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Properties of Critical Points of the Dinew-Popovici Energy Functional

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Abstract: Recently, Dinew and Popovici introduced and studied an energy functional F acting on the metrics in the Aeppli cohomology class of a Hermitian-symplectic metric and showed that in dimension 3 its critical points (if any) are Kähler. In this article we further investigate the critical points of this functional in higher dimensions and under holomorphic deformations. We first prove that being a critical point for F is a closed property under holomorphic deformations. We then show that the existence of a Kähler metric ω in the Aeppli cohomology class is an open property under holomorphic deformations. Furthermore, we consider the case when the $(2,\ 0)$-torsion form $\rho_\omega^{2,\ 0}$ of ω is ∂-exact and prove that this property is closed under holomorphic deformations. Finally, we give an explicit formula for the differential of F when the $(2,\ 0)$-torsion form $\rho_\omega^{2,\ 0}$ is ∂-exact.

Keywords: Deformation theory, Hermitian-symplectic metrics, Kähler manifolds

MSC: 51M15, 51M16

1 Introduction

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$ and ω a Hermitian metric on X. This means that $\omega \in C^\infty_{1,\ 1}(X, \mathbb{C})$ and $\omega > 0$. Let us recall the following standard definitions.

Definition 1.1. *(i) ω is called Kähler if $d\omega = 0$. We say that X is a Kähler manifold if there exists a Kähler metric ω on X.*

(ii) ω is called Hermitian-symplectic (H-s) if there exists $\rho^{2,\ 0} \in C^\infty_{2,\ 0}(X,\ \mathbb{C})$ such that

$$d(\rho^{2,\ 0} + \omega + \rho^{0,\ 2}) = 0, \tag{1.1}$$

*where $\rho^{0,\ 2} := \overline{\rho^{2,\ 0}}$. We denote $\Omega = \rho^{2,\ 0} + \omega + \rho^{0,\ 2}$ the **corresponding completion** of ω. We say that X is a Hermitian-symplectic manifold if there exists a Hermitian-symplectic metric ω on X.*

(iii) ω is called SKT (pluriclosed) if $\partial\bar\partial\omega = 0$. We say that X is a SKT manifold if there exists a SKT metric ω on X.

(iv) ω is called balanced if $d\omega^{n-1} = 0$. We say that X is a balanced manifold if there exists a balanced metric ω.

By a holomorphic family of compact complex manifolds we mean a proper holomorphic submersion $\pi : \mathcal{X} \to B$ between complex manifolds $\mathcal{X}$ and B. This means that for every $t \in B$, $X_t = \pi^{-1}(t)$ is a compact complex submanifold of $\mathcal{X}$. From now on we denote $(X_t)_{t\in B}$ as a holomorphic family of compact complex manifolds instead of referring to $\pi : \mathcal{X} \to B$. If B is simply connected then by Ehresmann's theorem (see [7]) all fibers X_t are diffeomorphic. So $\mathcal{X}$ can be considered as a C^∞ manifold X equipped with a holomorphic family $(J_t)_{t\in B}$ of complex structures $((X,\ (J_t)_{t\in B}))$. From now on B is an open ball containing the origin in $\mathbb{C}^m$.

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One of the key theorems in deformation theory is the following statement by Kodaira and Spencer. The statement of the theorem is as follows.

Theorem 1.2. *([14], Theorem 15) Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds. If X_0 is a Kähler manifold, then any X_t for all t close enough to 0 is again a Kähler manifold.*

Two important and commonly used concepts in deformation theory are openness and closedness properties. The following terminology was introduced in [16]. Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds.

(i) A given property (P) of a compact complex manifold is said to be **open** under holomorphic deformations if

$$X_0 \text{ has property } (P) \Rightarrow X_t \text{ has property } (P), \text{ for } t\in B \text{ sufficiently close to } 0.$$

(i) A given property (P) of a compact complex manifold is said to be **closed** under holomorphic deformations if

$$X_t \text{ has property } (P) \text{ for } t\in B\setminus\{0\} \text{ sufficiently close to } 0 \Rightarrow X_0 \text{ has property } (P).$$

So by Theorem 1.2, the property of being a Kähler manifold is open under holomorphic deformations. However it is proved by H. Hironaka in [9] and [10] that the Kähler property is not closed under holomorphic deformations.

The class of Kähler metrics is not the only class that is open under holomorphic deformations. In [18], Popovici showed that the strongly Gauduchon property is open under holomorphic deformations as well, but it is not a closed property by Proposition 3.4 in [23]. The notion of a strongly Gauduchon manifold was introduced by Popovici in [17]. Recall that ω is called strongly Gauduchon if $\partial\omega^{n-1}$ is $\bar\partial$-exact and, we say that X is said to be strongly Gauduchon manifold if there exists a strongly Gauduchon metric on X.

However, the openness property for an arbitrary class of metrics does not always hold. As a famous example, consider a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$, and suppose ω_0 is a balanced metric on X_0.
In [1], it is shown that the balanced property is not open under holomorphic deformations. Alessandrini and Bassanelli pointed out the counter-example of the Iwasawa manifold endowed with the holomorphically parallelizable complex structure.

The closedness property for balanced metrics does not hold either. In [3], M. Ceballos, A. Otal, L. Ugarte and R. Villacampa constructed a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ such that for $t\in B\setminus\{0\}$, X_t is a balanced manifold, but the central fiber X_0 does not admit any strongly Gauduchon metric (so it doest not admit any balanced metric).

Another example is that of class SKT manifolds. The behavior of this class of manifolds under holomorphic deformations is the same as the class of balanced manifolds. This means the SKT property is neither closed nor open under holomorphic deformations. See for example Proposition 3.1 for openness and Proposition 3.4 for closedness in [23] (see also [8] for closedness).

Another class of manifolds that has drawn a lot of attention is the one of $\partial\bar\partial$-manifolds because they satisfy the Hodge decomposition and the Hodge symmetry. Recall that X is called a $\partial\bar\partial$-manifold if and only if for every d-closed pure-type form u on X the following exactness properties are equivalent (the conclusion of $\partial\bar\partial$-lemma):

$$u \text{ is } d\text{-exact} \Leftrightarrow u \text{ is } \partial\text{-exact} \Leftrightarrow u \text{ is } \bar\partial\text{-exact} \Leftrightarrow u \text{ is } \partial\bar\partial\text{-exact}.$$

In [24] C.C. Wu proved that the $\partial\bar\partial$-property is open under holomorphic deformations.
In fact, if one considers a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ and supposes that

the central fiber X_0 is a $\partial\bar{\partial}$-manifold then both the SKT and the balanced properties become open under holomorphic deformations.

In a more general setting, we do not consider our manifolds to be $\partial\bar{\partial}$-manifolds. The main class of metrics that we discuss in this article is that of Hermitian-symplectic metrics. In dimension 2 any Hermitian-symplectic manifold is Kähler (see [21]) but in higher dimensions, the following question is still open.

Question 1.3. *([21], Question 1.7]) Do there exist non-Kähler Hermitian-symplectic complex manifolds X with $dim_{\mathbb{C}} X \geqslant 3$?*

In [25], S. Yang proved that the property of having a Hermitian-symplectic metric is open under holomorphic deformations (see also [2]). But it is not a closed property under holomorphic deformations by Theorem 3.8 of [23].
In Definition 1.1 (ii) Ω is not of type $(1, 1)$ and $\rho^{2,0}$ is not unique. One can find a unique $(2, 0)$-form such that has the minimal L^2_ω-norm among such all forms, which we call the $(2, 0)$-**torsion form** of ω and it is denoted by $\rho^{2,0}_\omega$.

