $C^\infty_{\mathbb{G}_{q-1,N}}$.

3.2 Inversion of the Chow transformation of currents

Recall that the injectivity of the integral transformation

$$C : \{\text{currents of bidegree}\,(q, q) \quad \text{on } \mathbb{P}_N\} \to \{\text{currents of bidegree } (1, 1) \quad \text{on } \mathbb{G}_{q-1,N}\}$$

can be proved by using a dual integral transformation C^*.

Let Ω be the fundamental $(1, 1)$-form of the Hermitian metric induced in $\mathbb{G}_{q-1,N} = G(q, V)$, which satisfies $\Omega = dd^c \log \|z^1 \wedge \ldots \wedge z^q\|$ for $s = \mathrm{vect}(z^1, \ldots, z^q)$. We denote by $d_G = \dim G(q, V) = q(N + 1 - q)$ the dimension of $\mathbb{G}_{q-1,N}$.

If $2q \leq N$ and Θ is a current of bidegree $(1, 1)$ on $\mathbb{G}_{q-1,N}$, we set

$$C^*(\Theta) = \varphi_* \psi^*\big(\Omega^{d_G - 2 - N + 2q} \wedge \Theta\big), \tag{16}$$

which is a current of bidegree (q, q) on $\mathbb{P}_N$.

If $2q \geq N$, the current

$$\varphi_* \psi^*\big(\Omega^{d_G - 2} \wedge \Theta\big)$$

is of bidegree $(N - q, N - q)$ on $\mathbb{P}_N$ and the integral transformation that associates to T the $(N - q, N - q)$-current

$$\varphi_* \psi^*(\Omega^{d_G - 2} \wedge C(T)) = \varphi_*(\psi^* \Omega^{d_G - 2} \wedge \psi^* C(T))$$

on $P(V)$ is injective and has a left inverse, which is a linear differential operator $\mathcal{P}$ with smooth coefficients.

We denote by L_ω the operator of multiplication with the Fubini-Study form ω in $\mathbb{P}_N = P(V)$. Since $2q \geq N$, the operator of multiplication

$$L_\omega^{2q-N}: \bigwedge^{N-q,N-q} T^*P(V) \to \bigwedge^{q,q} T^*P(V)$$

is an isomorphism, and therefore, the integral transformation that associates to T the (q, q)-current

$$\omega^{2q-N} \wedge \varphi_* \psi^*(\Omega^{d_G-2} \wedge C(T)) = \varphi_*(\varphi^* \omega^{2q-N} \wedge \psi^* \Omega^{d_G-2} \wedge \psi^* C(T))$$

on $P(V)$ is injective too. So, in this case, we set

$$C^*(\Theta) = \omega^{2q-N} \wedge \varphi_* \psi^* \big(\Omega^{d_G-2} \wedge \Theta\big). \tag{17}$$

In the case $2q \le N$, recall now the proof of the injectivity of C^*C by using the expression of the coefficients of $C^*C(u)$ in terms of the coefficients $C_{jl}^{km}(u)(z)$ for u a smooth differential (q, q)-form on $P(V)$.

For $|I| = |I'| = q$, let $F_{I,I'}$ be the smooth function on $G(q, V)$ defined by

$$F_{I,I'}(s) = F_{I,I'}(\mathrm{vect}(z^0, z^1, \ldots, z^{q-1})) = \sum_{1 \le \alpha,\beta \le q} (-1)^{\alpha+\beta} \det\, (M_{ii'})_{\substack{i \in I, i \ne i_\alpha \\ i' \in I', i' \ne i'_\beta}} C^{00}_{i_\alpha i'_\beta}(u)(z) + \det\, (M_{ii'})_{i \in I, i' \in I'} \sum_{\substack{1 \le k \le q-1 \\ 0 \le l \le N}} C^{kk}_{ll}(u)(z),$$

where the Hermitian matrix $(M_{ii'})_{i,i'}$ depending only on $s = \mathrm{vect}(z^0, \ldots, z^{q-1})$ is provided as follows:

$$d(x, s)^2 = \frac{\Delta(x, z^0, \ldots, z^{q-1})}{\Delta(z^0, \ldots, z^{q-1})} = \sum_{0 \le i, i' \le N} x_i \overline{x}_{i'} M_{ii'}$$

with $d(x, s)$ the distance of x to the vector subspace s of $\mathbb{C}^{N+1}$ and with $\Delta(z^0, \ldots, z^{q-1})$ the Gram determinant of $z^0, \ldots, z^{q-1}$. Then, the expression in coordinates of $C^*C(u)$ is given by the following result.

Lemma 3. *For all $x \in \mathbb{C}^{N+1} = V$ such that $\|x\| = 1$, we have*

$$\pi^*(C^*C(u))_x = K' \sum_{|I|=|I'|=q} \mathrm{d}x_I \wedge \mathrm{d}\overline{x}_{I'} \int_{s \in \varphi^{-1}([x])} F_{I,I'} \Omega^d = K' \sum_{|I|=|I'|=q} \mathrm{d}x_I \wedge \mathrm{d}\overline{x}_{I'} ({}^t\mathcal{R}_{q-1})(F_{I,I'})([x]),$$

where ${}^t\mathcal{R}_{q-1}$ is the dual transform of the projective $(q-1)$-dimensional Radon transform $\mathcal{R}_{q-1}$ and K' is a constant factor.

Now, we express the functions $F_{I,I'}$ for $|I| = |I'| = q$ in terms of the transforms $\tilde{u}_{K,K'}$, so in terms of the Radon transforms $\mathcal{R}_{q-1}(f_{K,K'})$ for $|K| = |K'| = q$.

First,

$$\begin{aligned} &\sum_{1 \le \alpha,\beta \le q} (-1)^{\alpha+\beta} \det\, (M_{ii'})_{\substack{i \in I, i \ne i_\alpha \\ i' \in I', i' \ne i'_\beta}} C^{00}_{i_\alpha i'_\beta}(u)(z) \\ &\quad = \sum_{1 \le \alpha,\beta \le q} (-1)^{\alpha+\beta} \det\, (M_{ii'})_{\substack{i \in I, i \ne i_\alpha \\ i' \in I', i' \ne i'_\beta}} (-1)^{0+0} \sum_{|J|=|J'|=q-1} \tilde{u}_{i_\alpha J, i'_\beta J'}(z) z_J^{\hat{0}} \overline{z}_{J'}^{\hat{0}} \\ &\quad = \sum_{|K|=|K'|=q} \tilde{u}_{K,K'}(z) \sum_{i_\alpha \in K, i'_\beta \in K'} (-1)^{\alpha+\beta} (-1)^{\alpha_0+\beta_0} \det\, (M_{ii'})_{\substack{i \in I, i \ne i_\alpha \\ i' \in I', i' \ne i'_\beta}} z^{\hat{0}}_{K \setminus \{i_\alpha\}} \overline{z}^{\hat{0}}_{K' \setminus \{i'_\beta\}}, \end{aligned}$$

where α_0 and β_0 are such that $i_\alpha = k_{\alpha_0}$ and $i'_\beta = k'_{\beta_0}$ when $K = (k_1, \ldots, k_q)$ with $0 \le k_1 < \ldots < k_q \le N$ and $K' = (k'_1, \ldots, k'_q)$ with $0 \le k'_1 < \ldots < k'_q \le N$.

Then,

$$\begin{aligned} &\det\, (M_{ii'})_{i \in I, i' \in I'} \sum_{\substack{1 \le k \le q-1 \\ 0 \le l \le N}} C^{kk}_{ll}(u)(z) \\ &\quad = \sum_{0 \le l \le N} \sum_{|J|=|J'|=q-1} \tilde{u}_{lJ,lJ'}(z) \sum_{1 \le k \le q-1} \det\, (M_{ii'})_{i \in I, i' \in I'} z_J^{\hat{k}} \overline{z}_{J'}^{\hat{k}} \\ &\quad = \sum_{|K|=|K'|=q} \tilde{u}_{K,K'}(z) \sum_{l \in K \cap K'} (-1)^{\alpha_0+\beta_0} \sum_{1 \le k \le q-1} \det\, (M_{ii'})_{i \in I, i' \in I'} z^{\hat{k}}_{K \setminus \{l\}} \overline{z}^{\hat{k}}_{K' \setminus \{l\}}, \end{aligned}$$

where α_0 and β_0 are such that $l = k_{\alpha_0} = k'_{\beta_0}$ when $K = (k_1, \dots, k_q)$ with $0 \le k_1 < \dots < k_q \le N$ and $K' = (k'_1, \dots, k'_q)$ with $0 \le k'_1 < \dots < k'_q \le N$.

With all that together, we conclude that

$$F_{I,I'}(s) = \sum_{|K|=|K'|=q} \mathcal{R}_{q-1}(f_{K,K'})(s) c^{I,I'}_{K,K'}(s), \tag{18}$$

where the coefficients $c^{I,I'}_{K,K'}(s)$ are independent of the smooth differential (q, q)-form u.

Set the matrix $C(s) = (c^{I,I'}_{K,K'}(s))_{II',KK'}$ and note that (K, K') and (I, I') belong to a set with $\binom{N+1}{q}^2$ elements.

Let λ_K be the Plücker coordinates of $(\lambda^0, \dots, \lambda^{q-1}) \in (\mathbb{C}^{N+1})^q$ defined by

$$\lambda^0 \wedge \dots \wedge \lambda^{q-1} = \sum_{|K|=q} \lambda_K e_K.$$

The coefficients $c^{I,I'}_{K,K'}(s)$ can be calculated by means of the relation

$$\begin{aligned} &\alpha_q \Delta(z) \sum_{|I|=|I'|=q} \sum_{|K|=|K'|=q} \bar{\lambda}_K \lambda_{K'} c^{I,I'}_{K,K'}(s) \mathrm{d}x_I \wedge \mathrm{d}\bar{x}_{I'} \\ &= D_1 p^*_{s^\perp}(i\partial\bar\partial(\|x\|^2))^{q-1} \wedge \partial\bar\partial(|\langle x \wedge z^1 \wedge \dots \wedge z^{q-1} | \lambda^0 \wedge \dots \wedge \lambda^{q-1}\rangle|^2) \\ &\quad + D_2 p^*_{s^\perp}(i\partial\bar\partial(\|x\|^2))^q \sum_{1\le k\le q-1} \mathrm{trace}(|\langle x \wedge z^0 \wedge \dots \wedge \widehat{z^k} \wedge \dots \wedge z^{q-1} | \lambda^0 \wedge \dots \wedge \lambda^{q-1}\rangle|^2) \end{aligned}$$

with $D_1 = (-1)^{q-1}(i^{(q-1)^2}(q-1)!)^{-1}$ and $D_2 = (i^{q^2} q!)^{-1}$ and with $p_{s^\perp}$ the orthogonal projection onto $s^\perp$.

Proposition 9. *In both cases, $2q \le N$ and $2q \ge N$, there are linear differential operators $Q^{I,I'}_{K,K'}$ on $P(V) = \mathbb{P}_N$ with smooth coefficients satisfying*

$$\pi^*(C^*C(u))_x = \|x\|^{-2q} \sum_{|I|=|I'|=q} \mathrm{d}x_I \wedge \mathrm{d}\bar{x}_{I'} \sum_{|K|=|K'|=q} Q^{I,I'}_{K,K'}({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f_{K,K'}).$$

Proof. The projective $(q-1)$-dimensional Radon transform $\mathcal{R}_{q-1}$ is injective, thus the integral transform ${}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1}$ is injective too. Then $\mathrm{im}({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1}) = (\ker({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1}))^\perp$ and ${}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1}$ is surjective too. In such a way, a transform of the $f_{K,K'}$ is a transform of the $({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f_{K,K'})$.

