Research Article

Open Access

Nicholas Buchdahl* and Georg Schumacher

Polystable bundles and representations of their automorphisms

Abstract: Using a quasi-linear version of Hodge theory, holomorphic vector bundles in a neighbourhood of a given polystable bundle on a compact Kähler manifold are shown to be (poly)stable if and only if their corresponding classes are (poly)stable in the sense of geometric invariant theory with respect to the linear action of the automorphism group of the bundle on its space of infinitesimal deformations.

Introduction

The aim of this paper is to shed light on the relationship between the notion of polystability for holomorphic vector bundles on a compact Kähler manifold and the classical GIT notion of polystability for points in a representation space of a reductive Lie group. More specifically, we study the action of the automorphism group of a polystable bundle on its space of infinitesimal deformations and relate this to an action on the parameter space of a semi-universal deformation. In so doing, it is shown that the relationship between the infinite-dimensional bundle-theoretic notion and the finite-dimensional representation-theoretic notion is more than just analogy.

In the sense of Kuranishi's approach to deformation theory, Miyajima [28] constructed a semi-universal deformation of a holomorphic vector bundle E_0 on a compact (Kähler) manifold from infinitesimal deformations, i.e., from elements of $H^1(X, End\, E_0)$. A parameter space S is given as the zero-set of a holomorphic map $\Psi : \mathcal{N} \to H^2(X, End\, E_0)$, where $\mathcal{N} \subseteq H^1(X, End\, E_0)$ is a certain neighbourhood of the origin.

Deformations of non-stable holomorphic vector bundles can be realised as pull-backs of semi-universal deformations by means of base change maps, but these need not be not uniquely determined and therefore an action of the automorphism group on a semi-universal deformation cannot be expected in general. Nevertheless, a holomorphic map Ψ with the following property exists:

Theorem 1. *Let E_0 be a polystable holomorphic vector bundle on a compact Kähler manifold (X, ω). Then there is a holomorphic map Ψ defined on an open neighbourhood $\mathcal{N}$ of zero in $H^1(X, End\, E_0)$ with values in $H^2(X, End\, E_0)$ that is equivariant with respect to the action of $Aut\, E_0$. The space $S = \Psi^{-1}(0)$ is the parameter space of a semi-universal deformation of E_0, and the action of $Aut(E_0)$ on $H^1(X, End\, E_0)$ induces a holomorphic action on the germ of $(S, 0)$. Points of S correspond to isomorphic bundles if and only if they lie in the same orbit of $Aut\, E_0$ acting on the ambient space $H^1(X, End\, E_0)$.*

*Corresponding Author: Nicholas Buchdahl: School of Mathematical Sciences, University of Adelaide, Adelaide, Australia 5005. E-mail: nicholas.buchdahl@adelaide.edu.au
Georg Schumacher: Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Lahnberge, Hans-Meerwein-Strasse, D-35032 Marburg, Germany. E-mail: schumac@mathematik.uni-marburg.de

The action of $Aut\,E_0$ on $H^q(X, End\,E_0)$ is by conjugation, and Ψ is equivariant with respect to this action in the sense that $\Psi(\gamma\alpha\gamma^{-1}) = \gamma\Psi(\alpha)\gamma^{-1}$ if both α and $\gamma\alpha\gamma^{-1}$ lie in $\mathcal{N}$.

Theorem 2. *With the same objects as in Theorem 1 and restricting $\mathcal{N}$ if necessary, every holomorphic bundle E corresponding to a point of $\Psi^{-1}(0)$ is semistable, and any destabilising subsheaf of E is a subbundle.*

Polystable bundles are direct sums of stable bundles, and the existence of an irreducible Hermite-Einstein connection on a stable holomorphic bundle was established by Uhlenbeck and Yau [35], generalising the result of Donaldson [11] when (X, ω) is algebraic of dimension two. The Hitchin-Kobayashi correspondence is the statement that a holomorphic bundle admits an irreducible Hermite-Einstein connection if and only if it is stable, for which the "only if" part was established earlier by Kobayashi [22] and Lübke [26].

As explained in the last section of [11], the Hitchin-Kobayashi correspondence can be viewed as an infinite-dimensional analogue of a theorem of Kempf and Ness [19] from classical geometric invariant theory, in which the central component of the curvature corresponds to the moment map and the Mumford-Takemoto definition of stability [33] corresponds to a numerical condition for stability derived from the Hilbert-Mumford criterion. In this context, the following result embodies the central idea of this paper, showing that this analogue goes further:

Theorem 3. *A class $\alpha \in \Psi^{-1}(0) \subseteq H^1(X, End\,E_0)$ is (poly)stable with respect to the action of $Aut\,E_0$ on $H^1(X, End\,E_0)$ if and only if the corresponding bundle E_α is (poly)stable with respect to ω.*

In [7], Theorem 3 is used to construct a classifying space for polystable bundles on X, a reduced analytic space for which the weak normalisation is shown in [8] to satisfy the axioms for a coarse moduli space taken in the category of weakly normal analytic spaces. The local models for the space are local analytic GIT quotients. The construction involves methods of local analytic geometry, as well as an analytic version of Luna's slice theorem [27].

In the context of moduli spaces of Higgs bundles on compact Riemann surfaces, analogous results have also been obtained recently by Fan [13], reducing infinite-dimensional quotients to finite-dimensional ones to which classical GIT applies. The analysis in that reference makes use of the Yang-Mills(-Higgs) flow, unlike the analysis used here.

The application of classical GIT to problems in infinite-dimensional analysis has been developed by Donaldson and his students, amongst others. In particular, in the context of constant scalar curvature Kähler metrics, Székelyhidi [32] shows that under the assumption of K-polystability, small deformations of cscK metrics are again of this type; see also [3]. Another example of such methods appears in [16], studying the coupled Kähler-Yang-Mills equations. Székelyhidi's proof relates K-polystability to polystability of points in a finite-dimensional representation space for a reductive group, for which the statement and proof of Theorem 3 given here are in many respects parallel. A notable difference in approaches results from the need to have uniform estimates for moving reference points, which occupies all of the eighth section here.

Strictly speaking, the (reductive) group with respect to which stability and its concomitant notions are applied (to elements of $H^1(X, End\,E_0)$) is the quotient of $Aut\,E_0$ by the subgroup $\mathbb{C}^\star \cdot 1$ of non-zero scalar multiples of the identity since that subgroup acts trivially. In general, the ineffectivity kernel of the representation of $Aut\,E_0$ may be larger, although in this case no small deformation of E_0 is stable; see the first of the Remarks at the end of §3.