The main discussion of this article is based on [6], where Dinew and Popovici introduced the **Dinew-Popovici energy functional**. Let ω_0 be a fixed Hermitian-symplectic metric on X. They define $\mathcal{S}_{\{\omega_0\}}$ as follows

$$\mathcal{S}_{\{\omega_0\}} := \{\omega_0 + \partial\bar{u}_0 + \bar{\partial}u_0 \mid u_0 \in C^\infty_{1,0}(X, \mathbb{C}) \text{ such that } \omega_0 + \partial\overline{u_0} + \bar{\partial}u_0 > 0\}.$$

The definition of Dinew-Popovici energy functional F is given by

$$F : \mathcal{S}_{\{\omega_0\}} \to [0, +\infty), \qquad F(\omega) = \int_X |\rho^{2,0}_\omega|^2_\omega \, dV_\omega = ||\rho^{2,0}_\omega||^2_\omega, \tag{1.2}$$

where $\omega \in \mathcal{S}_{\{\omega_0\}}$ and $\rho^{2,0}_\omega$ is the $(2, 0)$-torsion form of ω, while $|\ \ |_\omega$ is the pointwise norm and $||\ \ ||_\omega$ is the L^2 norm induced by ω.
When the dimension of X is 3, the critical points for F are exactly the Kähler metrics in the Aeppli cohomology class of ω_0. In Theorem 1.4 we show that this property is open under holomorphic deformations in any dimension. In other words, we prove the following

Theorem 1.4. *Suppose B is an open ball in $\mathbb{C}^m$ containing the origin and $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds of complex dimension n satisfying the following conditions:*

1) *for every $t \in B$, X_t is equipped with a Hermitian-symplectic metric ω_t and the family $(\omega_t)_{t\in B}$ is a C^∞-family of $(1, 1)$-forms,*
2) *for $t = 0$, ω_0 is a Kähler metric on X_0.*

Then after possibly shrinking B about 0, there exists a family of $(1, 1)$-forms $(\tilde{\omega}_t)_{t\in B}$ such that

a) *$\tilde{\omega}_t \in \{\omega_t\}_A$, where $\{\omega_t\}_A$ is the Aeppli cohomology class of ω_t,*
b) *$\tilde{\omega}_t$ is a Kähler metric on X_t for every $t \in B$,*
c) *$\tilde{\omega}_0 = \omega_0$,*
d) *$(\tilde{\omega}_t)_{t\in B}$ is a C^∞ family of metrics.*

By Theorem 1.2, the open property for Kähler metrics is known. But the way that we constructed the C^∞ family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ is different. The new result of Theorem 1.4 is that we have constructed a Kähler metric in a specific Aeppli cohomology class.
In higher dimension, $\dim_{\mathbb{C}} X > 3$ the following question is still open

Question 1.5. *When $dim_{\mathbb{C}} X > 3$, are the critical points of the Dinew-Popovici energy functional $F : \mathcal{S}_{\{\omega_0\}} \to [0, +\infty)$ exactly the Kähler metrics in the Aeppli cohomology class of ω_0?*

We give a partial answer to this question in Proposition 1.6 and Corollary 1.7. Precisely we show that

Proposition 1.6. *Suppose that* (X, ω_0) *is a compact complex Hermitian-symplectic manifold of dimension* n. *Fix an* $\omega \in \mathcal{S}_{\{\omega_0\}}$. *If* $\rho_\omega^{2,0} = \partial\xi$, *for some* $(1,0)$*-form* ξ, *then the differential at* ω *of the Dinew-Popovici energy functional* F *defined in equation (1.2) evaluated on* $\gamma = \bar\partial\xi + \partial\bar\xi$ *is*

$$d_\omega F(\gamma) = 2\|\rho_\omega^{2,0}\|^2 + 2Re\int\limits_X \bar\partial\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3}. \tag{1.3}$$

From this we get the following

Corollary 1.7. *Under the assumptions of Proposition 1.6 if*

(i) ω *is a critical point for* F, *and*
(ii) the $(2, 0)$*-torsion form* $\rho_\omega^{2,0} = \partial\xi$ *such that* $\bar\partial\xi$ *is weakly semi-positive,*

then ω *is a Kähler metric on* X.

Moreover, in Proposition 1.8, we prove that the property of being a critical point for F is closed under holomorphic deformations. Precisely, we prove the following proposition.

Proposition 1.8. *Suppose* $(X_t)_{t\in B}$ *is a holomorphic family of compact complex manifolds,* $(\omega_t)_{t\in B}$ *is a* C^∞ *family of Hermitian-symplectic metrics on* $(X_t)_{t\in B}$ *and* $(F_t)_{t\in B}$ *is the associated family of Dinew-Popovici energy functionals* $F_t : \mathcal{S}_{\{\omega_t\}} \to [0, \infty)$ *(see section 2.3). If after possibly shrinking* B *about* 0,

(1) for every $t \in B \setminus \{0\}$, ω_t *is a critical point in* F_t,
(2) for every $t \in B$, $h_{BC,\,t} = h_{BC,\,0}$, *where* $h_{BC,\,t}$ *is the dimension of* $H_{BC}^{0,\,2}(X_t, \mathbb{C})$,

Then ω_0 *is a critical point for* F_0.

In the above statement, $H_{BC}^{0,\,2}(X_t, \mathbb{C})$ is the Bott-Chern cohomology group of bidegree $(0, 2)$ of X_t (see Definition 2.7). In section 2 we first give the definitions and tools to state the main results and in section 3 we state our new results and prove them.

2 Preliminaries

In this section, we recall the required definitions, lemmas, and propositions that will be frequently used in section 3.
Throughout this section, X is a compact complex manifold of dimension n equipped with a Hermitian metric ω. This means that ω is a C^∞ positive definite $(1, 1)$-form on X.

2.1 General background on complex geometry

This subsection contains some standard and well-known definitions and results in complex geometry. The reader is referred to [4], [11], and [22] for further details.
First, we recall four different notions of positivity for differential forms. Let V be a complex vector space of dimension n and $(z_1, \ldots, z_n)$ be a coordinate on V. We denote the corresponding basis of V by $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ and its dual basis in $V^\star$ by $(dz_1, \ldots, dz_n)$. Consider the exterior algebra

$$\Lambda V_{\mathbb{C}}^\star = \bigoplus \Lambda^{p,q} V^\star, \quad \Lambda^{p,q} V^\star = \Lambda^p V^\star \otimes \Lambda^q \overline{V^\star}.$$

Since V is a complex vector space, it has a canonical orientation, given by the (n, n)-form

$$\tau(z) = idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n = 2^n dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_j = x_j + iy_j$. In fact, if $(w_1, \dots, w_n)$ are other coordinates, we find

$$dw_1 \wedge \cdots \wedge dw_n = \det(\partial w_j/\partial z_k) dz_1 \wedge \cdots \wedge dz_n,$$

$$\tau(w) = |\det(\partial w_j/\partial z_k)|^2 \tau(z).$$

So one can define the notion of positivity as independent of local coordinates.

Definition 2.1. *(1) A (q, q)-form $v \in \Lambda^{q,q} V^\star$ is said to be* ***strongly semi-positive*** *(resp.* ***strongly strictly positive****) if v is a convex combination*

$$v = \sum \gamma_s i\alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \cdots \wedge i\alpha_{s,q} \wedge \bar{\alpha}_{s,q}$$

where $\alpha_{j,s} \in V^\star$ and $\gamma_s \geq 0$ (resp. $\gamma_s > 0$).

(2) A (p, p)-form $u \in \Lambda^{p,p} V^\star$ is said to be ***weakly semi-positive*** *(resp.* ***weakly strictly positive****) if for all $\alpha_j \in V^\star$, $1 \leq j \leq q = n - p$, then*

$$u \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \bar{\alpha}_q \geq 0 \ \ (\text{resp. } u \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_q \wedge \bar{\alpha}_q > 0, \quad \forall a_j \neg 0)$$

Remark 2.2. *Locally, any Hermitian metric ω is a strongly strictly positive $(1, 1)$-form. This means that at any point $x \in X$, we can choose a local holomorphic coordinate $(z_1, \dots, z_n)$ and an open set $x \in U \subset X$, such that ω has the following representation*

$$\omega = \sum idz_i \wedge d\bar{z}_i.$$

Fortunately, the concepts of weakly semi-positive (resp. weakly strictly positive) and strongly semi-positive (resp. strongly strictly positive) coincide in bidegree $(1, 1)$ and $(n - 1, n - 1)$.

Proposition 2.3. *([4], Chapter III, Proposition 1.11) If $u_1, \dots, u_s$ are strongly semi-positive (resp. strongly strictly positive) forms, then $u_1 \wedge \cdots \wedge u_s$ is strongly semi-positive (resp. strongly strictly positive) form.*

For simplicity we recall the following notation.

