Complex Manifolds 2022; 9:114–137 DE GRUYTER

Research Article **Open Access**

Stephane Tchuiaga*, Franck Houenou, and Pierre Bikorimana

On Cosymplectic Dynamics I

https://doi.org/10.1515/coma-2021-0132
Received January 28, 2022; accepted March 24, 2022

Abstract: This paper is an introduction to cosymplectic topology. Through it, we study the structures of the group of cosymplectic diffeomorphisms and the group of almost cosymplectic diffeomorphisms of a cosymplectic manifold (M, ω, η) : (i)– we define and present the features of the space of almost cosymplectic vector fields (resp. cosymplectic vector fields); (ii)– we prove by a direct method that the identity component in the group of all cosymplectic diffeomorphisms is C^0–closed in the group $\mathrm{Diff}^\infty(M)$ (a rigidity result), while in the almost cosymplectic case, we prove that the Reeb vector field determines the almost cosymplectic nature of the C^0–limit ϕ of a sequence of almost cosymplectic diffeomorphisms (a rigidity result). A sufficient condition based on Reeb's vector field which guarantees that ϕ is a cosymplectic diffeomorphism is given (a flexibility condition), the cosymplectic analogues of the usual symplectic capacity-inequality theorem are derived and the cosymplectic analogue of a result that was proved by Hofer-Zehnder follows.

Keywords: Rigidity results, Convergence in general topology (sequences, filters, limits, convergence spaces, nets, etc.), Dynamical systems involving smooth mappings and diffeomorphisms, Dynamics in general topological spaces

MSC2020: 53C24, 54A20, 37C05, 37B02.

1 Introduction

In most of the numerous formulations of time-dependent mechanics, cosymplectic manifolds play a major role: either in the classical descriptions of regular Lagrangian systems (see [11]) and of Hamiltonian systems [6, 11], or in more detailed descriptions of Lagrangian submanifolds (see [9, 12]). These manifolds have been at the center of interest of several other authors starting from the works of Liberman [14] to those of Li [13]. An interesting result due to Li [13] shows how moving from a cosymplectic structure, one can always generate a symplectic structure, and vice-versa.

On the other hand, a common aspect to both symplectic geometry and cosymplectic geometry is that the symplectic (resp. the cosymplectic) structure induces an isomorphism between the space of vector fields and the space of 1–forms of the underlined manifold : this is a fundamental aspect. In the context of symplectic geometry, the corresponding isomorphism is used to describe some structures of the group of symplectomorphisms from the space of closed 1–forms of the symplectic manifold.

As far as we know, almost nothing is known about the structure of the group of cosymplectic diffeomorphisms of a cosymplectic manifold (one doesn't know how to fragment an arbitrary co-Hamiltonian vector field of a cosymplectic manifold with respect to an open cover of the manifold; how to describe accurately the dynamics of a cosymplectic manifold from its group of cosymplectic diffeomorphisms; how to quantify

***Corresponding Author: Stephane Tchuiaga:** Department of Mathematics, University of Buea, Po.Box 63, Buea- Cameroon, Cameroon, E-mail: tchuiagas@gmail.com
Franck Houenou: Department of Mathematics, University of Abomey Calavi, Bénin, E-mail: rdjeam@gmail.com
Pierre Bikorimana: Institut de Mathématiques et de Sciences Physiques, University of Abomey Calavi, Po.box 613, Benin, Bénin, E-mail: pierrebikorimana@gmail.com

the energy needed by a cosymplectic diffeomorphism in order to displace a given open subset; how rigid is the cosymplectic structure with respect to the C^0–limit and so on.)

The goal of the present paper is to adapt some methods from symplectic topology to characterize, and then to study several subgroups of diffeomophisms of a cosymplectic manifold.

We organize this paper as follows. In Section 2, we recall the definition of cosymplectic vector space, cosymplectic linear group, and cosymplectic manifolds. Section 3 deals with vector fields of a cosymplectic manifold. Mainly, in Subsection 3.1, we define and study cosymplectic vector fields and in Subsection 3.2, we introduce and study the algebra of almost cosymplectic vector fields. Section 4 deals with the study of almost cosymplectic diffeomorphisms, cosymplectic diffeomorphisms, almost cosymplectic isotopies, and cosymplectic isotopies. Lemma 4.1 shows how the Reeb vector field of a cosymplectic manifold transforms under the push forward by a cosymplectic diffeomorphism (resp. by an almost cosymplectic diffeomorphism), while Lemma 4.2 is a slight generalization of Lemma 4.1 between two arbitrary cosympletic manifolds, and Proposition 4.1 characterizes the Lie algebras of some cosympletic and almost cosymplectic subspaces. In Subsection 4.1 we show how cosymplectic (resp. almost cosymplectic) geometry varies under the composition and inversion of cosymplectic (resp. almost cosymplectic) isotopies. In Subsection 4.2, we compare cosymplectic isotopies with symplectic isotopies, and characterize periodic orbits in weakly Hamiltonian dynamical systems. In Subsection 4.3, we compare almost cosymplectic isotopies with symplectic isotopies (Propositions 4.2, 4.3, 4.4, and 4.5) prove a theorem showing that the Reeb vector field determines the almost cosymplectic nature of a uniform limit of a sequence of almost cosymplectic diffeomorphisms (Theorem 4.1), and we characterize periodic orbits in almost co-Hamiltonian dynamical systems. In Section 5, we define and study the cosymplectic setting of Hofer and Hofer-like geometries with respect to the group of all cosymplectic diffeomorphisms isotopic to the identity map. Here, we first start from a comparison of the uniform sup norm of a closed 1–form and that of its pull-back with respect to a projection map, next we study the co-Hofer norms, co-Hofer-like norms, we establish the cosymplectic setting of the energy-capacity-inequality from symplectic geometry, discuss on displacement energies in the fibers of a certain Cartesian product, and prove that the group of all cosympletic diffeomorphisms isotopic to the identity map is C^0–closed inside the group of all smooth diffeomorphisms (Theorems 5.1, 5.2, 5.4, and 5.5).

2 Preliminaries

2.1 Cosymplectic vector spaces

A bilinear form on a vector space V is a map $b : V \times V \longrightarrow \mathbb{R}$ which is linear in each variable. When a bilinear map b satisfies $b(u, v) = -b(v, u)$, for all $u, v \in V$, then b is called antisymmetric (or skew symmetric).

Theorem 2.1. *(***Standard form for antisymmetric bilinear maps***)*

Consider a skew-symmetric bilinear map b on V. Then there is a basis $(u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n)$ *of V such that*

1. $b(u_i, v) = 0$, *for all i and for all v ;*
2. $b(e_i, e_j) = 0 = b(f_i, f_j)$, *for all i, j ;*
3. $b(e_i, f_j) = \delta_{ij}$, *for all i, j.*

Given any non-trivial linear map $\psi : V \longrightarrow \mathbb{R}$, together with a bilinear map $b : V \times V \longrightarrow \mathbb{R}$, one defines a linear map

$$\begin{array}{rccl} \widetilde{I}_{\psi,b} : & V & \longrightarrow & V^{\star} \\ & Y & \longmapsto & \widetilde{I}_{\psi,b}(Y) := \imath_Y b + \psi(Y)\psi \end{array}$$

so that $\widetilde{I}_{\psi,b}(Y)(X) = b(Y, X) + \psi(Y)\psi(X)$, for all $X, Y \in V$.

Definition 2.1.

1. A pair (b,ψ) consisting of an antisymmetric bilinear map $b: V\times V\longrightarrow\mathbb{R}$ and a non-trivial linear map $\psi: V\longrightarrow\mathbb{R}$ is called a cosymplectic structure if the map $\widetilde{I}_{\psi,b}$ is a bijection.
2. A cosymplectic vector space is a triple (V,b,ψ) where V is a vector space and (b,ψ) is a cosymplectic structure on V.

Proposition 2.1. *Let (V,b,ψ) be a cosymplectic vector space. Then,* $\dim(V)=2n+1$.

Proof. By Theorem 2.1, we have $\dim(V)=2n+k$. So, it is enough to show that $k=1$. Let $(u_1,\cdots,u_k,e_1,\cdots,e_n,f_1,\cdots,f_n)$ be a basis of V as in Theorem 2.1. Since, $b(u_i,v)=0$, for all $1\leqslant i\leqslant k$, and for all v, we derive that $\widetilde{I}_{\psi,b}(u_i)=\psi(u_i)\psi$, for each i. This implies that

$$u_i=\widetilde{I}^{-1}_{\psi,b}(\psi(u_i)\psi)=\psi(u_i)\widetilde{I}^{-1}_{\psi,b}(\psi)=:\psi(u_i)\xi, \tag{2.1}$$

for each i, where $\xi:=\widetilde{I}^{-1}_{\psi,b}(\psi)$. Since $(u_1,\cdots,u_k,e_1,\cdots,e_n,f_1,\cdots,f_n)$ is a basis of V, then (2.1) implies that $k=1$. This completes the proof. □

Proposition 2.2. *Let (V,b,ψ) be a cosymplectic vector space. Then, there exists a unique vector $\xi\in V$ called the Reeb vector such that:*

1. $\psi(\xi)=1$,
2. $b(\xi,v)=0$, *for all* $v\in V$.

Proof. From the proof of Proposition 2.1, take $\xi:=\frac{u_1}{\psi(u_1)}$, and verify that $\psi(\xi)=\frac{\psi(u_1)}{\psi(u_1)}=1$ and $b(\xi,v)=\frac{1}{\psi(u_1)}b(u_1,v)=0$, for all $v\in V$. This completes the proof. □

Remark 2.2. Let (V,b,ψ) be a cosymplectic vector space of dimension $(2n+1)$. Then, the $(2n+1)$–multilinear map B, defined by $B:=\psi\wedge\underbrace{b\wedge\cdots\wedge b}_{n\text{-factors}}$, is non-trivial. Indeed, by Proposition 2.1, let $(\xi,e_1,\cdots,e_n,f_1,\cdots,f_n)$ be a basis of V as in Theorem 2.1 and set $v_1=\xi$, $v_{i+1}=e_i$, and $v_{n+i+1}=f_i$, $\quad 1\leqslant i\leqslant n$. We have

$$\begin{aligned} B(v_1,\cdots,v_n,v_{n+1},\cdots,v_{2n+1}) &= (\psi\wedge b\wedge\cdots\wedge b)(v_1,\cdots,v_{2n+1})\\ &= \sum_{\sigma\in S_{(2n+1)}} sign(\sigma)\psi(v_{\sigma(1)})(b\wedge\cdots\wedge b)(v_{\sigma(2)},\cdots v_{\sigma(2n+1)})\\ &= \sum_{\sigma\in S_{(2n+1)},\sigma(1)=1} sign(\sigma)(b\wedge\cdots\wedge b)(v_{\sigma(2)},\cdots v_{\sigma(2n+1)})\\ &= \varepsilon\cdot 1\cdot 2\cdots(n-1)\cdot n \end{aligned} \tag{2.2}$$

with $\varepsilon\in\{-1,1\}$, where $S_{(2n+1)}$ stands for the set of all permutations of $(2n+1)$ elements.

2.2 Cosymplectic Manifolds

Hereafter, we recall the definition of cosymplectic manifold and its link with a symplectic manifold.

Definition 2.2.
1. A cosymplectic structure on a smooth manifold M is a pair (ω,η) consisting of closed 2–form ω and closed 1–form η such that for each $x\in M$, the triple (T_xM,ω_x,η_x) is a cosymplectic vector space. Equivalently ω and η are closed forms such that $\eta\wedge\omega^n$ is nowhere vanishing top form on M.
2. A cosymplectic manifold is a triple (M,ω,η) where M is a smooth manifold and (ω,η) is a cosymplectic structure on M

For further details, we refer to [13, 14] and references therein.

In particular, Remark 2.2 tells us that any cosymplectic manifold (M, ω, η) is oriented with respect to the volume form $\eta \wedge \omega^n$, while by Proposition 2.2, any cosymplectic manifold (M, ω, η) admits a vector field ξ called the Reeb vector field such that $\eta(\xi) = 1$ and $\imath_\xi \omega = 0$.

Remark 2.3. Not all odd dimensional manifolds admits a cosymplectic structure. In fact, let $M^{(2k+1)}$ be any closed manifold with $k \neq 0$ such that $H^\star(M^{(2k+1)}, \mathbb{R})$ denotes its $\star$–th de Rham group with real coefficients. If $H^1(M^{(2k+1)}, \mathbb{R}) = 0$, or $H^2(M^{(2k+1)}, \mathbb{R}) = 0$, then $M^{(2k+1)}$ has no cosymplectic structure. In particular, since $H^1(S^{(2k+1)}, \mathbb{R}) = 0$, then the unit spheres $S^{(2k+1)}$ have no cosymplectic structures, for any integer k: this is a consequence of the usual Stoke's theorem.

In order to well describe in more details some features of cosymplectic manifolds we shall need the following result due to Li [13].

Lemma 2.1. *([13])*
Let M be a manifold and η, ω be two differential forms on M with degrees 1 and 2 respectively. Consider $\widetilde{M} = M \times \mathbb{R}$ equipped with the 2–form $\widetilde{\omega} := p^\star(\omega) + p^\star(\eta) \wedge \pi_2^\star(du)$ where u is the coordinate function on $\mathbb{R}$, $p : \widetilde{M} \longrightarrow M$, and $\pi_2 : \widetilde{M} \longrightarrow \mathbb{R}$, are canonical projections. Then, (M, ω, η) is a cosymplectic manifold if and only if $(\widetilde{M}, \widetilde{\omega})$ is a symplectic manifold.

2.3 The C^0–topology

Hereafter is a brief description of the C^0–topology on the group of homeomophisms and on the set of paths of homeomorphisms starting from identity. Let $Homeo(M)$ denotes the group of all homeomorphisms of M equipped with the C^0– compact-open topology. This is the metric topology induced by the following distance

$$d_0(f, h) = \max\left(d_{C^0}(f, h), d_{C^0}(f^{-1}, h^{-1})\right), \tag{2.3}$$

where

$$d_{C^0}(f, h) = \sup_{x \in M} d(h(x), f(x)). \tag{2.4}$$

On the space of all continuous paths $\lambda : [0, 1] \longrightarrow Homeo(M)$ such that $\lambda(0) = id_M$, we consider the C^0–topology as the metric topology induced by the following metric

$$\bar{d}(\lambda, \mu) = \max_{t \in [0,1]} d_0(\lambda(t), \mu(t)). \tag{2.5}$$

3 Vector fields and (almost) cosymplectic structure

Let $\Omega^1(M)$ (resp. $\mathfrak{X}(M)$) be the space of all 1–forms (resp. smooth vector fields) of a cosymplectic manifold (M, ω, η). The cosymplectic structure induces an isomorphism of $C^\infty(M, \mathbb{R})$–modules

$$\begin{array}{cccl} \widetilde{I}_{\eta,\omega} : & \mathfrak{X}(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & \widetilde{I}_{\eta,\omega}(X) = \imath_X \omega + \eta(X)\eta. \end{array}$$

The vector field $\xi := \widetilde{I}_{\eta,\omega}^{-1}(\eta)$ is called the Reeb vector field of (M, ω, η) and is characterized by : $\eta(\xi) = 1$ and $\imath_\xi \omega = 0$.

Proposition 3.1 ((ω, η)–decomposition)).
Let (M, ω, η) be a cosymplectic manifold. Then, any vector field X on M decomposes in a unique way as : $X = X_\omega + X_\eta$, where $X_\omega := \widetilde{I}_{\eta,\omega}^{-1}(\imath_X \omega)$ and $X_\eta := \widetilde{I}_{\eta,\omega}^{-1}(\eta(X)\eta)$.

3.1 Cosymplectic vector fields

In this subsection we study vector fields X of a cosymplectic manifold (M, ω, η) whose generating flows preserve the forms η and ω.

Definition 3.1. Let (M, ω, η) be a cosymplectic manifold. A vector field X is said to be cosymplectic if $\mathcal{L}_X\eta = 0$ and $\mathcal{L}_X\omega = 0$.

We will denote by $\mathfrak{X}_{\eta,\omega}(M)$ the space of all cosymplectic vector fields of (M, ω, η).

