

Research Article

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Pluricanonical Maps and the Fujita Conjecture

<https://doi.org/10.1515/coma-2021-0136>

Received June 11, 2022; accepted June 26, 2022

Abstract: We describe examples showing the sharpness of Fujita’s conjecture on adjoint bundles also in the general type case, and use these examples to formulate related bold conjectures on pluricanonical maps of varieties of general type.

Keywords: Pluricanonical maps, adjoint bundles, Fujita conjecture, weighted hypersurfaces, Cartier canonical divisor

MSC: 14E05, 14E25, 32Q40

Introduction and history of the problem.

The celebrated Fujita’s conjecture on adjoint bundles [13] predicts that, if H is an ample divisor on a smooth projective variety X of dimension n , then $D := K_X + (n + 1)H$ is spanned (that is, $\mathcal{O}_X(D)$ is generated by global sections), and $K_X + (n + 2)H$ is very ample. Fujita observed that, if X is projective space $\mathbb{P}^n$, and H is the divisor of a hyperplane, then $K_X = -(n + 1)H$ hence his conjecture is sharp.

I describe here some series of examples that I discovered some 20 years ago, ¹ which show the sharpness of the celebrated Fujita’s conjecture on adjoint bundles also in the case where X is a variety of general type with ample canonical divisor K_X (here we take $H := K_X$).

The motivation for this short note came from reading the review of the paper [5], which shows the birationality of the pentacanonical map for threefolds of general type with a Gorenstein minimal model.

Our examples are constructed just as an extension to higher dimensions of some surfaces classified by Enriques in Chapter VIII of his book [11], and which, by work of Bombieri [4], are characterized exactly as the surfaces of general type with the worst possible behaviour of the pluricanonical maps.

These examples show the sharpness of some bold question-conjectures which we propose here, which are easily seen to be some sort of generalization of the Fujita conjecture [13] on adjoint bundles in the general type case (part b) below is due to Meng Chen).

Conjecture 1. Let X be a variety of general type, which is minimal, of dimension n and with at worst $\mathbb{Q}$ -factorial terminal singularities and such that the canonical divisor K_X is a Cartier divisor: then

- a) the pluricanonical map associated to $|mK_X|$ is birational onto its image as soon as $m \geq n + 3$,
- a’) the m -th pluricanonical map yields an embedding of the canonical model as soon as $m \geq n + 3$
- b) if $K_X^n = \text{Vol}(X) \geq 2$, then the pluricanonical map associated to $|mK_X|$ is birational onto its image as soon as $m \geq n + 2$.
- b’) if $K_X^n = \text{Vol}(X) \geq 2$, then the m -th pluricanonical map yields an embedding of the canonical model as soon as $m \geq n + 2$.

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¹ and which I communicated verbally to several people

Remark 2. (I) Assume that X is smooth of dimension n and that K_X is an ample divisor.

Then the above conjecture is implied by the Fujita conjecture on adjoint bundles: if H is an ample divisor, then $K_X + (n + 2)H$ is very ample.

(II) Alternatively, one may formulate the conjecture with the assumption that X is the canonical model of a variety of general type, and that K_X is a Cartier divisor.

Recall that Hacon and McKernan [14] proved that for each dimension n there is a constant $c(n)$ such that for each variety X of general type of dimension n the pluricanonical map associated to $|mK_X|$ is birational onto its image as soon as $m \geq c(n)$.

Moreover, we know from [3] that the canonical ring

$$\mathcal{R}(X) = \bigoplus_m H^0(X, \mathcal{O}_X(mK_X))$$

of a variety of general type is finitely generated, hence there exists the canonical model $Y := \text{Proj}(\mathcal{R}(X))$, which has canonical singularities, and the volume $\text{Vol}(X)$ is also equal to K_Y^n (Y is $\mathbb{Q}$ -Gorenstein). We also know from ibidem that X admits a minimal model, that is, a model with terminal singularities, in particular $\mathbb{Q}$ -Gorenstein.

In general, see [15] page 359 for examples in dimension 3, the question to determine $c(n)$ is complicated by the fact that the volume need not be an integer: for this reason one should first of all restrict oneself to the case where $\text{Vol}(X)$ is an integer.