For bundles near E_0 that are semistable but not polystable, the following analogue of the standard GIT result holds:

Theorem 4. *If $\bar{\alpha}$ is a polystable point in the closure of the orbit of $\alpha \in \Psi^{-1}(0)$ under the action of $Aut\,E_0$, the polystable bundle $E_{\bar{\alpha}}$ is isomorphic to the graded object $Gr(E_\alpha)$ associated to a Seshadri filtration of E_α.*

The manuscript is organised as follows. In the next section notation is fixed, based largely on [23]. The second section contains a construction for a slice of the complexified gauge group $\mathcal{G}$ acting on the space of hermitian connections on a given bundle, leading to the construction of the function Ψ of Theorem 1.

In the third and fourth sections semistability and polystability are taken into consideration respectively, including the estimates required for the subsequent solving of the Hermite-Einstein equations. The fifth section includes proofs of Theorem 2 and of the "if" part of Theorem 3, as well as proving the equivariance of Ψ and the last statement of Theorem 1. In the sixth section, using ellipticity of the equations and by restricting to slices, the action of $\mathcal{G}$ on the space of integrable semi-connections is reduced to an action of $Aut\,E_0$ on $H^1(X, End\,E_0)$, this making critical use of the existence of a Hermite-Einstein connection on E_0. The process of comparing the two notions of polystability is commenced in the seventh section, and this is continued over the course of the eighth section, concluding with the completion of the proof of Theorem 3 in §8.

In §9, consideration is given to those bundles that are semistable but not polystable, particularly those for which the closure of their orbit under $\mathcal{G}$ is the polystable bundle E_0, with the conclusions summarised by Theorem 4. The eleventh and final section presents several general remarks and observations concerning the methods and results in the paper.

1 Preliminaries

Following Kobayashi [23], a *semi-connection* on a complex vector bundle E is a $\mathbb{C}$-linear map $\bar{\partial}$ on differentiable local sections of E taking values in $\Lambda^{0,1} \otimes E$ and satisfying the $\bar{\partial}$-Leibniz rule; here $\Lambda^{p,q}$ is the space of (p,q)-forms on X. A semi-connection has natural prolongations $\Lambda^{0,q} \otimes E \to \Lambda^{0,q+1} \otimes E$, and it is by definition integrable if $\bar{\partial} \circ \bar{\partial} = 0$. A semi-connection on a bundle induces semi-connections on associated bundles in the usual ways, these being integrable if the original semi-connection is integrable. The set of semi-connections on E is an affine space.

Let $\bar{\partial}_0$ be an integrable semi-connection on E. Then every semi-connection $\bar{\partial}$ on the bundle can be written $\bar{\partial} = \bar{\partial}_0 + a''$ for some unique $(0,1)$-form a'' with coefficients in End E, and the integrability condition for $\bar{\partial}$ is $\bar{\partial}_0 a'' + a'' \wedge a'' = 0$. The notation, which will be used throughout, is derived from the often-used convention to denote by a' and a'' respectively the $(1,0)$- and $(0,1)$-components of a 1-form a, where that 1-form may take values in some vector bundle.

The group $\mathcal{G}$ of (differentiable) complex automorphisms of E acts on the affine space of semi-connections as a "complex gauge group". This action, which preserves the integrability condition, is denoted by $g \cdot \bar{\partial} := g \circ \bar{\partial} \circ g^{-1}$. A holomorphic structure is defined by an integrable semi-connection, and two such structures are isomorphic if and only if they lie in the same orbit of $\mathcal{G}$. By virtue of the Newlander-Nirenberg theorem, this view of holomorphic structures is equivalent to the more usual one of holomorphic vector bundles being described by systems of holomorphic transition functions.

Denote by $A^{p,q}(E)$ the global smooth (p,q)-forms with coefficients in E. For an integrable semi-connection $\bar{\partial}_0$ defining a holomorphic structure E_0, the Dolbeault cohomology groups

$$H^q_{\bar{\partial}_0}(X, E) = \left(\ker \bar{\partial}_0 :\ A^{0,q}(E) \to A^{0,q+1}(E)\right) / \left(\operatorname{im} \bar{\partial}_0 :\ A^{0,q-1}(E) \to A^{0,q}(E)\right)$$

are denoted by $H^q(X, E_0)$, these being finite-dimensional spaces with $H^1(X, End\,E_0)$ being by definition the space of infinitesimal deformations of E_0.

Analysis of the small deformations of E_0 is achieved with the introduction of metrics on both the bundle E and the manifold X. A hermitian metric h is fixed once and for all on the bundle E, which is henceforth denoted E_h. The group $\mathcal{G}$ is the complexification of the group $\mathcal{U}$ of unitary gauge transformations. The hermitian structure on E_h gives a one-to-one correspondence between semi-connections $\bar{\partial}$ and hermitian connections $d = \partial + \bar{\partial}$ on E_h, and if $d = d_0 + a$ for some skew-adjoint $a \in A^1(End\,E_h)$, then $\partial = \partial_0 + a'$ and $\bar{\partial} = \bar{\partial}_0 + a''$ where $a = a' + a''$ and $a' = -(a'')^\star$. Henceforth, all connections are taken to be hermitian.

The action of $\mathcal{G}$ on the space of semi-connections extends to an action on the space of connections via $g \cdot d := g^{\star-1} \circ \partial \circ g^\star + g \circ \bar{\partial} \circ g^{-1} = d + g^{\star-1} \partial g^\star - \bar{\partial} g\, g^{-1}$. The curvature $F(d) = d \circ d \in A^2(End\,E_h)$ of a connection

is a skew-adjoint 2-form with coefficients in End E_h, and the connection (i.e., the associated semi-connection) is integrable if and only if $F(d)$ is of type $(1, 1)$. Since $F(d_0 + a) = F(d_0) + d_0 a + a \wedge a$, for $g \in \mathcal{G}$ it follows that

$$F(g \cdot d) = g\, F^{0,2}(d)\, g^{-1} + g^{\star -1}\, F^{2,0}(d)\, g^{\star} \tag{1.1}$$
$$+ \left(F^{1,1}(d) + \bar{\partial}(g^{\star -1}\partial g^{\star}) - \partial(\bar{\partial}g\, g^{-1}) + (g^{\star -1}\partial g^{\star}) \wedge (\bar{\partial}g\, g^{-1}) + (\bar{\partial}g\, g^{-1}) \wedge (g^{\star -1}\partial g^{\star})\right) .$$