Notation 2.4. *For any $k \in \mathbb{N}$,*

$$\omega_k = \frac{\omega^k}{k!}.$$

It is obvious that $\partial\omega_k = \partial\omega \wedge \omega_{k-1}$ and $\bar{\partial}\omega_k = \bar{\partial}\omega \wedge \omega_{k-1}$. Also, it is a well-known fact that for a Hermitian metric ω we have

$$\star_\omega \omega_k = \omega_{n-k}, \qquad \forall\, k \in \{1, \dots, n\}, \tag{2.1}$$

where $\star_\omega$ is the Hodge star operator induced by ω. The following proposition plays an important role in our discussion later.

Proposition 2.5. *([22], Proposition 6.29) If $u \in C^\infty_{p,\,q}(X, \mathbb{C})$ is primitive then*

$$\star u = (-1)^{\frac{(p+q)^2+p+q}{2}} i^{p-q} \omega_{n-q-p} \wedge u. \tag{2.2}$$

Recall that a $(p,\ q)$-form u is primitive if $L^\star_\omega(u) = 0$, where $L^\star_\omega$ is the adjoint of the Lefschetz operator $L_\omega(u) = \omega \wedge u$.

Now we mention four equations which one can easily check from equation (1.1).

Observation 2.6. *If ω is a Hermitian-symplectic metric then*

(i) $\partial\omega = -\bar\partial\rho_\omega^{2,0}$ *and* $\bar\partial\omega = -\partial\rho_\omega^{0,2}$.
(ii) $\partial\rho_\omega^{2,0} = 0$ *and* $\bar\partial\rho_\omega^{0,2} = 0$.
(iii) ω *is Kähler if and only if* $\rho_\omega^{2,0} = 0$.
(iv) $\partial\bar\partial\omega = 0$.

Note that (i) and (ii) imply that if ω is a Hermitian-symplectic metric then $\partial\omega$ and $\bar\partial\omega$ are d-closed.
In order to define suitable cohomology groups for Hermitian-symplectic and SKT metrics, we recall the definitions of the Bott-Chern cohomology and the Aeppli cohomology groups.

Definition 2.7. *For every* $p, q \in \{1, \dots n\}$ *one defines:*

(i) the Bott-Chern cohomology group of bidegree (or type) (p, q) *of* X *as*

$$H_{BC}^{p,q}(X, \mathbb{C}) = \frac{\ker\partial \cap \ker\bar\partial}{\operatorname{Im}(\partial\bar\partial)},$$

(ii) the Aeppli cohomology group of bidegree (or type) (p, q) of X as

$$H_{A}^{p,q}(X, \mathbb{C}) = \frac{\ker(\partial\bar\partial)}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial},$$

where all the kernels and images are considered as $\mathbb{C}$*-vector subspaces of* $C_{p,q}^\infty(X, \mathbb{C})$ *according to the case.*

From definitions 1.1 and 2.7 one can see if ω is a Hermitian-symplectic (resp. SKT) metric then the Aeppli (resp. Bott-Chern) cohomology class of ω (resp. $\partial\omega$), which will be denoted by $\{\omega\}_A$ (respectively $\{\partial\omega\}_{BC}$), is well-defined.
In the following definition, we recall formal definitions of two elliptic self-adjoint operators and mention the Hodge decompositions for $C_{p,q}^\infty(X, \mathbb{C})$ of these operators.

Definition 2.8. *Fix* p, $q \in \{1, \dots n\}$ *then*

(i) The Bott-Chern Laplacian operator $\Delta_{BC}^{p,q} : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})$ *is defined as follows*

$$\Delta_{BC}^{p,q} := \partial^\star\partial + \bar\partial^\star\bar\partial + (\partial\bar\partial)^\star(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^\star + (\partial^\star\bar\partial)^\star(\partial^\star\bar\partial) + (\partial^\star\bar\partial)(\partial^\star\bar\partial)^\star,$$

(ii) The Dolbeault Laplacian operator $\Delta_{\bar\partial}^{p,q} : C_{p,q}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})$ *is defined as follows*

$$\Delta_{\bar\partial}^{p,q} := \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial.$$

It is worth mentioning that by the definition of the Bott-Chern Laplacian operators, it is a real self-adjoint operator but the Dolbeault operator is not.
For each of the above operators, we have the following L_ω^2 two-space orthogonal decomposition for $C_{p,q}^\infty(X, \mathbb{C})$

(i)
$$C_{p,q}^\infty(X, \mathbb{C}) = \ker\Delta_{\mathrm{BC}}^{p,q} \oplus \operatorname{Im}\Delta_{\mathrm{BC}}^{p,q},$$

(ii)
$$C_{p,q}^\infty(X, \mathbb{C}) = \ker\Delta_{\bar\partial}^{p,q} \oplus \operatorname{Im}\Delta_{\bar\partial}^{p,q}.$$

2.2 Background on deformation of complex structures

This subsection is a summary of some basic definitions and results on deformation of complex structures of compact complex manifolds. Our main references for this part are [12], [13] and [14]. Also there are series of papers published by D. Popovici [6], [15], [19], [16], [17] and [20] which play a crucial role in this article, so we recall some lemmas and propositions from them.

We recall the green operator of a self-adjoint elliptic operator. For every fixed $p, q \in \{1, \dots, n\}$ suppose E is a self-adjoint elliptic operator on $C^\infty_{p,q}(X, \mathbb{C})$, since X is a compact manifold $\ker E$ is a finite-dimensional complex vector space. We denote by $P_E : C^\infty_{p,q}(X, \mathbb{C}) \to \ker E$ the L^2_ω orthogonal projection. One can define the **Green operator** of E, $E^{-1} : C^\infty_{p,q}(X, \mathbb{C}) \longrightarrow \operatorname{Im} E$, such that

$$E^{-1}E(\gamma) = EE^{-1}(\gamma) = \gamma - P_E(\gamma), \qquad \gamma \in C^\infty_{p,q}(X, \mathbb{C}).$$

If we restrict E to $\operatorname{Im} E$ then E is a bijection and so $E^{-1} : \operatorname{Im} E \to \operatorname{Im} E$ is the inverse of this restriction. In particular one can define, P_{BC}, $P_{\bar\partial}$, $\Delta^{-1}_{\mathrm{BC}}$, and $\Delta^{-1}_{\bar\partial}$.
Thanks to [6] we have all tools to give the explicit formula for the $(2, 0)$-torsion form $\rho^{2,0}_\omega$ for any Hermitian-symplectic metric ω. Recall that, $\rho^{2,0}_\omega$ is the unique $(2, 0)$-form such that has the minimal L^2_ω-norm among such all forms that satisfy equation (1.1).

Lemma 2.9. *([6], Lemma and Definition 3.1) Suppose ω is a Hermitian-symplectic metric on X and $\rho^{2,0}_\omega$ is the $(2, 0)$-torsion form of ω. Then*

$$\rho^{2,0}_\omega = -\Delta^{-1}_{BC}[\bar\partial^\star\partial\omega + \bar\partial^\star\partial\partial^\star\partial\omega]. \tag{2.3}$$

Notice that in (2.3) by Δ^{-1}_{BC} we mean $(\Delta^{2,0}_{BC})^{-1} : C^\infty_{2,0}(X, \mathbb{C}) \longrightarrow \operatorname{Im}\Delta_{BC}$. Also since Δ_{BC} is a real operator (so Δ^{-1}_{BC} s also real) we thus have

$$\rho^{0,2}_\omega = -\Delta^{-1}_{BC}[\partial^\star\bar\partial\omega + \partial^\star\bar\partial\bar\partial^\star\bar\partial\omega].$$

The following theorem gives us a criteria to determine whether these families are C^∞ family of linear operators. This is the main key to proving Theorem 1.4.

Theorem 2.10. *([14]) Kodaira-Spencer fundamental theorem.*
Suppose that $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds.