So there are linear differential operators $Q^{I,I'}_{K,K'}$ on $\mathbb{P}_N$ with smooth coefficients satisfying

$$({}^t\mathcal{R}_{q-1})\left(\sum_{|K|=|K'|=q} \mathcal{R}_{q-1}(f_{K,K'}) c^{I,I'}_{K,K'}\right) = \sum_{|K|=|K'|=q} Q^{I,I'}_{K,K'}({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f_{K,K'})$$

for each system of functions $f_{K,K'} \in C^\infty(\mathbb{P}_N)$. □

Theorem 3. *The transformation $C^*C : C^\infty_{q,q}(\mathbb{P}_N) \to C^\infty_{q,q}(\mathbb{P}_N)$ is an isomorphism, and therefore, C is injective and C^* is surjective.*

Proof. The multiplication map $L^{d_G-2}_\Omega : \bigwedge^{1,1} T^*\mathbb{G}_{q-1,N} \to \bigwedge^{d_G-1,d_G-1} T^*\mathbb{G}_{q-1,N}$ by Ω^{d_G-2} is an isomorphism. The expression $\varphi_*\psi^* L^{d_G-2}_\Omega \psi_*\varphi^*$ is invariant by transposition and to see that C^*C is injective, we can say that

$$\sum_{|K|=|K'|=q} Q^{I,I'}_{K,K'}({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f_{K,K'}) = 0 \Rightarrow ({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1}) f_{K,K'} = 0 \Rightarrow f_{K,K'} = 0.$$ □

Now we reprove Theorem 3 by using the point of view of sheaves of left $\mathcal{D}$-modules, i.e., by using the fact that for each smooth differential (q, q)-form u on $\mathbb{P}_N$, the transform $C(u)$ belongs to the solutions space

$$H^0(\mathbb{G}_{q-1,N}, \mathrm{Hom}_{\mathcal{D}_{\mathbb{G}_{q-1,N}}}(\textstyle\bigwedge^{1,1} T\mathbb{G}_{q-1,N} / \mathcal{M}, C^\infty_{\mathbb{G}_{q-1,N}})).$$

We calculate the $\mathcal{D}_{\mathbb{G}_{q-1,N}}$-module $\varphi_*\psi^*(\bigwedge^{1,1}T\mathbb{G}_{q-1,N}/\mathcal{M})$, which satisfies the inclusion of

$$H^0\big(\mathbb{P}_N,\mathcal{A}_{\mathbb{P}_N}\big)\otimes\varphi_*\psi^*(\bigwedge^{1,1}T\mathbb{G}_{q-1,N}/\mathcal{M})$$

into

$$\varphi_*\psi^*(\det Q\otimes\overline{\det Q}\otimes\psi_*\varphi^*(\mathcal{D}_{\mathbb{P}_N}\otimes\bigwedge^{q,q}T\mathbb{P}_N\otimes O_{\mathbb{P}_N}(-1)\otimes\overline{O_{\mathbb{P}_N}(-1)})).$$

When U is an open subset of $\mathbb{P}_N$, we have to calculate the space of continuous sections on $U\times\mathbb{G}_{q-1,N}$ of

$$\big(\mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}/\mathcal{I}\big)\otimes\left\{\left(\bigoplus_{|I|=|J|=q}\mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}\otimes\left(\frac{\partial}{\partial P_I}\wedge\frac{\partial}{\partial\overline{P}_J}\right)\right)/\operatorname{pr}_2^*(\langle\Delta Q,\Delta\overline{R}\rangle+\langle E_{KL},\overline{E}_{KL}\rangle)\right\}.$$

Lemma 4. *For every linear differential operator $P\in\mathcal{D}_{\mathbb{P}_N}\otimes\mathcal{D}_{\mathbb{G}_{q-1,N}}$ on $U\times\mathbb{G}_{q-1,N}$, there are real-analytic functions h_α on $\mathbb{G}_{q-1,N}$ and linear differential operators $Q_\alpha\in\mathcal{D}_{\mathbb{P}_N}(U)$, for α in an infinite set, such that $P-\sum_\alpha h_\alpha Q_\alpha\in\langle E,\overline{E}\rangle(U)$.*

Proof. Let F be the inverse image in V^q of a smooth function on $G(q,V)$, i.e., a smooth function in V^q satisfying the homogeneity property $F(A.\,z)=F(z)$ for all $A\in\mathrm{GL}_q(\mathbb{C})$. Then, Lemma 1 gives that

$$\partial\overline{\partial}F=\frac{1}{((q-1)!)^2}\sum_{|I|=|J|=q}(\partial_I\overline{\partial}_JF)\mathrm{d}P_I\wedge\mathrm{d}\overline{P}_J,$$

where we have denoted here by ∂_I the linear differential operator

$$\det\left(\frac{\partial}{\partial z_j^k}\right)_{\substack{0\le k\le q-1\\ j\in I}}$$

for $|I|=q$ and by P_I the Plücker coordinates of z. As a consequence, $\partial_j^k\overline{\partial}_l^m$ decompose on $\partial_I\overline{\partial}_J$ with coefficients, that are polynomials in z and $\overline{z}$. But ∂_I decompose on $\partial_j^k\partial_{j'}^{k'}-\partial_{j'}^k\partial_j^{k'}$ so $\partial_I\overline{\partial}_J$ decompose on

$$(\partial_j^k\partial_{j'}^{k'}-\partial_{j'}^k\partial_j^{k'})(\overline{\partial}_l^k\overline{\partial}_{l'}^{k'}-\overline{\partial}_{l'}^k\overline{\partial}_l^{k'}).$$ □

Let $Q^0_{K'L}\in\oplus_{|I|=|J|=q}\mathcal{D}_{\mathbb{P}_N}\otimes(e_I\wedge\overline{e}_J)$ defined by $Q^0_{K'L}(u_{IJ})=\sum_j x_ju_{jK',L}$. Then, formula (15) shows that

$$\Delta Q^0_{K'L}\in\mathcal{I}\left(\bigoplus_{|I|=|J|=q}\mathcal{D}_{\mathbb{P}_N\times\mathbb{G}_{q-1,N}}\otimes\left(\frac{\partial}{\partial P_I}\wedge\frac{\partial}{\partial\overline{P}_J}\right)\right)+\langle\Delta Q,\Delta\overline{R}\rangle.$$

Therefore, the morphism

$$H^0\big(\mathbb{G}_{q-1,N},\mathcal{A}_{\mathbb{G}_{q-1,N}}\big)\otimes\big\{(\bigoplus_{|I|=|J|=q}\mathcal{D}_{\mathbb{P}_N}\otimes(e_I\wedge\overline{e}_J))/\langle Q^0_{K'L},\overline{R}^0_{KL'}\rangle\big\}(U)\to\varphi_*\psi^*(\bigwedge^{1,1}T\mathbb{G}_{q-1,N}/\mathcal{M})(U)$$

is injective; in other words, we have the inclusion of

$$H^0\big(\mathbb{P}_N,\mathcal{A}_{\mathbb{P}_N}\big)\otimes H^0\big(\mathbb{G}_{q-1,N},\mathcal{A}_{\mathbb{G}_{q-1,N}}\big)\otimes\bigwedge^{q,q}T\mathbb{P}_N$$

into

$$\varphi_*\psi^*(\det Q\otimes\overline{\det Q}\otimes\psi_*\varphi^*(\mathcal{D}_{\mathbb{P}_N}\otimes\bigwedge^{q,q}T\mathbb{P}_N\otimes O_{\mathbb{P}_N}(-1)\otimes\overline{O_{\mathbb{P}_N}(-1)})).$$

Proposition 10. *There is a left $\mathcal{D}_{\mathbb{P}_N}$-module $\mathcal{N}$ isomorphic to $\bigwedge^{q,q}T\mathbb{P}_N$ satisfying*

(*i*) $H^0(\mathbb{P}_N,\mathcal{A}_{\mathbb{P}_N})\otimes H^0(\mathbb{G}_{q-1,N},\mathcal{A}_{\mathbb{G}_{q-1,N}})\otimes\mathcal{N}$ *is contained in* $\varphi_*\psi^*(\det Q\otimes\overline{\det Q}\otimes\psi_*\varphi^*(\mathcal{D}_{\mathbb{P}_N}\otimes\bigwedge^{q,q}T\mathbb{P}_N\otimes O_{\mathbb{P}_N}(-1)\otimes\overline{O_{\mathbb{P}_N}(-1)}))$, (*ii*) *for every smooth differential* (q, q)*-form* u *on* $\mathbb{P}_N$*, the differential* (q, q)*-form* $C^*C(u)$ *is a global solution on* $\mathbb{P}_N$ *of* $\mathcal{N}$ *with values in* $C^\infty_{\mathbb{P}_N}$.

Proof. By taking the solutions, we have an isomorphism of

$$C^\infty_{q,q}(\mathbb{P}_N) = H^0(\mathbb{P}_N, \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}_N}}\big(\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \overset{q,q}{\bigwedge} T\mathbb{P}_N, C^\infty_{\mathbb{P}_N}\big))$$

onto the space of global solutions on $\mathbb{P}_N$ of $\mathcal{N}$ with values in $C^\infty_{\mathbb{P}_N}$, which is nothing than Im C^*C. □

To calculate the above left $\mathcal{D}_{\mathbb{P}_N}$-module, we can first replace $\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \bigwedge^{q,q} T\mathbb{P}_N \otimes_{\mathcal{A}_{\mathbb{P}_N}} \mathcal{O}_{\mathbb{P}_N}(-1) \otimes_{\mathcal{A}_{\mathbb{P}_N}} \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}$ by $\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \mathcal{O}_{\mathbb{P}_N}(-1) \otimes_{\mathcal{A}_{\mathbb{P}_N}} \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}$. So with the same way, we have the following result.

Proposition 11. *There is a left* $\mathcal{D}_{\mathbb{P}_N}$*-module* $\mathcal{N}$ *isomorphic to* $\mathcal{D}_{\mathbb{P}_N}$ *satisfying*

(*i*) $H^0(\mathbb{P}_N, \mathcal{A}_{\mathbb{P}_N}) \otimes H^0(\mathbb{G}_{q-1,N}, \mathcal{A}_{\mathbb{G}_{q-1,N}}) \otimes \mathcal{N}$ *is contained in* $\varphi_*\psi^*(\det Q \otimes_{\mathcal{A}_{\mathbb{G}_{q-1,N}}} \overline{\det Q} \otimes_{\mathcal{A}_{\mathbb{G}_{q-1,N}}} \psi_*\varphi^* (\mathcal{D}_{\mathbb{P}_N} \otimes_{\mathcal{A}_{\mathbb{P}_N}} \mathcal{O}_{\mathbb{P}_N}(-1) \otimes_{\mathcal{A}_{\mathbb{P}_N}} \overline{\mathcal{O}_{\mathbb{P}_N}(-1)}))$,

(*ii*) *for every* $f \in C^\infty(\mathbb{P}_N)$*, the transform* $({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f)$ *is a global solution on* $\mathbb{P}_N$ *of* $\mathcal{N}$ *with values in* $C^\infty_{\mathbb{P}_N}$.