Now fix a positive $(1, 1)$-form ω on X. If $\dim X = n$, the associated volume form is $dV = \omega^n$, where the convention is adopted throughout that $\omega^q := (1/q!)\, \omega \wedge \cdots \wedge \omega$ (q times). If d is an integrable connection on E_h, standard Hodge theory on compact manifolds gives a unique $\bar{\partial}$-harmonic representative in each Dolbeault cohomology class, where $\tau \in A^{0,q}(E_h)$ is $\bar{\partial}$-harmonic if $\bar{\partial}\tau = 0 = \bar{\partial}^{\star}\tau$ for $\bar{\partial}^{\star} = -\star \partial \star$, the formal adjoint of $\bar{\partial}$. So τ is $\bar{\partial}$-harmonic if and only if it lies in the kernel of the $\bar{\partial}$-Laplacian $\triangle'' = \bar{\partial}\bar{\partial}^{\star} + \bar{\partial}^{\star}\bar{\partial}$. In general, there is an L^2-orthogonal decomposition

$$A^{0,q}(E_h) = (\ker \bar{\partial})^{\perp} \oplus (\ker \bar{\partial}^{\star})^{\perp} \oplus H^{0,q} = \overline{\operatorname{im} \bar{\partial}^{\star}} \oplus \overline{\operatorname{im} \bar{\partial}} \oplus H^{0,q}$$

where $H^{0,q} = H^{0,q}(\bar{\partial})$ is the space of $\bar{\partial}$-harmonic $(0, q)$-forms. Here, notation has been abused in that $A^{0,q}(E_h)$ is no longer denoting the space of smooth sections of E_h, but rather the space of global sections of $\Lambda^{0,q} \otimes E_h$ that are square integrable, and the closures on the right are the closures in L^2 of the images under $\bar{\partial}$ and $\bar{\partial}^{\star}$ of the spaces of smooth global sections. Standard elliptic regularity implies that the $\bar{\partial}$-harmonic sections are smooth, at least if the connections are.

This abuse of notation will be employed throughout, so that $A^{0,q}(E_h)$ will always denote a space of global sections of $\Lambda^{0,q} \otimes E_h$ but with the degree of differentiability and/or integrability to be specified in the respective context. Sobolev spaces of functions are denoted by L^p_k, meaning all weak derivatives up to and including those of order k lie in L^p. Having fixed a base connection on E_h once and for all, the spaces of L^p_k elements of $A^{0,q}(E_h)$ acquire norms that make them Banach spaces.

Henceforth, a number $p > 2n$ (for $n = \dim X$) will be fixed, so by the Sobolev embedding theorem there are compact embeddings $L^p_1 \subset C^0$ and $L^p_2 \subset C^1$. By standard elliptic theory on compact manifolds, for an integrable connection $d = \partial + \bar{\partial}$ there is a constant $C > 0$ (depending upon d) such that

$$\|\tau\|_{L^p_1} \leq C\left(\|\bar{\partial}\tau\|_{L^p} + \|\bar{\partial}^{\star}\tau\|_{L^p} + \|\Pi^{0,q}\tau\|_{L^2}\right) \quad \text{for } \tau \in A^{0,q}(E_h) , \tag{1.2}$$

where $\Pi^{0,q}\tau$ is the L^2-orthogonal projection of $\tau \in A^{0,q}(E_h)$ in $H^{0,q}(\bar{\partial})$. Connections on E_h will be permitted to have coefficients in L^p_1, and the fact that an integrable L^p_1 connection defines a holomorphic structure in the usual way follows from Lemma 8 of [5]. When not indicated by a subscript on the norm symbol, $\|\tau\|$ will always mean $\|\tau\|_{L^2}$.

If d is an integrable semi-connection and $g \in \mathcal{G}$, the Dolbeault cohomology groups defined by $\bar{\partial}$ and by $g \cdot \bar{\partial}$ are isomorphic, the isomorphism induced by mapping a $\bar{\partial}$-closed $(0, q)$-form $\tau \in A^{0,q}(E_h)$ to the $(g \cdot \bar{\partial})$-closed $(0, q)$-form $g\,\tau$. This isomorphism does not preserve harmonic representatives in general, unless $g \in \mathcal{U}$ in which case it also preserves L^2 norms.

Subsequently in this paper it will be useful to consider connections that are not integrable, in which case the Dolbeault cohomology groups are not defined. However, one can still define the spaces $H^{0,q}(\bar{\partial})$ of $\bar{\partial}$-harmonic $(0, q)$-forms as null spaces of the appropriate Laplacians, these still being finite-dimensional spaces consisting only of smooth forms (if $\bar{\partial}$ is itself smooth).

If $d\omega = 0$, the formal adjoints $\partial^{\star}$ and $\bar{\partial}^{\star}$ have alternative expressions in terms of the Kähler identities:

$$\partial^{\star} = i(\Lambda\bar{\partial} - \bar{\partial}\Lambda) , \quad \bar{\partial}^{\star} = -i(\Lambda\partial - \partial\Lambda) ,$$

where $\Lambda : \Lambda^{p+1,q+1} \to \Lambda^{p,q}$ is the adjoint of $\omega\wedge$, so $\Lambda\omega = n$. For an integrable connection d on E_h with curvature $F = F(d)$, the Bianchi and Kähler identities imply that the Yang-Mills equations $d^{\star}F = 0$ are equivalent to the equation $d\widehat{F} = 0$, where $\widehat{F} := \Lambda F$, the central component of the curvature. Under these circumstances, the bundle and connection split into eigenspaces of the covariantly constant self-adjoint endomorphism $i\widehat{F}$, and when restricted to any such eigenspace, the curvature of the restricted connection has central component that is a constant multiple of the identity. That is, it is a *Hermite-Einstein* connection [21].

2 A neighbourhood of a holomorphic bundle

As in the previous section, let X be a compact complex manifold and let E_0 be a holomorphic vector bundle on X. A construction of a semi-universal deformation has been given by Forster and Knorr [14] using power series methods. More in the spirit of Kuranishi's construction [24] is Miyajima's construction [28] (cf. also [15]). Either way, there is a holomorphic function Ψ defined in a neighbourhood of $0 \in H^1(X, End\, E_0)$ such that $\Psi^{-1}(0)$ is a complete family of small deformations of E_0. In this section, the construction of a particular such function Ψ will be presented in a manner to suit the purposes of the remainder of the paper. The entire discussion is essentially an n-dimensional version of the 2-dimensional case presented in §6.4 of [12].

Fix a positive $(1, 1)$-form ω on X, which at this stage is not assumed to be d-closed. Let E_h be the complex bundle underlying E_0 equipped with a fixed hermitian structure, and let d_0 be a connection on E_h inducing the holomorphic structure E_0.