(i) If the dim $\ker\Delta_{BC,t} : C^\infty_{p,q}(X_t, \mathbb{C}) \to C^\infty_{p,q}(X_t, \mathbb{C})$ (resp. dim $\ker\Delta_{\bar\partial,t}$) is independent of $t \in B$, then the family $(P_{BC,t})_{t\in B}$ (resp. $(P_{\bar\partial,t})_{t\in B}$) is a C^∞ family of linear operators.
(ii) If the dim $\ker\Delta_{BC,t} : C^\infty_{p,q}(X_t, \mathbb{C}) \to C^\infty_{p,q}(X_t, \mathbb{C})$ (resp. dim $\ker\Delta_{\bar\partial,t}$) is independent of $t \in B$, then the family $(\Delta^{-1}_{BC,t})_{t\in B}$ (resp. $(\Delta^{-1}_{\bar\partial,t})_{t\in B}$) is a C^∞ family of linear operators.

2.3 Background on the Dinew-Popovici energy functional

This subsection is devoted to some definitions and results based on [6]. Throughout this section ω_0 is a Hermitian-symplectic metric on X which we consider as our background metric. The main goal of this subsection is to find the explicit formula for differential of Dinew-Popovici energy functional F in equation (1.2) at ω, where $\omega \in \mathcal{S}_{\{\omega_0\}}$ is a fixed Hermitian-symplectic metric.

Proposition 2.11. *([6], Proposition 3.5) The differential at ω for F is given by the formula:*

$$(d_\omega F)(\gamma) = -2\,Re\,\langle\langle u, \bar\partial^\star\omega\rangle\rangle_\omega + 2\,Re\int_X u\wedge\rho^{2,0}_\omega\wedge\overline{\rho^{2,0}_\omega}\wedge\bar\partial\omega_{n-3} \tag{2.4}$$

for every $(1, 1)$-form $\gamma = \bar\partial u + \partial\bar u$.

In dimension 3, the term $2\operatorname{Re}\int_X u\wedge\rho^{2,0}_\omega\wedge\overline{\rho^{2,0}_\omega}\wedge\bar\partial\omega_{n-3}$ vanishes and we have $(d_\omega F)(\gamma) = -2\operatorname{Re}\langle\langle u, \bar\partial^\star\omega\rangle\rangle_\omega$. If ω is a critical point for F and $u = \bar\partial^\star\omega$ then one can see that $\bar\partial^\star\omega = 0$. This means that ω is a balanced metric, on the other hand ω is SKT. So we conclude that ω is Kähler. Therefore in dimension 3 the critical points of F

are exactly Kähler metrics in the Aeppli cohomology class of ω_0.
Now consider a holomorphic family of compact complex manifolds $(X_t)_{t\in B}$ and $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian-symplectic metrics on $(X_t)_{t\in B}$. This means that for every $t \in B$, ω_t is a Hermitian-symplectic metric on X_t. Therefore like (1.2) one can define a family of Dinew-Popovici energy functionals $(F_t)_{t\in B}$, $F_t : \mathcal{S}_{\{\omega_t\}} \to [0, \infty)$, as follows

$$F_t : \mathcal{S}_{\{\omega_t\}} \to [0, +\infty), \qquad F_t(\bar{\omega}_t) = \int\limits_{X_t} |\rho^{2,0}_{\bar{\omega}_t}|^2_{\bar{\omega}_t}\, dV_{\bar{\omega}_t} = ||\rho^{2,0}_{\bar{\omega}_t}||^2_{\bar{\omega}_t},$$

where like the Equation (1.2) $\bar{\omega}_t \in \mathcal{S}_{\{\omega_t\}}$ and $\rho^{2,0}_{\bar{\omega}_t}$ is the $(2, 0)$-torsion form of $\bar{\omega}_t$, while $|\ |_{\bar{\omega}_t}$ is the pointwise norm and $||\ ||_{\bar{\omega}_t}$ is the L^2 norm induced by $\bar{\omega}_t$.
Henceforth if we fix any Hermitian symplectic $\bar{\omega}_t \in \mathcal{S}_{\{\omega_t\}}$ then for every $t \in B$ one can define the differential at $\bar{\omega}_t$ of F_t exactly like Proposition 2.11.

3 Results

This section is devoted to our new results based on [6]. We give a proof for Theorem 1.4. This theorem shows that if a compact complex manifold X admits a Hermitian-symplectic metric ω_0, then the existence of a Kähler metric $\tilde{\omega}_0$ in the Aeplli cohomology class of ω_0 is an open property under holomorphic deformations. Before we present the proof of Theorem 1.4, we mention three theorems which play a crucial role in our proof.

Theorem 3.1. *([15], Theorem 4.1) Fix a compact Hermitian manifold (X, ω). For any C^∞ (p, q)-form $v \in Im(\partial\bar{\partial})$, the (unique) minimal L^2-norm solution of the equation*

$$\partial\bar{\partial}u = v$$

is given by the formula

$$u = (\partial\bar{\partial})^\star \Delta^{-1}_{BC} v,$$

where Δ^{-1}_{BC} is the Green operator of the Bott-Chern Laplacian Δ_{BC} induced by ω.

Theorem 3.2. *([24], Theorem 5.12) Let $(X_t)_{t\in B}$ be a holomorphic family of compact complex manifolds of complex dimension n. If the central fiber X_0 is a $\partial\bar{\partial}$-manifold, then after possibly shrinking B about 0, X_t is a $\partial\bar{\partial}$-manifold for all $t \in B$.*

Theorem 3.3. *([5], Section 6) Every compact Kähler manifold is a $\partial\bar{\partial}$-manifold.*

Proof of Theorem 1.4. Since ω_0 is a Kähler metric on X_0 by Theorem 3.3, X_0 is a $\partial\bar{\partial}$-manifold, therefore by Theorem 3.2 after possibly shrinking B about 0 one can assume that X_t is a $\partial\bar{\partial}$-manifold for every $t \in B$. Let us fix a $t \in B$, ω_t is a Hermitian-symplectic metric on X_t then by Observation 2.6 in Section 2.1 one implies that $\partial_t\omega_t$ is d-closed and ∂_t-exact. Since X_t is a $\partial\bar{\partial}$-manifold, $\partial_t\omega_t$ is $\partial_t\bar{\partial}_t$-exact. So the following equation

$$-\partial_t\bar{\partial}_t u_t = \partial_t\omega_t. \tag{3.1}$$

has at least one solution, u_t, for $t \in B$. By Theorem 3.1 we are able to choose the minimal L^2-norm solution with respect to ω_t among all such u_t. The minimal $L^2_{\omega_t}$-norm solution of equation (3.1) is given by

$$u_t^{min} = -(\partial_t\bar{\partial}_t)^\star \Delta^{-1}_{BC,t}(\partial_t\omega_t), \tag{3.2}$$

where $\Delta^{-1}_{BC,t}$ is the Green operator of the Bott-Chern Laplacian $\Delta_{BC,\,t}$ induced by ω_t, mentioned in Section 2.2. Now we define,

$$\tilde{\omega}_t = \omega_t + \partial_t\overline{u_t^{min}} + \bar{\partial}_t u_t^{min},$$

for all $t \in B$.
By the construction of $\tilde{\omega}_t$, one can see that

$$\partial_t\bar{\partial}_t\tilde{\omega}_t = \partial_t\bar{\partial}_t(\omega_t + \partial_t\overline{u_t^{min}} + \bar{\partial}_t u_t^{min}) = \partial_t\bar{\partial}_t\omega_t = 0.$$

Therefore $\{\tilde{\omega}_t\}_A$ is well-defined and by the definition of the Aeppli cohomology group, adding $\partial_t\overline{u_t^{min}}$ and $\bar{\partial}_t u_t^{min}$ to ω_t does not change the Aeppli cohomology class of ω_t. Hence $\tilde{\omega}_t \in \{\omega_t\}_A$, this proves (a).
Also for every $t \in B$, $\tilde{\omega}_t$ is d-closed because

$$d\tilde{\omega}_t = d(\omega_t + \partial_t\overline{u_t^{min}} + \bar{\partial}_t u_t^{min}) = \partial_t\omega_t + \bar{\partial}_t\omega_t + \bar{\partial}_t\partial_t\overline{u_t^{min}} + \partial_t\bar{\partial}_t u_t^{min}. \tag{3.3}$$