Proof. When $P \in \mathcal{D}(\mathbb{P}_N)$ and $Q \in \mathcal{D}(\mathbb{G}_{q-1,N})$, we define the action of $P \otimes Q$ on $({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f)$ by

$$(P \otimes Q)\cdot({}^t\mathcal{R}_{q-1}\mathcal{R}_{q-1})(f) = P({}^t\mathcal{R}_{q-1})({}^tQ)\mathcal{R}_{q-1}(f),$$

which is equal to 0 when Q is a global section of $\langle E, \overline{E}\rangle$. On the other hand, $({}^t\mathcal{R}_{q-1})(F)$ is calculated by integration on the fibers of φ and $\langle \xi^k, x\rangle = 0 = \overline{\langle \xi^k, x\rangle}$ if $s = \mathrm{vect}(\xi^0, \ldots, \xi^p)^\perp \ni x \Leftrightarrow s \in \varphi^{-1}([x])$. □

3.3 Inversion of the Chow transformation of closed currents by means of conormal currents

For $1 \le q \le N$ and Θ, a current of bidegree $(1, 1)$ on $\mathbb{G}_{q-1,N} = G(q, V)$, we set here

$$C^*(\Theta) = \varphi_*\psi^*\big(\Omega^{d_G-1} \wedge \Theta\big), \tag{19}$$

which is a current of bidegree $(N + 1 - q, N + 1 - q) = (p + 1, p + 1)$ on $\mathbb{P}_N = P(V)$.

We calculate the conormal current $\mathrm{con}(C^*([\Sigma]))$, where $\Sigma = C(Z)$ is the Chow form of an algebraic cycle Z of codimension q in $\mathbb{P}_N$.

Proposition 12. *When* T *is a closed* (q, q)*-current in* $\mathbb{P}_N$*, the conormal current* $\mathrm{con}(T)$ *is well defined as a closed current of bidegree* (N, N) *on* $T^*\mathbb{P}_N$.

Proof. We write $T = \lim\limits_{\nu}[Z_\nu]$ with Z_ν algebraic cycle of codimension q in $P(V)$, whose irreducible components are projective subspaces of $P(V)$.

We use the surjective morphism

$$\alpha : T^*P(V)\backslash 0_{P(V)} \to \{([x], [\lambda]) \in P(V) \times P(V^*) \text{ such that } \lambda(x) = 0\},$$

which associates $\alpha(\xi) = ([x], [\lambda])$ to $\xi \in T^*_{[x]}P(V) = \mathrm{Hom}(V/\mathbb{C}x, \mathbb{C}x)$ equal to $\lambda \otimes x$ with $\lambda(x) = 0$.

For Z, an irreducible algebraic subvariety of codimension q of $P(V)$, we have $\xi \in N^*_{[x]}Z$ if and only if $\ker\lambda \supset d\pi_x^{-1}(T_{[x]}Z)$, i.e., $\alpha(\xi)$ belongs to W_Z with

$$W_Z = \{([x], [\lambda]) \in P(V) \times P(V^*) \text{ such that } [x] \in Z \quad \text{and} \quad P(\ker\lambda) \supset \mathbb{T}_{[x]}Z\},$$

where $\mathbb{T}_{[x]}Z = P(\mathrm{d}\pi_x^{-1}(T_{[x]}Z)) \subset P(V)$. In other words, $(N^*Z)\setminus 0_Z = \alpha^{-1}(W_Z)$ and W_Z is an algebraic subvariety of $P(V) \times P(V^*)$ with $\dim W_Z = N - 1$. The image by pr_2 of W_Z is the dual subvariety $\mathrm{pr}_2(W_Z) = Z^* \subset P(V^*)$, which satisfies

$$\operatorname{codim} Z^* = p + 1 \Leftrightarrow \dim Z^* = q - 1 = \operatorname{codim} Z - 1$$

if Z is a projective subspace of $P(V)$.

When $T = \lim_\nu [Z_\nu]$ is positive, then W_{Z_ν} has bounded degree. Therefore, we can assume that $[W_{Z_\nu}]$ converges to a closed positive current on $P(V) \times P(V^*)$, with support in the incidence manifold

$$\{([x], [\lambda]) \in P(V) \times P(V^*) \quad \text{such that } \lambda(x) = 0\}.$$

This current is a direct image of a closed positive (N, N)-current W_T defined on this manifold.

When T is a closed smooth differential (q, q)-form on $P(V)$, the closed current W_T is still defined on the incidence manifold, since T is the difference of closed positive smooth differential (q, q)-forms on $P(V)$. Finally, the conormal current $\mathrm{con}(T)$ is the closed (N, N)-current on $T^*P(V)$ defined by $\mathrm{con}(T) = \alpha^*(W_T)$. □

Note that W_T satisfies the relations

$$\mathrm{pr}_{2*}\left(\frac{i_{0*}W_T}{\deg T} \wedge \mathrm{pr}_1^*\omega^{N-q}\right) = \frac{T^*}{\deg T^*}$$

and

$$\mathrm{pr}_{1*}\left(\frac{i_{0*}W_T}{\deg T^*} \wedge \mathrm{pr}_2^*(\omega^*)^{q-1}\right) = \frac{T}{\deg T},$$

where we equip $P(V^*)$ with the dual metric ω^* and i_0 is the canonical injection of the incidence manifold into $P(V) \times P(V^*)$. Here, T^* is the closed $(N + 1 - q, N + 1 - q)$-current on $P(V^*)$ dual of T. Therefore, the transformation $T \to W_T$ is continuous for the weak topology.

When $T = [Z]$ for Z an irreducible algebraic subvariety of codimension q of $P(V)$, we have only $\mathrm{supp}(T^*) = \mathrm{supp}([Z]^*) = Z^*$ since the dual subvariety Z^* satisfies only $\dim Z^* \geq q - 1$. But $\mathrm{con}(T)$ is a closed positive (N, N)-current with support in $\mathrm{con}(Z)$, and therefore, $\mathrm{con}([Z]) = [\mathrm{con}(Z)] = [N^*Z]$.

We use the formula between the conormals (see [18])

$$N^*Z = \Phi_*\Psi^*(N^*\Sigma)$$

by denoting by $\Phi : N^*\Gamma \to T^*\mathbb{P}_N$ and $\Psi : N^*\Gamma \to T^*\mathbb{G}_{q-1,N}$ the restrictions to the conormal $N^*\Gamma \subset T^*\mathbb{P}_N \times T^*\mathbb{G}_{q-1,N}$ of the canonical projections. In other words, when we transform $\mathrm{con}(\Sigma) = N^*\Sigma$ by means of the double fibration

$$\begin{array}{ccc} & N^*\Gamma & \\ \swarrow & & \searrow \\ T^*\mathbb{P}_N & & T^*\mathbb{G}_{q-1,N}, \end{array}$$

we obtain $\mathrm{con}(Z) = N^*Z$.

Proposition 13. *There is a linear differential operator with smooth coefficients*

$$\mathcal{P} : \{(p+1, p+1)\text{-currents on } \mathbb{P}_N\} \to \{(q, q)\text{-currents on } \mathbb{P}_N\}$$

satisfying

(*i*) $\mathrm{con}(C^*([\Sigma])) = \mathrm{con}(\mathcal{P}^{-1}([Z]))$ *for all algebraic cycles Z of codimension q in $\mathbb{P}_N$,*

(*ii*) $\mathcal{P}$ *transforms closed $(p + 1, p + 1)$-currents on $\mathbb{P}_N$ into closed (q, q)-currents on $\mathbb{P}_N$.*

Proof. The transformation $Z \to (C^*C)([Z])$ is injective, and there is a linear differential operator with smooth coefficients on $\mathbb{P}_N$ such that $\mathcal{P}(C^*([\Sigma])) = (\mathcal{P}C^*C)([Z]) = [Z]$ for all algebraic cycles Z of codimension q in $P(V)$. So $\operatorname{con}(C^*C)([Z]) = \operatorname{con}(\mathcal{P}^{-1}([Z]))$, which can be proved by using the Crofton formula.

In effect, let $V_{q-1} \subset V_{q+1}$ be vectorial subspaces of $V = \mathbb{C}^{N+1}$ of respective dimensions $q-1$ and $q+1$. We set

$$\sigma = \sigma_{V_{q-1},V_{q+1}} = \{s \in G(q,V),\ V_{q-1} \subset s \subset V_{q+1}\} = P(V_{q+1}/V_{q-1}) \simeq \mathbb{P}_1.$$

Then, the Crofton formula (see [34]) can be stated as follows:

$$\Omega^{d_G-1} = K \int\limits_{V_{q-1}\subset V_{q+1}} [\sigma]\mathrm{d}\nu = K \int\limits_{V_{q-1}\subset V_{q+1}} \left[\sigma_{V_{q-1},V_{q+1}}\right]\mathrm{d}\nu(V_{q-1}, V_{q+1}),$$

where K is a constant factor and ν is the measure on

$$\{(V_{q-1}, V_{q+1}) \in G(q-1,V) \times G(q+1,V) \text{ such that } V_{q-1} \subset V_{q+1}\}$$

associated to the Hermitian metric induced by that of $G(q-1,V) \times G(q+1,V)$. With $i : \sigma \to G(q,V)$ the canonical injection, we can express

$$K^{-1}C^*C([Z]) = \int\limits_{V_{q-1}\subset V_{q+1}} \varphi_*\psi^*([\sigma] \wedge [\Sigma])\mathrm{d}\nu = \int\limits_{V_{q-1}\subset V_{q+1}} \varphi_*\psi^* i_* i^*[\Sigma]\mathrm{d}\nu.$$

By definition of the Chow form, we have

$$\Sigma = \{s \in G(q,V),\ P(s) \cap Z \neq \varnothing\} \Rightarrow i^*\Sigma = \{s \in G(q,V),\ V_{q-1} \subset s \subset V_{q+1} \quad \text{and} \quad P(s) \cap Z \neq \varnothing\},$$

and we can assume that $P(V_{q+1}) \cap Z$ is a finite set of $\deg Z$ points $[x_1], [x_2], \dots$, so

$$i^*\Sigma = \{V_{q-1} \oplus \mathbb{C}x_1, V_{q-1} \oplus \mathbb{C}x_2, \dots\} \Rightarrow \varphi_*\psi^* i_* i^*\Sigma = \sum_{1\le l\le \deg Z} [P(V_{q-1} \oplus \mathbb{C}x_l)].$$

Finally, we calculate the dual of $K^{-1}C^*C([Z])$, which is a closed (q,q)-current in $P(V^*)$. We use that

$$\sum_{1\le l\le \deg Z} [P(V_{q-1} \oplus \mathbb{C}x_l)]^* = [P(V_{q-1})]^* \wedge ([Z] \wedge [P(V_{q+1})])^*,$$

which allows us to express this dual as follows:

$$K^{-1}(C^*C([Z]))^* = \int\limits_{V_{q+1}\in G(q+1,V)} A_{V_{q+1}} \wedge ([Z] \wedge [P(V_{q+1})])^*\mathrm{d}\mu(V_{q+1}),$$

where the integral is calculated with respect to some invariant measure μ on $G(q+1,V)$ and

$$A_{V_{q+1}} = \int\limits_{V_{q+1}^\perp \subset V_{q-1}^\perp} [P(V_{q-1}^\perp)] = \sigma^* \int [P(V_{q-1}^\perp / V_{q+1}^\perp)] = \sigma^*((\omega^*)^{q-1}_{|P(V_{q+1}^*)})$$

since $V_{q-1}^\perp$ is the inverse image of $V_{q-1}^\perp / V_{q+1}^\perp$ by the projection $\sigma : V^* \to V^*/V_{q+1}^\perp \simeq V_{q+1}^*$.