Corollary 3.1. *Let (M, ω, η) be a cosymplectic manifold. For any $X \in \mathfrak{X}_{\eta,\omega}(M)$, the 1–form $\widetilde{I}_{\eta,\omega}(X)$ is closed.*

We have the following fact.

Lemma 3.1. *Let (M, ω, η) be a cosymplectic manifold. Consider the symplectic manifold $\widetilde{M} = M \times \mathbb{R}$ equipped with the symplectic form $\Omega := p^\star(\omega) + p^\star(\eta) \wedge \pi_2^\star(du)$ where u is the coordinate function on $\mathbb{R}$, $p : \widetilde{M} \longrightarrow M$, and $\pi_2 : \widetilde{M} \longrightarrow \mathbb{R}$ are projection maps. Let α be any closed 1–form on M, and set $X_\alpha := \widetilde{\Omega}^{-1}(p^\star(\alpha))$, where $\widetilde{\Omega}$ is the isomorphism induced by the symplectic form Ω defined from the space of all vector fields on $\widetilde{M}$ onto the space of all 1–forms on $\widetilde{M}$. Then, the vector field $Y_\alpha := p_\star(X_\alpha)$ is a cosymplectic vector field, if and only if, $d\Big(\big(du((\pi_2)_\star(X_\alpha))\big)(1)\Big) = \alpha(\xi)\eta$, where ξ is the Reeb vector field of (M, ω, η).*

Proof. Consider the symplectic manifold $\widetilde{M} = M \times \mathbb{R}$ equipped with the symplectic form

$$\Omega := p^\star(\omega) + p^\star(\eta) \wedge \pi_2^\star(du)$$

where u is the coordinate function on $\mathbb{R}$, $p : \widetilde{M} \longrightarrow M$ and $\pi_2 : \widetilde{M} \longrightarrow \mathbb{R}$, are projection maps. Let α be any closed 1–form on M and put $X_\alpha := \widetilde{\Omega}^{-1}(p^\star(\alpha))$. Since by definition we have $\imath_{X_\alpha}\Omega = p^\star(\alpha)$, by computing

$$\imath_{X_\alpha}\Omega = p^\star\left(\imath_{p_\star(X_\alpha)}\omega\right) + p^\star\left(\imath_{p_\star(X_\alpha)}\eta\right)\pi_2^\star(du) - \big(du((\pi_2)_\star(X_\alpha))\big) \circ \pi_2 p^\star(\eta),$$

we derive that $p^\star\left(\imath_{p_\star(X_\alpha)}\omega\right) + p^\star\left(\imath_{p_\star(X_\alpha)}\eta\right)\pi_2^\star(du) - \big(\big(du((\pi_2)_\star(X_\alpha))\big) \circ \pi_2\big)\, p^\star(\eta) = p^\star(\alpha)$. We apply the vector field $\frac{\partial}{\partial u}$ to both sides of the above equality to obtain: $0 + p^\star\left(\imath_{p_\star(X_\alpha)}\eta\right) - 0 = 0$ since $p_\star(\frac{\partial}{\partial u}) = 0$, and $(\pi_2)_\star(\frac{\partial}{\partial u}) = \frac{\partial}{\partial u}$: this gives $\eta(Y_\alpha) = 0$. Besides, let S_l be the corresponding section of the projection p and fix $l \in \mathbb{R}$. Composing the equality

$$p^\star\left(\imath_{p_\star(X_\alpha)}\omega\right) + p^\star\left(\imath_{p_\star(X_\alpha)}\eta\right)\pi_2^\star(du) - \big(du((\pi_2)_\star(X_\alpha))\big) \circ \pi_2 p^\star(\eta) = p^\star(\alpha),$$

in both sides by $S_l^\star$, yields:

$$\imath_{Y_\alpha}\omega - \big(du((\pi_2)_\star(X_\alpha))\big)(l)\eta = \alpha. \tag{3.1}$$

Therefore, it follows from (3.1) that $\big(du((\pi_2)_\star(X_\alpha)\big)(l) = \alpha(\xi)$, and so, $0 = d\left(\imath_{Y_\alpha}\omega\right) = d\left(\big(\big(du((\pi_2)_\star(X_\alpha))(l)\big)\big)\eta + \alpha\right)$, whenever $d\big(\big(du((\pi_2)_\star(X_\alpha))\big)(l)\big) = \alpha(\xi)\eta$ which implies that $\mathcal{L}_{Y_\alpha}\omega = 0$, and $\mathcal{L}_{Y_\alpha}\eta = 0$. Conversely, if Y_α is cosymplectic, then from (3.1) we derive that $d\big(\big(du((\pi_2)_\star(X_\alpha)\big)(l)\eta\big) = 0$, which implies that $d\big(du((\pi_2)_\star(X_\alpha))(l)\big) = \alpha(\xi)\eta$. □

However, we do not know whether for any $\alpha \in \mathcal{Z}^1(M)$, the vector field $X := \widetilde{I}_{\eta,\omega}^{-1}(\alpha)$ is a cosymplectic vector field or not. Therefore, let us consider the set $C^{ste}(M)$ consisting of all constant function on M, and put

$$\mathcal{Z}_\xi^1(M) := \left\{\beta \in \mathcal{Z}^1(M)\,;\, \beta(\xi) \in C^{ste}(M)\right\} \tag{3.2}$$

where ξ is the Reeb vector field. The set $\mathcal{Z}_\xi^1(M)$ is non-empty, since $\eta(\xi) = 1$. Also, for any vector field X on M such that $d(\imath_X\omega) = 0$, we have $(\imath_X\omega)(\xi) = 0$, i.e., $\imath_X\omega \in \mathcal{Z}_\xi^1(M)$.

Proposition 3.2. *Let (M, ω, η) be a compact cosymplectic manifold. Let $\alpha \in \mathcal{Z}_\xi^1(M)$ and let $X := \widetilde{I}_{\eta,\omega}^{-1}(\alpha)$. If $\{\psi_t\}_t$ is the flow generated by X then for each t we have : $\psi_t^\star(\omega) = \omega$ and $\psi_t^\star(\eta) = \eta$.*

Proof. Since $d\widetilde{I}_{\eta,\omega}(X) = d\alpha = 0$, then $\mathcal{L}_X(\omega) = -\mathcal{L}_X(\eta) \wedge \eta$. But, $\imath_X\omega + \eta(X)\eta = \alpha$, and $\alpha \in \mathcal{Z}^1_\xi(M)$ imply that $\eta(X) = \alpha(\xi) = cte$. So, $d(\eta(X)) = d(\alpha(\xi)) = 0$ which infers that $\mathcal{L}_X(\eta) = d(\eta(X)) = 0$ and $\mathcal{L}_X(\omega) = -\mathcal{L}_X(\eta) \wedge \eta = 0$. □

Remark 3.1. The map

$$\begin{array}{cccc} \sharp_{\eta,\omega}: & \mathfrak{X}_{\eta,\omega}(M) & \longrightarrow & \mathcal{Z}^1_\xi(M) \\ & X & \longmapsto & \sharp_{\eta,\omega}(X) := \widetilde{I}_{\eta,\omega}(X) \end{array}$$

is a linear isomorphism. Furthermore, for each closed 1–form α, the vector field $Z_\alpha := Y_\alpha - \alpha(\xi)\xi$ satisfies $\widetilde{I}_{\eta,\omega}(Z_\alpha) = \alpha$, $\mathcal{L}_{Z_\alpha}\eta = d\left(\imath(Y_\alpha)\eta - \alpha(\xi)\imath(\xi)\eta\right) = 0$ and $\mathcal{L}_{Z_\alpha}\omega = d\left(\alpha(\xi)\right) \wedge \eta$, where Y_α is as in Lemma 3.1.

The following result is a consequence of the (ω, η)–decomposition of vector fields on a cosymplectic manifold.

Proposition 3.3. *Let X be a cosymplectic vector field with (ω, η)–decomposition $X = X_\omega + X_\eta$. Then,*

1. *the vector fields X_ω and X_η are cosymplectic,*
2. $[X_\omega, X_\eta] = 0$,
3. *for any $Y \in \mathfrak{X}_{\eta,\omega}(M)$, we have $[X, Y] \in \mathfrak{X}_{\eta,\omega}(M)$*
4. *when it exists, the flow $\Phi_\omega = \{\phi^t_\omega\}_t$ (resp. $\Phi_\eta = \{\phi^t_\eta\}_t$) generated by the vector field X_ω (resp. X_η) preserves the cosymplectic structure,*
5. *$\phi^s_\eta \circ \phi^t_\omega = \phi^t_\omega \circ \phi^s_\eta$, for each s, t,*
6. *the flow Φ generated by X decomposes as: $\Phi = \Phi_\omega \circ \Phi_\eta = \Phi_\eta \circ \Phi_\omega$.*

Proof. For (1), let $X \in \mathfrak{X}_{\eta,\omega}(M)$. Since $\mathcal{L}_X\omega = 0$ and $X = X_\omega + X_\eta$, then $\mathcal{L}_{X_\eta}\omega = -\mathcal{L}_{X_\omega}\omega$. We claim that $\mathcal{L}_{X_\eta}\omega = 0$. In fact, since $\widetilde{I}_{\eta,\omega}(X_\omega) = \imath_{X_\omega}\omega$, and $\widetilde{I}_{\eta,\omega}(X_\eta) = \eta(X)\eta$, then we compose in both sides with respect to the Reeb vector field ξ to obtain $\eta(X_\omega) = 0$, and $\eta(X_\eta) = \eta(X)$. Thus, it follows that $\mathcal{L}_{X_\omega}\omega = 0$, $\mathcal{L}_{X_\omega}\eta = 0$, $\mathcal{L}_{X_\eta}\omega = 0$, and $\mathcal{L}_{X_\eta}\eta = 0$. For (2), by the means of the formula $\imath_{[X_\omega,X_\eta]} = \mathcal{L}_{X_\omega} \circ \imath_{X_\eta} - \imath_{X_\eta} \circ \mathcal{L}_{X_\omega}$, we compute:

$$\imath_{[X_\omega,X_\eta]}\omega \quad = \quad \mathcal{L}_{X_\omega}\left(\imath_{X_\eta}\omega\right) - \imath_{X_\eta}\left(\mathcal{L}_{X_\omega}\omega\right) = 0 - 0,$$

and

$$\imath_{[X_\eta,X_\omega]}\eta \quad = \quad \mathcal{L}_{X_\eta}\left(\imath_{X_\omega}\eta\right) - \imath_{X_\omega}\left(\mathcal{L}_{X_\eta}\eta\right) = \mathcal{L}_{X_\omega}(0) - \imath_{X_\omega}(\mu\eta) = 0 - 0,$$

and then, we derive that $\widetilde{I}_{\eta,\omega}([X_\eta, X_\omega]) = \imath_{[X_\omega,X_\eta]}\omega - \left(\imath_{[X_\eta,X_\omega]}\eta\right)\eta = 0 - 0$. The non-degeneracy of $\widetilde{I}_{\eta,\omega}$, implies that $[X_\eta, X_\omega] = 0$. For (3), let $X, Y \in \mathfrak{X}_{\eta,\omega}(M)$. From the formulas $\imath_{[X,Y]} = \mathcal{L}_X \circ \imath_Y - \imath_Y \circ \mathcal{L}_X$, and $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$, we derive that $\mathcal{L}_{[X,Y]}\omega = d(\imath_{[X,Y]}\omega) = 0$ and $\mathcal{L}_{[X,Y]}\eta = d(\imath_{[X,Y]}\eta) = 0$. Hence, $[X, Y] \in \mathfrak{X}_{\eta,\omega}(M)$. For (4), we derive from the first item that when it exists, the 1–parameter family of diffeomorphisms $\Phi_\omega = \{\phi^t_\omega\}_t$, preserves the cosymplectic structure. From the formula

$$\mathcal{L}_{(\psi^t_\omega)^{-1}_*(X_\eta)}\eta = (\psi^t_\omega)^*\left(\mathcal{L}_{X_\eta}\left(\left(\psi^t_\omega\right)^{-1}\right)^*(\eta)\right) = (\psi^t_\omega)^*\left(\mathcal{L}_{X_\eta}\eta\right),$$

we derive that when it exists, the 1–parameter family of diffeomorphisms Ψ_η, also preserves that cosymplectic structure. The last items easily follow. □

Definition 3.2. Let (M, ω, η) be a cosymplectic manifold. An element $Y \in \mathfrak{X}_{\eta,\omega}(M)$ is called

- an η–vector field if the 1–form $\imath_Y\omega = 0$,
- an ω–vector field if the function $\eta(Y)$ is trivial.

Example 3.1. Let Y be any cosymplectic vector field. Then in the decomposition, $Y = Y_\eta + Y_\omega$, we have that Y_η is an ω–vector field while Y_ω is an η–vector field.

Definition 3.3. Let (M, ω, η) be a cosymplectic manifold. An element $Y \in \mathfrak{X}_{\eta,\omega}(M)$ is called a weak Hamiltonian vector field if the 1–form $\widetilde{I}_{\eta,\omega}(Y)$ is exact.

We shall denote by $ham_{\eta,\omega}(M)$ the space of all weak Hamiltonian vector fields of (M, ω, η).

Proposition 3.4. *[3] Let (M, ω, η) be a cosymplectic manifold. For any $X, Y \in \mathfrak{X}_{\eta,\omega}(M)$, we have $[X, Y] \in ham_{\eta,\omega}(M)$.*

Definition 3.4. Let (M, ω, η) be a cosymplectic manifold. An element $Y \in \mathfrak{X}_{\eta,\omega}(M)$ is called a co-Hamiltonian vector field if the 1–form $\imath_Y\omega$ is exact and $\eta(Y) = 0$.

We shall denote by $ham^0_{\eta,\omega}(M)$ the space of all co-Hamiltonian vector fields of (M, ω, η). Observe that

$$ham^0_{\eta,\omega}(M) \subset ham_{\eta,\omega}(M).$$

Example 3.2. Let (M, ω) be a symplectic manifold. Consider the Cartesian product $\widetilde{M} := M\times\mathbb{R}$ equipped with the 2–form $\widetilde{\omega} := p^\star(\omega)$ and the 1–form $\widetilde{\eta} := \pi_2^\star(du)$ where u is the coordinate function on $\mathbb{R}$, $p : \widetilde{M} \longrightarrow M$, and $\pi_2 : \widetilde{M} \longrightarrow \mathbb{R}$ are the canonical projections on each factor of $\widetilde{M}$ respectively. Let l be fixed in $\mathbb{R}$, take S_l to be a smooth section of the projection p, and let X be any Hamiltonian vector field of (M, ω) with generating function H. The vector field $\widetilde{X} := (S_l)_\star(X)$ of $\widetilde{M}$ satisfies $\imath_{\widetilde{X}}\widetilde{\omega} = p^\star(\imath_X\widetilde{\omega}) = d(H \circ p)$ since $p \circ S_l = id_M$. Also, we have $\widetilde{\eta}(\widetilde{X}) = du((\pi_2 \circ S_l)_\star(X)) = 0$ because $\pi_2 \circ S_l$ is a constant map. Thus, we have that $\widetilde{X} \in ham^0_{\widetilde{\eta},\widetilde{\omega}}(\widetilde{M})$. Besides, let Y be a vector field on $\mathbb{R}$ such that the function $du(Y)$ is non-trivial, let $x \in M$ be fixed, and let S_x be a smooth section of the projection π_2. The vector field $\widetilde{Y} := (S_x)_\star(Y)$ of $\widetilde{M}$ satisfies $\imath_{\widetilde{Y}}\widetilde{\omega} = p^\star(\imath_0\widetilde{\omega}) = 0$ since $\pi \circ S_x$ is the constant map.

Also, we have $\widetilde{\eta}(\widetilde{Y}) = du(Y) \neq 0$ because $\pi_2 \circ S_x = id_{\mathbb{R}}$. Thus, we have that $\widetilde{Y} \in \left(ham_{\widetilde{\eta},\widetilde{\omega}}(\widetilde{M}) \setminus ham^0_{\widetilde{\eta},\widetilde{\omega}}(\widetilde{M})\right)$.

Proposition 3.5. *Let (M, ω, η) be a cosymplectic manifold. For any $X, Y \in ham^0_{\eta,\omega}(M)$, we have $[X, Y] \in ham^0_{\eta,\omega}(M)$.*

3.2 Almost cosymplectic vector fields

In this subsection, we define and study those vector fields X of a cosymplectic manifold (M, ω, η) whose generating flows preserve the differential forms η, and ω up to a multiplicative smooth function.