Equation (3.2) implies that $\partial\bar{\partial}_t u_t^{min} = -\partial_t\omega_t$, put this in the equation (3.3) one can see that $\tilde{\omega}_t$ is d-closed. On the other hand, the strict positivity of ω_0 implies strict positivity of $\tilde{\omega}_t$ for all $t \in B$ sufficiently close to 0, henceforth $\tilde{\omega}_t$ is a strictly positive d-closed $(1,\ 1)$-form on X_t, this proves (b).
So we have a family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ on $(X_t)_{t\in B}$. At $t = 0$, there are two Kähler metrics on X_0. One of them is ω_0, which is given by assumption (2) and the other one is $\tilde{\omega}_0$ by our construction. Since ω_0 is a Kähler metric on X_0, $\partial_0\omega_0 = 0$. Hence

$$u_0^{min} = -(\partial_0\bar{\partial}_0)^\star \Delta_{BC,0}^{-1}(\partial_0\omega_0) = 0.$$

So,

$$\tilde{\omega}_0 = \omega_0 + \partial_0\overline{u_0^{min}} + \bar{\partial}_0 u_0^{min} = \omega_0.$$

This means that these two metrics coincide on X_0 which proves (c).
For every $t \in B$ we denote by $h_{\mathrm{BC},\,t}(X_t)$ the dimension of $\ker \Delta_{\mathrm{BC},\,t}$ ($\Delta_{\mathrm{BC},\,t} : C^\infty_{2,\,1}(X_t,\ \mathbb{C}) \to C^\infty_{2,\,1}(X_t,\ \mathbb{C})$). Since $(X_t)_{t\in B}$, after possibly shrinking B about 0, is a holomorphic family of compact complex $\partial\bar{\partial}$-manifolds, by Theorem 5.12 in [24], $h_{\mathrm{BC},\,t}(X_t) = h_{\mathrm{BC},\,0}(X_0)$ for every $t \in B$. By Theorem 2.10 (ii) the family of linear operators $(\Delta_{BC,\,t}^{-1})_{t\in B}$ acting on $(2,\ 1)$-forms is a C^∞ family of linear operators, therefore the family $(u_t^{min})_{t\in B}$ is a C^∞ family of $(1,\ 0)$-forms, and since $(\omega_t)_{t\in B}$ is a C^∞ family of metrics one can say $(\tilde{\omega}_t)$ is a C^∞ family of metrics, this proves (d). □

We saw that in dimension 3, Kähler metrics are the critical points for the Dinew-Popovici energy functional F. So as a consequence of Theorem 1.4 one can get the following corollary.

Corollary 3.4. *Suppose $(X_t)_{t\in B}$ is a holomorphic family of compact complex manifolds of dimension 3, $(\omega_t)_{t\in B}$ is a C^∞ family of Hermitian-symplectic metrics on $(X_t)_{t\in B}$, and $(F_t)_{t\in B}$ is a family of Dinew-Popovici energy functionals mentioned in section 2.3. If for $t = 0$, ω_0 is a critical point of F_0, then after possibly shrinking B about 0 there exists a C^∞ family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ such that for every $t \in B$, $\tilde{\omega}_t \in \mathcal{S}_{\{\omega_t\}}$ and $\tilde{\omega}_t$ is a critical point of F_t and $\tilde{\omega}_0 = \omega_0$.*

Proof. The existence of a C^∞ family of Kähler metrics $(\tilde{\omega}_t)_{t\in B}$ such that for every $t \in B$ each $\tilde{\omega}_t \in \mathcal{S}_{\{\omega_t\}}$ and $\tilde{\omega}_0 = \omega_0$ come directly from Theorem 1.4 and since the dimension of each X_t is 3, $\tilde{\omega}_t$ being a Kähler for each $t \in B$ implies that $\tilde{\omega}_t$ is a critical point of F_t . □

By Corollary 4.2 of [6], in dimension 3 if ω is a Hermitian-symplectic metric and the given Aeppli class $\{\omega\}_A$ contains a Kähler metric ω_k, then its $(0,\ 2)$-torsion form $\rho_\omega^{0,\,2}$ is $\bar{\partial}$-exact. Therefore by Theorem 1.4 if the given Aeppli class $\{\omega\}_A$ contains a Kähler metric ω_k, then the $\bar{\partial}$-exactness for $\rho_\omega^{0,\,2}$ is an open property under holomorphic deformations.
So it is natural to investigate the openness and the closedness properties of the $(0,\ 2)$-torsion form $\rho_\omega^{0,\,2}$ in higher dimensions.
In the following proposition we show that for a Hermitian-symplectic metric ω, the $\bar{\partial}$-exactness for the $(0,\ 2)$-torsion form $\rho_\omega^{0,\,2}$ is a closed property under small holomorphic deformations in any dimension.
First, we fix some notations for next proposition. For every $t \in B$ let $h_{BC,\,t} = \dim\ker\Delta_{BC,\,t}^{0,2}$ and $h_{\bar{\partial},\,t} = \dim\ker\Delta_{\bar{\partial},\,t}^{0,2}$

Proposition 3.5. *Suppose that* $(X_t)_{t\in B}$ *is a holomorphic family of compact complex manifolds,* $(\omega_t)_{t\in B}$ *is a* C^∞ *family of Hermitian-symplectic metrics on* $(X_t)_{t\in B}$. *If*

(1) *for every* $t \in B$ *sufficiently close to* 0, $h_{BC,\,t} = h_{BC,\,0}$,
(2) *for every* $t \in B$ *sufficiently close to* 0, $h_{\bar\partial,\,t} = h_{\bar\partial,\,0}$,
(3) *for every* $t \in B\setminus\{0\}$ *and sufficiently close to* 0, *the* $(0,\ 2)$*-torsion form* $\rho^{0,\,2}_{\omega_t}$ *is* $\bar\partial_t$*-exact.*

Then

(a) *the family* $(\rho^{0,\,2}_{\omega_t})_{t\in B}$ *is a* C^∞ *family of* $(0,\ 2)$*-forms,*
(b) *for* $t = 0$, *the* $(0,\ 2)$*-torsion form* $\rho^{0,\,2}_{\omega_0}$ *is* $\bar\partial_0$*-exact.*

Before giving the proof of Proposition 3.5, we recall the following lemma which will be used in the proof.

Lemma 3.6. *([19], p 11-12) Let* (X, ω) *be an n-dimensional compact Hermitian manifold. For every* $\rho \in \bar\partial(C^\infty_{0,2}(X, \mathbb{C})$, *the minimal* L^2*-norm solution of the equation*

$$\bar\partial\varphi = \rho$$

is given by the following Neumann formula

$$\varphi = \bar\partial^\star(\Delta_{\bar\partial})^{-1}\rho, \tag{3.4}$$

where $(\Delta_{\bar\partial})^{-1}$ *is the Green operator of the* $\bar\partial$*-Laplacian* $\Delta_{\bar\partial}$ *induced by* ω.

Proof of Proposition 3.5. From equation (2.3) one sees that the $\rho^{2,\,0}_{\omega_t}$ has the following form

$$\rho^{2,\,0}_{\omega_t} = -\Delta^{-1}_{BC,\,t}[\bar\partial^\star_t\partial_t\omega_t + \bar\partial^\star_t\partial_t\partial^\star_t\partial_t\omega_t],$$

for all $t \in B$. By conjugating the above equation one can see that

$$\rho^{0,\,2}_{\omega_t} = -\Delta^{-1}_{BC,\,t}[\partial^\star_t\bar\partial_t\omega_t + \partial^\star_t\bar\partial_t\bar\partial^\star_t\bar\partial_t\omega_t], \tag{3.5}$$

for all $t \in B$. Note that in (3.5) we used the fact that $\Delta_{BC} = \overline{\Delta_{BC}}$. Since $h_{BC,\,t} = h_{BC,\,0}$ for t sufficiently close to the origin, by Theorem 2.10.(ii), the family $(\Delta^{-1}_{BC,\,t})_{t\in B}$ of linear operators, acting on $(0,\ 2)$-form, is a C^∞ family of linear operators. This means that the family $(\rho^{0,\,2}_{\omega_t})_{t\in B}$ is a C^∞ family of $(0,\ 2)$-forms. In particular $\rho^{0,2}_{\omega_t} \to \rho^{0,2}_{\omega_0}$, when $t \longrightarrow 0$. This proves (a).
By assumption (3) for every $t \in B\setminus\{0\}$, $\rho^{0,\,2}_{\omega_t}$ is $\bar\partial_t$-exact. So after possibly shrinking B about the origin the following equation

$$\rho^{0,\,2}_{\omega_t} = \bar\partial_t\beta_t \tag{3.6}$$

has at least one solution β_t in $C^\infty_{0,1}(X_t, \mathbb{C})$ for every $t \in B\setminus\{0\}$. By Lemma 3.6, we are able to choose the unique solution among such β_t with the minimal L^2-norm induced by ω_t. Hence by equation (3.4), the minimal L^2-norm solution of equation (3.6) has the following form

$$\beta^{\min}_t = \bar\partial^\star_t(\Delta_{\bar\partial,\,t})^{-1}\rho^{0,\,2}_{\omega_t} \overset{(I)}{=} (\Delta_{\bar\partial,\,t})^{-1}\bar\partial^\star_t\rho^{0,\,2}_{\omega_t}. \tag{3.7}$$