We conclude that $(C^*C([Z]))^*$ is the dual of a pseudodifferential operator evaluated in $[Z]$. □

As a consequence, we have $T = (\mathcal{P}C^*C)(T)$ for all (q,q)-currents T that satisfy $\mathrm{d}T = 0$ on $\mathbb{P}_N$. In particular, we have

$$C^*C(T) = 0 \quad \text{and} \quad \mathrm{d}T = 0 \Leftrightarrow T = 0.$$

In the general case, $C^*C(T) = 0$ if and only if

$$\Lambda_\omega T \in \operatorname{im}\partial + \operatorname{im}\bar\partial, \tag{20}$$

where $\Lambda_\omega = L_\omega^*$ is the contraction operator by ω. This is a consequence of the adjunction formula

$$\int_{P(V)} C^*C(T) \wedge \Lambda_\omega \overline{T} = \frac{1}{d_G} \int_{s \in G(q,V)} \|C(\Lambda_\omega T)(s)\|^2 \Omega^{d_G}.$$

The condition (20) is equivalent to

$$\partial\bar{\partial}\Lambda_\omega T = 0 \quad \text{and} \quad \deg T = \int_{P(V)} T \wedge \omega^{N-q} = 0.$$

If $q \geq \frac{N+1}{2}$, note that $\Lambda_\omega T = 0 \Leftrightarrow T = 0$. For any q, the equation $\partial\bar{\partial}\Lambda_\omega T = 0$ is equivalent to

$$\Lambda_\omega \partial\bar{\partial}T - i\bar{\partial}^*\bar{\partial}T + i\partial\partial^* T = 0.$$

If $\mathrm{d}T = 0$, it becomes $\Delta' T = 0 \Leftrightarrow T = \beta\omega^q$ for some constant β. If $\bar{\partial}T = 0$, it becomes $\partial\partial^*T = 0 \Leftrightarrow \partial^*T = 0$ since $\int_{P(V)} \langle \partial\partial^* T, T\rangle \omega^N = \int_{P(V)} \|\partial^* T\|^2 \omega^N$. Here, $\partial^* = - * \bar{\partial} *$ is the adjoint operator of ∂. Thus, if $T \in \operatorname{im} \partial^*\bar{\partial} = \operatorname{im} \bar{\partial}\partial^*$, then T satisfies (20).

More generally, we can express the solutions T of $C^*C(T) = 0$ as the images of a linear differential operator with smooth coefficients. In effect, when U is a smooth function on $G(q, V)$, we have

$$\int_{G(q,V)} C(T) \wedge \Omega^{d_G - 1} U = \int_{P(V)} T \wedge \varphi_* \psi^*\left(U \Omega^{d_G-1}\right)$$

and $\varphi_*\psi^*(U\Omega^{d_G-1})$ is solution in $P(V)$ of $\mathcal{E} = 0$, where $\mathcal{E}$ is a linear differential operator of order 2. In such a way $C(T) \wedge \Omega^{d_G - 1} = 0$ implies $T \in \operatorname{im}({}^t\mathcal{E})$.

Proposition 14.

(*i*) *When* $T = [Z]$ *is the integration current associated to a projective subspace* $Z = \mathbb{P}_{N-q}$, *the conormal current* $\operatorname{con}(T)$ *can be obtained by means of an integral transform from* T.

(*ii*) *When* T *is any* (q, q)*-current in* $\mathbb{P}_N$ *that is not necessarily closed*, $\operatorname{con}(T)$ *is still defined as a* (N, N)*-current in* $T^*\mathbb{P}_N$.

Proof. In the case $Z = \mathbb{P}_{N-q}$, the inverse image of W_Z in $V \times V^*$ is

$$\{(x,\xi) \in V \times V^*, x \in \pi^{-1}(Z) \quad \text{and} \quad \operatorname{Ker} \xi \supset T_x \pi^{-1}(Z)\} = \pi^{-1}(Z) \times (V/\pi^{-1}(Z))^*.$$

Let us assume $\pi^{-1}(Z) \subset V = \mathbb{C}^{N+1}$ defined by the equations $0 = g_0 = \ldots = g_{q-1}$ with the $g_i \in V^*$ and set $g = (g_0, \ldots, g_{q-1})$. On the another hand, let us assume $\pi^{-1}(Z) = \operatorname{vect}(v_q, \ldots, v_N)$ with the $v_j \in V$. We arrive at

$$\{(x,\xi) \in V \times V^*, 0 = g_0(x) = \ldots = g_{q-1}(x), 0 = \langle \xi, v_q\rangle = \ldots = \langle \xi, v_N\rangle\} \subset \{(x,\xi) \in V \times V^*, \langle \xi, x\rangle = 0\},$$

whose integration current in $V \times V^*$ is proportional to

$$\delta(g_0(x), \ldots, g_{q-1}(x), \langle \xi, v_q\rangle, \ldots, \langle \xi, v_N\rangle) \partial g_0(x) \wedge \ldots \wedge \partial g_{q-1}(x) \wedge \\ \overline{\partial g_0(x) \wedge \ldots \wedge \partial g_{q-1}(x)} \wedge \partial\langle \xi, v_q\rangle \wedge \ldots \wedge \partial\langle \xi, v_N\rangle \wedge \overline{\partial\langle \xi, v_q\rangle \wedge \ldots \wedge \partial\langle \xi, v_N\rangle}$$

with δ the Dirac mass at 0 in $\mathbb{C}^{N+1}$. This current is also proportional to

$$\delta(g_0(x), \ldots, g_{q-1}(x), \langle \xi, v_q\rangle, \ldots, \langle \xi, v_N\rangle) \sum_{|K|=|L|=q} g_K \overline{g_L} \mathrm{d}x_K \wedge \mathrm{d}\overline{x}_L \wedge \sum_{|I|=|J|=p+1} v_I \overline{v_J} \mathrm{d}\xi_I \wedge \mathrm{d}\overline{\xi}_J,$$

where $p = N - q$ and with the determinants $g_K = \det(g_{kk'})_{\substack{0 \leq k \leq q-1 \\ k' \in K}}$ and $v_I = \det(v_{ij})_{\substack{i \in I \\ q \leq j \leq N}}$.

Let A be the matrix of type $(N+1, N+1)$ such that $v_j = Ae_j$ for $q \leq j \leq N$ and ${}^tAg_k = e_k^*$ for $0 \leq k \leq q-1$. In other words, it is proportional to the inverse image by $(x, \xi) \to (A^{-1}x, {}^tA\xi)$ of

$$\delta(x_0, \ldots, x_{q-1}, \xi_q, \ldots, \xi_N) \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_{q-1} \wedge \overline{\mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_{q-1}} \wedge \mathrm{d}\xi_q \wedge \ldots \wedge \mathrm{d}\xi_N \wedge \overline{\mathrm{d}\xi_q \wedge \ldots \wedge \mathrm{d}\xi_N}.$$

Now we recover the distribution $\delta(x_0,\dots,x_{q-1},\xi_q,\dots,\xi_N)$ in $V\times V^*$ from $\delta(x_0,\dots,x_{q-1})$ in V, by means of an integral transform. First, the inverse Fourier transform with respect to x of the inverse image of $\delta(x_0,\dots,x_{q-1},\xi_q,\dots,\xi_N)$ by the map $(x,\xi)\to(x,\xi-x)$ is equal to the function

$$\mathbb{C}^{N+1}\ni s\to e^{2\pi i\mathrm{Re}(s_q\bar{\xi}_q+\dots+s_N\bar{\xi}_N)}$$

in other words, the value at $x=(x_0,\dots,x_N)$ of the Fourier transform of this function is equal to

$$\delta(x_0,\dots,x_{q-1},\xi_q-x_q,\dots,\xi_N-x_N).$$

Second, there is $w_\xi(s,.)\in C^\infty(\mathbb{P}_N)$ satisfying

$$\begin{aligned}e^{2\pi i\mathrm{Re}(s_q\bar{\xi}_q+\dots+s_N\bar{\xi}_N)}&=\int\limits_{[x]\in\mathbb{P}_N}\delta(x_0,\dots,x_{q-1})\|x\|^{2q+2}\delta(\langle\xi,x\rangle)w_\xi(s,[x])\omega^N\\&=\int\limits_{\mathbb{P}_{N-q}}\delta(\xi_qx_q+\dots+\xi_Nx_N)(|x_q|^2+\dots+|x_N|^2)w_\xi(s,[0,\dots,0,x_q,\dots,x_N])\omega^{N-q}.\end{aligned}$$

We can write

$$e^{2\pi i\mathrm{Re}(s_q\bar{\xi}_q+\dots+s_N\bar{\xi}_N)}=\int\limits_{\mathbb{P}_{N-q}}\delta(\xi_qx_q+\dots+\xi_Nx_N)(|x_q|^2+\dots+|x_N|^2)u_{(\xi_q,\dots,\xi_N)}(s_q,\dots,s_N,[x_q,\dots,x_N])\omega^{N-q}$$

and choose $w_\xi(s,.)$ such that

$$w_\xi(s,[0,\dots,0,x_q,\dots,x_N])=u_{(\xi_q,\dots,\xi_N)}(s_q,\dots,s_N,[x_q,\dots,x_N]).$$

Consequently, $\delta(x_0,\dots,x_{q-1},\xi_q,\dots,\xi_N)$ is the value at x of the Fourier transform of

$$\mathbb{C}^{N+1}\ni s\to\int\limits_{[y]\in\mathbb{P}_N}\delta(y_0,\dots,y_{q-1})\|y\|^{2q+2}\delta(\langle x+\xi,y\rangle)w_{x+\xi}(s,[y])\omega^N$$

and taking the matrix A unitary, we arrive at the relation

$$\delta(g_0(x),\dots,g_{q-1}(x),\langle\xi,v_q\rangle,\dots,\langle\xi,v_N\rangle)=\int\limits_{[y]\in\mathbb{P}_N}\delta(g_0(y),\dots,g_{q-1}(y))\|y\|^{2q+2}\delta(\langle x+\xi,y\rangle)\tilde{w}_{x+\xi}(x,[y])\omega^N$$

with $\tilde{w}_{x+\xi}(x,[y])$ the value at x of the Fourier transform of

$$\mathbb{C}^{N+1}\ni s\to w_{x+\xi}(s,[y]).$$

More generally, if the system $\{v_q,\dots,v_N\}\subset\{0=g_0=\dots=g_{q-1}\}$ of vectors is not necessarily orthonormal, the previous relation becomes

$$\begin{aligned}&\delta(g_0(x),\dots,g_{q-1}(x),\langle\xi,v_q\rangle,\dots,\langle\xi,v_N\rangle)\|v_q\wedge\dots\wedge v_N\|^2\\&=\int\limits_{[y]\in\mathbb{P}_N}\delta(g_0(y),\dots,g_{q-1}(y))\|y\|^{2q+2}\delta(\langle x+\xi,y\rangle)\tilde{w}_{x+\xi}(x,[y])\omega^N\\&=\int\limits_{[y]\in\mathbb{P}_N}\delta(g_0(y),\dots,g_{q-1}(y))\mathcal{K}(x,\xi,y)\omega^N\end{aligned}$$

for an integral kernel $\mathcal{K}$, when a Hermitian scalar product $\langle .|.\rangle$ on $V=\mathbb{C}^{N+1}$ is fixed, where $x\in V$ is identified with $\langle .|x\rangle\in V^*$.