Definition 3.5. Let (M, ω, η) be a cosymplectic manifold. A vector field X is said to be almost cosymplectic if $\mathcal{L}_X\omega = 0$, and there is a smooth function μ_X on M, non-identically trivial such that $\mathcal{L}_X\eta = \mu_X\eta$.

We shall denote by $\mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$ the space of all almost cosymplectic vector fields of (M, ω, η).

Proposition 3.6. *Let (M, ω, η) be a cosymplectic manifold. For any $X, Y \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$, we have $[X, Y] \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$.*

Proof. Let $X, Y \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$, from the formulas $\imath_{[X,Y]} = \mathcal{L}_X \circ \imath_Y - \imath_Y \circ \mathcal{L}_X$, and $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$, we derive that $\mathcal{L}_{[X,Y]}\omega = d\imath_{[X,Y]}\omega = 0$ and

$$\begin{aligned}\mathcal{L}_{[X,Y]}\eta &= d\imath_{[X,Y]}\eta = d\big(\mu_Y\eta(X) - \mu_X\eta(Y)\big) = \eta(X)d\mu_Y + \mu_Y d\eta(X) - \eta(Y)d\mu_X - \mu_X d\eta(Y)\\ &= (d\mu_Y)(X)\eta + \mu_Y\mu_X\eta - (d\mu_X)(Y)\eta - \mu_X\mu_Y\eta,\end{aligned} \tag{3.3}$$

since $d(\eta(X)) = \mathcal{L}_X\eta = \mu_X\eta$ and $d(\eta(Y)) = \mathcal{L}_Y\eta = \mu_Y\eta$. Hence,

$$\mathcal{L}_{[X,Y]}\omega = (d\mu_Y)(X)\eta - (d\mu_X)(Y)\eta = f_{X,Y}\eta,$$

with $f_{X,Y} := \big((d\mu_Y)(X) - (d\mu_X)(Y)\big) \in C^\infty(M)$, i.e., $[X, Y] \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$. □

Definition 3.6. Let (M, ω, η) be a cosymplectic manifold. An element $Y \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$ is called an almost co-Hamiltonian vector field if the 1–form $\imath_Y\omega$ is exact.

We shall denote by $\mathcal{A}ham_{\eta,\omega}(M)$ the space of all almost co-Hamiltonian vector fields of (M, ω, η).

Proposition 3.7. *Let (M, ω, η) be a cosymplectic manifold. For any $X, Y \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$, we have $[X, Y] \in \mathcal{A}ham_{\eta,\omega}(M)$.*

Proof. Since $[X, Y] \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$, we derive that $\imath_{[X,Y]}\omega = \mathcal{L}_X \circ \imath_Y\omega = d(\pm\omega(X, Y))$. □

Corollary 3.2. *Let (M, ω, η) be a cosymplectic manifold. For any $X \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$, the 1–form $\widetilde{I}_{\eta,\omega}(X)$ is closed.*

Proof. For each $X \in \mathcal{A}\mathfrak{X}_{\eta,\omega}(M)$, since $\mathcal{L}_X = \imath_X \circ d + d \circ \imath_X$, we derive from the equalities $\mathcal{L}_X(\omega) = 0$ and $\mathcal{L}_X(\eta) = \mu_X\eta$ that $d(\imath_X\omega) = 0$ and $d(\eta(X)) = \mu_X\eta$. Thus,

$$d(\widetilde{I}_{\eta,\omega}(X)) = d(\imath_X\omega) + d(\eta(X)\eta) = d(\imath_X\omega) + d(\eta(X)) \wedge \eta = 0 + 0.$$

□

Here is a consequence of the (ω, η)–decomposition of vector fields on a cosymplectic manifold.

Proposition 3.8. *Let Y be an almost cosymplectic vector field such that $\mathcal{L}_Y(\eta) = \mu\eta$ with (ω, η)–decomposition $Y = Y_\omega + Y_\eta$. Then,*

1. *the vector field Y_ω (resp. Y_η) is cosymplectic (resp. almost cosymplectic),*
2. *the flow $\Psi_\omega = \{\psi^t_\omega\}$ (resp. $\Psi_\eta = \{\psi^t_\eta\}_t$) generated by the vector field Y_ω (resp. Y_η) is cosymplectic (resp. almost cosymplectic with $\mathcal{L}_{\dot{\psi}^t_\eta}\eta = \mu\eta$) when it exists,*
3. *$[Y_\omega, Y_\eta] = 0$,*
4. *$\psi^s_\eta \circ \psi^t_\omega = \psi^t_\omega \circ \psi^s_\eta$, for each s, t, and*
5. *the flow Ψ generated by Y decomposes as $\Psi = \Psi_\omega \circ \Psi_\eta = \Psi_\eta \circ \Psi_\omega$.*

Proof. Let Y be an almost cosymplectic vector field such that $\mathcal{L}_Y(\eta) = \mu\eta$ with (ω, η)–decomposition $Y = Y_\omega + Y_\eta$. For (1), we derive from $\widetilde{I}_{\eta,\omega}(Y_\omega) = \imath_Y\omega$, by composing with respect to the Reeb vector field that, $\eta(Y_\omega) = 0$, and so we also have $\imath_{Y_\omega}\omega = \imath_Y\omega$, which show that Y_ω is a cosymplectic vector field. Similarly, from $\widetilde{I}_{\eta,\omega}(Y_\eta) = \eta(Y)\eta$, we derive that $\eta(Y_\eta) = \eta(Y)$, and so we also have $\imath_{Y_\eta}\omega = 0$. This implies that $\mathcal{L}_{Y_\eta}\omega = 0$, and $\mathcal{L}_{Y_\eta}\eta = \mu\eta$. To prove (2), we derive from the first item that when the isotopy generated by Y_ω exists, then the latter is a cosymplectic isotopy. From the formula

$$\mathcal{L}_{(\psi^t_\omega)^{-1}_\star(Y_\eta)}\eta = (\psi^t_\omega)^\star\left(\mathcal{L}_{Y_\eta}\left((\psi^t_\omega)^{-1}\right)^\star(\eta)\right) = (\psi^t_\omega)^\star\left(\mathcal{L}_{Y_\eta}\eta\right),$$

we derive that Ψ_η, is an almost cosymplectic isotopy when it exists. By the means of the formula $\imath_{[Y_\omega,Y_\eta]} = \mathcal{L}_{Y_\omega} \circ \imath_{Y_\eta} - \imath_{Y_\eta} \circ \mathcal{L}_{Y_\omega}$, we compute:

$$\imath_{[Y_\omega,Y_\eta]}\omega = \mathcal{L}_{Y_\omega}\left(\imath_{Y_\eta}\omega\right) - \imath_{Y_\eta}\left(\mathcal{L}_{Y_\omega}\omega\right) = 0 - 0,$$

and

$$\imath_{[Y_\omega,Y_\eta]}\eta = \mathcal{L}_{Y_\eta}\left(\imath_{Y_\omega}\eta\right) - \imath_{Y_\omega}\left(\mathcal{L}_{Y_\eta}\eta\right) = \mathcal{L}_{Y_\omega}(0) - \imath_{Y_\omega}(\mu\eta), = 0 - 0,$$

and then, we derive that $\widetilde{I}_{\eta,\omega}([Y_\eta, Y_\omega]) = \imath_{[Y_\omega,Y_\eta]}\omega - \left(\imath_{[Y_\eta,Y_\omega]}\eta\right)\eta = 0 - 0$. The non-degeneracy of $\widetilde{I}_{\eta,\omega}$, implies that $[Y_\eta, Y_\omega] = 0$. □

4 Diffeomorphisms of a cosymplectic manifold

Definition 4.1. Let (M, ω, η) be a cosymplectic manifold.

1. A diffeomorphism $\phi : M \longrightarrow M$ is called an almost cosymplectic diffeomorphism (or almost cosymplectomorphism) diffeomorphism if: $\phi^\star(\omega) = \omega$ and there exists a smooth function $f \in C^\infty(M)$ such that $\phi^\star(\eta) = e^f\eta$.
2. A diffeomorphism $\phi : M \longrightarrow M$ is called a cosymplectic diffeomorphism (or cosymplectomorphism) diffeomorphism if: $\phi^\star(\omega) = \omega$ and $\phi^\star(\eta) = \eta$.

We shall denote by $\mathcal{A}\mathrm{Cosymp}_{\eta,\omega}$(M) the space of all almost cosymplectomorphisms of $(M, \omega, \eta,)$ and by $\mathrm{Cosymp}_{\omega,\eta}(M)$ the space of all cosymplectomorphisms of (M, ω, η).

Definition 4.2. Let (M, ω, η) be a cosymplectic manifold. An isotopy $\Phi = \{\phi_t\}_t$ is called an almost cosymplectic (resp. cosymplectic) isotopy if for each time t, we have $\phi_t \in \mathcal{A}\mathrm{Cosymp}_{\eta,\omega}$(M) (resp. $\phi_t \in \mathrm{Cosymp}_{\eta,\omega}(M)$).

We shall denote by $\mathcal{A}Iso_{\eta,\omega}(M)$ (resp. $Iso_{\eta,\omega}(M)$) the space of all almost cosymplectic (resp. the space of all cosymplectic) isotopies of (M, ω, η).

We then define the following important subgroups:

$$\mathcal{A}G_{\eta,\omega}(M) := ev_1\left(\mathcal{A}Iso_{\eta,\omega}(M)\right) \quad \text{and} \quad G_{\eta,\omega}(M) := ev_1\left(Iso_{\eta,\omega}(M)\right),$$

where ev_1 is a time 1–map (namely the map that associates to an isotopy $\{\phi^t\}_{0\leqslant t\leqslant 1}$ the map ϕ^1). We equip both groups $\mathcal{A}G_{\eta,\omega}(M)$ and $G_{\eta,\omega}(M)$ with the C^∞–compact-open topology [7].

Lemma 4.1. *Let (M, ω, η) be a cosymplectic manifold and let ξ be its Reeb vector field. We have the following properties.*

1. *If $\phi \in G_{\eta,\omega}(M)$, then $\phi_\star(\xi) = \xi$.*
2. *If $\psi \in \mathcal{A}G_{\eta,\omega}(M)$ with $\psi^\star(\eta) = e^f\eta$, then $\psi_\star(\xi) = e^{f\circ\psi^{-1}}\xi$.*

Proof. Since for all diffeomorphism $\varphi \in \mathrm{Diff}(M)$, we have $\widetilde{I}_{\eta,\omega}(\varphi_\star(\xi)) = (\varphi^{-1})^\star(\imath_\xi\varphi^\star(\omega)) + (\varphi^{-1})^\star(\imath_\xi\varphi^\star(\eta))\eta$, then we derive that:

1. If $\phi \in G_{\eta,\omega}(M)$, then for $\varphi = \phi$, we have
$$\begin{aligned}\widetilde{I}_{\eta,\omega}(\phi_\star(\xi)) &= (\phi^{-1})^\star(\imath_\xi\phi^\star(\omega)) + (\phi^{-1})^\star(\imath_\xi\phi^\star(\eta))\eta = (\phi^{-1})^\star(\imath_\xi\omega) + (\phi^{-1})^\star(\imath_\xi\eta)\eta \\ &= \eta \quad = \quad \widetilde{I}_{\eta,\omega}(\xi).\end{aligned} \tag{4.1}$$
Thus, $\widetilde{I}_{\eta,\omega}(\phi_\star(\xi)) = \widetilde{I}_{\eta,\omega}(\xi)$ which implies that $\phi_\star(\xi) = \xi$, since $\widetilde{I}_{\eta,\omega}$ is non-degenerate.
2. If $\psi \in \mathcal{A}G_{\eta,\omega}(M)$ with $\psi^\star(\eta) = e^f\eta$, then
$$\begin{aligned}\widetilde{I}_{\eta,\omega}(\psi_\star(\xi)) &= (\psi^{-1})^\star(\imath_\xi\psi^\star(\omega)) + (\psi^{-1})^\star(\imath_\xi\psi^\star(\eta))\eta = (\psi^{-1})^\star(\imath_\xi\omega) + (\psi^{-1})^\star(e^f\imath_\xi\eta)\eta \\ &= 0 + e^{f\circ\psi^{-1}}\eta \quad = \quad \widetilde{I}_{\eta,\omega}(e^{f\circ\psi^{-1}}\xi).\end{aligned} \tag{4.2}$$
Thus, $\widetilde{I}_{\eta,\omega}(\psi_\star(\xi)) = \widetilde{I}_{\eta,\omega}(e^{f\circ\psi^{-1}}\xi)$ from which we derive $\psi_\star(\xi) = e^{f\circ\psi^{-1}}\xi$, since $\widetilde{I}_{\eta,\omega}$ is non-degenerate.

□

The following result generalizes Lemma 4.1 and its proof follows immediately from the one of Lemma 4.1.

Lemma 4.2. *Let (M_i, ω_i, η_i), $i = 1, 2$ be two cosymplectic manifolds and let ξ_i, $i = 1, 2$ be their Reeb vector fields respectively. We have the following properties.*

1. *If $\phi \in \mathrm{Diff}(M_1, M_2)$ such that $\phi^\star(\omega_2) = \omega_1$, and $\phi^\star(\eta_2) = \eta_1$, then $\phi_\star(\xi_1) = \xi_2$.*
2. *If $\psi \in \mathrm{Diff}(M_1, M_2)$ such that $\psi^\star(\omega_2) = \omega_1$, and $\psi^\star(\eta_2) = e^h\eta_1$, then $\psi_\star(\xi_1) = e^{h\circ\psi^{-1}}\xi_2$.*

Definition 4.3. Let (M, ω, η) be a cosymplectic manifold. An isotopy $\Psi := \{\psi_t\}_t$ is called an almost co-Hamiltonian isotopy, if for each t, the vector field $\dot{\psi}_t$ is an almost co-Hamiltonian vector field, i.e., $\dot{\psi}_t \in \mathcal{A}ham_{\eta,\omega}(M)$, for each t.

We shall denote by $\mathcal{A}H_{\eta,\omega}(M)$ the space of all almost co-Hamiltonian isotopies of (M, ω, η), and put

$$\mathcal{A}Ham_{\eta,\omega}(M) := ev_1\left(\mathcal{A}H_{\eta,\omega}(M)\right). \tag{4.3}$$

The elements of the set $\mathcal{A}H_{\eta,\omega}(M)$ are called almost co-Hamiltonian diffeomorphisms of (M, ω, η).

Definition 4.4. Let (M, ω, η) be a cosymplectic manifold. An isotopy $\Psi := \{\psi_t\}_t$ is called a weakly Hamiltonian isotopy, if for each t, the vector field $\dot{\psi}_t$ is a weakly Hamiltonian vector field, i.e., $\dot{\psi}_t \in ham_{\eta,\omega}(M)$, for each t.

We shall denote by $H_{\eta,\omega}(M)$ the space of all weakly Hamiltonian isotopies of (M, ω, η), and put

$$Ham_{\eta,\omega}(M) := ev_1\left(H_{\eta,\omega}(M)\right). \tag{4.4}$$

The elements of the set $Ham_{\eta,\omega}(M)$ are called weakly Hamiltonian diffeomorphisms of (M, ω, η).

Definition 4.5. Let (M, ω, η) be a cosymplectic manifold. An isotopy $\Psi := \{\psi_t\}_t$ is called a co-Hamiltonian isotopy, if for each t, the vector field $\dot{\psi}_t$ is a co-Hamiltonian vector field, i.e., $\dot{\psi}_t \in ham^0_{\eta,\omega}(M)$, for each t.

We shall denote by $H^0_{\eta,\omega}(M)$ the space of all co-Hamiltonian isotopies of (M, ω, η), and put

$$Ham^0_{\eta,\omega}(M) := ev_1\left(H^0_{\eta,\omega}(M)\right). \tag{4.5}$$

The elements of the set $Ham^0_{\eta,\omega}(M)$ are called co-Hamiltonian diffeomorphisms of (M, ω, η).