Where (I) is implied as follows

$$\bar\partial^\star\Delta_{\bar\partial} = \bar\partial^\star(\bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial) = \bar\partial^\star\bar\partial\bar\partial^\star = \bar\partial^\star\bar\partial\bar\partial^\star + \bar\partial\bar\partial^\star\bar\partial^\star = (\bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial)\bar\partial^\star = \Delta_{\bar\partial}\bar\partial^\star.$$

By assumption (2) after possibly shrinking B about the origin, $h_{\bar\partial,\,t} = h_{\bar\partial,\,0}$ for every $t \in B$. Therefore by Theorem 2.10 (ii) the family $(\Delta^{-1}_{\bar\partial,\,t})_{t\in B}$ is a C^∞ family of linear operators acting on $(0,\ 1)$-forms. On the other hand from (a) one can imply that the family $(\rho^{0,\,2}_{\omega_t})_{t\in B}$ is a C^∞ family of $(0,\ 2)$-forms. Hence there exists a $\beta_0 = (\Delta_{\bar\partial,\,0})^{-1}\bar\partial^\star_0\rho^{0,\,2}_{\omega_0} \in C^\infty_{0,\,1}(X_0, \mathbb{C})$ such that the family $(\beta^{\min}_t)_{t\in B}$ is a C^∞ family of $(0,\ 1)$-forms. In other words

$$\lim_{t\longrightarrow 0}\beta^{\min}_t = \beta_0. \tag{3.8}$$

From equations (3.7) and (3.8) we get

$$\bar\partial_0\beta_0 = \bar\partial_0 \lim_{t\to 0}\beta_t^{\min} \overset{(I)}{=} \lim_{t\to 0}\bar\partial_t\beta_t^{\min} \overset{(II)}{=} \lim_{t\to 0}\bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} \overset{(III)}{=} \lim_{t\to 0}\rho_{\omega_t}^{0,2}.$$

In the above equation, (I) comes from the fact that the family $(\bar\partial_t)_{t\in B}$ is a C^∞ family of smooth linear operators so it commutes with lim, (II) comes from the definition of $\beta_t^{\min}$ in equation (3.7) and finally, we have (III) because

$$\begin{aligned}\rho_{\omega_t}^{0,2} &= \Delta_{\bar\partial,t}(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} = (\bar\partial_t^\star\bar\partial_t + \bar\partial_t\bar\partial_t^\star)(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2}\\ &= \bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} + \bar\partial_t^\star\bar\partial_t(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2},\end{aligned} \tag{3.9}$$

first note that

$$\bar\partial^\star\bar\partial\Delta_{\bar\partial} = \bar\partial^\star\bar\partial(\bar\partial^\star\bar\partial + \bar\partial\bar\partial^\star) = \bar\partial^\star\bar\partial\bar\partial^\star\bar\partial = \bar\partial^\star\bar\partial\bar\partial^\star\bar\partial + \bar\partial\bar\partial^\star\bar\partial^\star\bar\partial = (\bar\partial^\star\bar\partial + \bar\partial\bar\partial^\star)\bar\partial^\star\bar\partial = \Delta_{\bar\partial}\bar\partial^\star\bar\partial,$$

so $(\Delta_{\bar\partial,t})^{-1}$ commutes with $\bar\partial^\star\bar\partial$, hence in equation (3.9) one gets

$$\bar\partial_t^\star\bar\partial_t(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} = (\Delta_{\bar\partial,t})^{-1}\bar\partial_t^\star\bar\partial_t\rho_{\omega_t}^{0,2}$$

and since $\bar\partial_t\rho_{\omega_t}^{0,2} = 0$, $(\Delta_{\bar\partial,t})^{-1}\bar\partial_t^\star\bar\partial_t\rho_{\omega_t}^{0,2}$ vanishes, so

$$\bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} + \bar\partial_t^\star\bar\partial_t(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2} = \bar\partial_t\bar\partial_t^\star(\Delta_{\bar\partial,t})^{-1}\rho_{\omega_t}^{0,2}.$$

From (a) one can see that the family $(\rho_{\omega_t}^{0,2})_{t\in B}$ is a C^∞ family of $(0,\ 2)$-forms. Which means that

$$\lim_{t\to 0}\rho_{\omega_t}^{0,2} = \rho_{\omega_0}^{0,2}.$$

This proves (b). □

In Proposition 3.5, not only did we prove that the $\bar\partial$-exactness for the family $\rho_{\omega_t}^{0,2}$ is a closed property under holomorphic deformations but also we showed that the family $(\beta_t^{\min})_{t\in B}$ of minimal $L^2_{\omega_t}$ solutions is a C^∞ family of $(1,\ 0)$-forms and the existence of a minimal $L^2_{\omega_t}$ solution is closed property under holomorphic deformations.

From now on we focus on the Dinew-Popovici energy functional F defined in section 1 and its critical points. In the following we give a proof for Proposition 1.8. We show that for a fix Hermitian-symplectic metric ω being a critical point for Dinew-Popovici energy functional F is a closed property under holomorphic deformations.

Proof of Proposition 1.8. From Proposition 2.11 for every $t\in B$ and for every $(1,1)$-form $\gamma_t = \bar\partial_t u_t + \partial_t \bar u_t$, one gets

$$(d_{\omega_t}F_t)(\gamma_t) = -2\,\mathrm{Re}\,\langle\langle u_t,\ \bar\partial_t^\star\omega_t\rangle\rangle_{\omega_t} + 2\,\mathrm{Re}\int_{X_t} u_t\wedge\rho_{\omega_t}^{2,0}\wedge\overline{\rho_{\omega_t}^{2,0}}\wedge\bar\partial_t\frac{\omega_t^{n-3}}{(n-3)!}.$$

Since ω_t is a critical point of F_t for $t\in B\backslash\{0\}$, $(d_{\omega_t}F_t)(\gamma_t) = 0$ for every $\gamma_t\in C^\infty_{1,1}(X_t,\mathbb{C})$. Also by assumption (2) and Proposition 3.5 the family $(\rho_{\omega_t}^{0,2})_{t\in B}$ is a C^∞ family of $(0,\ 2)$-forms. It is obvious that the smooth J_t-$(1,0)$ forms u_t on X_t determine $d_{\omega_t}F_t$. Define for every $t\in B$,

$$T_t : C^\infty_{1,0}(X_t,\mathbb{C}) \longrightarrow \mathbb{R} \qquad\qquad T_t(u_t) = G_t(u_t) + H_t(u_t),$$

where

$$G_t(u_t) = -2\,\mathrm{Re}\,\langle\langle u_t,\ \bar\partial_t^\star\omega_t\rangle\rangle_{\omega_t}$$

and

$$H_t(u_t) = 2\,\mathrm{Re}\int_{X_t} u_t\wedge\rho_{\omega_t}^{2,0}\wedge\overline{\rho_{\omega_t}^{2,0}}\wedge\bar\partial_t\frac{\omega_t^{n-3}}{(n-3)!}.$$

In order to prove that ω_0 is a critical point for F_0, it is sufficient to show that the family $(T_t)_{t\in B}$ is a C^∞ family of linear operators. Therefore it is sufficient to consider a C^∞ family of $(1,0)$-forms $(u_t)_{t\in B}$ and show that $T_t(u_t) = 0$ for all $t \in B$.