By using that $\|v_q\wedge\dots\wedge v_N\|^2=\sum_{|I|=|J|=p+1}v_I\bar{v}_J\langle e_I|e_J\rangle$ and by identifying, we obtain the existence of integral transformations $\mathcal{L}_{IJ}$ such that

$$\delta(g_0(x),\dots,g_{q-1}(x),\langle\xi,v_q\rangle,\dots,\langle\xi,v_N\rangle)v_I\bar{v}_J=\mathcal{L}_{IJ}(\delta(g_0,\dots,g_{q-1}))(x,\xi).$$

As a conclusion, we can write the integration current in $V \times V^*$ of the inverse image of W_Z in $V \times V^*$ as being proportional to

$$\sum_{|I|=|J|=p+1} \sum_{|K|=|L|=q} \mathcal{L}_{IJ}(\delta(g_0, \ldots, g_{q-1}) g_K \overline{g}_L)(x, \xi) dx_K \wedge d\overline{x}_L \wedge d\xi_I \wedge d\overline{\xi}_J$$

while $\pi^*[Z]$ is proportional to

$$\sum_{|K|=|L|=q} \delta(g_0(x), \ldots, g_{q-1}(x)) g_K \overline{g}_L dx_K \wedge d\overline{x}_L.$$

So W_T and $\mathrm{con}(T) = \alpha^*(W_T)$ are still defined, when we replace the integration current $[Z]$ on the projective subspace $Z = \mathbb{P}_{N-q} \subset \mathbb{P}_N$ by any (q, q)-current T on $\mathbb{P}_N$ not necessarily closed. □

4 Approximation by algebraic cycles of *X*

4.1 Chow transform $\hat{C}(T)$ defined on the space of cycles of *X*

Let X be a complex projective manifold of dimension d_X and let T be a smooth differential form on X of bidimension (p, p) with $p = d_X - q$. The Chow transform $\hat{C}(T)$ of T is defined in Section 2.3 as a current of bidegree $(1, 1)$ on the space $C_{q-1}(X)$ of effective algebraic cycles in X of dimension $q - 1$.

We assume T closed in X and we recall the condition for writing $T = \lim_k [Z_k]$ weakly in X with Z_k algebraic cycle of codimension q in X with complex coefficients. This is equivalent to

$$\hat{C}(T) = \lim_k [\hat{C}(Z_k)]$$

weakly in $C_{q-1}(X)$, thus equivalent to the orthogonality relations on $C_{q-1}(X)$

$$\int_{C_{q-1}(X)} \hat{C}(T) \wedge \Phi = 0$$

for every smooth differential form Φ of bidimension $(1, 1)$ on $C_{q-1}(X)$, which satisfies

$$\int_{C_{q-1}(X)} [\hat{C}(Z)] \wedge \Phi = 0$$

for every algebraic cycle Z of codimension q in X.

We fix $c \in C_{q-1}(X)$, then there is an open neighborhood $W \subset X$ of supp c such that $T_{|W} = dd^c S$ with a smooth differential $(q - 1, q - 1)$-form S in W. Let $\mathcal{W}$ be an open neighborhood of c in $C_{q-1}(X)$ such that every cycle element of $\mathcal{W}$ has support in W.

Then, $\hat{C}(T)_{|\mathcal{W}} = dd^c \hat{U}$ with the potential $\hat{U}(c') = \int_{c'} S$, which is only continuous with respect to $c' \in \mathcal{W}$ (see [1–3,35]).

In the above condition on Φ, we can assume supp $\Phi \subset \mathcal{W}$. By applying the characterization of the algebraic cycles with the Chow transformation, we know that

$$dd^c \Phi = \sum_j ({}^t Q_j)(\beta_j)$$

for some measures β_j with supp $\beta_j \subset \mathcal{W}$, where Q_j denotes linear differential operators with coefficients generically smooth in $\mathcal{W}$ satisfying

$$Q_j(\hat{U}) = 0.$$

The obstructions result from the fact that the β_j are not C^∞ in general. We can state the following property (see [31]).

Proposition 15. *If $\hat{C}(T)$ is globally in $C_{q-1}(X)$, the weak limit of a sequence of smooth closed differential $(1,1)$-forms, whose local potentials $\hat{U}_m$ in $\mathcal{W}$ satisfy $Q_j(\hat{U}_m) \to 0$ in the C^0 sense in $\mathcal{W}$ for all j and all $\mathcal{W}$, then T can be approximated by algebraic cycles of X with complex coefficients.*

Proof. In general, $\hat{U}$ is only continuous because $\hat{\psi}$ is equidimensional but not a submersion and because $C_{q-1}(X)$ has singularities. As being a fiber-integral, it has an asymptotic expansion near each $c \in C_{q-1}(X)$.

When we fix $c \in C_{q-1}(X)$, we can replace T by a closed (q,q)-current $\tilde{T}$ in the same cohomology class such that $\hat{C}(\tilde{T})$ is smooth near c. But we can repeat this argument for all c simultaneously only if T is dd^c-exact in X.

So we cannot assume $\hat{C}(T)$ smooth and have to use a regularization to calculate

$$\int_{\mathcal{W}} dd^c\hat{U} \wedge \Phi = \int_{\mathcal{W}} \hat{U} dd^c\Phi = \lim_m \sum_j \int_{\mathcal{W}} \hat{U}_m({}^tQ_j)(\beta_j)$$

when β_j are measures with compact support in $\mathcal{W}$. If $Q_j(\hat{U}_m)$ converges in the C^0 sense, we arrive at

$$\lim_m \sum_j \int_{\mathcal{W}} Q_j(\hat{U}_m)(\beta_j) = 0$$

since $Q_j(\hat{U}) = 0$. □

Since $Q_j(\hat{U}_m)$ weakly converges to 0, we have always the convergence of $Q_j(\hat{U}_m)(c)$ for almost every c. Therefore, the assumption in the above proposition is equivalent to $\lim_m Q_j(\hat{U}_m)(c) = 0$ for special c. Note that the Q_j are smooth only generically.

In such a way, we retrieve the orthogonality conditions of [31,32], which can be written as $\int_X T \wedge \Delta = 0$ for an infinite set of currents Δ of bidegree (p,p), which are dd^c-closed in X. When $\{T\}$ is rational, then the obstruction $\int_X T \wedge \Delta$ is constant with respect to Δ and even equal to 0.

So the transformation $\hat{C}$ can be used to solve the problem of approximating closed (q,q)-currents in X by algebraic cycles.

More precisely, we fix an irreducible component M of the cycle space $C_{q-1}(X)$, which covers X and such that the Chow transformation $T \to \hat{C}(T)_{|M}$ is injective, which is equivalent to the surjectivity of

$$\hat{\varphi}_*\hat{\psi}^* : \{\text{currents of bidegree } (d_M - 1, d_M - 1) \text{ on } M\} \to \{\text{currents of bidegree } (p,p) \text{ on } X\}.$$

Then, $\hat{C}(T)_{|M}$ is a closed $(1,1)$-current on M of order 0, and if, moreover, $\{T\}$ is rational, $\hat{C}(T)_{|M}$ is a weak limit of divisors with complex coefficients. We can write

$$\hat{C}(T)_{|M} = \int_{D \in \operatorname{Div} M} \lambda(D)[D], \tag{21}$$

where $\operatorname{Div} M$ is the space of divisors of $M \subset C_{q-1}(X)$ and λ is a measure on $\operatorname{Div} M$. In this integral, we can restrict ourselves to D such that $[D]$ is cohomologous to $\hat{C}(T)_{|M}$ in M.

The formula (21) can be written by using the Radon transformation

$$\mathcal{R} : \{C^\infty \text{ differential forms of bidimension } (1,1) \text{ on } M\} \to \{C^\infty \text{ functions on } \operatorname{Div} M\},$$

which is defined by integration on the divisors D. So we have

$$\hat{C}(T)_{|M} = ({}^t\mathcal{R})(\lambda)$$

and λ is not uniquely determined. But we can project λ on $(\mathrm{Ker}{}^t\mathcal{R})^\perp = \mathrm{Im}\,\mathcal{R}$. With the condition $\lambda \in \mathrm{Im}\,\mathcal{R}$, then λ is unique.

The Lebesgue-Nikodym decomposition of λ is

$$\lambda = \lambda_0 + \lambda_1$$

with λ_0 a L^1_{loc} function and λ_1 a measure such that $\mathrm{supp}\lambda_1$ is negligible. We now determine $\mathrm{supp}\lambda_1$ by using the injection $C_p(X) \hookrightarrow \mathrm{Div}\,M$, which associates to $Z \in C_p(X)$ the incidence divisor,

$$D_{Z|M} = \{c \in M, Z \cap c \neq \varnothing\}$$

of Z.

Lemma 5. *There is a current w of bidimension $(1,1)$ on M such that*

$$\int_{D\in \mathrm{Div}\,M} \lambda\mathcal{R}(w) = \int_{D\in \mathrm{Div}\,M} \lambda\delta_{D_0}$$

for all $\lambda \in \mathrm{Im}\,\mathcal{R}$ smooth.

Proof. The current w should satisfy

$$\langle \Phi, ({}^t\mathcal{R}\mathcal{R})(w)\rangle = \langle \mathcal{R}(\Phi), \mathcal{R}(w)\rangle = \langle \mathcal{R}(\Phi), \delta_{D_0}\rangle = \langle \Phi, ({}^t\mathcal{R})\big(\delta_{D_0}\big)\rangle = \langle \Phi, [D_0]\rangle = \int_{D_0}\Phi$$

for all smooth differential forms Φ of bidimension (1,1) on M. In other words, we should have

$$({}^t\mathcal{R}\mathcal{R})(w) = [D_0].$$

But

$$(\mathrm{Im}\ {}^t\mathcal{R}\mathcal{R})^\perp = \mathrm{Ker}{}^t\mathcal{R}\mathcal{R} = \mathrm{Ker}\mathcal{R} = (\mathrm{Ker}\mathcal{R} \cap \mathrm{Ker d}) + \mathrm{Im}\ \partial + \mathrm{Im}\ \bar{\partial}$$

in such a way that

$$\mathrm{Im}\ {}^t\mathcal{R}\mathcal{R} = \{\text{closed (1,1)-currents on } M \text{ orthogonal to } \mathrm{Ker}\mathcal{R} \cap \mathrm{Ker d}\}.$$ □

We fix $D_0 \in M$ and consider w a current of bidimension $(1,1)$ on M such that $D \to \int_D w$ is the Dirac measure at D_0, when testing on $\mathrm{Im}\,\mathcal{R}$. Thus,

$$\langle T, \hat{\varphi}_*\hat{\psi}^* w\rangle = \int_{D\in \mathrm{Div}\,M} \lambda(D)\int_D w$$

is a value of $\int_{\mathrm{Div}\,M} \lambda\delta_{D_0} = \lambda_0(D_0) + \lambda_1(D_0)$ in a generalized sense. Actually, we decompose $\hat{\varphi}_*\hat{\psi}^* w = \hat{\varphi}_*\hat{\psi}^* w_0 + \hat{\varphi}_*\hat{\psi}^* w_1$. Then, we identify $\lambda_0(D_0) = \langle T, \hat{\varphi}_*\hat{\psi}^* w_0\rangle$ and $\lambda_1(D_0) = \langle T, \hat{\varphi}_*\hat{\psi}^* w_1\rangle$, which are well defined when $D_0 \notin C_p(X)$.