This is not enough, as observed by Chen Jiang, who gave the following example.

Example 1. (Chen Jiang)

Take X to be the Reid-Fletcher hypersurface $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$, which has volume $\text{Vol}(X) = \frac{1}{420}$ and canonical stability index 27.

Take further a smooth curve C of genus 421.

Let $Y := X \times C$. Then Y is a minimal 4-fold of general type with canonical volume $\text{Vol}(Y) = 8$, an integer. However $7K_Y$ does not yield a birational map. More high dimensional examples can be produced by the same procedure.

Our first series of examples are varieties of general type which show that both conjectures are sharp: questions a) a') and the Fujita conjecture (for the latter, we have mentioned that it is sharp for projective space, but it is useful to produce other examples, of general type, where it is sharp).

Indeed, for even dimension n , we get smooth varieties X for which K_X is ample and $K_X + (n + 1)H = (n + 2)K_X$ is not birational onto its image.

For odd dimension instead $(n + 2)K_X$ is not birational onto its image, but here X is canonical with terminal singularities which are not Gorenstein, they are only 2-Gorenstein.

In particular, we get threefolds of general type with volume equal to 1, such that the 5-canonical map is not birational. The main result of [5] is that the 5-canonical map is birational if X is Gorenstein.

The result shows the subtlety of the higher dimensional geometry, where minimal models need not be Gorenstein: as observed by Meng Chen, under the Gorenstein assumption, the volume must be even because of the Riemann-Roch theorem, hence $\text{Vol}(X) \geq 2$ and the result confirms part b) of the conjecture.

Our 'conjecture' is a theorem obviously for dimension $n = 1$, here the smooth curves of genus 2 show that the bicanonical map need not be birational, while the tricanonical map is always an embedding.

For surfaces, as already mentioned, the conjecture is again a theorem, due to Bombieri [4], and the surfaces which show that the 4-canonical map need not be a birational embedding are exactly the double covers of a quadric cone branched on the section with a degree 5 surface (these examples are to be found in section 14 of Chapter VIII, pages 303-305 of [11]) while the other surfaces for which the 3-canonical map need not be a birational embedding are the double covers of the plane branched on a curve of degree 8 (see section 17, Chapter VIII, pages 311-312 of [11]).

Our second series of examples are smooth varieties of general type X with K_X ample, and show that conjectures b), b') are sharp. They also show that for Fujita's conjecture one may also ask whether $K_X + (n+1)H$ is very ample once we know that $H^n \geq 2$.

In higher dimension, the general case of varieties of general type is quite complicated: already for threefolds of general type whose minimal model is not Gorenstein the situation is intricate: improving on [7], [8] proves birationality for $m \geq 57$, else one needs some other hypotheses, for instance the hypothesis of positive irregularity [6] [9].

Also concerning the Fujita conjecture there are until now no counterexamples, and the best result was obtained by Angehrn and Siu [1], showing that $K_X + mH$ is very ample for $m \geq \frac{1}{2}(n^2 + n + 2)$.

1 A series of hypersurfaces in weighted projective space

For simplicity we stick to algebraic varieties defined over $\mathbb{C}$, since the above conjectures are meant for geometry over the complex numbers: but the examples make perfect sense over any algebraically closed field k of $\text{char}(k) \neq 2$ ($\text{char}(k) = 2$ requires some minor modification).

Example 2. For each n , consider the weighted projective space $\mathbb{P}(1^n, 2, n+3)$, with coordinates