Now suppose that the C^∞ family of $(1,0)$-forms $(u_t)_{t\in B}$ is given and B is sufficiently shrunk about the origin. For every $t \in B$,

$$\bar\partial_t^\star : C^\infty_{1,1}(X_t, \mathbb{C}) \longrightarrow C^\infty_{1,0}(X_t, \mathbb{C}),$$

is a smooth linear operator and the family $(\bar\partial_t^\star)_{t\in B}$ is a C^∞ family of linear operators. Also for every $t \in B$, the map

$$\langle\langle\ ,\ \omega_t\rangle\rangle_{\omega_t} : C^\infty_{1,1}(X_t, \mathbb{C}) \longrightarrow \mathbb{C}, \qquad \langle\langle\ ,\ \omega_t\rangle\rangle_{\omega_t}(\alpha) = \langle\langle\alpha,\ \omega_t\rangle\rangle_{\omega_t}$$

is a smooth linear map and the family $(\langle\langle\ ,\ \omega_t\rangle\rangle_{\omega_t})_{t\in B}$ is a C^∞ family of linear operators. So for every $t \in B$, G_t is a smooth linear operator and the family $(G_t)_{t\in B}$ is a C^∞ family of linear operators. In other words

$$\lim_{t\to 0} G_t(u_t) = G_0(u_0) = -2\,\mathrm{Re}\,\langle\langle u_0,\ \bar\partial_0^\star\omega_0\rangle\rangle_{\omega_0}. \tag{3.10}$$

We show that the family $(H_t)_{t\in B}$ is a C^∞ family of linear operators. First it is obvious that for every $t \in B$, the map

$$\bar\partial_t : C^\infty_{n-3,n-3}(X_t, \mathbb{C}) \longrightarrow C^\infty_{n-3,n-2}(X_t, \mathbb{C}),$$

is a smooth linear operator and the family $(\bar\partial_t)_{t\in B}$ is a C^∞ family of linear operators. On the other hand, the family $(\omega_t)_{t\in B}$ is a C^∞ family of metrics, henceforth the family $(\frac{\bar\partial\omega_t^{n-3}}{(n-3)!})_{t\in B}$ is a C^∞ family of $(n-3,\ n-2)$-forms. Also, assumption (2) allows us to employ Proposition 3.5 and say that both families $(\rho^{0,2}_{\omega_t})_{t\in B}$ and $(\overline{\rho^{0,2}_{\omega_t}})_{t\in B}$ are C^∞ family of $(2,0)$-forms and $(0,2)$-forms respectively. Therefore for every $t \in B$ the map H_t is a smooth real-valued linear map and the family $(H_t)_{t\in B}$ is a C^∞ family of linear operators. In other words,

$$\lim_{t\to 0} H_t(u_t) = H_0(u_0) = \ \mathrm{Re}\int_{X_0} u_0 \wedge \rho^{2,0}_{\omega_0} \wedge \overline{\rho^{2,0}_{\omega_0}} \wedge \bar\partial_0 \frac{\omega_0^{n-3}}{(n-3)!}. \tag{3.11}$$

The smoothness of T_t for every $t \in B$ is implied by the smoothness of G_t and H_t, and by equations (3.10) and (3.11) one can get

$$T_0(u_0) = \lim_{t\in B} T_t(u_t) = \lim_{t\in B} G_t(u_t) + \lim_{t\in B} H_t(u_t) = 0.$$

Which means that ω_0 is a critical point of F_0. □

In section 2.3 we saw that in dimension 3 the explicit formula for differential of the Dinew-Popovici energy functional F a ω is simpler than in higher dimensions. In the next result of this article we give a proof to Proposition 1.6, where we compute the differential of F at ω, when ω is a fixed Hermitian-symplectic metric on compact complex manifold X of dimension n and the $(2,0)$-torsion form $\rho^{2,0}_\omega$ is ∂-exact.

Proof of Proposition 1.6. First note that since $\omega \in \mathcal{S}_{\{\omega_0\}}$, it is a Hermitian-symplectic metric on X so the $(2,0)$-torsion form $\rho^{2,0}_\omega$ satisfies, $\bar\partial\omega = -\partial\overline{\rho^{2,0}_\omega}$ and $\partial\omega = -\bar\partial\rho^{2,0}_\omega$ and $\bar\partial\overline{\rho^{2,0}_\omega} = 0$. On the other hand, since X is a compact complex manifold it has no boundary so for every $(n-1,\ n)$-form α and every $(n,\ n-1)$-form β

$$\int_X \partial\alpha = 0 \quad \text{and} \quad \int_X \bar\partial\beta = 0$$

by the Stokes' theorem. From (2.4), one can observe that when $\gamma = \bar\partial\xi + \partial\bar\xi$ the differential at ω of F evaluated on γ is

$$(d_\omega F)(\gamma) = -2\,\mathrm{Re}\,\langle\langle\xi,\ \bar\partial^\star\omega\rangle\rangle_\omega + 2\,\mathrm{Re}\int_X \xi \wedge \rho^{2,0}_\omega \wedge \overline{\rho^{2,0}_\omega} \wedge \bar\partial\omega_{n-3}.$$

First we compute $\langle\langle\xi,\ \bar\partial^\star\omega\rangle\rangle_\omega$. By the definition of the L^2_ω inner product ($u \wedge \star\bar v = \langle u,\ v\rangle_\omega dV_\omega$), we have

$$\langle\langle\xi,\ \bar\partial^\star\omega\rangle\rangle_\omega = \int_X \langle\xi,\ \bar\partial^\star\omega\rangle_\omega dV_\omega = \int_X \xi \wedge \star\partial^\star\omega. \tag{3.12}$$

By standard computation for the Hodge star operator $-\star\star = id$ on odd-degree forms and by equation (2.1), one gets

$$\star\partial^\star\omega = -\star\star\bar{\partial}\star\omega = \bar{\partial}\omega_{n-1}, \tag{3.13}$$

hence by equations (3.12) and (3.13),

$$\langle\langle\xi, \bar{\partial}^\star\omega\rangle\rangle_\omega = \int_X \xi\wedge\bar{\partial}\omega_{n-1}. \tag{3.14}$$

Now equation (3.14) allows us to compute $\langle\langle\xi, \bar{\partial}^\star\omega\rangle\rangle_\omega$. We get

$$\langle\langle\xi, \bar{\partial}^\star\omega\rangle\rangle_\omega = \int_X \xi\wedge\bar{\partial}\omega_{n-1} = \int_X \xi\wedge\bar{\partial}\omega\wedge\omega_{n-2} = -\int_X \xi\wedge\partial\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2}. \tag{3.15}$$

By the Stokes' theorem $0 = \int_X \partial(\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2})$, so

$$0 = \int_X \partial\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2} - \int_X \xi\wedge\partial\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2} - \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega_{n-2}.$$

Therefore,

$$-\int_X \xi\wedge\partial\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2} = \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega_{n-2} - \int_X \partial\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2}.$$

By assumption $\rho_\omega^{2,0} = \partial\xi$ so,

$$\int_X \partial\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2} = \int_X \rho_\omega^{2,0}\wedge\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2}.$$

Since $\rho_\omega^{2,0}$ is a primitive form of bidegree $(2, 0)$, we can apply (2.2) and we get:

$$\int_X \rho_\omega^{2,0}\wedge\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2} = \int_X \rho_\omega^{2,0}\wedge\star\overline{\rho_\omega^{2,0}} = \langle\langle\rho_\omega^{2,0}, \rho_\omega^{2,0}\rangle\rangle_\omega = \|\rho_\omega^{2,0}\|_\omega^2.$$

Hence

$$-\int_X \xi\wedge\partial\overline{\rho_\omega^{2,0}}\wedge\omega_{n-2} = \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega_{n-2} - \|\rho_\omega^{2,0}\|_\omega^2. \tag{3.16}$$

Now, the goal is to compute $\int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega_{n-2}$ in equation (3.16). We get:

$$\int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega_{n-2} = \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega\wedge\omega_{n-3} = -\int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\bar{\partial}\rho_\omega^{2,0}\wedge\omega_{n-3}. \tag{3.17}$$