Proposition 4.1. *Let (M, ω, η) be a cosymplectic manifold. The following properties hold.*

1. *The set $\mathcal{A}Ham_{\eta,\omega}(M)$ is a Lie group whose Lie algebra is the space $\mathcal{A}ham_{\eta,\omega}(M)$.*
2. *The set $\mathcal{A}Ham_{\eta,\omega}(M)$ is a normal subgroup in the group $\mathcal{A}G_{\eta,\omega}(M)$.*
3. *The set $Ham_{\eta,\omega}(M)$ is a Lie group whose Lie algebra is the space $ham_{\eta,\omega}(M)$.*
4. *The set $Ham_{\eta,\omega}(M)$ is a normal subgroup in the group $G_{\eta,\omega}(M)$.*
5. *The set $Ham^0_{\eta,\omega}(M)$ is a Lie group whose Lie algebra is the space $ham^0_{\eta,\omega}(M)$.*
6. *The set $Ham^0_{\eta,\omega}(M)$ is a normal subgroup in the group $Ham_{\eta,\omega}(M)$.*
7. *The group $Ham_{\eta,\omega}(M)$ is not simple provided $Ham^0_{\eta,\omega}(M) \neq \{id_M\}$.*

4.1 Cosymplectic and almost cosymplectic flows

In this subsection, we give some relations for the cosymplectic and almost cosymplectic flows. To this end, we shall need the following fact: to any smooth isotopy $\Phi = \{\phi_t\}_t$ with $\phi_0 = id_M$, is attached a smooth family of smooth vector fields $\{\dot{\phi}_t\}_t$, defined by

$$\dot{\phi}_t := X_t \circ \phi_t^{-1} \tag{4.6}$$

where

$$X_t(x) := \frac{d\phi_t(x)}{dt}, \qquad \text{for each } t \text{ and } \forall\, x \in M. \tag{4.7}$$

Furthermore, if $\Phi = \{\phi_t\}$ is cosymplectic then we have for each t

$$\phi_t^*\left(\widetilde{I}_{\eta,\omega}(\dot{\phi}_t)\right) = \phi_t^*\left(\widetilde{I}_{\eta,\omega}(X_t)\right). \tag{4.8}$$

Hereafter are these relations (items 1 – 10)

1. Let $\{\phi_t\} \in H_{\eta,\omega}(M)$ such that $\widetilde{I}_{\eta,\omega}(\dot{\phi}_t) = dF_t$, for each t and smooth function F_t. Then, from the relation $\dot{\phi}_{-t} = -(\phi_t^{-1})_\star(\dot{\phi}_t)$, for each t and $\imath_{(\phi_t^{-1})_\star\dot{\phi}_t}\alpha = -\phi_t^\star\left(\imath_{(\dot{\phi}_t\circ\phi_t^{-1})}\alpha\right)$, for all p–form α, we derive that
$$\widetilde{I}_{\eta,\omega}(\dot{\phi}_{-t}) = -\phi_t^\star(\imath_{\dot{\phi}_t}\omega) - \phi_t^\star(\eta(\dot{\phi}_t))\eta = -\phi_t^\star(\widetilde{I}_{\eta,\omega}(\dot{\phi}_t)) = d(-F_t\circ\phi_t) \qquad \forall\, t. \tag{4.9}$$
Hence, $\{\phi_t^{-1}\} \in H_{\eta,\omega}(M)$ and $\widetilde{I}_{\eta,\omega}(\dot{\phi}_{-t}) = d(-F_t\circ\phi_t)$, where $\phi_t^{-1} =: \phi_{-t}$, for each t.
2. If $\Phi_F = \{\phi_t\}_t$ is a weakly Hamiltonian isotopy such that $\widetilde{I}_{\eta,\omega}(\dot{\phi}_t) = dF_t$, for all t, then for all $\rho \in G_{\eta,\omega}(M)$, the isotopy $\Psi = \{\psi_t\}_t$ with $\psi_t := \rho^{-1}\circ\phi_t\circ\rho$ is also weakly Hamiltonian : in fact, from $\dot{\psi}_t = \rho_\star^{-1}(\dot{\phi}_t)$, we derive that
$$\widetilde{I}_{\eta,\omega}(\dot{\psi}_t) = \rho^\star(\widetilde{I}_{\eta,\omega}(\dot{\phi}_t)) = d(F_t\circ\rho), \quad \text{for each } t.$$
3. Similarly, if $\{\psi_t\}_t$ and $\{\phi_t\}_t$ are two elements of $H_{\eta,\omega}(M)$ such that $\widetilde{I}_{\eta,\omega}(\dot{\phi}_t) = dF_t$ and $\widetilde{I}_{\eta,\omega}(\dot{\psi}_t) = dK_t$, for each t, then we have
$$\widetilde{I}_{\eta,\omega}(\dot{\overbrace{\phi_t\circ\psi_t}}) = d(F_t + K_t\circ\phi_t^{-1}), \qquad \text{for each } t. \tag{4.10}$$
4. Let $\{\phi_t\}_t \in Iso_{\eta,\omega}(M)$. Then, for each t, we have $\widetilde{I}_{\eta,\omega}(\dot{\phi}_{-t}) = -\phi_t^\star(\widetilde{I}_{\eta,\omega}(\dot{\phi}_t))$.
5. If $\{\psi_t\}_t$ and $\{\phi_t\}_t$ are two elements of $Iso_{\eta,\omega}(M)$, then for each t, we have
$$\widetilde{I}_{\eta,\omega}(\dot{\overbrace{\phi_t\circ\psi_t}}) = \widetilde{I}_{\eta,\omega}(\dot{\phi}_t) + (\phi_t^{-1})^\star(\widetilde{I}_{\eta,\omega}(\dot{\psi}_t)).$$
6. Let $\{\phi_t\}_t \in \mathcal{A}Iso_{\eta,\omega}(M)$ such that $\mathcal{L}_{\dot{\phi}_t}\eta = \mu_t\eta$, (or $\phi_t^\star(\eta) = e^{f_t}\eta$), for each t and smooth function f_t. Compute,
$$\begin{aligned}\mathcal{L}_{\dot{\phi}_{-t}}\eta &= \phi_t^\star\left(\frac{d}{dt}\left((\phi_t^{-1})^\star(\eta)\right)\right) = \phi_t^\star\left(\frac{d}{dt}\left(e^{-f_t\circ\phi_t^{-1}}\eta\right)\right) &(4.11)\\ &= \phi_t^\star\left(\eta\frac{d}{dt}\left(e^{-f_t\circ\phi_t^{-1}}\right)\right) = \left(-\dot{\overbrace{(f_t\circ\phi_t^{-1})}}\circ\phi_t\right)\eta, &(4.12)\end{aligned}$$
i.e., $\mathcal{L}_{\dot{\phi}_{-t}}\eta = \vartheta_t\eta$, for all t, with $\vartheta_t := -\dot{\overbrace{(f_t\circ\phi_t^{-1})}}\circ\phi_t$.
7. If $\{\psi_t\}_t$ and $\{\phi_t\}_t$ are two elements of $\mathcal{A}Iso_{\eta,\omega}(M)$ such that $\mathcal{L}_{\dot{\phi}_t}\eta = \mu_t\eta$, (or $\phi_t^\star(\eta) = e^{f_t}\eta$), and $\mathcal{L}_{\dot{\psi}_t}\eta = \mu_t^{'}\eta$, (or $\psi_t^\star(\eta) = e^{q_t}\eta$), for each t, then we have
$$\begin{aligned}\mathcal{L}_{\dot{\overbrace{\phi_t\circ\psi_t}}}\eta &= ((\phi_t\circ\psi_t)^{-1})^\star\left(\frac{d}{dt}\left((\phi_t\circ\psi_t)^\star\eta\right)\right) = ((\phi_t\circ\psi_t)^{-1})^\star\left(\frac{d}{dt}\left(e^{f_t\circ\psi_t}e^{q_t}\eta\right)\right)\\ &= \left(\dot{\overbrace{f_t\circ\psi_t}} + \dot{q}_t\right)\circ(\phi_t\circ\psi_t)^{-1}\eta, \quad \text{for each } t. &(4.13)\end{aligned}$$
Thus, $\mathcal{L}_{\dot{\overbrace{\phi_t\circ\psi_t}}}\eta = \varrho_t\eta$, with $\varrho_t := \left(\dot{\overbrace{f_t\circ\psi_t}} + \dot{q}_t\right)\circ(\phi_t\circ\psi_t)^{-1}$, for all t.
8. If $\Phi = \{\phi_t\}_t$ is an almost co-Hamiltonian isotopy such that $\imath_{\dot{\phi}_t}\omega = dF_t$, for all t, and $\mathcal{L}_{\dot{\phi}_t}\eta = \mu_t\eta$, (or $\phi_t^\star(\eta) = e^{f_t}\eta$), for each t, then for all $\rho \in \mathcal{A}G_{\eta,\omega}(M)$ such that $\rho^\star(\eta) = e^{f^\rho}\eta$, the isotopy $\Psi = \{\psi_t\}_t$ with $\psi_t := \rho^{-1}\circ\phi_t\circ\rho$ is almost co-Hamiltonian: in fact, from $\dot{\psi}_t = \rho_\star^{-1}(\dot{\phi}_t)$, we derive that $\imath_{\dot{\psi}_t}\omega = \rho^\star(\imath_{\dot{\phi}_t}\omega) = d(F_t\circ\rho)$, for each t. Besides, we have
$$\begin{aligned}\mathcal{L}_{\rho_\star^{-1}(\dot{\phi}_t)}\eta &= \mathcal{L}_{\dot{\psi}_t}\eta = (\psi_t^{-1})^\star\left(\frac{d}{dt}\left((\rho^{-1}\circ\phi_t\circ\rho)^\star(\eta)\right)\right)\\ &= (\psi_t^{-1})^\star\frac{d}{dt}\left(e^{-f^\rho\circ\rho^{-1}\circ\phi_t\circ\rho}e^{f_t\circ\rho}e^{f^\rho}\eta\right), \qquad \text{for all } t. &(4.14)\end{aligned}$$
Hence, $\mathcal{L}_{\rho_\star^{-1}(\dot{\phi}_t)}\eta = H_t(\rho)\eta$, with
$$\begin{aligned}H_t(\rho) &:= e^{-f^\rho\circ\rho^{-1}\circ\phi_t\circ\rho}e^{f_t\circ\rho}e^{f^\rho}\frac{d}{dt}\left(e^{-f^\rho\circ\rho^{-1}\circ\phi_t\circ\rho}e^{f_t\circ\rho}e^{f^\rho}\eta\right)\circ(\rho^{-1}\circ\phi_t\circ\rho)^{-1},\\ &= \left(-df^\rho((\rho^{-1})_\star(\dot{\phi}_t)) + \dot{f}_t\right)\circ(\rho^{-1}\circ\phi_t\circ\rho)^{-1} \quad \text{for each } t. &(4.15)\end{aligned}$$

9. For any isotopy $\Phi = \{\phi_t\}_t$, we will denote by $C^t_{\Phi,\eta}$ the smooth function $x \longmapsto \eta(\dot{\phi}_t)(\phi_t(x))$. If $\Phi = \{\phi_t\}_t$ and $\Psi = \{\psi_t\}_t$ are two elements of $Iso_{\eta,\omega}(M)$ such that $\mathcal{L}_{\dot{\phi}_t}\eta = \mu_t\eta$, (or $\phi_t^\star(\eta) = e^{f_t}\eta$), for each t, then we have

$$\begin{aligned} C^t_{\Phi\circ\Psi,\eta} &= \eta(\dot{\phi}_t + (\phi_t)_\star(\dot{\psi}_t)) \circ (\phi_t \circ \psi_t) = C^t_{\Phi,\eta} \circ \psi_t + \eta((\phi_t)_\star(\dot{\psi}_t)) \circ (\phi_t \circ \psi_t) \\ &= C^t_{\Phi,\eta} \circ \psi_t + \left((\phi_t^{-1})^\star(e^{f_t}\eta(\dot{\psi}_t))\right) \circ (\phi_t \circ \psi_t) = C^t_{\Phi,\eta} \circ \psi_t + e^{f_t\circ\psi_t} C^t_{\Psi,\eta}, \qquad \text{for each } t \end{aligned}$$

namely

$$C^t_{\Phi\circ\Psi,\eta} = C^t_{\Phi,\eta} \circ \psi_t + e^{f_t\circ\psi_t} C^t_{\Psi,\eta}, \quad \text{for all } t. \tag{4.16}$$

10. So, from (4.16), we derive that if $\Phi = \{\phi_t\}_t$ is an almost cosymplectic isotopy such that $\phi_t^\star(\eta) = e^{f_t}\eta$, for each t, then, we have

$$C^t_{\Phi^{-1},\eta} = -e^{f_t} C^t_{\Phi,\eta} \circ \phi_t^{-1}, \quad \text{for all } t.$$

4.2 Cosymplectic geometry and Symplectic geometry

The following facts give some relationships between symplectic and cosymplectic geometries.

From Lemma 2.1 we have :

Fact 1: For any cosymplectic isotopy $\Phi = \{\phi_t\}_t$, one defines an isotopy $\widetilde{\Phi} = \{\widetilde{\phi}_t\}_t$ of the symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ as follows: For each t,

$$\begin{array}{rccc} \widetilde{\phi}_t: & M \times \mathbb{R} & \longrightarrow & M \times \mathbb{R} \\ & (x,u) & \longmapsto & \left(\phi_t(x), \mathcal{R}_{\Lambda_t(\Phi)}(x,u)\right), \end{array}$$

where $\mathcal{R}_{\Lambda_t(\Phi)}(x,u) := u - \int_0^t C^s_{\Phi,\eta}(x)ds \mod 2\pi$. Furthermore, if we consider the canonical projection $p : \widetilde{M} \longrightarrow M$, then for each t, we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\phi}_t} & \widetilde{M} \\ p\downarrow & & \downarrow p \\ M & \xrightarrow{\phi_t} & M, \end{array}$$

namely $p \circ \widetilde{\phi_t} = \phi_t \circ p$. The isotopy $\widetilde{\Phi} = \{\widetilde{\phi}_t\}_t$ is in fact symplectic: since we have

$$\begin{aligned} \widetilde{\phi_t}^\star(\widetilde{\omega}) &= (p \circ \widetilde{\phi_t})^\star\omega + (p \circ \widetilde{\phi_t})^\star\eta \wedge \widetilde{\phi_t}^\star(\pi_2^\star(du)) = (\phi_t \circ p)^\star\omega + (\phi_t \circ p)^\star\eta \wedge \pi_2^\star(du) \\ &= p^\star\omega + p^\star\eta \wedge \pi_2^\star(du) = \widetilde{\omega}. \end{aligned} \tag{4.17}$$

Fact 2: For any cosymplectic isotopy $\Phi = \{\phi_t\}_t$, one defines an isotopy $\widetilde{\Phi} = \{\widetilde{\phi}_t\}_t$ as in Fact 1, we can easily compute, $\dot{\widetilde{\phi_t}} = \dot{\phi}_t - p^\star(C^t_{\Phi,\eta})\frac{\partial}{\partial u}$, for each t, which implies that

$$\imath_{\dot{\widetilde{\phi}}_t}\widetilde{\omega} = p^\star(\imath_{\dot{\phi}_t}\omega) + p^\star(C^t_{\Phi,\eta})\pi_2^\star(du) + p^\star(C^t_{\Phi,\eta}\eta), \quad \text{for each } t. \tag{4.18}$$

4.3 Almost cosymplectic structure and Symplectic structure

Proposition 4.2. *Let (M, ω, η) be a cosymplectic manifold. If $\Psi = \{\phi_t\}_t$ is any almost cosymplectic isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, then the isotopy $\widetilde{\Psi} = \{\widetilde{\psi}_t\}_t$ defined by*

$$\begin{array}{rccc} \widetilde{\psi}_t: & M \times \mathbb{R} & \longrightarrow & M \times \mathbb{R} \\ & (x,u) & \longmapsto & \left(\psi_t(x), ue^{-f_t(x)}\right) \end{array}$$

is a symplectic isotopy of the symplectic manifold $(\widetilde{M}, \widetilde{\omega})$.