Proposition 2.1. *There is a number $\epsilon > 0$ depending on d_0 with the property that for any integrable hermitian connection $d_a = d_0 + a$ with $\|a\|_{L_1^p} < \epsilon$ the L^2 orthogonal projection $H^{0,q}(\bar\partial_a) \ni \tau \mapsto \Pi^{0,q}\tau \in H^{0,q}(\bar\partial_0)$ is injective.*

Proof. Write $a = a' + a''$ for $a'' \in A^{0,1}(End\, E_h)$. Suppose $\tau \in A^{0,q}(E_h)$ satisfies

$$\bar\partial_0 \tau + a'' \wedge \tau = 0 = \bar\partial_0^{\star}\tau - i\Lambda(a' \wedge \tau),$$

so $\tau \in H^{0,q}(\bar\partial_a)$. If $\Pi^{0,q}\tau = 0 \in H^{0,q}(d_0)$, then $\tau = \bar\partial_0\mu + \bar\partial_0^{\star}\nu$ for some $\mu \in A^{0,q-1}(E_h)$ and some $\nu \in A^{0,q+1}(E_h)$, with μ and ν respectively orthogonal to the kernels of $\bar\partial_0$ and $\bar\partial_0^{\star}$. Then

$$-a'' \wedge \tau = \bar\partial_0\tau = \bar\partial_0\bar\partial_0^{\star}\nu \quad \text{and} \quad i\Lambda(a' \wedge \tau) = \bar\partial_0^{\star}\tau = \bar\partial_0^{\star}\bar\partial_0\mu,$$

so on taking inner products with ν and μ respectively it follows that

$$\|\bar\partial_0^{\star}\nu\|^2 \le \|\nu\|\,\|a'' \wedge \tau\| \quad \text{and} \quad \|\bar\partial_0\mu\|^2 \le \|\mu\|\,\|a' \wedge \tau\|,$$

where $\|\cdot\|$ is the L^2 norm. Since $p > 2n$, the Sobolev embedding theorem gives $\sup |a| \le C\,\|a\|_{L_1^p}$ for some constant C independent of a and d_0, so after adding the last two inequalities it follows that

$$\|\tau\|^2 = \|\bar\partial_0\mu\|^2 + \|\bar\partial_0^{\star}\nu\|^2 \le C\,\|a\|_{L_1^p}\,\|\tau\|\big(\|\mu\| + \|\nu\|\big).$$

Since μ and ν are orthogonal to $\ker\bar\partial_0$ and $\bar\partial_0^{\star}$ respectively, ellipticity of $\triangle_0''$ implies that $\|\mu\| \le C_1\|\bar\partial_0\mu\|$ and $\|\nu\| \le C_2\|\bar\partial_0^{\star}\nu\|$ for some constants C_1, C_2, so $\|\tau\|^2 \le C_3\|a\|_{L_1^p}\|\tau\|^2$ for some new constant $C_3 = C_3(d_0)$, giving the stated result. □

This proposition clearly implies the well-known and standard semi-continuity of cohomology. It also has the following useful consequence, resulting from equality of dimensions of cohomology groups:

Corollary 2.2. *Under the hypotheses of Proposition 2.1, suppose in addition that d_0 and d_a lie in the same $\mathcal{G}$-orbit. Then the map $H^{0,q}(\bar\partial_a) \to H^{0,q}(\bar\partial_0)$ induced by orthogonal projection is an isomorphism.* □

Regardless of the integrability or otherwise of $\bar\partial_a = \bar\partial_0 + a''$, if $\|a''\|_{L_1^p}$ is sufficiently small, the Sobolev embedding theorem combined with (1.2) gives the following perturbed version of that estimate:

Lemma 2.3. *There exists $\epsilon > 0$ and $C > 0$ depending on d_0 with the properties that if $a'' \in A^{0,1}(End\, E_h)$ satisfies $\|a''\|_{L_1^p} < \epsilon$, then*

$$\|\tau\|_{L_1^p} \le C\big(\|\bar\partial_0^{\star}\tau\|_{L^p} + \|\bar\partial_0\tau + a'' \wedge \tau\|_{L^p} + \|\Pi^{0,q}\tau\|_{L^2}\big), \qquad \tau \in A^{0,q}(E_h), \tag{2.1}$$

where $\Pi^{0,q} : A^{0,q}(End\, E_h) \to H^{0,q}(\bar\partial_0)$ is L^2 orthogonal projection. □

Replacing E_h by $\operatorname{End} E_h$, d_0 by the induced connection d_0 on $\operatorname{End} E_h$, taking $q = 1$ and $\tau = a''$ gives:

Corollary 2.4. *There exists $\epsilon > 0$ and $C > 0$ with the properties that each $a'' \in A^{0,1}(\operatorname{End} E_h)$ with $\|a''\|_{L_1^p} < \epsilon$ satisfies*

$$\|a''\|_{L_1^p} \leq C \left(\|\vec{\bar{\partial}}_0^* a''\|_{L^p} + \|\bar{\partial}_0 a'' + a'' \wedge a''\|_{L^p} + \|\Pi^{0,1} a''\|_{L^2} \right), \tag{2.2}$$

where $\Pi^{0,1} : A^{0,1}(\operatorname{End} E_h) \to H^{0,1}(\bar{\partial}_0)$ is L^2 orthogonal projection. $\qquad\square$

If $a'' \in A^{0,1}(\operatorname{End} E_h)$ and $g \in \mathcal{G}$, then $g \cdot (\bar{\partial}_0 + a'') = \bar{\partial}_0 - \bar{\partial}_0 g\, g^{-1} + g a'' g^{-1}$. The map

$$\mathcal{G} \times A^{0,1}(\operatorname{End} E_h) \ni (g, a'') \mapsto \vec{\bar{\partial}}_0^*(-\bar{\partial}_0 g\, g^{-1} + g a'' g^{-1}) \in A^{0,0}(\operatorname{End} E_h)$$

maps into the subspace of $A^{0,0}(\operatorname{End} E_h)$ orthogonal to the kernel of $\bar{\partial}_0$, and its linearisation at $(1, 0)$ in the $\mathcal{G}$-direction is $A^{0,0}(\operatorname{End} E_h) \ni \gamma \mapsto -\triangle_0'' \gamma$. This is an isomorphism from the space of L_2^p sections in $A^{0,0}(\operatorname{End} E_h)$ orthogonal to $\ker \bar{\partial}_0 = H^{0,0}$ to the space of L^p sections in $A^{0,0}(\operatorname{End} E_h)$ orthogonal to $\ker \bar{\partial}_0$. The implicit function theorem for Banach spaces now implies:

Lemma 2.5. *There exist ϵ, $C > 0$ with the property that for each $a'' \in A^{0,1}(\operatorname{End} E_h)$ with $\|a''\|_{L_1^p} < \epsilon$ there is a unique $\varphi \in A^{0,0}(\operatorname{End} E_h) \cap (\ker \bar{\partial}_0)^\perp$ with $\|\varphi\|_{L_2^p} \leq C \|\vec{\bar{\partial}}_0^* a''\|_{L^p}$ such that $\vec{\bar{\partial}}_0^*(-\bar{\partial}_0 g\, g^{-1} + g\, a''\, g^{-1}) = 0$, where $g = \exp(\varphi)$.* $\qquad\square$

This is a complex analogue of fixing a unitary gauge for hermitian connections near a given such connection, corresponding to the linear operation of projecting a $(0, 1)$-form orthogonal to the range of $\bar{\partial}_0$. If $\bar{\partial}_0 + a''$ is an integrable semi-connection with $\|a''\|_{L_1^p}$ sufficiently small as dictated by this lemma, after applying an appropriate complex gauge transformation so that the new semi-connection $\bar{\partial}_0 + \tilde{a}''$ satisfies $\vec{\bar{\partial}}_0^* \tilde{a}'' = 0$, Corollary 2.4 gives an estimate of the form $\|\tilde{a}''\|_{L_1^p} \leq C \|\Pi^{0,1} \tilde{a}''\|_{L^2}$. Consequently, the well-known result that if $H^1(X, \operatorname{End} E_0) = 0$ then every small deformation of E_0 is isomorphic to E_0 follows immediately. A simple but pertinent example is given by that of a trivial bundle on $\mathbb{P}_1$.