Again, by the Stokes' theorem, $\int_X \bar{\partial}(\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\omega_{n-3}) = 0$ and because $\bar{\partial}\overline{\rho_\omega^{2,0}} = 0$, we have

$$0 = \int_X \bar{\partial}\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\omega_{n-3} - \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\bar{\partial}\rho_\omega^{2,0}\wedge\omega_{n-3} - \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\bar{\partial}\omega_{n-3}. \tag{3.18}$$

Therefore, from (3.18) and (3.17) one can deduce the following equation

$$\int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\partial\omega_{n-2} = -\int_X \bar{\partial}\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\omega_{n-3} + \int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\bar{\partial}\omega_{n-3}. \tag{3.19}$$

By putting equations (3.15), (3.16) and (3.19) together we see that

$$-2\,\mathrm{Re}\langle\langle\xi, \bar{\partial}^\star\omega\rangle\rangle_\omega = 2\,\mathrm{Re}\int_X \bar{\partial}\xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\omega_{n-3} - 2\,\mathrm{Re}\int_X \xi\wedge\overline{\rho_\omega^{2,0}}\wedge\rho_\omega^{2,0}\wedge\bar{\partial}\omega_{n-3} + 2\|\rho_\omega^{2,0}\|_\omega^2. \tag{3.20}$$

By adding $2\,\mathrm{Re}\int_X \xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \bar\partial\omega_{n-3}$ to equation (3.20) and by using (2.4) with $u = \xi$ we get the formula (1.3). This proves the proposition. □

In formula (1.3), $2\mathrm{Re}\int_X \bar\partial\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3}$ is signless in general. However, if it supposes to be non-negative one sees immediately that ω is a Kähler metric whenever it is a critical point for F. In the following proof, we show that if $\bar\partial\xi$ is a weakly semi-positive $(1,\ 1)$-form then $2\mathrm{Re}\int_X \bar\partial\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3}$ is non-negative.

Proof of Corollary 1.7. Since positivity is a pointwise property, one can fix a point $x \in X$ and local coordinates $(z_1, \dots, z_n)$ centered at x such that ω has the following shape

$$\omega = \sum i dz_i \wedge d\bar z_i \qquad \text{at } x$$

In particular, ω is a strongly strictly positive $(1,\ 1)$-form. By Definition 2.1 (1) and Proposition 2.3, ω_{n-3} is a strongly strictly positive $(n-3,\ n-3)$-form. On the other hand by Example 1.2 of [4], for every $p \in \{1, \dots, n\}$ and any $(p,\ 0)$-form β, the $(p,\ p)$-form $i^{p^2}\beta \wedge \bar\beta$ is weakly strictly positive. Hence the $(2,\ 2)$-form

$$i^4\rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} = \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}}$$

is weakly strictly positive.
Since $\bar\partial\xi$ is weakly semi-positive $(1,\ 1)$-form, there exist real non negative functions $c_1, \dots, c_n$ and $(1,\ 0)$-forms $\alpha_1, \dots, \alpha_n$ such that

$$\bar\partial\xi = \sum c_k i\alpha_k \wedge \bar\alpha_k.$$

Therefore

$$\begin{aligned}\bar\partial\xi \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3} &= \sum c_k i\alpha_k \wedge \bar\alpha_k \wedge \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge \omega_{n-3} \\ &= \sum c_k \rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar\alpha_k \wedge \omega_{n-3}.\end{aligned}$$

Note that by Definition 2.1 $\alpha_k \wedge \bar\alpha_k$ is strongly strictly positive $(1,\ 1)$-form for all $k \in \{1, \dots,\ n\}$. By Definition 2.1 (2), $c_k\rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar\alpha_k \wedge \omega_{n-3}$ is a weakly semi-positive $(n,\ n)$-form. Hence

$$2\mathrm{Re}\int_X \sum c_k\rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar\alpha_k \wedge \omega_{n-3} = \sum 2\mathrm{Re}\int_X c_k\rho_\omega^{2,0} \wedge \overline{\rho_\omega^{2,0}} \wedge i\alpha_k \wedge \bar\alpha_k \wedge \omega_{n-3} \geqslant 0$$

This proves the Corollary. □

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References

[1] L. Alessandrini, G. Bassanelli, *Small deformations of a class of compact non-Kähler manifolds*, Proc. Amer. Math. Soc. 109, no. 4, (1990), 1059–1062.

[2] H. Bellitir, *Deformation stability of p-SKT and p-HS manifolds*. European Journal of Mathematics 6, (2020), 1403–1423.

[3] M. Ceballos, A. Otal, L. Ugarte, R. Villacampa, *Invariant complex structures on 6-nilmanifolds: classification*, Frölicher spectral sequence and special Hermitian metrics. J. Geom. Anal. 26, (2016), 252–286.

[4] J. P. Demailly *Complex Analytic and Algebraic Geometry*, http://www-fourier.ujf-grenoble.fr/ demailly/books.html.

[5] P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. 29, no. 3, (1975), 245–274.

[6] S. Dinew, D. Popovici, *A Generalized Volume Invariant for Aeppli Cohomology Classes of Hermitian-symplectic Metrics*, Advances in Mathematics, Vol 393, 24 December (2021), 108056.
[7] C. Ehresmann *Sur les espaces fibrés différentiables*, C. R. Acad. Sci. Paris 224, (1947), 1611-1612.
[8] A. Fino, M. Parton, and S, Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv.79, no. 2, (2004), 317–340.
[9] H. Hironaka, *On the theory of birational blowing up*, Thesis, Harvard, (1960).
[10] H. Hironaka, *An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures.* Ann. Math. 75, (1962), 190–208.
[11] D. Huybrechts, *Complex Geometry: An Introduction*, Universitext, Springer-Verlag, New York, (2005).
[12] K. Kodaira, *Complex Manifolds and Deformations of Complex Structures*, Grundlehrender Math. Wiss. 283, Springer, (1986).
[13] K. Kodaira, D. C. Spencer, *On Deformations of Complex Analytic Structures, I*, Annals of Mathematics , Mar., 1958, second series, Vol. 67, no. 2 (Mar., 1958), 328-401.
[14] K. Kodaira, D. C. Spencer, *On Deformations of Complex Analytic Structures, III. Stability Theorems for Complex Structures*, Annals of Mathematics , Jan., (1960), Second Series, Vol. 71, no. 1 (Jan., 1960), 43-76.
[15] D. Popovici, *Aeppli Cohomology Classes Associated with Gauduchon Metrics on Compact Complex Manifolds*, Bull. Soc. Math. Fr. 143, (2015).
[16] D. Popovici, *Deformation Openness and Closedness of Various Classes of Compact Complex Manifolds; Examples*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XIII, (2014), 255-305.
[17] D. Popovici, *Deformation Limits of Projective Manifolds: Hodge Numbers and Strongly Gauduchon Metrics*, Invent. Math. 194, (2013), 515-534.
[18] D. Popovici, *Limits of Moishezon Manifolds under Holomorphic Deformations*, arXiv e-print math.AG/1003.3605v1m, (2010).
[19] D. Popovici, *Limits of Projective Manifolds Under Holomorphic Deformations*, arXiv:0910.2032, (2020).
[20] D. Popovici, J. Stelzig, and L. Ugarte. *Higher-page Bott–Chern and Aeppli cohomologies and applications*, für die reine und angewandte Mathematik (Crelles Journal), no. 777, (2021), 157-194.
[21] J. Streets, G. Tian, *A Parabolic Flow of Pluriclosed Metrics*, Int. Math. Res. Notices, 16, (2010), 3101-3133.
[22] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, Cambridge Studies in Advanced Mathematics, 76, Cambridge University Press, Cambridge, (2002).
[23] A. Otal, L. Ugarte, R. Villacampa, Hermitian metrics on compact complex manifolds and their deformation limits. *Special metrics and group actions in geometry*, 269–290, Springer INdAM Ser., 23, Springer, Cham, (2017).
[24] C. C. Wu, *On the Geometry of Superstrings with Torsion*, PhD Thesis, Harvard University, (2006).
[25] S. Yang, *On blow-ups and resolutions of Hermitian-symplectic and strongly Gauduchon metrics*. Arch. Math. (Basel) 104, (2015), 441–450.