Proof. Assume $\Psi = \{\psi_t\}_t$ is any almost cosymplectic isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, and consider the projection maps $p : \widetilde{M} \longrightarrow M$, and $\pi_2 : \widetilde{M} \longrightarrow \mathbb{R}$. For each t, we have $p \circ \widetilde{\psi}_t = \psi_t \circ p$, and also

$$\begin{aligned} \widetilde{\psi}_t^\star(\widetilde{\omega}) &= \widetilde{\psi}_t^\star(p^\star(\omega) + p^\star(\eta) \wedge \pi_2^\star(d\theta)) = (p \circ \widetilde{\psi}_t)^\star \omega + (p \circ \widetilde{\psi}_t)^\star \eta \wedge (\pi_2 \circ \widetilde{\psi}_t)^\star du \\ &= (\psi_t \circ p)^\star \omega + (\psi_t \circ p)^\star \eta \wedge e^{-f_t \circ p} \pi_2^\star du = p^\star((\psi_t)^\star \omega) + p^\star((\psi_t)^\star \eta) \wedge e^{-f_t \circ p}\pi_2^\star du \\ &= p^\star(\omega) + e^{f_t \circ p} p^\star(\eta) \wedge e^{-f_t \circ p} \pi_2^\star(du) \\ &= \widetilde{\omega}. \end{aligned}$$

Thus, $\widetilde{\Psi} = \{\widetilde{\psi}_t\}_t := \left\{(\psi_t \circ p, \pi_2 e^{-f_t \circ p})_t\right\}_t$ is a symplectic isotopy of $(\widetilde{M}, \widetilde{\omega})$. □

Proposition 4.3. *Let (M, ω, η) be a cosymplectic manifold. If $\Psi = \{\psi_t\}_t$ is any almost cosymplectic isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, then we have*

$$d\left(\pi_2 \eta(\dot{\psi}_t) \circ p\right) = \left(\eta(\dot{\psi}_t) \circ p\right) \pi_2^\star(du) + \left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) p^\star(\eta), \quad \textit{for each } t.$$

Proof. Assume $\Psi = \{\psi_t\}_t$ is any almost cosymplectic isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, and consider the projection maps $p : \widetilde{M} \longrightarrow M$, and $\pi_2 : \widetilde{M} \longrightarrow \mathbb{R}$. From the previous Proposition 4.2, the isotopy $\widetilde{\Psi} := \{(\psi_t \circ p, \pi_2 e^{-f_t \circ p})_t\}_t$ is a symplectic isotopy of $(\widetilde{M}, \widetilde{\omega})$; namely, the 1−form $\imath_{\dot{\widetilde{\psi}}_t} \widetilde{\omega}$ is closed for each t. We also have, $\dot{\widetilde{\psi}}_t = \dot{\psi}_t - \left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) \frac{\partial}{\partial u}$, which implies that

$$\imath_{\dot{\widetilde{\psi}}_t} \widetilde{\omega} = p^\star(\imath_{\dot{\psi}_t} \omega) + \eta(\dot{\psi}_t) \circ p \pi_2^\star(du) + \left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) p^\star(\eta), \qquad \text{for each } t. \tag{4.19}$$

Therefore, differentiating (4.19) gives:

$$d\imath_{\dot{\widetilde{\psi}}_t} \widetilde{\omega} = d\left(p^\star(\imath_{\dot{\psi}_t}\omega)\right) + d\left(\eta(\dot{\psi}_t) \circ p \pi_2^\star(du) + \left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) p^\star(\eta)\right),$$

i.e., $0 = 0 + d\left(\eta(\dot{\psi}_t) \circ p \pi_2^\star(du) + \left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) p^\star(\eta)\right)$, for each t. That is,

$$d\left(\eta(\dot{\psi}_t) \circ p\right) \wedge \pi_2^\star(d\theta) = -d\left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) \wedge p^\star(\eta), \qquad \text{for each } t.$$

Taking the interior derivative in the above equality with respect to the vector field $\frac{\partial}{\partial u}$, yields $-d\left(\eta(\dot{\psi}_t) \circ p\right) = -\left(\dot{f}_t \circ p e^{-f_t \circ p}\right) p^\star(\eta)$, for each t. Finally, we compute for each t

$$d\left(\pi_2 \eta(\dot{\psi}_t) \circ p\right) = \left(\eta(\dot{\psi}_t) \circ p\right) \pi_2^\star(du) + d\left(\eta(\dot{\psi}_t) \circ p\right) \pi_2 = \left(\eta(\dot{\psi}_t) \circ p\right) \pi_2^\star(du) + \left(\dot{f}_t \circ p e^{-f_t \circ p} \pi_2\right) p^\star(\eta).$$

□

The following is a consequence of Proposition 4.3.

Proposition 4.4. *Let (M, ω, η) be a cosymplectic manifold. If $\Psi = \{\psi_t\}_t$ is any almost cosymplectic isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ (or $\mathcal{L}_{\dot{\psi}_t}\eta = \mu_t \eta$) for all t, then we have*

$$\mu_t = f_t' e^{-f_t}, \qquad \textit{or equivalently} \qquad f_t' \circ \psi_t = f_t' e^{f_t}, \qquad \textit{for each} \quad t. \tag{4.20}$$

Proof. Assume $\Psi = \{\psi_t\}$ to be any almost cosymplectic isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ (or $\mathcal{L}_{\dot{\psi}_t}\eta = \mu_t\eta$) for all t. From the proof of Proposition 4.3, we derive that $d\left(\eta(\dot{\psi}_t) \circ p\right) = \left(f_t' \circ p e^{-f_t \circ p}\right) p^\star(\eta)$, and composing the latter equality with any smooth section of the projection p, yields $d\left(\eta(\dot{\psi}_t)\right) = \left(f_t' e^{-f_t}\right)\eta$, for each t. On the other hand, from $d\left(\eta(\dot{\psi}_t)\right) = \mathcal{L}_{\dot{\psi}_t}\eta = \mu_t\eta$, for all t, we derive that $\left(f_t' e^{-f_t}\right)\eta = \mu_t\eta$, for each t. Applying the Reeb vector field in both sides of the latter equality implies $\mu_t = f_t' e^{-f_t}$, for each t. From $f_t = \int_0^t \mu_s \circ \psi_s ds$, it follows that $f_t' \circ \psi_t = f_t' e^{f_t}$, for each t. □

Proposition 4.5. *Let (M, ω, η) be a cosymplectic manifold. If $\Psi = \{\psi_t\}_t$ is any almost co-Hamiltonian isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, and $\iota_{\dot{\psi}_t}\omega = dH_t$, then the isotopy $\widetilde{\Psi} = \{\widetilde{\psi}_t\}_t$ defined by*

$$\begin{array}{cccc} \widetilde{\psi}_t: & M\times\mathbb{R} & \longrightarrow & M\times\mathbb{R} \\ & (x,u) & \longmapsto & \left(\psi_t(x), ue^{-f_t(x)}\right) \end{array}$$

is a Hamiltonian isotopy of the symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ such that for each t.

$$\iota_{\dot{\widetilde{\psi}}_t}\widetilde{\omega} = d\left(H_t\circ p + \pi_2\eta(\dot{\psi}_t)\circ p\right). \tag{4.21}$$

The following theorem shows that the Reeb vector field determines the almost cosymplectic nature of a uniform limit of a sequence of almost cosymplectic diffeomorphisms.

Theorem 4.1. *Let (M, ω, η) be a compact cosymplectic manifold with Reeb vector field ξ. Let $\{\psi_i\}_i$ be any sequence of almost co-symplectic diffeomorphisms that uniformly converges to a diffeomorphism ψ, and suppose that $\psi_i^\star(\eta) = e^{f_i}\eta$. The following assertions hold.*

1. *The smooth function $\psi^\star(\eta)(\xi)$, is non-negative.*
2. *If the smooth function $\psi^\star(\eta)(\xi)$, is positive, then the sequence $\{f_i\}_i$ uniformly converges to $F_\psi^\xi := \ln\left(\psi^\star(\eta)(\xi)\right)$.*
3. *If the smooth function $\psi^\star(\eta)(\xi)$ is equal to the constant function 1, then ψ is a cosymplectic diffeomorphism: a flexibility result.*
4. *If the smooth function $\psi^\star(\eta)(\xi)$ is positive and different from the constant function 1, then ψ is an almost cosymplectic diffeomorphism with $\psi^\star(\eta) = e^{F_\psi^\xi}\eta$: a rigidity result.*

To prove the above theorem, we need the following lemma and its corollary.

Lemma 4.3. *Let (M, ω, η) be a compact cosymplectic manifold. If $\{\psi_i\}_i$ is a sequence of almost cosymplectic diffeomorphisms that uniformly converges to a diffeomorphism ψ, with $\psi_i^\star(\eta) = e^{f_i}\eta$, then for each fixed $x \in M$, the sequence of positive real numbers $\left\{e^{f_i(x)}\right\}_i$ converges to $(\psi^\star(\eta)(\xi))(x)$, where ξ is the Reeb vector field.*

Corollary 4.1. *Let (M, ω, η) be a compact cosymplectic manifold. If $\{\psi_i\}_i$ is a sequence of almost co-symplectic diffeomorphisms that uniformly converges to a diffeomorphism ψ, with $\psi_i^\star(\eta) = e^{f_i}\eta$, then for any smooth curve $\gamma \subset M$, we have $\lim\limits_{i\longrightarrow\infty}\int_\gamma e^{f_i}\eta = \int_\gamma \psi^\star(\eta)$.*

Proof. Assume M to be equipped with a Riemannian metric g, with injectivity radius $r(g)$. Let γ be any smooth curve in M. Since $\psi_i \xrightarrow{C^0} \psi$, then for i sufficiently large we may assume that $d_{C^0}(\psi_i, \psi) \leqslant \frac{r(g)}{2}$, and derive that, for each $t \in [0, 1]$, the point $\psi_i(\gamma(t))$ can be connected to the point $\psi(\gamma(t))$ through a minimizing geodesic $\mathfrak{X}_i^t$. This means that the curves $\mathfrak{X}_i^0, \mathfrak{X}_i^1, \psi_i\circ\gamma$, and $\psi\circ\gamma$ form the boundary of a smooth 2–chain $\clubsuit(\gamma, \psi_i, \psi) \subset M$. Since $d\eta = 0$, we obtain, by using Stockes' theorem that $\int_{\clubsuit(\gamma,\psi_i,\psi)} d\eta = 0$, i.e., $\int_{\psi_i\circ\gamma}\eta - \int_{\psi\circ\gamma}\eta = \int_{\mathfrak{X}_i^1}\eta - \int_{\mathfrak{X}_i^0}\eta$, for i sufficiently large. Furthermore, from $\psi_i^\star(\eta) - \psi^\star(\eta) = e^{f_i}\eta - \psi^\star(\eta)$, we derive that for all i sufficiently large,

$$\left|\int_\gamma\left(e^{f_i}\eta - \psi^\star(\eta)\right)\right| = \left|\int_\gamma\left(\psi_i^\star(\eta) - \psi^\star(\eta)\right)\right| = \left|\int_{\mathfrak{X}_i^1}\eta - \int_{\mathfrak{X}_i^0}\eta\right| \leqslant 2|\eta|_0 d_{C^0}(\psi_i, \psi), \tag{4.22}$$

where $|\cdot|_0$ stands for the uniform sup norm on the space of 1–forms of a compact manifold [16]. The top right hand side of the above estimates tends to zero as i goes to infinity. □

Proof of Lemma 4.3. Let $\{\phi_t\}_t$ be the cosymplectic flow generated by the Reeb vector field ξ. For each fixed $t \in]0,1[$, and each $x \in M$, consider the smooth curve $\bar{\gamma}_{x,t} : s \longmapsto \phi_{st}(x)$, and derive from Corollary 4.1 that $\lim_{i\longrightarrow\infty} \int_{\bar{\gamma}_{x,t}} \left(e^{f_i}\eta\right) = \int_{\bar{\gamma}_{x,t}} \psi^\star(\eta)$, for each fixed $t \in]0,1[$, i.e., $\lim_{i\longrightarrow\infty} \int_0^t \left(e^{f_i(\phi_u(x))}du\right) = \int_0^t (\psi^\star(\eta)(\xi))(\phi_u(x))du$, for each fixed $t \in]0,1[$.

Therefore, considering the sequence of smooth functions $U_k : s \longmapsto \int_0^s e^{f_k(\phi_u(x))}du$. By definition $U_k'(t_0) := \lim_{s\longrightarrow t_0} \frac{U_k(s) - U_k(t_0)}{s - t_0}$, so, for all $t_0 \in]0,1[$, and for all $x \in M$, we have

$$\begin{aligned}\lim_{k\longrightarrow\infty} U_k'(t_0) &:= \lim_{k\longrightarrow\infty}\lim_{s\longrightarrow t_0} \frac{U_k(s) - U_k(t_0)}{s - t_0} = \lim_{s\longrightarrow t_0}\left(\lim_{k\longrightarrow\infty} \frac{U_k(s) - U_k(t_0)}{s - t_0}\right),\\ &= \lim_{s\longrightarrow t_0}\left[\frac{1}{s-t_0}\left(\int_0^s (\psi^\star(\eta)(\xi))(\phi_u(x))du - \int_0^{t_0} (\psi^\star(\eta)(\xi))(\phi_u(x))du\right)\right] = (\psi^\star(\eta)(\xi))(\phi_{t_0}(x)).\end{aligned}$$

This implies that $\lim_{i\longrightarrow\infty} e^{f_i(x)} = (\psi^\star(\eta)(\xi))(x)$, for each $x \in M$. □

Proof. of Theorem 4.1. By Lemma 4.3, the smooth function $x \longmapsto (\psi^\star(\eta)(\xi))(x)$, is the uniform limit of a sequence of positive functions, hence the latter is non-negative. Assume that the smooth function $x \longmapsto (\psi^\star(\eta)(\xi))(x)$, is positive. Therefore, as in Proposition 4.2, we define a sequence of symplectic isotopies of the symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ by $\widetilde{\psi}_i := (\psi_i \circ p, \pi_2 e^{-f_i\circ p})$, for each i. Since by assumption, $\psi_i \xrightarrow{C^0} \psi$, and by Lemma 4.3 we have $\lim_{i\longrightarrow\infty} e^{-f_i\circ p} = e^{-\ln((\psi^\star(\eta)(\xi)))\circ p}$, then, it follows that $\widetilde{\psi}_i \xrightarrow{C^0} (\psi\circ p, \pi_2 e^{-F_\xi^\psi\circ p})$, with $F_\xi^\psi := \ln((\psi^\star(\eta)(\xi)))$. Since the map $(\psi\circ p, \pi_2 e^{-F_\xi^\psi\circ p})$ is a diffeomorphism, then it follows from the celebrated rigidity theorem of Elishberg-Gromov that the diffeomorphism $\Phi_\psi := (\psi\circ p, \pi_2 e^{-F_\xi^\psi\circ p})$, is a symplectic diffeomorphism of $(\widetilde{M}, \widetilde{\omega})$. Finally, the fact that $\Phi_\psi^\star(\widetilde{\omega}) = \widetilde{\omega}$, obviously implies that $\psi^\star(\omega) = \omega$, and $\psi^\star(\eta) = e^{F_\xi^\psi}\eta$, provided the positive function $x \longmapsto (\psi^\star(\eta)(\xi))(x)$ is different from the constant function 1: that is, ψ is an almost cosymplectic diffeomorphism. Otherwise, we have $\psi^\star(\omega) = \omega$, and $\psi^\star(\eta) = \eta$ meaning that ψ is a cosymplectic diffeomorphism. □

4.3.1 Almost co-Hamiltonian dynamical systems

In this subsection, we derive a consequence of Arnold's conjecture from symplectic geometry.