In effect, when D_0 is irreducible, we set

$$Z_0 = \{x \in X, \hat{\varphi}^{-1}(x) \subset \hat{\psi}^{-1}(D_0)\} = \{x \in X, (c \ni x \quad \text{and} \quad c \in M) \Rightarrow c \in D_0\} \subset X,$$

which is of dimension $\leq p$ and satisfies $\dim Z_0 = p$ when D_0 is an incidence divisor. Assume, moreover, $D_0 \notin C_p(X)$, i.e., $\dim Z_0 < p$, then the singularities of the current $\hat{\varphi}_*\hat{\psi}^* w$ are of a lower order and $\int_{\mathrm{Div}\,M} \lambda\delta_{D_0}$ is the effective value of λ at D_0.

As a conclusion, the (p,p)-currents $\hat{\varphi}_*\hat{\psi}^* w_1$ are orthogonal to all T such that $\{T\}$ is algebraic in X and their supports satisfy $\dim \mathrm{supp}\hat{\varphi}_*\hat{\psi}^* w_1 = d_X - p$.

Lemma 6. *The absolutely continuous part λ_0 is smooth on* Div M.

Proof. For $D_0 \notin \operatorname{supp}\lambda_1$, thus for almost every D_0, we have $\lambda_0(D_0) = \langle T, \hat{\varphi}_* \hat{\psi}^* w_0\rangle$ and $D_0 \to \langle T, \hat{\varphi}_* \hat{\psi}^* w_0\rangle$ is smooth. In effect, w depends smoothly on D_0 and since $\hat{\psi}$ is equidimensional, $\hat{\psi}^*$ is continuous for the weak topology. As a conclusion, the equality $\lambda_0(D_0) = \langle T, \hat{\varphi}_* \hat{\psi}^* w_0\rangle$ is valid everywhere. □

Theorem 4. *If $\{T\}$ is rational, then T can be approximated in X by algebraic cycles of X.*

Proof. The obstruction $\hat{\varphi}_* \hat{\psi}^* w_1$ is continuous with respect to $D_0 \notin C_p(X)$ and constant and even equal to 0 when $\{T\}$ is rational. In other words $D_0 \notin C_p(X) \Rightarrow \lambda_1(D_0) = 0$, so supp $\lambda_1 \subset C_p(X)$.

The linear differential equations $Q_j(\hat{U}) = 0$ locally in $C_{q-1}(X)$ characterize the Chow transforms of smooth differential (q, q)-forms on X. Since supp $\lambda_1 \subset C_p(X)$, we can introduce the closed (q, q)-current T_0 on X such that

$$\hat{C}(T_0) = \int_{D \in \operatorname{Div} M} \lambda_0(D)[D].$$

On the one hand, since λ_0 is smooth, $\hat{C}(T_0)$ is smooth and $Q_j(\hat{U}) = 0$ everywhere in $\mathcal{W}$. By the Proposition 15, T_0 can be approximated by algebraic cycles. On the other hand, $\operatorname{supp}\lambda_1 \subset C_p(X)$ implies that $T_1 = T - T_0$ can also be approximated by algebraic cycles. □

Remark 1. We denote by $v : P(V) \to P(S^k V)$ the Veronese embedding and by C_k the Chow transformation on $P(S^k V)$, where $k \geq 1$. When T is a current of bidimension $(0, 0)$ on $P(V)$, the Chow transform $C_k(v_* T)$ of $v_* T$ is a closed current of bidegree $(1, 1)$ on $P(S^k V^*)$, obtained by integrating T on the algebraic hypersurfaces of $P(V)$ of degree k. This integral transformation is a particular case of the transformation $\mathcal{R}$ of the Lemma 5.

In effect, assume T smooth and write $T = (\deg T)\omega^N + dd^c w$ with w a smooth differential $(N-1, N-1)$-form on $P(V)$. Then,

$$C_k(v_* T) = (\deg T) C_k(v_* \omega^N) + dd^c u,$$

where u is the smooth function on $P(S^k V^*)$, defined by $u(D) = \int_D w$, for every algebraic hypersurface D of $P(V)$ of degree k. So we have the following image characterization.

Proposition 16. *The integral transformation that associates $C_k(v_* T)$ to $T \in C^{\infty}_{N,N}(P(V))$ is injective and $\Theta \in \{C^{\infty}_{1,1}(P(S^k V^*)), d\Theta = 0\}$ belongs to its image if and only if $\Theta = (\deg T) C_k(v_* \omega^N) + dd^c u$, where the smooth function u satisfies a system of linear differential equations with smooth coefficients on $P(S^k V^*)$.*

Proof. The transposition of the transformation $w \to u$ associates $\int_{D \in P(S^k V^*)} \mu(D)[D]$ to a measure μ on $P(S^k V^*)$. We have to determine the kernel of this transformation. For this purpose, we write $D = f^{-1}(0)$ with $[f] \in P(S^k V^*)$ and we use the Poincaré-Lelong formula. It is equivalent to determine the kernel of

$$\mu \to \left([x] \to \int_{[f] \in P(S^k V^*)} \mu([f]) \log\left(\frac{|f(x)|^2}{\|f\|_0^2 \|x\|^{2k}} \right) \right),$$

where $\|f\|_0$ is the norm of f. Assume $\mu = dd^c \Phi$ with Φ a smooth differential form of bidimension $(1, 1)$ on $P(S^k V^*)$. This integral is equal up to a constant to

$$\int_{Y_{[x]}} \Phi = \mathcal{R}(\Phi)(Y_{[x]}),$$

where $Y_{[x]} = \{f \in S^k V^*, \langle f, x^k\rangle = f(x) = 0\}$ and

$$\mathcal{R} : \{\text{smooth differential forms of bidimension } (1,1) \text{ on } P(S^k V^*)\} \to C^{\infty}(P(S^k V))$$

is the hyperplane Radon transformation. Then, μ is in the kernel if and only if the function $\mathcal{R}(\Phi)$ is 0 on the Veronese submanifold of $P(S^kV)$. This occurs when $\Phi \in \operatorname{Im}({}^tE_0)$, where E_0 is a linear differential operator with smooth coefficients. This is equivalent to $\mu \in \operatorname{Im}({}^tE)$, where E is a linear differential operator with smooth coefficients. □

4.2 Chow transform $C(j_*T)$ defined on the Grassmannian $\mathbb{G}_{q-1,N}$

In this subsection, we reprove Theorem 4 by using an embedding $j : X \to P(V)$ of X into $P(V) = \mathbb{P}_N$ and the induced meromorphic map

$$\rho : G(p+1, V^*) \to C_{q-1}(X),$$

which associates $X \cap P(\ker \xi_0) \cap \ldots \cap P(\ker \xi_p)$ to $\operatorname{vect}(\xi_0, \ldots, \xi_p) \in G(p+1, V^*)$. We have the equality

$$\rho^* \hat{C}(T) = C(j_*T)$$

between (1, 1)-currents on $G(p+1, V^*) = G(N-p, V)$, with C the Chow transformation on $P(V)$.

Assume T smooth closed in X and write

$$C(j_*T) = (\deg T)\Omega + dd^c U$$

with U a distribution in $G(q, V)$, Ω the fundamental differential (1, 1)-form of the metric in $G(q, V)$ and $\deg T$ the degree of T with respect to the metric induced in X by the Fubini-Study form ω in $P(V)$.

According to [9,10], every closed (1, 1)-current in $\mathbb{G}_{N-p-1,N}$ is a weak limit of divisors with complex coefficients. Thus, we can write the (1, 1)-current $C(j_*T)$, which is of order 0, in the following way:

$$C(j_*T) = \int_{H \in \operatorname{Div}(\mathbb{G}_{N-p-1,N})} \lambda(H)[H],$$

where $\operatorname{Div}(\mathbb{G}_{N-p-1,N})$ is the space of divisors of $\mathbb{G}_{N-p-1,N}$ and λ is a measure on $\operatorname{Div}(\mathbb{G}_{N-p-1,N})$.

Thanks to the Poincaré-Lelong formula, up to a constant, we have

$$U(s) = \int_{[f] \in \operatorname{Div}(\mathbb{G}_{N-p-1,N})} \lambda([f]) \log\left(\frac{\|f(s)\|}{\|f\|_0}\right),$$

where $\|f\|_0$ is the norm of the polynomial form f.

The local differential equations $Q_j(\hat{U}) = 0$ on $C_{q-1}(X)$ imply the equations $\mathcal{P}_j(U) = (\deg T)\psi_j$ on $G(q, V)$, with smooth functions ψ_j. Here, $\mathcal{P}_j$ denotes linear differential operators with smooth coefficients satisfying $\mathcal{P}_j(1) = 0$.

For $m \in \mathbb{N}^*$, we consider

$$U_m(s) = \frac{1}{2} \int_{[f] \in \operatorname{Div}(\mathbb{G}_{N-p-1,N})} \lambda([f]) \log\left(\frac{\|f(s)\|^2}{\|f\|_0^2} + \frac{1}{m}\right),$$

which is C^∞ on $\mathbb{G}_{N-p-1,N}$ and weakly converging to $U(s)$ up to a constant. As a consequence $\mathcal{P}_j(U_m)(s)$ is weakly converging to $(\deg T)\psi_j(s)$ in $G(q, V)$.

We set $R_s([f]) = \frac{\|f(s)\|^2}{\|f\|_0^2}$ and $\varepsilon = \frac{1}{m} > 0$ in such a way that

$$\mathcal{P}_j\left(\log\left(\frac{\|f(s)\|^2}{\|f\|_0^2} + \frac{1}{m}\right)\right)(s) = \sum_{1 \le l \le m_j} \frac{v_{j,s,l}([f])}{(R_s([f]) + \varepsilon)^l} = \int_0^\infty \left(\sum_{1 \le l \le m_j} v_{j,s,l} \frac{r^{l-1}}{(l-1)!}\right) e^{-rR_s} e^{-\varepsilon r} dr$$

by denoting by m_j the order of $\mathcal{P}_j$.

Proposition 17. *For almost every $s \in G(q,V)$, the simple limit*

$$\lim_m \mathcal{P}_j(U_m)(s) = \lim_m \int\limits_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})} \frac{\lambda([f])}{2}\mathcal{P}_j\left(\log(\frac{\|f(s)\|^2}{\|f\|_0^2}+\frac{1}{m})\right)$$
$$= \lim_m \int\limits_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})} \frac{\lambda([f])}{2}\mathcal{P}_j\left(\log(\|f(s)\|^2+\frac{1}{m}\|f\|_0^2)\right)$$

is equal to $(\deg T)\psi_j(s)$.

Proof. For $w(s)$, any smooth function of $s \in G(q,V)$, we know that

$$\int\limits_{s\in G(q,V)} \mathcal{P}_j(U_m)(s)w(s) \to (\deg T) \int\limits_{s\in G(q,V)} \psi_j(s)w(s)$$

when $m \to \infty$. On the other hand, this limit is equal to $\int_{s\in G(q,V)}(\lim_m \mathcal{P}_j(U_m)(s))w(s)$.