It follows from Corollary 2.4 that for integrable semi-connections in the "good" complex gauge lying in a sufficiently small neighbourhood of $\bar{\partial}_0$ in L_1^p, the projection onto the $\bar{\partial}_0$-harmonic component is a homeomorphism onto a closed subset of an open neighbourhood of 0 in $H^{0,1}$, where $H^{0,q}$ denotes $H^{0,q}(\bar{\partial}_0, \operatorname{End} E_h)$ in this section. This closed subset is the zero-set of the holomorphic function Ψ mentioned in the introduction, as will now be discussed.

If $a'' \in A^{0,1}(\operatorname{End} E_h)$ is $\bar{\partial}_0$-closed, the semi-connection $\bar{\partial}_0 + a''$ is not integrable in general, since $a'' \wedge a''$ need not be zero. But one might attempt to perturb a'' in such a way that the corresponding perturbed semi-connection is integrable.

The derivative of the map

$$\begin{array}{ccc} \left(A^{0,1} \times A^{0,2}\right)(\operatorname{End} E_h) & \longrightarrow & A^{0,2}(\operatorname{End} E_h) \\ \cup\!\!\!| & & \cup\!\!\!| \\ (a'', \beta) & \mapsto & \bar{\partial}_0(a'' + \vec{\bar{\partial}}_0^* \beta) + (a'' + \vec{\bar{\partial}}_0^* \beta) \wedge (a'' + \vec{\bar{\partial}}_0^* \beta) \end{array} \tag{2.3}$$

in the $A^{0,2}$-direction at $(0, 0)$ is $A^{0,2}(\operatorname{End} E_h) \ni b \mapsto \bar{\partial}_0 \vec{\bar{\partial}}_0^* b \in A^{0,2}(\operatorname{End} E_h)$, which is an isomorphism from the closed subspace of the L_2^p sections in $A^{0,2}(\operatorname{End} E_h)$ that are orthogonal in L^2 to the kernel of $\vec{\bar{\partial}}_0^*$ onto the closed subspace of the L^p sections in $A^{0,2}(\operatorname{End} E_h)$ orthogonal in L^2 to the kernel of $\vec{\bar{\partial}}_0^*$. The non-linear mapping (2.3) does not map into the latter closed subspace in general, but if Π is the L^2 orthogonal projection of $A^{0,2}(\operatorname{End} E_h)$ onto the closed subspace $\ker \vec{\bar{\partial}}_0^*$ and $\Pi_\perp = 1 - \Pi$, the map (2.3) can be replaced by

$$(a'', \beta) \mapsto \Pi_\perp \left[\bar{\partial}_0(a'' + \vec{\bar{\partial}}_0^* \beta) + (a'' + \vec{\bar{\partial}}_0^* \beta) \wedge (a'' + \vec{\bar{\partial}}_0^* \beta) \right]$$

to obtain a map with the same linearisation in the $A^{0,2}$-direction at $(0, 0)$.

Given $\bar{\partial}_0$-closed $a'' \in A^{0,1}(\text{End}\, E_h)$ in L_1^p and $\beta \in A^{0,2}(\text{End}\, E_h)$ in L_2^p, the form

$$\tau := \bar{\partial}_0(a'' + \vec{\partial}_0^*\beta) + (a'' + \vec{\partial}_0^*\beta) \wedge (a'' + \vec{\partial}_0^*\beta)$$

is an L^p section in $A^{0,2}(\text{End}\, E_h)$. Since

$$\Pi_\perp \tau = \bar{\partial}_0(a'' + \vec{\partial}_0^*\beta) + \Pi_\perp \left((a'' + \vec{\partial}_0^*\beta) \wedge (a'' + \vec{\partial}_0^*\beta) \right)$$

and $\vec{\partial}_0^*((a'' + \vec{\partial}_0^*\beta) \wedge (a'' + \vec{\partial}_0^*\beta))$ lies in L^p (again using $p > 2n$ and the Sobolev embedding theorem), it follows that $\Pi_\perp \tau$ lies in L^p, and that the composition $(a'', \beta) \mapsto \Pi_\perp \tau$ is continuous with respect to the $L_1^p \times L_2^p$ topology on the domain and the L^p topology on the codomain.

By ellipticity of $\triangle_0''$ on $A^{0,2}(\text{End}\, E_h)$, there is a constant K such that $\sup_X |e| \leq K$ for any $e \in H^{0,2}$ with $\|e\|_{L^2} = 1$. If $\bar{\partial}_0 a'' = 0$, then $\bar{\partial}_0(a'' \wedge a'') = 0$, which implies that

$$\Pi_\perp\left(\bar{\partial}_0 a'' + a'' \wedge a''\right) = \Pi_\perp(a'' \wedge a'') = \Pi_\perp^{0,2}(a'' \wedge a'') \,,$$

and hence that the L^p norm of $\Pi_\perp\left(\bar{\partial}_0 a'' + a'' \wedge a''\right)$ is uniformly bounded by a constant multiple of the L^p norm of $a'' \wedge a''$ in this case. Another application of the implicit function theorem now yields:

Proposition 2.6. *There exist $\epsilon > 0$ and $C > 0$ with the properties that for any $\bar{\partial}_0$-closed $a'' \in A^{0,1}(\text{End}\, E_h)$ satisfying $\|a''\|_{L_1^p} < \epsilon$ there is a unique $\beta \in (\ker \vec{\partial}_0^*)^\perp \subseteq A^{0,2}(\text{End}\, E_h)$ satisfying $\|\beta\|_{L_2^p} \leq C \|a'' \wedge a''\|_{L^p}$ such that $\bar{\partial}_0(a'' + \vec{\partial}_0^*\beta) + (a'' + \vec{\partial}_0^*\beta) \wedge (a'' + \vec{\partial}_0^*\beta)$ is weakly $\vec{\partial}_0^*$-closed.* $\qquad\square$

Remarks.