Proposition 4.6. *Let (M, ω, η) be a closed cosymplectic manifold. If $\Psi = \{\phi_t\}_t$ is any almost co-Hamiltonian isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, and $\imath_{\dot{\psi}_t}\omega = dH_t$, then for each t, the map ψ_t has at least one fixed point x_t satisfying $f_t(x_t) = 0$.*

Proof. If $\Psi = \{\phi_t\}_t$ is any almost co-Hamiltonian isotopy such that $\psi_t^\star(\eta) = e^{f_t}\eta$ for all t, then the isotopy $\widetilde{\Psi} = \{\widetilde{\psi}_t\}_t$ defined by $\widetilde{\psi}_t : M\times\mathbb{R} \longrightarrow M\times\mathbb{R}, (x,u) \longmapsto (\psi_t(x), ue^{-f_t(x)})$, is a Hamiltonian isotopy of the symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ (Proposition 4.5). Thus, by Arnold's conjecture, for each t, the map $\widetilde{\psi}_t$ has at least one fix point (x_t, u_t). That is, $\psi_t(x_t) = x_t$, and $u_t e^{-f_t(x_t)} = u_t$, namely, $\psi_t(x_t) = x_t$, and $f_t(x_t) = 0$, for each t. □

Proposition 4.7. *Let (M, ω, η) be a closed cosymplectic manifold. Let ψ be an almost co-Hamiltonian diffeomorphism such that $\psi^\star(\eta) = e^f\eta$. If $\Psi = \{\psi_t\}_t$ is any almost co-Hamiltonian isotopy with time-one map ψ, then any fix point of ψ is a critical point for the function $x \longmapsto \left(\int_0^1 \psi_s^\star\left(\eta(\dot{\psi}_s)\right)ds\right)(x)$.*

Proof. From the formula

$$e^f\eta - \eta \quad = \quad \psi^\star(\eta) - \eta = d\left(\int_0^1 \psi_s^\star\left(\eta(\dot\psi_s)\right) ds\right), \tag{4.23}$$

we derive that, for each $x \in \mathcal{F}ix(\psi)$, we have

$$e^{f(x)}\eta_{|_x} - \eta_{|_x} \quad = \quad \left(e^f\eta - \eta\right)_{|_x} = d\left(\int_0^1 \psi_s^\star\left(\eta(\dot\psi_s)\right) ds\right)\Bigg|_x, \tag{4.24}$$

i.e., $0 = d\left(\int_0^1 \psi_s^\star\left(\eta(\dot\psi_s)\right) ds\right)\Bigg|_x$, since by Proposition 4.6, we have $f(y) = 0$, whenever $y \in \mathcal{F}ix(\psi)$. □

5 Geometry of cosymplectic diffeomorphisms

5.1 Comparison of norms

Let (M, g^M) and (N, g^N) be two smooth compact Riemannian manifolds, and consider $\widetilde{M} := M \times N$. Let $p : \widetilde{M} \longrightarrow M$ and $q : \widetilde{M} \longrightarrow N$ be the projection maps and denote by $\widetilde{g}$ the corresponding induced product metric on $\widetilde{M}$. We recall in this section some norms on the set of closed 1–forms (see. [16]).

5.1.1 Comparison of the norms $|p^\star(\alpha)|_0$ and $|\alpha|_0$ with $\alpha \in \Omega^1(M)$

Consider a 1– form α on M and let us recall the definition of the supremum norm (i.e., the uniform sup norm) of α: for each $x \in M$, we know that α induces a linear map $\alpha_x : T_xM \longrightarrow \mathbb{R}$, whose norm is given by

$$\|\alpha_x\|^{g^M} := \sup\left\{|\alpha_x(X)| \;:\; X \in T_xM,\; \|X\|_{g^M} = 1\right\}, \tag{5.1}$$

where $\|\cdot\|_{g^M}$ is the norm induced on each tangent space T_xM (at the point x) by the Riemannian metric g^M. Therefore, the uniform sup norm of α, say $|\cdot|_0$, is defined as

$$|\alpha|_0 := \sup_{x\in M}\|\alpha_x\|^{g^M} = \sup_{x\in M}\sup_{X\in(S^1TM)_x}|\alpha_x(X)|. \tag{5.2}$$

Moreover, since $p^\star\alpha$ is a 1–form on $\widetilde{M}$, then for each $(x, y) \in \widetilde{M}$, we have

$$\begin{aligned}
\|p^\star(\alpha)_{|(x,y)}\|^{\widetilde{g}} \quad &= \quad \sup\left\{|\alpha_x(p_\star(Y))| \;:\; Y \in T_{(x,y)}\widetilde{M}, \|Y\|_{\widetilde{g}} = 1\right\}\\
&= \quad \sup\left\{|\alpha_x(Y_1)| \;:\; (Y_1 + Y_2) \in T_xM \oplus T_yN \text{ and } \|Y_1\|_{g^M} + \|Y_2\|_{g^N} = 1\right\}\\
&\leqslant \quad \sup\left\{|\alpha_x(Y_1)| \;:\; Y_1 \in T_xM, \|Y_1\|_{g^M} \leqslant 1\right\},
\end{aligned}$$

where $\|\cdot\|_{g^M}$ (resp. $\|\cdot\|_{g^N}$) is the norm induced on each tangent space T_xM (resp. T_yN) by the Riemannian metric g^M (resp. g^N). Therefore, we have $\|p^\star(\alpha)_{|(x,y)}\|^{\widetilde{g}} \leqslant \|\alpha_x\|^{g^M}$, for each $(x, y) \in \widetilde{M}$, which implies that $|p^\star(\alpha)|_0 \leqslant |\alpha|_0$.

5.1.2 Splitting of closed 1-forms and the uniform sup norm

Let $H^1(M,\mathbb{R})$ (resp. $H^1(\widetilde{M},\mathbb{R})$) denote the first de Rham cohomology group (with real coefficients) of M (resp. $\widetilde{M}$) and let $\mathcal{Z}^1(M)$ (resp. $\mathcal{Z}^1(\widetilde{M})$) denote the space of all closed 1–forms on M (resp. $\widetilde{M}$). Consider the map

$$\mathcal{S} : H^1(M,\mathbb{R}) \longrightarrow \mathcal{Z}^1(M), \tag{5.3}$$

to be a fixed linear section of the natural projection

$$\pi_M : \mathcal{Z}^1(M) \longrightarrow H^1(M,\mathbb{R}). \tag{5.4}$$

Each $\alpha \in \mathcal{Z}^1(M)$ splits as:

$$\alpha = \mathcal{S}(\pi_M(\alpha)) + (\alpha - \mathcal{S}(\pi_M(\alpha))). \tag{5.5}$$

We shall call the 1–form $(\alpha - \mathcal{S}(\pi_M(\alpha)))$ the exact part of α and throughout the paper, for simplicity, when this will be necessary, the latter 1–form will be denoted $df_{\alpha,\mathcal{S}}$ to mean that it is the differential of a certain function that depends on α and $\mathcal{S}$; while we shall call the 1–form $\mathcal{S}(\pi_M(\alpha))$ the $\mathcal{S}$–form of α. Let $\mathbb{H}^1(M,\mathcal{S})$ denote the space of all $\mathcal{S}$–forms and define the set $\mathbb{B}^1(M) := \left(\mathcal{Z}^1(M) \setminus \mathbb{H}^1(M,\mathcal{S})\right) \cup \{0\}$.

We then have the following direct sum: $\mathcal{Z}^1(M) = \mathbb{H}^1(M,\mathcal{S}) \oplus \mathbb{B}^1(M)$, with $\dim(\mathbb{H}^1(M,\mathcal{S})) = \dim(H^1(M,\mathbb{R})) < \infty$, for each linear section $\mathcal{S}$ (see [16]).

Denote by $\mathcal{P}\mathbb{H}^1(M,\mathcal{S})$ the space of all smooth mappings $\mathcal{H} : [0,1] \longrightarrow \mathbb{H}^1(M,\mathcal{S})$. Since both spaces $\mathbb{H}^1(M,\mathcal{S})$ and $H^1(M,\mathbb{R})$ are isomorphic and $H^1(M,\mathbb{R})$ is a finite dimensional vector space whose dimension is the first Betti number $b_1(M)$, then $\mathbb{H}^1(M,\mathcal{S})$ is of finite dimension [15]. Thus, there exists a positive constants $K_1(g)$ and $k_2(g)$ which depend on the Riemannian metric g on M such that

$$k_1(g)\|\alpha\|_{L^2} \leqslant |\alpha|_0 \leqslant k_2(g)\|\alpha\|_{L^2}, \tag{5.6}$$

for all $\alpha \in \mathbb{H}^1(M,\mathcal{S})$. On the other hand, consider the projection $p : \widetilde{M} \longrightarrow M$, and let

$$\pi_{\widetilde{M}} : \mathcal{Z}^1(\widetilde{M}) \longrightarrow H^1(\widetilde{M},\mathbb{R}), \tag{5.7}$$

be the canonical projection, where $\mathcal{Z}^1(\widetilde{M})$ is the set of all closed 1–forms on $\widetilde{M}$: we have the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^1(M) & \xrightarrow{p^\star} & \mathcal{Z}^1(\widetilde{M}) \\ \pi_M \downarrow & & \downarrow \pi_{\widetilde{M}} \\ H^1(M,\mathbb{R}) & \xrightarrow{p^\star} & H^1(\widetilde{M},\mathbb{R}), \end{array}$$

namely $p^\star \circ \pi_M = \pi_{\widetilde{M}} \circ p^\star$. The following composition of linear mappings

$$\begin{array}{ccccc} \mathbb{H}^1(M,\mathcal{S}) & \xrightarrow{\pi_M} & H^1(M,\mathbb{R}) & \xrightarrow{p^\star} & H^1(\widetilde{M},\mathbb{R}) \\ \alpha & \longmapsto & \pi_M(\alpha) & \longmapsto & p^\star(\pi_M(\alpha)), \end{array}$$

is continuous, then there is a constant κ_0 such that $\|\pi_{\widetilde{M}}(p^\star(\alpha))\|_{L^2} = \|p^\star(\pi_M(\alpha))\|_{L^2} \leqslant \kappa_0|\alpha|_0$, since from the commutation of the previous diagram we have $p^\star \circ \pi_M = \pi_{\widetilde{M}} \circ p^\star$. Let $\widetilde{S} : H^1(\widetilde{M},\mathbb{R}) \longrightarrow \mathcal{Z}^1(\widetilde{M})$ be any fixed linear section of $\pi_{\widetilde{M}}$, then there exists a positive constant υ_0 such that

$$|\widetilde{S}(\pi_{\widetilde{M}}(\theta))|_0 \leqslant \upsilon_0\|\pi_{\widetilde{M}}(\theta)\|_{L^2}, \qquad \forall\, \theta \in \mathcal{Z}^1(\widetilde{M}). \tag{5.8}$$

Summarizing the above inequalities gives for all $\alpha \in \mathcal{Z}^1(M)$:

$$\left|\widetilde{S}\left(\pi_{\widetilde{M}}(p^\star(\mathcal{S}(\pi_M(\alpha))))\right)\right|_0 \leqslant \upsilon_0\|\pi_{\widetilde{M}}(p^\star(\mathcal{S}(\pi_M(\alpha))))\|_{L^2} \leqslant \upsilon_0\kappa_0\left|\mathcal{S}(\pi_M(\alpha))\right|_0. \tag{5.9}$$

5.2 Co-Hofer-like geometries

For any $X \in \mathfrak{X}_{\eta,\omega}(M)$, the closed 1–forms $\imath_X\omega$ and $\eta(X)\eta$ split as follows:

$$\imath_X\omega = \mathcal{H}_\omega + dU_\omega, \qquad \text{and} \qquad \eta(X)\eta = \mathcal{K}_\eta + dV_\eta. \tag{5.10}$$

Hence, the closed 1–form, $\widetilde{I}_{\eta,\omega}(X)$ splits as $\widetilde{I}_{\eta,\omega}(X) = (\mathcal{K}_\eta + \mathcal{H}_\omega) + d(U_\omega + V_\eta)$, from which, one defines a norm $\|\cdot\|_C^{\mathcal{S}}$ on $\mathfrak{X}_{\eta,\omega}(M)$ as follows :

$$\|X\|_C^{\mathcal{S}} := \|\mathcal{K}_\eta + \mathcal{H}_\omega\|_{L^2} + \nu^B(d(U_\omega + V_\eta)) + |\eta(X)|, \quad \forall X \in \mathfrak{X}_{\eta,\omega}(M) \tag{5.11}$$

where $\|\cdot\|_{L^2}$ is the L^2–Hodge norm and ν^B is any norm on $\mathbb{B}^1(M)$ which we assume to be equivalent to the oscillation norm (see [16])

$$osc(f) = \max_x f(x) - \min_x f(x), \qquad \forall f \in C^\infty(M).$$

Theorem 5.1. *Let (M, ω, η) be a compact cosymplectic manifold, let $\mathcal{S}$ and $\mathcal{T}$ be two linear sections of the projection $\pi : \mathcal{Z}^1(M) \longrightarrow H^1(M, \mathbb{R})$. Then, the two norms $\|\cdot\|_C^{\mathcal{S}}$ and $\|\cdot\|_C^{\mathcal{T}}$ are equivalent.*

Proof. Let X be a strict cosymplectic vector field such that $\widetilde{I}_{\eta,\omega}(X) = \left(\mathcal{K}_\eta^{\mathcal{S}} + \mathcal{H}_\omega^{\mathcal{S}}\right) + d\left(U_\omega^{\mathcal{S}} + V_\eta^{\mathcal{S}}\right)$, with respect to the $\mathcal{S}$–decomposition, and $\widetilde{I}_{\eta,\omega}(X) = \left(\mathcal{K}_\eta^{\mathcal{T}} + \mathcal{H}_\omega^{\mathcal{T}}\right) + d\left(U_\omega^{\mathcal{T}} + V_\eta^{\mathcal{T}}\right)$, with respect to the $\mathcal{T}$–decomposition. It is enough to show that there exists $C_1 > 0$, and $C_2 > 0$ such that

$$C_1\|X\|_C^{\mathcal{T}} \leqslant \|X\|_C^{\mathcal{S}} \leqslant C_2\|X\|_C^{\mathcal{T}}.$$

Since $\dim(\mathbb{H}^1(M, \mathcal{S})) = \dim(H^1(M, \mathbb{R})) = \dim(\mathbb{H}^1(M, \mathcal{T})) < \infty$, then all the norms on each of the spaces $H^1(M, \mathcal{S})$ and $H^1(M, \mathcal{T})$ are equivalent. We shall equip $H^1(M, \mathcal{S})$ with a basis $\mathbf{B}$ (resp. $H^1(M, \mathcal{T})$ with a basis $\mathbf{B}'$) and denote by $\|\cdot\|_{\mathbf{B}}$ (resp. $\|\cdot\|_{\mathbf{B}'}$) the corresponding norm. So, we only have to show that

$$C_1(\nu^B(d(U_\omega^{\mathcal{T}} + V_\eta^{\mathcal{T}})) + \|\mathcal{K}_\eta^{\mathcal{T}} + \mathcal{H}_\omega^{\mathcal{T}}\|_{\mathbf{B}'} + |\eta(X)|) \leqslant (\nu^B(d(U_\omega^{\mathcal{S}} + V_\eta^{\mathcal{S}})) + \|\mathcal{K}_\eta^{\mathcal{S}} + \mathcal{H}_\omega^{\mathcal{S}}\|_{\mathbf{B}} + |\eta(X)|), \tag{5.12}$$

and

$$(\nu^B(d(U_\omega^{\mathcal{S}} + V_\eta^{\mathcal{S}})) + \|\mathcal{K}_\eta^{\mathcal{S}} + \mathcal{H}_\omega^{\mathcal{S}}\|_{\mathbf{B}} + |\eta(X)|) \leqslant C_2(\nu^B(d(U_\omega^{\mathcal{T}} + V_\eta^{\mathcal{T}})) + \|\mathcal{K}_\eta^{\mathcal{T}} + \mathcal{H}_\omega^{\mathcal{T}}\|_{\mathbf{B}'} + |\eta(X)|). \tag{5.13}$$

The inequalities (5.12) and (5.13) follow from similar arguments to those used in Banyaga [1] for Hodge's decomposition. But, here the uniqueness of the harmonic part in Hodge's decomposition is replaced by the fact that $\mathbb{H}^1(M, \mathcal{S}) \cap \mathbb{B}^1(M) = \{0\}$ (resp. $\mathbb{H}^1(M, \mathcal{T}) \cap \mathbb{B}^1(M) = \{0\}$). □

Base on Theorem 5.1, we shall denote the norm $\|\cdot\|_C^{\mathcal{S}}$, simply by $\|\cdot\|_C$ no matter the choice of the linear section $\mathcal{S}$.