Since the measure λ on $\mathrm{Div}(\mathbb{G}_{N-p-1,N})$ depends on T, we can transpose and write

$$\frac{1}{2}\int\limits_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})} \lambda([f])\log\left(\frac{\|f(s)\|^2}{\|f\|_0^2}+\varepsilon\right) = \int\limits_{[x]\in X} T([x])\wedge \tilde{\gamma}_{s,\varepsilon}([x])$$

for all $s \in \mathbb{G}_{N-p-1,N}$ and all $\varepsilon > 0$, with $\tilde{\gamma}_s = \lim_{\varepsilon\to 0^+}\tilde{\gamma}_{s,\varepsilon}$ which is singular along $X \cap P(s)$.

By inverting the Laplace transform, we write the following function of $\varepsilon > 0$

$$\mathcal{P}_j(U_m)(s) = \int\limits_X T\wedge \mathcal{P}_j(\tilde{\gamma}_{s,\varepsilon}) = \int\limits_0^\infty e^{-\varepsilon t}h_s(t)\mathrm{d}t,$$

where $h_s(t)$ is integrable on $]0,+\infty[$ and we take the limit when $\varepsilon \to 0^+$. In such a way, we obtain

$$\lim_m \mathcal{P}_j(U_m)(s) = \int\limits_0^\infty h_s(t)\mathrm{d}t = \int\limits_{X\setminus X\cap P(s)} T\wedge (\lim_{\varepsilon\to 0^+}\mathcal{P}_j(\tilde{\gamma}_{s,\varepsilon})) = \int\limits_{X\setminus X\cap P(s)} T\wedge \mathcal{P}_j(\tilde{\gamma}_s),$$

in other words, we can take $\varepsilon = 0$ when calculating the $\lim_{\varepsilon\to 0^+}$.

If T is a closed current of order 0 and of bidimension (p,p) on X, the potential $U(s) = \int_{X\setminus X\cap P(s)} T\wedge\tilde{\gamma}_s$ is L^1_{loc} on $G(q,V)$ and still satisfies the conditions $\mathcal{P}_j(U) = (\deg T)\psi_j$ on $G(q,V)$. This implies that the differential form $\mathcal{P}_j(\tilde{\gamma}_s)$, which is defined on $X\setminus X\cap P(s)$, has L^1_{loc} coefficients on X, for almost every $s \in G(q,V)$. In effect, these differential equations are equivalent to $\mathcal{P}_j([X\cap P(s)]) = 0$. Since the distribution $\mathcal{P}_j([X\cap P(s)])$ is *a priori* of higher order, these equations imply some compensations when calculating $\mathcal{P}_j(\tilde{\gamma}_s)$ near each point of $X\cap P(s)$, which provide that $\mathcal{P}_j(\tilde{\gamma}_s)$ can be extended by a current of order 0 in X. □

Let $\rho_s : X \to [0,1]$ be a real-analytic function such that $\rho_s^{-1}(0) = X\cap P(s)$ with multiplicity 2. As a consequence, we have

$$\lim_m \mathcal{P}_j(U_m)(s) = \int\limits_0^1 \frac{\alpha_j(t,s)}{t^{m_j}}\mathrm{d}t,$$

where $\alpha_j(t,s) = \rho_{s*}(T\wedge \rho_s^{m_j}\mathcal{P}_j(\tilde{\gamma}_s))$ has the usual asymptotic expansion when $t\to 0^+$.

In effect, for $(t,s) \in \left]0,\frac{1}{2}\right] \times G(q,V)$, according to [2,3], we can write

$$\alpha_j(t,s) = \sum_{r,r'}\mathcal{L}_{r,r'}(s)t^r|\log t|^{r'} + \beta(t,s)t^{m_j},$$

where β is bounded and $(r, r') \in \mathbb{Q}^2$. Because of the convergence of the integral on $]0,1[$, if $\mathcal{L}_{r,r'}(s) \neq 0$ for some s, then $r > m_j - 1$ or $r = m_j - 1$ with $r' < -1$. In this asymptotic expansion (r, r') belongs to a finite set independent of s.

Thus, we knew *a priori* that the sequence Re $\mathcal{P}_j(U_m)(s)$ (respectively, Im $\mathcal{P}_j(U_m)(s)$) converges when $m \to \infty$ or $\lim_{m\to\infty}\mathrm{Re}\, \mathcal{P}_j(U_m)(s) = \pm\infty$ (respectively, $\lim_{m\to\infty}\mathrm{Im}\, \mathcal{P}_j(U_m)(s) = \pm\infty$).

Proposition 18. *When $m \to \infty$, the simple limit $\lim_m \mathcal{P}_j(U_m)(s)$ exists* a priori *for every $s \in \mathbb{G}_{N-p-1,N}$.*

Proof. We set $\rho_s([x]) = d(\frac{x}{\|x\|}, s)^2$ for $[x] \in X$ and consider the differential (p,p)-form $A_s = \rho_s^{p+\frac{1}{2}} \mathcal{P}_j(\tilde{y}_s)$, which is defined in X. Let h_s be the function defined almost everywhere in X by

$$T \wedge \mathcal{P}_j(\tilde{y}_s) = \frac{T \wedge A_s}{\rho_s^{p+\frac{1}{2}}} = h_s \omega_{|X}^{d_X}.$$

Let $s \in G(q, V)$ be such that $\dim X \cap P(s) = d_X - p$ and consider a sequence $s_k \to s$ with $k \in \mathbb{N}$ such that $\dim X \cap P(s_k) = d_X - p - 1$. Then, $h_{s_k} \to h_s$ almost everywhere in X. Let $[x]$ be a generic point of $X \cap P(s)$ and let B be an open neighborhood of $[x]$ in X satisfying $\int_B \mathrm{Re}\, h_s = +\infty$. We can apply the Fatou lemma that gives

$$\int_B \mathrm{Re}\, h_s \leq \lim_k \int_B \mathrm{Re}\, h_{s_k}.$$

By blowing up X along $X \cap P(s)$ with view to make flat the family of cycles, we see that $\lim_k \int_B \mathrm{Re}\, h_{s_k}$ is finite. This is a contradiction, thus $\int_B \mathrm{Re}\, h_s < +\infty$ and in the same way $\int_B \mathrm{Re}\, h_s > -\infty$. We conclude that $\int_{X\setminus X\cap P(s)} T \wedge \mathcal{P}_j(\tilde{y}_s)$ is finite.

So $\frac{T\wedge A_s}{\omega_{|X}^{d_X}}$ can be divided by ρ_s. Since the dimension of $\rho_s^{-1}(0)$ changes, the quotient $\frac{(T\wedge A_s)/\omega_{|X}^{d_X}}{\rho_s}$ is not continuous with respect to s. □

Consequently, the product $\lambda([f])\mathcal{P}_j(\log\|f(s)\|)$ of distributions is defined in $\mathrm{Div}(G(q,V))$ as follows:

$$\lim_{\varepsilon\to 0^+} \frac{\lambda([f])}{2} \mathcal{P}_j\left(\log(\frac{\|f(s)\|^2}{\|f\|_0^2} + \varepsilon)\right) = \lim_{\varepsilon\to 0^+} \int_0^\infty \frac{\lambda}{2}\left(\sum_{1\leq l\leq m_j} v_{j,s,l}\frac{r^{l-1}}{(l-1)!}\right) e^{-rR_s} e^{-\varepsilon r}\mathrm{d}r$$

which provides the following expression for the limit

$$\lim_m \mathcal{P}_j(U_m)(s) = \int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})} \lambda([f])\mathcal{P}_j(\log\|f(s)\|).$$

The product $\lambda([f])\mathcal{P}_j(\log\|f(s)\|)$ can also be defined, as in [32], from the divisions $\frac{\lambda v_{j,s,l}}{2R_s^l}$. These divisions exist, thanks to the Hörmander-Lojasiewicz theorem, but are not unique and for each l appears a residual distribution with support in $f^{-1}(0) \cap \mathrm{supp}\lambda$.

In effect, by denoting by $(N-p)d(f)$ the degree of the polynomial form f, since $\mathrm{dd^c}\log|f| = [f^{-1}(0)]$, we have

$$\mathcal{P}_j(\log\|f(s)\|) = d(f)\psi_j(s) + \text{ distribution with support in } f^{-1}(0) \tag{22}$$

in $\mathbb{G}_{N-p-1,N}$, in such a way that $\|f(s)\|^{2m_j}\mathcal{P}_j(\log\|f(s)\|)\psi_j(s)^{-1}$ is C^∞ in $\mathbb{G}_{N-p-1,N}$.

The decomposition (22) implies that $\lim_m \mathcal{P}_j(U_m)(s)$ appears as the sum of two terms and that we can also obtain in the following way. Write

$$\mathcal{P}_j(U_m)(s) = \frac{1}{2}\sum_{1\leq l\leq m_j}\int \frac{\lambda v_{j,s,l}}{(R_s+\varepsilon)^l} = \frac{1}{2}\sum_{1\leq l\leq m_j}\int_{[0,1]} \frac{R_{s*}(\lambda v_{j,s,l})}{(t+\varepsilon)^l} = \sum_{1\leq l\leq m_j}\int_0^1 \frac{g_l(t)}{(t+\varepsilon)^l}\mathrm{d}t, \tag{23}$$

where

$$g_l(t) = \frac{1}{2} R_{s*}(\lambda v_{j,s,l} / \mathrm{d}R_s)(t) = \frac{1}{2} \int \delta_0(R_s - t)\lambda v_{j,s,l}$$

are measures on $[0, 1]$, since $R_{s*}(\lambda v_{j,s,l})$ are of order 0 on $[0, 1]$.

Note that, for every, $s \in G(q, V)$, we have

$$\lim_m \mathcal{P}_j(U_m)(s) = \lim_{\varepsilon \to 0^+} \int_0^\infty \left\{ \sum_{1 \le l \le m_j} \mathcal{L}(g_l)(r) \frac{r^{l-1}}{(l-1)!} \right\} e^{-\varepsilon r} \mathrm{d}r$$

with the Laplace transform $\mathcal{L}(g_l)(r) = \int_0^1 g_l(t) e^{-tr} \mathrm{d}t$, but we cannot take $\varepsilon = 0$.

We set $u_l(t) = \int_t^1 \frac{g_l(r)}{r^l} \mathrm{d}r$ and use the Lebesgue-dominated convergence theorem to prove that

$$\lim_{t \to 0^+} (t^l u_l(t)) = 0.$$

Then, by integrating by parts, we obtain

$$\sum_{1 \le l \le m_j} \int_0^1 \frac{g_l(t)}{(t + \varepsilon)^l} \mathrm{d}t = \sum_{1 \le l \le m_j} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{t^l}{(t + \varepsilon)^l} \right) u_l(t) \mathrm{d}t.$$

For each $1 \le l \le m_j$, the family $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{t^l}{(t+\varepsilon)^l})$ weakly converges to the Dirac mass δ_0 when $\varepsilon \to 0^+$. We conclude that

$$\lim_m \mathcal{P}_j(U_m)(s) = \lim_{t \to 0^+} \left(\sum_{1 \le l \le m_j} u_l(t) \right) + \text{residual limit} \tag{24}$$

since the $\lim_{t \to 0^+} u_l(t)$ do not necessarily exist separately for each $1 \le l \le m_j$.