1. If β is as in this lemma, then $\|a'' + \vec{\partial}_0^*\beta\|_{L_1^p}$ is uniformly bounded by a constant multiple of $\|a''\|_{L_1^p}$. In fact, if $\tilde{a}'' := a'' + \vec{\partial}_0^*\beta$, then $\tau := \bar{\partial}_0\tilde{a}'' + \tilde{a}'' \wedge \tilde{a}''$ satisfies

$$\bar{\partial}_0\tau + \tilde{a}'' \wedge \tau - \tau \wedge \tilde{a}'' = 0 = \vec{\partial}_0^*\tau$$

weakly, so by elliptic regularity it follows that τ in fact lies in L_1^p. Hence by Lemma 2.3, if $\|a''\|_{L_1^p}$ (and hence $\|\tilde{a}''\|_{L_1^p}$) is sufficiently small, there is a constant $C = C(d_0)$ such that $\|\tau\|_{L_1^p} \leq C \|\Pi^{0,2}\tau\|_{L^2}$.

2. Proposition 2.6 remains valid even if d_0 is not integrable—all that is required is a uniform C^0 bound on $F^{0,2}(d_0)$. The proof as given only needs minor modification by noting that $\bar{\partial}_0\tau$ involves an extra term $F^{0,2}(d_0) \wedge a'' - a'' \wedge F^{0,2}(d_0)$, this lying in L^p if $|F(d_0)|$ is bounded in C^0.

3. If γ is a complex automorphism of E_h satisfying $d_0\gamma = 0$, then by the uniqueness statement of the lemma, $\beta(\gamma a'' \gamma^{-1}) = \gamma\beta(a'')\gamma^{-1}$, at least if $\|\gamma a'' \gamma^{-1}\|_{L^4}$ is sufficiently small.

From (1.2) there is a constant $c > 0$ depending only on d_0 such that any $\alpha \in A^{0,1}$ with $\bar{\partial}_0\alpha = 0 = \vec{\partial}_0^*\alpha$ satisfies $\|\alpha\|_{L_1^p} \leq c \|\alpha\|$. Thus there is a number $\delta > 0$ depending only on d_0 such that $\|\alpha\|_{L_1^p} < \epsilon$ if $\|\alpha\| < \delta$, and for such α there is a unique $\beta \in A^{0,2}(\text{End}\, E_h)$ orthogonal to $\ker \vec{\partial}_0^*$ with $\|\beta\|_{L_2^p} \leq C' \|\alpha\|^2$ for which the form $\tau = \bar{\partial}_0 a'' + a'' \wedge a''$ is $\vec{\partial}_0^*$-closed, where $a'' := \alpha + \vec{\partial}_0^*\beta$. Moreover, $\|\tau\|_{L_1^p} \leq C\|\Pi^{0,2}\tau\|$ for some constant $C = C(d_0)$.

Define the function Ψ on the set of $\bar{\partial}_0$-harmonic forms $\alpha \in A^{0,1}(\text{End}\, E_h)$ with $\|\alpha\| < \delta$ that takes values in $H^{0,2}(\bar{\partial}_0, \text{End}\, E_h)$ by

$$\Psi(\alpha) := \Pi^{0,2}(a'' \wedge a'') \quad \text{for } a'' = \alpha + \vec{\partial}_0^*\beta \text{ with } \beta = \beta(\alpha) \text{ as in Proposition 2.6.} \tag{2.4}$$

Then the zero set of Ψ parameterises precisely the integrable connections in the good complex gauge in this L_1^p neighbourhood of d_0, in the sense of defining a semi-universal deformation. By fixing orthonormal bases for each of $H^{0,1}$ and $H^{0,2}$ and working in these bases, it is immediate that Ψ is holomorphic. As will be shown later (Corollary 4.3), in the special case that d_0 is Hermite-Einstein, Ψ is equivariant with respect to the action of $\text{Aut}(E_0)$ on these spaces, a consequence of the third remark above.

3 A neighbourhood of a semistable bundle

With the same notation and conventions as in the previous section, we assume from now on that (X, ω) is Kähler. The results in this section on bounds of slopes of subsheaves, (which can be seen as analogues of the standard fact that *Hermite-Einstein implies semistable)*, may be of independent interest.

Lemma 3.1. *Let E be a holomorphic bundle defined by an integrable semi-connection $\bar{\partial}$ on* E_h *and let $d = \bar{\partial} + \partial$ be the associated hermitian connection. Then there is a constant C depending only on ω such that*

$$\deg(A) \leq C \,\|\widehat{F}(d)\|_{L^1} \quad \text{and} \quad \|c_1(A)\| \leq C \left(\|F(d)\|_{L^1} + |\deg(A)| \right)$$

for any coherent analytic sheaf $A \subset E$ with torsion-free quotient. Here $\|c_1(A)\|$ denotes the L^2 norm of the harmonic $(1, 1)$-form representing the image of $c_1(A)$ in $H^2_{dR}(X)$.

Proof. Suppose first that $A \subset E$ is a holomorphic sub-*bundle* of E and let $B := E/A$ be the quotient bundle. In a unitary frame for $E_h = A_h \oplus B_h$, the connection $d =: d_E$ has the form

$$d_E = \begin{bmatrix} d_A & \beta \\ -\beta^\star & d_B \end{bmatrix}$$

where d_A and d_B are the induced hermitian connections on A and B, and $\beta \in A^{0,1}(Hom(B, A))$ is a $\bar{\partial}$-closed $(0, 1)$-form representing the extension $0 \to A \to E \to B \to 0$. The curvature $F_E = F(d_E)$ has the form

$$F_E = \begin{bmatrix} F_A - \beta \wedge \beta^\star & d_{BA}\beta \\ -d_{AB}\beta^\star & F_B - \beta^\star \wedge \beta \end{bmatrix} \,,$$

where d_{BA} here is the connection on $Hom(B, A)$ induced by d_A and d_B. So if Π_A is pointwise-orthogonal projection $E \to A$, it follows that $F_A = \Pi_A F_E \Pi_A + \beta \wedge \beta^\star$. Since β is a $(0, 1)$-form, $i \operatorname{tr} \beta \wedge \beta^\star$ is a *non-positive* $(1, 1)$-form and therefore $i \operatorname{tr} F_A \leq \operatorname{tr} (\Pi_A i F_E \Pi_A)$. Applying $\omega^{n-1} \wedge$ and integrating over X, it follows that $c_1(A) \cdot [\omega^{n-1}]$ is bounded above by a fixed multiple of $\|\widehat{F}_E\|_{L^1}$.