5.2.1 Co-Hofer-like lengths

Let $\Phi = \{\phi_t\} \in Iso_{\eta,\omega}(M)$, for each t, we have $\|\dot{\phi}_t\|_C := \|\mathcal{K}_\eta^t + \mathcal{H}_\omega^t\|_{L^2} + osc(U_\omega^t + V_\eta^t) + |C_{\Phi,\eta}^t|$. Therefore, we define the $L^{(1,\infty)}$–version of the co-Hofer-like length of $\Phi := \{\phi_t\}$ as:

$$l_{Co}^{(1,\infty)}(\Phi) := \int_0^1 \|\dot{\phi}_t\|_C dt, \tag{5.14}$$

and, L^∞–version of the co-Hofer-like length of Φ as:

$$l_{Co}^{\infty}(\Phi) := \max_{t\in[0,1]} \|\dot{\phi}_t\|_C. \tag{5.15}$$

Using the relation $\widetilde{I}_{\eta,\omega}(\dot{\phi}_t) = \left(\mathcal{K}_\eta^t + \mathcal{H}_\omega^t\right) + d\left(U_\omega^t + V_\eta^t\right)$, for each t, we have $\widetilde{I}_{\eta,\omega}(\dot{\phi}_{-t}) = -\phi_t^\star\left(\mathcal{K}_\eta^t + \mathcal{H}_\omega^t\right) - d\left(U_\omega^t \circ \phi_t + V_\eta^t \circ \phi_t\right)$, i.e., $\widetilde{I}_{\eta,\omega}(\dot{\phi}_{-t}) = -\left(\mathcal{K}_\eta^t + \mathcal{H}_\omega^t\right) - d\left(U_\omega^t \circ \phi_t + V_\eta^t \circ \phi_t + \Delta_t(\mathcal{K}_\eta + \mathcal{H}_\omega, \Phi)\right)$, with $\Delta_t(\alpha, \Phi) := \int_0^t \alpha_t(\dot{\phi}_s) \circ \phi_s ds$. Hence, we see that in general, we may have

$$l_{Co}^{(1,\infty)}(\Phi) \neq l_{Co}^{(1,\infty)}(\Phi^{-1}) \qquad \text{or} \qquad l_{Co}^{\infty}(\Phi) \neq l_{Co}^{\infty}(\Phi^{-1}). \tag{5.16}$$

The restriction of the above lengths to the group $H_{\eta,\omega}(M)$ will be called co-Hofer lengths, and denoted l_{CH}^∞, and $l_{CH}^{(1,\infty)}$. Indeed, if $\Phi_F = \{\phi_t\}_t$ is a weakly Hamiltonian isotopy such that $\widetilde{I}_{\eta,\omega}(\dot{\phi}_t) = dF_t$, for all t, then

$$l_{CH}^{(1,\infty)}(\Phi_F) = \int_0^1 \left(osc(F_t) + |C_{\Phi_F,\eta}^t|\right) dt, \tag{5.17}$$

and

$$l_{CH}^{\infty}(\Phi_F) = \max_t \left(osc(F_t) + |C_{\Phi_F,\eta}^t|\right). \tag{5.18}$$

Observe that the lengths l_{CH}^∞, and $l_{CH}^{(1,\infty)}$ are symmetric.

5.2.2 Displacement energy of fibers

Assume that $\Phi = \{\phi_t\}$ and $\widetilde{\Phi} = \{\widetilde{\phi}_t\}$ are as defined in Subsection 4.2 with $\phi_1 \neq id_M$; let l_{HL} denote the Hofer-like length and by E_S, we denote the symplectic displacement energy defined on the closed symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ [1, 16]. Since $\phi_1 \neq id_M$, then $\widetilde{\phi}_1 \neq id_{\widetilde{M}}$, i.e., there exists a compact subset $\mathbf{B}_0 \subset \widetilde{M}$ such that $\widetilde{\phi}_1(\mathbf{B}_0) \cap \mathbf{B}_0 = \emptyset$. We may assume that $\mathbf{B}_0$ is of the form $\mathbf{B} \times \mathbf{C}$, with $\mathbf{B}$ a compact subset of M and $\mathbf{C}$ a compact subset of $\mathbb{R}$. Thus, for each fixed $\theta \in \mathbf{C}$, the compact fiber $\mathbf{B} \times \{\theta\}$ is also completely displaced by $\widetilde{\phi}_1$. Therefore, we have

$$0 < E_S(\mathbf{B} \times \{\theta\}) \leqslant l_{HL}(\widetilde{\Phi}), \qquad \forall \theta \in \mathbf{C}. \tag{5.19}$$

Since the map $\theta \longmapsto E_S(\mathbf{B} \times \{\theta\})$ is bounded and positive on $\mathbf{C}$, we derive that

$$0 < \frac{1}{2\pi} \int_{\mathbf{C}} E_S(\mathbf{B} \times \{\theta\}) d\theta \leqslant l_{HL}(\widetilde{\Phi}). \tag{5.20}$$

On the other hand, from (4.18), if $\widetilde{I}_{\eta,\omega}(\dot{\Phi}_t) = \mathcal{H}_\omega^t + \mathcal{K}_\eta^t + d(U_\eta^t + U_\omega^t)$, for all t, then we derive that

$$\iota_{\dot{\widetilde{\Phi}}_t} \widetilde{\omega} = p^\star(\iota_{\dot{\phi}_t}\omega) + p^\star(C_{\Phi,\eta}^t) d\theta + p^\star(C_{\Phi,\eta}^t \eta) = p^\star(\mathcal{H}_\omega^t + \mathcal{K}_\eta^t) + d(C_{\Phi,\eta}^t \pi_2 + U_\eta^t + U_\omega^t), \tag{5.21}$$

for each t. Thus,

$$\begin{aligned} l_{HL}^\infty(\widetilde{\Phi}) &= \max_t \left(\|\pi_{\widetilde{M}}(p^\star(\mathcal{H}_\omega^t + \mathcal{K}_\eta^t))\|_{L^2} + osc(C_{\Phi,\eta}^t \pi_2 + U_\eta^t + U_\omega^t)\right) \\ &\leqslant \max_t \left(\|\pi_{\widetilde{M}}(p^\star(\mathcal{H}_\omega^t + \mathcal{K}_\eta^t))\|_{L^2} + osc(U_\eta^t + U_\omega^t)\right) + 2\pi \max_t \left(|C_{\Phi,\eta}^t|\right). \end{aligned} \tag{5.22}$$

By (5.9), we have $\|\pi_{\widetilde{M}}(p^\star(\mathcal{H}_\omega^t + \mathcal{K}_\eta^t))\|_{L^2} \leqslant \kappa_0 |\mathcal{H}_\omega^t + \mathcal{K}_\eta^t|_0$, whereas by (5.6), we have $|\mathcal{H}_\omega^t + \mathcal{K}_\eta^t|_0 \leqslant k_2(g)\|\mathcal{H}_\omega^t + \mathcal{K}_\eta^t\|_{L^2}$. So, it follows that

$$l_{HL}^\infty(\widetilde{\Phi}) \leqslant 2 \max\{(1 + \kappa_0 k_2(g)), 2\pi\} l_{Co}^\infty(\Phi). \tag{5.23}$$

Thus, (5.19) and (5.23) imply that

$$0 < \frac{1}{4\pi \max\{(1 + \kappa_0 k_2(g)), 2\pi\}} \int_{\mathbf{C}} E_S(\mathbf{B} \times \{\theta\}) d\theta \leqslant l_{Co}^\infty(\Phi), \tag{5.24}$$

In the rest of the paper, we refer to (5.24) as the Co-energy-inequality.

5.2.3 Gromov area of fibers

Assume that $\Phi_F = \{\phi_t\}_t$ is a weakly Hamiltonian isotopy such that $\widetilde{I}_{\eta,\omega}(\dot{\phi}_t) = dF_t$, for all t, and $\widetilde{\Phi}_F = \{\widetilde{\phi}_t\}$ is defined via Φ_F as in the Subsection 4.2 with $\phi_1 \neq id_M$, let $C_W(\overline{B})$ represents the Gromov area of a ball $\overline{B}$ on the closed symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ [10]. Since $\phi_1 \neq id_M$, then $\widetilde{\phi}_1 \neq id_{\widetilde{M}}$, i.e., there exists a compact subset $\mathbf{B}_0 \subset \widetilde{M}$ such that $\widetilde{\phi}_1(\mathbf{B}_0) \cap \mathbf{B}_0 = \emptyset$. We may assume that $\mathbf{B}_0$ is of the form $\mathbf{B}\times\mathbf{C}$, with $\mathbf{B}$ a compact subset of M, and $\mathbf{C}$ a compact subset of $\mathbb{R}$. Thus, for each fixed $\theta \in \mathbf{C}$, the compact fiber $\mathbf{B} \times \{\theta\}$ is also completely displaced by $\widetilde{\phi}_1$. Therefore,

$$0 < \frac{1}{(2\pi)^2} \int_{\mathbf{C}} C_W(\mathbf{B} \times \{\theta\}) d\theta \leqslant l^{\infty}_{CH}(\Phi_F). \tag{5.25}$$

In the rest of this paper, we shall refer to (5.25) as the co-capacity-inequality.

Here is the cosymplectic analogues of Theorem 6–[8].

Theorem 5.2. *Let (M, η, ω) be a closed cosymplectic manifold. Let $\Phi = \{\phi_i^t\}$ be a sequence of cosymplectic isotopies, $\Psi = \{\psi^t\}_t$ be another cosymplectic isotopy, and $\phi : M \longrightarrow M$ be a map such that*

- *(ϕ_i^1) converges uniformly to ϕ, and*
- *$l^{\infty}_C(\Psi^{-1} \circ \{\phi_i^t\}) \longrightarrow 0, i \longrightarrow \infty$.*

Then we must have $\phi = \psi^1$.

Proof. Let us assume that $\phi \neq \psi^1$, i.e., there exists a compact subset $\mathbf{B}_0 \subseteq M$ which is completely displaced by $(\psi^1)^{-1} \circ \phi$, and since the convergence $\phi_i^1 \longrightarrow \phi$, is uniform, then we may assume that $(\psi^1)^{-1} \circ \phi_i^1$, completely displace $\mathbf{B}_0$, for all i sufficiently large. Fix i_0 to be a sufficiently large natural number. We have a sequence of cosymplectic isotopies $\{\Psi^{-1} \circ \{\phi_j^t\}\}_{j \geqslant i_0}$ with time-one map $(\psi^1)^{-1} \circ \phi_j^1$, for all $j \geqslant i_0$. Then, we derive from the Co-energy-inequality that

$$0 < \frac{1}{4\pi \max\{(1 + \kappa_0 k_2(g)), 2\pi\}} \int_{\mathbf{C}_0} E_S(\mathbf{B}_0 \times \{\cdot\}) d\theta \leqslant l^{\infty}_{Co}(\Psi^{-1} \circ \{\phi_j^t\}), \tag{5.26}$$

for some non-trivial compact subset $\mathbf{C}_0$ of $\mathbb{R}$, and for all $j \geqslant i_0$. Since the right-hand side in (5.24) tends to zero as j tend to infinity, then (5.24) yields a contradiction. □

The following result is an immediate consequence of Theorem 5.2. It can support the existence of a C^0–counterpart of cosymplectic geometry (see [2]).

Corollary 5.1. *Let $\Phi_i = (\{\phi_i^t\}_t)_i$ be a sequence of symplectic isotopies, $\Psi = \{\psi_t\}_t$ be another symplectic isotopy, and $\Xi : t \longmapsto \Xi_t$ be a family of maps $\Xi_t : M \longrightarrow M$, such that the sequence Φ_i converges uniformly to Ξ and $l^{\infty}_C(\Psi^{-1} \circ \Phi_i) \longrightarrow 0, i \longrightarrow \infty$. Then $\Xi = \Psi$.*

Proof. Assume the contrary, i.e., that $\Psi \neq \Xi$. This is equivalent to say that there exists $t \in]0, 1]$ such that $\Xi_t \neq \psi_t$. Therefore, the sequence of symplectic paths $\Phi_{t,i} : s \longmapsto \phi_i^{st}$ contradicts Theorem 5.2. □

5.2.4 Co-Hofer norm

For any weakly Hamiltonian diffeomorphism ψ, we define the $L^{(1,\infty)}$–version of its co-Hofer norm and the L^{∞}–version of its co-Hofer norm respectively as follows:

$$\|\psi\|^{(1,\infty)}_{CH} := \inf(l^{(1,\infty)}_{CH}(\Psi)) \quad \text{and} \quad \|\psi\|^{\infty}_{CH} := \inf(l^{\infty}_{CH}(\Psi)), \tag{5.27}$$

where each infimum is taken over the set of all weakly Hamiltonian isotopies Ψ with time-one maps equal to ψ.

Theorem 5.3. *Let (M, ω, η) be a compact cosymplectic manifold. Then, each of the rules $\|\cdot\|_{CH}^{(1,\infty)}$ and $\|\cdot\|_{CH}^{\infty}$ induces a bi-invariant norm on $Ham_{\eta,\omega}(M)$.*

5.2.5 Co-Hofer-like energies

For $\phi \in G_{\eta,\omega}(M)$, we define its $L^{(1,\infty)}$–energy and its L^{∞}–energy as follows:

$$e_{Co}^{(1,\infty)}(\phi) := \inf(l_{Co}^{(1,\infty)}(\Phi)), \tag{5.28}$$

and

$$e_{Co}^{\infty}(\phi) := \inf(l_{Co}^{\infty}(\Phi)), \tag{5.29}$$

where each infimum is taken over the set of all cosymplectic isotopies Φ with time-one maps equal to ϕ.

5.2.6 Co-Hofer-like norms

The $L^{(1,\infty)}$–version and the L^{∞}–version of the co-Hofer-like norms of $\phi \in G_{\eta,\omega}(M)$ are respectively defined by,

$$\|\phi\|_{Co}^{(1,\infty)} := \frac{1}{2}(e_{Co}^{(1,\infty)}(\phi) + e_{Co}^{(1,\infty)}(\phi^{-1})), \tag{5.30}$$

and

$$\|\phi\|_{Co}^{\infty} := \frac{1}{2}(e_{Co}^{\infty}(\phi) + e_{Co}^{\infty}(\phi^{-1})). \tag{5.31}$$

Theorem 5.4. *Let (M, ω, η) be a compact cosymplectic manifold. Then, each of the rules $\|\cdot\|_{Co}^{(1,\infty)}$ and $\|\cdot\|_{Co}^{\infty}$ induces a right-invariant norm on $G_{\eta,\omega}(M)$.*

Proof. Since checking the other properties of a norm are straight calculations, we shall just prove the non-degeneracy of the norm $\|\cdot\|_{Co}^{\infty}$ if $\phi \in G_{\eta,\omega}(M)$ such that $\|\phi\|_{Co}^{\infty} = 0$, then from the definition of the norm $\|\cdot\|_{Co}^{\infty}$, we derive that there exists a sequence of cosymplectic isotopies $\{\Phi_i\}$, each of which has time-one map ϕ such that $l_{Co}^{\infty}(\Phi_i) < \frac{1}{i}$, for each positive integer i. That is,

- $\lim_{C^0}(\Phi_i(1)) = \phi$ and
- $l_{Co}^{\infty}(\{Id\}^{-1} \circ \Phi_i) \longrightarrow 0$, as $i \longrightarrow \infty$,

where Id is the constant path identity. Hence, by Theorem 5.2, we must have $\phi = id_M$. □

Here is a direct proof of the cosymplectic analogue of the C^0–rigidity result of Eliashberg-Gromov [5]. This proof follows as a direct consequence of Theorem 5.2, and here we adapt the proof of a similar result proved by Buhosky [4].

Theorem 5.5. *The group $G_{\eta,\omega}(M)$ is C^0–closed inside the group $Diff^{\infty}(M)$.*

We need the following rigidity lemma.

Lemma 5.1. *Let (M, ω, η) be a compact connected cosymplectic manifold, and let X_H be a weak Hamiltonian vector field such that $\widetilde{I}_{\eta,\omega}(X_H) = dH$. Let $\{\psi_i\} \subset G_{\eta,\omega}(M)$ such that $\psi_i \xrightarrow{C^0} \psi$. If $\psi \in Diff^{\infty}(M)$, then*

1. *$\psi_\star(\xi)(H) = \xi(H)$, and*
2. *$\eta(\psi_\star(\xi)) = 1$,*

where ξ stands for the Reeb vector field of (M, ω, η).