Lemma 7. *Each integral along the fibers $g_l(t)$ has an usual asymptotic expansion when $t \to 0^+$.*

Proof. We write

$$g_l(t) = \frac{1}{2} \int_{R_s = t} \lambda v_{j,s,l} / \mathrm{d}R_s = - \int_{R_s \ge t} \alpha$$

with $\alpha = \frac{1}{2} \mathrm{d}(\lambda v_{j,s,l} / \mathrm{d}R_s)$ in such a way that $g_l'(t)\mathrm{d}t = R_{s*}\alpha$. We use the meromorphic continuation

$$J(z) = \int R_s^z \alpha = -\frac{z}{2} \int R_s^{z-1} \lambda v_{j,s,l}$$

for $z \in \mathbb{C}$, which satisfies

$$-\frac{J(z)}{z} = \int_{-\infty}^0 e^{rz} g_l(e^r) \mathrm{d}r.$$

For $z = a + iy$ with a and r real, by the inversion formula for the Fourier transform, we arrive at

$$g_l(e^r) = \frac{e^{-ra}}{2\pi} \int_{-\infty}^{+\infty} e^{-iry} \frac{J(a + iy)}{a + iy} \mathrm{d}y.$$

We obtain the classical asymptotic expansion by using the Cauchy residue formula. □

The first term becomes

$$\lim_{t\to 0^+}\left(\sum_{1\le l\le m_j} u_l(t)\right) = \lim_{t\to 0^+}\int_t^1 \sum_{1\le l\le m_j}\frac{g_l(r)}{r^l}\mathrm{d}r$$
$$= \psi_j(s)\int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\lambda([f])d(f)$$
$$= \int_{\substack{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N}),\\ f(s)\neq 0}}\lambda([f])\mathcal{P}_j(\log\|f(s)\|).$$

We now use the function $\lim_m \mathcal{P}_j(U_m)(s)$ for $s\in G(q,V)$ to characterize the closed currents T of bidimension (p,p) in X, that can be approximated by algebraic cycles of X.

The Lebesgue-Nikodym decomposition of λ on $\mathrm{Div}(\mathbb{G}_{N-p-1,N})$ is

$$\lambda = \lambda_0 + \nu$$

with λ_0 a L^1_{loc} function and ν a measure such that $\mathrm{supp}\nu$ is negligible. Actually, we can assume λ_0 smooth on $\mathrm{Div}(\mathbb{G}_{N-p-1,N})$. The proof is analogous to that of Lemma 6, by replacing the formula $\lambda_0(D_0) = \langle T, \hat{\varphi}_*\hat{\psi}^* w_0\rangle$ by the formula $\lambda_0(H_0) = \langle T, j^*\varphi_*\psi^* w_0\rangle$.

Then, we set

$$h_{0,j}(s) = \psi_j(s)^{-1}\lim_m \int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\frac{\lambda_0([f])}{2}\mathcal{P}_j\left(\log\left(\frac{\|f(s)\|^2}{\|f\|_0^2}+\frac{1}{m}\right)\right) - \int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\lambda_0([f])d(f).$$

Proposition 19. *Since λ_0 is L^1_{loc}, the function $h_{0,j}(s)$ is continuous in $G(q,V)$, thus equal to 0 for every $s\in G(q,V)$.*

Proof. Because of the asymptotic expansion of $g_l(t)$, we can calculate the $\lim_{\varepsilon\to 0^+}$ of (23) by taking $\varepsilon = 0$. Actually, λ_0 is smooth on $\mathrm{Div}(\mathbb{G}_{N-p-1,N})$, so the part $\int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\frac{\lambda_0([f])}{2}\log R_s([f])$ is smooth on $\mathbb{G}_{N-p-1,N}$ and does not create any obstruction, when regularizing $U(s)$. ☐

We set

$$h_j(s) = \lim_{m\to\infty}\left(\frac{\mathcal{P}_j(U_m)(s)}{\psi_j(s)} - \deg T\right)$$

for $s\in G(q,V)$ and $A_1 = U^{-1}(\infty)$ the polar set of U, which is an algebraic subvariety of $G(q,V)$. Then, h_j is 0 on $G(q,V)\backslash A_1$ and continuous on A_1.

Theorem 5. *If $h_j(s)$ is constant for all $s\in A_1$, then $\mathcal{P}_j(\log\|f(s)\|) = d(f)\psi_j(s)$ for all $s\in A_1$, for each $[f]\in$ sing supp λ.*

Proof. We set

$$h_{1,j}(s) = \psi_j(s)^{-1}\lim_m \int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\frac{\nu([f])}{2}\mathcal{P}_j\left(\log\left(\frac{\|f(s)\|^2}{\|f\|_0^2}+\frac{1}{m}\right)\right) - \int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\nu([f])d(f)$$
$$= \psi_j(s)^{-1}\int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\nu([f])\mathcal{P}_j(\log\|f(s)\|) - \int_{[f]\in \mathrm{Div}(\mathbb{G}_{N-p-1,N})}\nu([f])d(f),$$

where the distribution $\nu([f])\mathcal{P}_j(\log\|f(s)\|)$ is defined by the Hörmander-Lojasiewicz division theorem.

We use homogeneous coordinates on the Grassmannian, i.e., we use the map

$$\tau : (\mathbb{C}^{N+1})^{N-p} \to \mathbb{G}_{N-p-1,N}$$

defined by $\tau(z^1,\dots,z^{N-p}) = \mathrm{Vect}\,(z^1,\dots,z^{N-p})$. Then,

$$(\mathcal{P}_j(\log\|f\|))\circ\tau = \mathcal{R}_j(\log|f|) + d(f)\psi_j \circ \tau$$

with $\mathcal{R}_j$ a linear differential operator with smooth coefficients in z, such that

$$\mathcal{R}_j(\log|f|) = \sum_{l,m} B_{j,l,m}(f,s)\frac{\partial^{l+m}\delta}{\partial t^l \partial \bar{t}^m} \circ f,$$

where $\delta(t)$ is the Dirac mass at 0 in $\mathbb{C}$ (see [32]).

We can express $\psi_j(s)^{-1}\mathcal{P}_j(\log\|f(s)\|) - d(f)$ as a linear differential operator $\mathcal{N}_j$ in $[f]$ with smooth coefficients in $([f], s)$ acting on $\log R_s([f])$. To this hand, we introduce the Plücker coordinates P_I of z, which are defined as follows:

$$z^1 \wedge \dots \wedge z^{N-p} = \sum_{|I|=N-p} P_I e_I \in \bigwedge^{N-p} V,$$

where $e_I = e_{i_1} \wedge \dots \wedge e_{i_{N-p}}$ for $I = (i_1,\dots,i_{N-p})$ with $0 \le i_1 < \dots < i_{N-p} \le N$, denoting by $e_0,\dots,e_N$ an orthonormal basis of V. Then,

$$f(z) = f((P_I)_I) = \sum_\alpha f_\alpha P^\alpha,$$

where $\alpha_I \in \mathbb{N}$ for each I and $P^\alpha = \prod_I P_I^{\alpha_I}$ with $|\alpha| = \sum_I \alpha_I = d(f)$. By applying, the Poincaré-Lelong formula to the hypersurface $\{[f] \in \mathrm{Div}(G(q,V)), f(z) = 0\}$, it relies on the fact that

$$\pi\delta(f(z)) = \frac{1}{P_\alpha \overline{P_\beta}}\frac{\partial^2}{\partial f_\alpha \partial \overline{f_\beta}}\log(|f(z)|^2) = \frac{1}{\overline{P_\beta}}\frac{\partial}{\partial \overline{f_\beta}}\left(\frac{1}{f(z)}\right)$$

is $= \frac{1}{P_\alpha \overline{P_\beta}}\frac{\partial^2}{\partial f_\alpha \partial \overline{f_\beta}} \log R_s([f])$ modulo a smooth function of the f_α.

As a consequence, by using the division by the function $f(z)$ of the f_α, the products $\nu([f])\frac{\partial^{l+m}\delta}{\partial t^l \partial \bar{t}^m}(f(z))$ of distributions are defined in $\mathrm{Div}(G(q,V))$ whole.

When $h_{1,j}(s)$ is constant on A_1, the functions $B_{j,l,m}(f,s)$ and their partial derivatives satisfy some relations over $f^{-1}(0) \cap A_1$ for $[f] \in \mathrm{supp}\nu$. These relations are precisely the differential equations on $f^{-1}(0) \cap A_1$ characterizing the fact that $[f] \in C_p(X)$.

In other words,

$$h_{1,j}(s) = \int_{[f]\in \mathrm{Div}(G(q,V))} \nu([f])\mathcal{N}_j(\log R_s([f])) = \int_{[f]\in \mathrm{Div}(G(q,V))} ({}^t\mathcal{N}_j)(\nu)([f]) \log R_s([f])$$

constant in A_1 implies $h_{1,j}(s) = 0$ in A_1, since the distribution $({}^t\mathcal{N}_j)(\nu)$ is of higher order, ν having a singular support.

In effect, A_1 is the poles set of $\int_{[f]\in \mathrm{Div}(G(q,V))} \nu([f]) \log R_s([f])$ and with $\mathcal{F}_s$ the space of smooth functions on $\mathrm{Div}(G(q,V))$ orthogonal to $\log R_s$, we should have $({}^t\mathcal{N}_j)(\nu) \in \mathcal{F}_s$ for each $s \in A_1$, which forces some annulations. □

We can reformulate the proof of the Theorem 5 in the following way. Thanks to the Proposition 19, $h_j(s)$ constant for all $s \in A_1$ implies $h_{1,j}(s)$ constant for all $s \in A_1$. Since

$$\mathcal{P}_j(\log\|f(s)\|) = d(f)\psi_j(s) \quad \text{for all } s \in A_1 \Leftrightarrow [f] \in C_p(X),$$

we can write

$$\psi_j(s)^{-1}\mathcal{P}_j(\log\|f(s)\|) = d(f) + \sum_{k=1}^{\infty} w_{j,k}([f])c_{j,k}(s) \tag{25}$$

on A_1, where the $c_{j,k}(s)$ are distributions on A_1 and the $w_{j,k}([f])$ are functions such that $w_{j,k}([f]) = 0 \Leftrightarrow [f] \in C_p(X)$. Formula (25) relies on the fact that

$$\delta\left(\sum_\alpha f_\alpha P^\alpha\right) = \sum_{k=1}^{\infty} \tilde{w}_k(f)\tilde{c}_k(z)$$

with some distributions $\tilde{c}_k(z)$ in $z \in V^{N-p}$.

If $\int \nu([f])w_{j,k}([f]) = 0$ for all k, since the measure ν is not a L^1_{loc} function when $\nu \neq 0$, we have $\text{supp}\nu \subset C_p(X)$. So we conclude that $h_{1,j}$ is 0 on A_1.

Corollary 3. *If $\{T\}$ is rational, then $h_j(s) = 0$ for all $s \in A_1$ and T can be approximated in X by algebraic cycles of X.*

Proof. If $\{T\}$ is rational, we can assume that the $\mathcal{P}_j$ are such that the h_j are constant on A_1. Theorem 5 implies that $h_j = 0$ on A_1, i.e., there is no obstruction to the approximation. □

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