   Now if A is only a subsheaf of E of rank $a > 0$ with torsion-free quotient B, replace E with $\Lambda^a E$, A by the maximal normal extension of $\Lambda^a A$ in $\Lambda^a E$ (which is a line bundle) and then after blowing up the zero set of the induced section of $Hom(\det A, \Lambda^a E)$ and resolving singularities, there is again an upper bound on the degree of the desingularised subsheaf in terms of the L^1 norm of F_E on the blowup. This upper bound depends on the metric used on the blowup, but as in §§2, 3 of [5], there is a family ω_ϵ of such metrics converging to the pullback of ω, and the resulting limit then gives the same bound: $\deg(A)$ is bounded above by a fixed multiple of $\|\widehat{F}(d)\|_{L^1}$ for any subsheaf $A \subset E$ with torsion-free quotient.

   To obtain uniform bounds on $\|c_1(A)\|$, suppose again initially that A is a subbundle of E. For notational simplicity, let $f := i \operatorname{tr} F_A$ and $g := i \operatorname{tr} (\Pi_A F_E \Pi_A)$, so from the preceding arguments, $g - f \geq 0$ as hermitian forms on TX. The space $H^{1,1}_{\mathbb{R}}(X)$ of real harmonic $(1, 1)$-forms is finite dimensional, so by picking an orthonormal basis, it is apparent that there is a constant $C > 0$ depending on ω such that $-C\,\omega \leq \varphi \leq C\,\omega$ for any $\varphi \in H^{1,1}_{\mathbb{R}}(X)$ with $\|\varphi\| = 1$; equivalently, $C\omega \pm \varphi \geq 0$. Therefore $(C\omega \pm \varphi) \wedge (g - f) \geq 0$, implying that

$$\pm\varphi \wedge f \leq (C\omega \pm \varphi) \wedge g - C\omega \wedge f$$

as real $(2, 2)$-forms pointwise on X. Applying $\omega^{n-2} \wedge$ and integrating over X, it follows that

$$\pm \int_X \omega^{n-2} \wedge \varphi \wedge f \leq \int_X \omega^{n-2} \wedge (C\omega \pm \varphi) \wedge g - C \int_X \omega^{n-1} \wedge f \,,$$

and by allowing φ to vary over the harmonic $(1, 1)$-forms of norm 1, it follows that

$$\|c_1(A)\|_{L^2} \leq C' \left(\|F(d_E)\|_{L^1} + |\deg(A)| \right)$$

for some new constant C' independent of d_E. This implies the second statement of the lemma when A is a sub-bundle.

In the case that A is only a subsheaf rather than a subbundle, the same method as earlier can be used to reduce to the case of a line subbundle on a blowup of X. It need only be checked that for metrics of the kind ω_ϵ mentioned earlier, $\lim_{\epsilon \to 0} \|c_1(L)\|_{L^2(\omega_\epsilon)} = 0$ for any line bundle L on the blowup that is trivial off the exceptional divisor, which is straight-forward to verify. □

Lemma 3.1 implies the following result, showing that semistability is an open condition.

Proposition 3.2. *Let E_0 be a semistable bundle on a compact Kähler manifold (X, ω), defined by an integrable connection d_0. Then there exists $\epsilon > 0$ such that any integrable connection $d_0 + a$ with $\|a\|_{L_1^2} + \|a\|_{L^4} < \epsilon$ defines a semistable holomorphic structure.*

Proof. If not, there is a sequence $(a_j) \in A^1(\mathrm{End}\,E_h)$ with $\|a_j\|_{L_1^2} + \|a_j\|_{L^4} \to 0$ and $d_j := d_0 + a_j$ integrable such that the holomorphic bundle E_j defined by d_j is not semistable, so there is a subsheaf $A_j \subset E_j$ with torsion-free quotient that strictly destabilises E_j. Passing to a subsequence, it can be assumed that the ranks of the sheaves A_j are constant, a say. The hypotheses on a_j imply that $\|F(d_j)\|_{L^2}$ is uniformly bounded and therefore so too is $\|F(d_j)\|_{L^1}$, and consequently Lemma 3.1 yields a uniform upper bound on $\deg(A_j)$. Since $\deg(A_j)$ is also uniformly bounded below, these bounds together with the bounds on $\|F(d_j)\|_{L^2}$ then give uniform bounds on the L^2 norms of the harmonic representatives of the forms representing $c_1(A_j)$ in $H_{dR}^2(X)$, and hence there is a convergent subsequence. Since the image of $H^2(X, \mathbb{Z})$ in $H_{dR}^2(X)$ is discrete, this convergent subsequence must be eventually constant, so after passing to another subsequence, it can be assumed that $c_1(A_j)$ is constant, c say. Since X is Kähler, $Pic^c(X)$ is a compact torus, so after passing to another subsequence, it can be supposed that $\det A_j$ converges to a holomorphic line bundle L on X. Since $(\det A_j)^\star \otimes \Lambda^a E_j$ has a non-zero holomorphic section for each j, so too does $L^\star \otimes \Lambda^a E_0$, and therefore $\Lambda^a E_0$ is strictly destabilised by L. But by Theorem 2 of [17], semistability of E_0 implies that of $\Lambda^a E_0$, giving the desired contradiction. □

Remark.

It is worth noting that this result is rather delicate in that the Kähler class must be fixed. For example, every non-split extension of the form $0 \to \mathcal{O}(-1, 1) \to E \to \mathcal{O}(1, 0) \to 0$ on $\mathbb{P}_1 \times \mathbb{P}_1$ is strictly stable with respect to $\omega_t := \pi_1^\star \omega_0 + t\,\pi_2^\star \omega_0$ for $t > 2$, is semistable but not polystable for $t = 2$, and is strictly unstable if $0 < t < 2$. The result also fails in the non-Kählerian case, at least when the degree fails to be topological. For example, if L is a non-trivial holomorphic line bundle on an Inoue surface with $L \otimes L$ trivial, the direct sum of L with the trivial line bundle **1** is polystable with respect to every Gauduchon metric, every small deformation is again a direct sum, and of these, the generic one is strictly unstable with respect to every Gauduchon metric. In particular, semistability is not a Zariski-open condition in this setting. In this case, the automorphism group of $E_0 = L \oplus \mathbf{1}$ acts trivially on $H^1(X, End\,E_0)$, so each of its orbits is closed.

4 A neighbourhood of a polystable bundle

Families of holomorphic bundles near a polystable but not stable fibre differ considerably from families near a stable fibre. Fibres near to a stable fibre are always stable, but the analogous fact does not hold (in general) in the polystable case: the implicit function theorem fails.