Proof. For (2), assume M to be equipped with a Riemannian metric g, with injectivity radius $r(g)$. Pick any smooth curve $\gamma \in M$. Using the uniform convergence $\psi_i \xrightarrow{C^0} \psi$, we derive that for i sufficiently large we may assume that $d_{C^0}(\psi_i, \psi) \leqslant \frac{r(g)}{2}$, and derive that, for each $t \in [0, 1]$, the points $\psi_i(\gamma(t))$ to $\psi(\gamma(t))$ can be connected through a minimizing geodesic $\mathfrak{X}_i^t$. This implies that the curves $\mathfrak{X}_i^0$, $\mathfrak{X}_i^1$, $\psi_i \circ \gamma$, and $\psi \circ \gamma$ form the boundary of a smooth 2–chain $\spadesuit(\gamma, \psi_i, \psi) \subset M$. Using the equation $d\eta = 0$, we derive from Stockes' theorem that $\int_{\spadesuit(\gamma,\psi_i,\psi)} d\eta = 0$, i.e., $\int_{\psi_i\circ\gamma} \eta - \int_{\psi\circ\gamma} \eta = \int_{\mathfrak{X}_i^1} \eta - \int_{\mathfrak{X}_i^0} \eta$, for i sufficiently large. That is, $\left| \int_{\psi_i\circ\gamma} \eta - \int_{\psi\circ\gamma} \eta \right| \leqslant 2|\eta|_0 d_{C^0}(\psi_i, \psi)$, for i sufficiently large because the length of any minimizing geodesic is bounded from above by the distance between its endpoints, i.e., $\lim_{i\longrightarrow\infty} \left(\int_{\psi_i\circ\gamma} \eta \right) = \int_{\psi\circ\gamma} \eta$. On the other hand, let $\{\phi_t\}$ be the cosymplectic flow generated by the Reeb vector field ξ. For each fixed $t \in]0, 1[$, and each $x \in M$, consider the smooth curve $\bar{\gamma}_{x,t} : s \longmapsto \phi_{st}(x)$, and derive from the previous limit that $\lim_{i\longrightarrow\infty} \int_{\psi_i(\bar{\gamma}_{x,t})} \eta = \int_{\psi(\bar{\gamma}_{x,t})} \eta$, for each fixed $t \in]0, 1[$, i.e., $t = \lim_{i\longrightarrow\infty} \int_0^t (du) = \int_0^t (\psi^\star(\eta)(\xi))(\phi_u(x))du$, for each fixed $t \in]0, 1[$ because by Lemma 4.1 we have $(\psi_i)_\star(\xi) = \xi$, for all i. Therefore, taking the derivative of the previous equality with respect to t gives: $1 = (\psi^\star(\eta)(\xi))(\phi_t(x))$, for each fixed $t \in]0, 1[$, for all $x \in M$, which implies $\eta(\psi_\star(\xi)) = 1$. For (1), since $\psi_i \xrightarrow{C^0} \psi$, and H is continuous, we derive that $\lim_{i\longrightarrow\infty} \left(H(\psi_i(\bar{\gamma}_{x,t}(1))) - H(\psi_i(x))\right) = \left(H(\psi(\bar{\gamma}_{x,t}(1))) - H(\psi(x))\right)$, for each fixed $t \in]0, 1[$, for all $x \in M$. We also have

$$\lim_{i\longrightarrow\infty} \left(H(\psi_i(\bar{\gamma}_{x,t}(1))) - H(\psi_i(x))\right) = \lim_{i\longrightarrow\infty} \left(\int_{\psi_i\circ\bar{\gamma}_{x,t}} dH \right) = \int_0^t dH((\psi_i)_\star(\xi)) \circ \bar{\gamma}_{x,t}(s)ds,$$

i,e.,

$$\lim_{i\longrightarrow\infty} \left(H(\psi_i(\bar{\gamma}_{x,t}(1))) - H(\psi_i(x))\right) = \int_0^t dH((\psi_i)_\star(\xi))(\psi_i(\phi_s(x))ds = \int_0^t dH(\xi)((\psi_i(\phi_s(x)))ds,$$

which implies that $\lim_{i\longrightarrow\infty} \left(H(\psi_i(\bar{\gamma}_{x,t}(1))) - H(\psi_i(x))\right) = t\xi(H)$, for each fixed $t \in]0, 1[$, for all $x \in M$ because the function $y \longmapsto \xi(H)(y)$ is constant. Thus, we have just proved that

$$t\xi(H) = \lim_{i\longrightarrow\infty} \left(H(\psi_i(\bar{\gamma}_{x,t}(1))) - H(\psi_i(x))\right) = \left(H(\psi(\bar{\gamma}_{x,t}(1))) - H(\psi(x))\right) = \int_{\psi\circ\bar{\gamma}_{x,t}} dH,$$

for each fixed $t \in]0, 1[$, for all $x \in M$, i.e., $t\xi(H) = \int_0^t \left(\psi_\star(\xi)(H)\right)(\psi(\phi_s(x))ds$, for each fixed $t \in]0, 1[$, for all $x \in M$. Thus, taking the derivative in the latter equality with respect to t, gives $\xi(H) = \psi_\star(\xi)(H)(\psi(\phi_t(x))$, for each fixed $t \in]0, 1[$, for all $x \in M$. □

Remark 5.6. Let (M, ω, η) be a compact connected cosymplectic manifold, and let X_H be a co-Hamiltonian vector field such that $\widetilde{I}_{\eta,\omega}(X_H) = dH$. For each smooth diffeomorphism ψ of M, as in Remark 3.1, define a vector field $X_{H\circ\psi} := Y_{H\circ\psi} - \xi(H \circ \psi)\xi$, which satisfies $\widetilde{I}_{\eta,\omega}(X_{H\circ\psi}) = d(H \circ \psi)$, $\mathcal{L}_{X_{H\circ\psi}}\eta = 0$, and $\mathcal{L}_{X_{H\circ\psi}}\omega = d\left(\xi(H \circ \psi)\right) \wedge \eta$, where $Y_{H\circ\psi}$ is the vector field constructed for the closed 1–form $d(H \circ \psi)$ as in Lemma 3.1.

If in addition, we have the information that there exists a sequence $\{\psi_i\}_i \subset G_{\eta,\omega}(M)$ such that $\psi_i \xrightarrow{C^0} \psi$, then with the help of Lemma 5.1, we can derive that $\psi_\star(\xi)(H) = \xi(H)$. This combined together with $\psi^\star(dH)(\xi) = \left(dH(\psi_\star(\xi))\right) \circ \psi$, implies that $\psi^\star(dH)(\xi) = \xi(H) \circ \psi$. Since the smooth function $\xi(H)$ is constant by assumption, then we have $\xi(H) = d(H \circ \psi)(\xi) = \xi(H \circ \psi)$. Thus, $\mathcal{L}_{X_{H\circ\psi}}\omega = d\left(\xi(H \circ \psi)\right) \wedge \eta = d(\xi(H)) \wedge \eta = 0$.

Therefore, $X_{H\circ\psi}$ is a weak Hamiltonian vector field such that $\widetilde{I}_{\eta,\omega}(X_{H\circ\psi}) = d(H\circ\psi)$, whenever ψ is the C^0–limit of a sequence of cosymplectic diffeomorphisms.

Poof of Theorem 5.5. We shall adapt the proof given by Buhosky [4] for similar result in symplectic geometry. Assume that M is equipped with a Riemannian metric g with injectivity radius $r(g)$. Let $\{\varphi_i\} \subseteq G_{\eta,\omega}(M)$ be a sequence of cosymplectic diffeomorphisms such that $\varphi_i \xrightarrow{C^0} \psi \in \mathrm{Diff}^\infty(M)$. Assume that ψ is not a cosymplectic diffeomorphism. Then, for any weak Hamiltonian vector field X_H such that $\widetilde{I}_{\eta,\omega}(X_H) = dH$, we have $\varphi_\star(X_H) \neq X_{H\circ\varphi^{-1}}$ (this is supported by Remark 5.6). This implies that, if Ψ_H is the cosymplectic flow generated by X_H, then we must have

$$\varphi\circ\Psi_H\circ\varphi^{-1} \neq \Psi_{H\circ\varphi^{-1}}, \tag{5.32}$$

where $\Psi_{H\circ\varphi^{-1}}$ is the cosymplectic flow generated by

$$Y_{H\circ\varphi^{-1}} := \widetilde{I}^{-1}_{\eta,\omega}\left(d\left(H\circ\varphi^{-1}\right)\right).$$

The sequence of weakly Hamiltonian isotopies $\varphi_i\circ\Psi_H\circ\varphi_i^{-1}$ converges uniformly to $\varphi\circ\Psi_H\circ\varphi^{-1}$, and we have

$$l^\infty_C(\Psi^{-1}_{H\circ\varphi^{-1}}\circ\{\varphi_i\circ\Psi_H\circ\varphi_i^{-1}\}) = osc(H\circ\varphi_i^{-1} - H\circ\varphi^{-1}) + \left|\eta(X_{H\circ\varphi^{-1}}) - \eta(X_{H\circ\varphi_i^{-1}})\right|,$$

for each i. On the other hand, for each fixed $x\in M$, consider the orbits $\mathcal{C}_{x,i} := \left(\varphi_i\circ\Psi_H\circ\varphi_i^{-1}\right)(x)$, and $\mathcal{C}_x := \left(\varphi\circ\Psi_H\circ\varphi^{-1}\right)(x)$. From the convergence $\varphi_i\circ\Psi_H\circ\varphi_i^{-1} \xrightarrow{C^0} \varphi\circ\Psi_H\circ\varphi^{-1}$, then for i sufficiently large we may assume that $\bar{d}(\varphi_i\circ\Psi_H\circ\varphi_i^{-1}, \varphi\circ\Psi_H\circ\varphi^{-1}) \leqslant \frac{r(g)}{2}$, and derive as in the proof of Lemma 5.1, that there exist two minimal geodesics γ_i (with endpoints x and $\left(\varphi_i\circ\Psi^1_H\circ\varphi_i^{-1}\right)(x)$) and γ (with endpoints x and $\left(\varphi\circ\Psi^1_H\circ\varphi^{-1}\right)(x)$) such that $\mathcal{C}_{x,i}$, $\mathcal{C}_x$, γ_i, and γ delimit a 2–chain in $\boxplus_{\mathcal{C}_{x,i},\mathcal{C}_x,\gamma_i,\gamma} \subset M$. Since $d\eta = 0$, it follows from Stoke's theorem that $\int_{\boxplus_{\mathcal{C}_{x,i},\mathcal{C}_x,\gamma_i,\gamma}} d\eta = 0$, i.e., $\int_{\mathcal{C}_{x,i}}\eta - \int_{\mathcal{C}_x}\eta = \int_{\gamma_i}\eta - \int_\gamma\eta$, for all i sufficiently large. That is, for each $x\in M$, we have

$$\begin{aligned}\left|\eta(X_{H\circ\varphi^{-1}})(x) - \eta(X_{H\circ\varphi_i^{-1}})(x)\right| &= \left|\int_{\mathcal{C}_{x,i}}\eta - \int_{\mathcal{C}_x}\eta\right| = \left|\int_{\gamma_i}\eta - \int_\gamma\eta\right| \\ &\leqslant 2|\eta|_0\bar{d}(\varphi_i\circ\Psi_H\circ\varphi_i^{-1}, \varphi\circ\Psi_H\circ\varphi^{-1}),\end{aligned} \tag{5.33}$$

for all i sufficiently large. Hence

$$l^\infty_C\left(\Psi^{-1}_{H\circ\varphi^{-1}}\circ\{\varphi_i\circ\Psi_H\circ\varphi_i^{-1}\}\right) = osc(H\circ\varphi_i^{-1} - H\circ\varphi^{-1}) + \left|\eta\left(X_{H\circ\varphi^{-1}}\right) - \eta\left(X_{H\circ\varphi_i^{-1}}\right)\right| \longrightarrow 0,\ i\longrightarrow\infty.$$

Similarly, one proves that: if we consider the reparametrized isotopies $\Psi_{t,H} : s \longmapsto \Psi^{ts}_H$, and $\Psi_{t,H\circ\varphi^{-1}} : s \longmapsto \Psi^{st}_{H\circ\varphi^{-1}}$, for each fixed t, then $l^\infty_C\left(\Psi^{-1}_{t,H\circ\varphi^{-1}}\circ\{\varphi_i\circ\Psi_{t,H}\circ\varphi_i^{-1}\}\right) \longrightarrow 0$, $i\longrightarrow\infty$, for each fixed t. Finally, we have proved that for each fixed t, we have

- $\varphi_i\circ\Psi_{t,H}\circ\varphi_i^{-1} \xrightarrow{C^0} \varphi\circ\Psi_{t,H}\circ\varphi^{-1}$, and
- $l^\infty_C(\Psi^{-1}_{t,H\circ\varphi^{-1}}\circ\{\varphi_i\circ\Psi^t_H\circ\varphi_i^{-1}\}) \longrightarrow 0,\ i\longrightarrow\infty$.

Thus, by Theorem 5.2 we must have, $\varphi\circ\Psi^t_H\circ\varphi^{-1} = \Psi^t_{H\circ\varphi^{-1}}$, for all t: This contradicts (5.32). □

Acknowledgements. The authors would like to thank Professor Banyaga for motivating us to investigate cosymplectic geometry and the anonymous referees for their contributive comments. Their gratitude also go to thank CEA-SMIA for their financial support.

Conflict of interest: Authors state no conflict of interest.

References

[1] A. Banyaga, *A Hofer-like metric on the group of symplectic diffeomorphisms,* Symplectic Topology and measure preserving dynamical systems, Contemp. Math **512** (2010), 1 – 23, Amer. Math.Soc., Providence, R.I.

[2] A. Banyaga and S. Tchuiaga, *The group of strong symplectic homeomorphisms in L^∞-metric*, Adv. Geom. **14** (2014), no. 3, 523 – 539.

[3] G. Bazzoni and O. Goertsches *k-Cosymplectic manifolds,* Ann. Glob. Anal. Geom, **47**(2015), no. 3, 239 – 270, doi:10.1007/s10455 – 014 – 9444 – y

[4] L. Buhovsky, *Variation on Eliashberg-Gromov theorem,* C^0–Symplectic Topology and Dynamical Systems (2014), IBS Center for Geometry and Physics, Korea, Lectures.

[5] Y. Eliashberg, *A theorem on the structure of wave fronts and its application in symplectic topology,* Funct. Anal. and Its Applications **21** (1987) 227 – 232.

[6] E. Guzmán and J. C. Marrero, *Time-dependent mechanics and Lagrangian submanifolds of presymplectic and Poisson manifolds,* J. Phys. A **43** (2010), no. 50, 505201.

[7] M. Hirsch, *Differential Topology,* Graduate Texts in Mathematics, no. 33, Springer Verlag, New York-Heidelberg. **3** (1976), corrected reprint (1994).

[8] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhauser Advanced Texts, Birkhauser Verlag (1994).

[9] I. Lacirasella, J. C. Marrero, and E. Padrón, *Reduction of symplectic principal R-bundles,* J. Phys. A **45** (2012), no. 32, 325202.

[10] F. Lalonde and D. Mc. Duff, *The geometry of symplectic energy*, Ann. of Math. **141** (1995), 711 – 727.

[11] M. de Léon and J. C. Marrero, *Constrained time-dependent Lagrangian systems and Lagrangian submanifolds,* J.Math. Phys. **34** (1993), no. 2, 622 – 644.

[12] M. de Léon, J. Marin-Solano and J. C. Marrero, *The constraint algorithm in the jet formalism,* Differential Geom. Appl. **6** (1996), no. 3, 275 – 300.

[13] H. Li, *Topology of cosymplectic/cokaehler manifolds,* Asian J. Math. **12** (2008) no 4, 527 – 544

[14] P. Libermann, *Les automorphismes infinitesimaux symplectiques et des structures de contact*, Centre Belge Rech. Math., Colloque Géom. Différ. Globale, Bruxelles du 19 au 22 Déc. 1958, 37-59 (1959).

[15] G. de Rham, *Form differentiali et loro integrali*, C.I.M.E 2 ciclo, Saltino, Vallombrossa, Agosto (1960)

[16] S. Tchuiaga, *On symplectic dynamics,* Differ. Geom. Appl. **61** (2018), 170 – 196.