$$(x, y, z) := (x_1, \dots, x_n, y, z),$$

and choose X to be a general hypersurface of degree $2(n+3)$,

$$X := \{(x, y, z) | z^2 = F(x, y)\}.$$

1. X is a double cover of $\mathbb{P}(1^n, 2)$, which is the projective cone over the quadratic Veronese embedding of $\mathbb{P}^{n-1}$, which therefore is smooth, except at the point $x = 0$, where we have the 'Kummer' singularity $\mathbb{C}^n / \pm 1$. The singularity is Gorenstein if and only if n is even; it is terminal for $n \geq 2$, since the minimal resolution is the quotient of the blow up of $\mathbb{C}^n$ at the origin, hence it carries an exceptional divisor $E \cong \mathbb{P}^{n-1}$ with normal bundle $= \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$. Therefore the canonical divisor of the resolution is the pull back of the canonical divisor of X plus $-\frac{1}{2}(n-2)E$.
2. The double covering is branched on the hypersurface $F(x, y) = 0$, which for general choice of F is smooth and does not contain the singular point of $\mathbb{P}(1^n, 2)$. Hence X is smooth except possibly for the points lying above the point $x = 0$.
3. X is generally smooth for n even, but it has two Kummer singularities if n is odd. In fact, over the point $x = 0$, we get the points defined by $z^2 = \lambda y^{n+3}$, where λ is the coefficient of y^{n+3} in F ; there seems to be two distinct solutions for $\lambda \neq 0$, $y = 1$, $z = \pm 1$, but, for n even, $n+3$ is odd, hence the diagonal action of -1 sends $y \mapsto y$, $z \mapsto -z$. Hence, we get two Kummer singularities if n is odd, but for n even we get a double covering ramified exactly at the singular point, hence we get a smooth point.
4. Adjunction (see [10]) shows that $\omega_X = \mathcal{O}_X(1)$, hence for X even we have smooth variety with K_X ample; for n odd we have that X is terminal and 2-Gorenstein, as seen above.
5. $H^0(\mathcal{O}_X(m))$ yields a birational map if and only if $m \geq n+3$, which is indeed an embedding.

The next series extends to higher dimension the other surfaces considered by Enriques, which require $m \geq 4$:

Example 3. For each n , consider the weighted projective space $\mathbb{P}(1^{n+1}, n+2)$, with coordinates

$$(x, z) := (x_1, \dots, x_n, x_{n+1}, z),$$

and choose X to be a general hypersurface of degree $2(n+2)$,

$$X := \{(x, z) | z^2 = F(x)\}.$$

1. X is a double cover of $\mathbb{P}^n$, branched on a hypersurface of degree $2n+4$, hence X is generally smooth.

2. Adjunction (see [10]) shows that $\omega_X = \mathcal{O}_X(1)$, hence we have smooth variety with K_X ample;
3. $H^0(\mathcal{O}_X(m))$ yields a birational map if and only if $m \geq n + 2$, which is indeed an embedding.

2 Post Scriptum

The Fujita conjecture is true for reduced curves, but not for non-reduced curves which are not numerically connected: take an elliptic curve E in a smooth algebraic surface S with selfintersection $E^2 = -d$, $d \geq 3$, set $C := 2D$, and take a divisor H with $H \cdot E = 1$ (S may just be the blow up of $E \times \mathbb{P}^1$).

Then on C we have $3H + K_C = 3H + K_S + C = 3H + K_S + 2E$ and its intersection with E equals $3 + E^2 = 3 - d$, hence this linear system is not even ample as soon as $d \geq 3$.

Added in proof: Burt Totaro mentioned to me that similar examples are to be found in Theorem 3 of [2], which contains also other examples where the minimal model does not have a Cartier canonical divisor K_X . Under the condition that the minimal models have a canonical divisor K_X which is not Cartier, then there are recent more extreme and striking examples by Esser, Wang, and Totaro; these examples are such that there are about $2^{2^{n/2}}$ vanishing plurigenera in dimension n , see Theorem 1.1 of [12] and ensuing discussion.

Acknowledgement: I would like to thank Gavin Brown for trying to answer many years ago my queries about the search for other hypersurfaces, respectively complete intersections, in weighted projective spaces, which would show the sharpness of Fujita's conjecture or even disprove it.

Many thanks to Meng Chen for very useful comments on a first draft of this note, and especially for contributing part b) of Conjecture 1. Thanks to Chen Jiang for the permission to include his example in this note. Thanks to the referee and to Burt Totaro for interesting comments on the first version of this article.

Conflict of interest: The author states that there is no conflict of interest.

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