A holomorphic bundle E that is a non-split extension $0 \to A \to E \to B \to 0$ by semistable bundles A, B of the same slope is semistable but not polystable, and cannot be separated from the direct sum $A \oplus B$ in the quotient topology on the space of integrable semi-connections modulo the complex gauge group. The corresponding holomorphic one-parameter family of extensions yields a family of holomorphic vector bundles being mutually isomorphic except for the direct sum. Consequently a coarse moduli space cannot exist. Given this, it makes sense to focus on polystable structures in a neighbourhood of a given polystable

bundle, in which case a great deal more can be said than in the previous section. The same objects and definitions as in the last section are used here, but now E_0 is assumed to be a *polystable* holomorphic bundle.

The following is a minor generalisation of a well-known result essentially due to Kobayashi [21]. Although its proof is elementary, it is presented here for the reason that in some respects, it is the pivotal result used in the paper.

Lemma 4.1. *Let d be a connection on a bundle E with $\widehat{F}(d) = 0$. If $s \in A^{0,0}(E_h)$ satisfies $\bar{\partial}s = 0$, then $ds = 0$.*

Proof. The equation $\bar{\partial}s = 0$ implies $\partial\langle s, s\rangle = \langle s, \partial s\rangle$, using the convention that $\langle \cdot, \cdot \rangle$ is conjugate-linear in the first variable. Therefore $\bar{\partial}\partial\langle s, s\rangle = \langle \partial s \overset{\wedge}{,} \partial s\rangle + \langle s, F^{1,1}(d)s\rangle$. Applying $i\Lambda$, it follows $\triangle''|s|^2 + |\partial s|^2 = 0$, so integration over X gives $\|\partial s\|^2 = 0$. $\qquad\qquad\square$

Note that it is not assumed that d should be integrable.

For a connection d_0 on E_h whose central curvature $\widehat{F}(d_0)$ is a constant multiple of the identity, the central curvature of the connection induced by d_0 on $\mathrm{End}\, E_h$ is identically zero, so Lemma 4.1 implies:

Corollary 4.2. *Suppose d_0 is a connection on E_h with $i\widehat{F}(d_0) = \lambda\, 1$ for some scalar λ. If $\sigma \in A^{0,0}(\mathrm{End}\, E_h)$ satisfies $\bar{\partial}_0\sigma = 0$, then $d_0\sigma = 0$.* $\qquad\qquad\square$

This corollary yields the equivariance property of the function Ψ asserted at the end of §2:

Corollary 4.3. *Under the hypotheses of Corollary 4.2, there is a constant $\epsilon = \epsilon(d_0) > 0$ with the property that for any $\bar{\partial}_0$-harmonic $\alpha \in A^{0,1}(\mathrm{End}\, E_h)$ and $\bar{\partial}_0$-closed $\gamma \in \mathcal{G}$ satisfying $\|\alpha\|_{L^2} + \|\gamma\alpha\gamma^{-1}\|_{L^2} < \epsilon$, the form $\beta = \beta(\alpha) \in A^{0,2}(\mathrm{End}\, E_h)$ of Proposition 2.6 satisfies $\beta(\gamma\alpha\gamma^{-1}) = \gamma\beta(\alpha)\gamma^{-1}$.*

Proof. Given $\bar{\partial}_0$-closed $a'' \in A^{0,1}(\mathrm{End}\, E_h)$ with $\|a''\|_{L_1^p}$ sufficiently small, Proposition 2.6 guarantees the existence of a unique $\beta = \beta(a'') \in A^{0,2}(\mathrm{End}\, E_h)$ orthogonal to $\ker \bar{\partial}_0^{\star}$ and with $\|\beta\|_{L_2^p} \leq C\|a'' \wedge a''\|_{L^p}$ such that $\tilde{a}'' := a'' + \bar{\partial}_0^{\star}\beta$ satisfies $\bar{\partial}_0^{\star}(\bar{\partial}_0\tilde{a}'' + \tilde{a}'' \wedge \tilde{a}'') = 0$. If $\gamma \in \mathcal{G}$ is $\bar{\partial}_0$-closed, then Corollary 4.2 implies that $\partial_0\gamma = 0$ and therefore conjugation by γ commutes with both $\bar{\partial}_0$ and $\bar{\partial}_0^{\star}$. Thus if β is orthogonal to $\ker \bar{\partial}_0^{\star}$ so too is $\gamma\beta\gamma^{-1}$ and also $\bar{\partial}_0^{\star}(\bar{\partial}_0(\gamma\tilde{a}''\gamma^{-1}) + \gamma(\tilde{a}'' \wedge \tilde{a}'')\gamma^{-1}) = 0$. Since $\gamma\tilde{a}''\gamma^{-1} = \gamma a''\gamma^{-1} + \bar{\partial}_0^{\star}(\gamma\beta\gamma^{-1})$, it follows from the uniqueness statement of Proposition 2.6 that $\beta(\gamma a''\gamma^{-1}) = \gamma\beta(a'')\gamma^{-1}$ if both a'' and $\gamma a''\gamma^{-1}$ are sufficiently small in L_1^p. If $a'' = \alpha$ is $\bar{\partial}_0$-harmonic, then so too is $\gamma\alpha\gamma^{-1}$, and the L_1^p norms of these forms are uniformly bounded by a fixed multiple of their L^2 norms. $\qquad\qquad\square$

The proof of Proposition 3.2 combined with Lemma 4.1 have the following useful consequence, which simplifies a number of subsequent arguments:

Proposition 4.4. *Suppose d_0 is an integrable connection with $i\widehat{F}(d_0) = \lambda\, 1$ defining a polystable holomorphic structure E_0. There exists $\epsilon > 0$ such that any integrable semi-connection $\bar{\partial} = \bar{\partial}_0 + a''$ with $\|a''\|_{L_1^p} < \epsilon$ defines a semistable holomorphic bundle E with the property that any subsheaf $A \subset E$ with $\mu(A) = \mu(E)$ and with E/A torsion-free is a sub-bundle of E.*

Proof. If not, then from the proof of Proposition 3.2, there is a sequence of integrable semi-connections $\bar{\partial}_j = \bar{\partial}_0 + a''_j$ with $\|a''_j\|_{L_1^p} \to 0$ such that the corresponding holomorphic bundle E_j has a subsheaf A_j with $\mathrm{rk}\, A_j = a$ independent of j, $\mu(A_j) = \mu(E_j) = \mu(E_0)$ for all j, and with E_j/A_j torsion-free but not locally free. Hence $\Lambda^a A_j$ is a rank 1 subsheaf of $\Lambda^a E_j$, and the bundle $(\det A_j)^{\star} \otimes \Lambda^a E_j$ has a holomorphic section that has a zero. After passing to a subsequence, the Hermite-Einstein connections on the line bundles $(\det A_j)^{\star}$ can be assumed to converge to define a holomorphic line bundle L on X, and compactness of the embedding of L_1^p in C^0 implies that a subsequence of the integrable connections defining $(\det A_j)^{\star} \otimes \Lambda^a E_j$ converges uniformly in C^0 to an Hermite-Einstein connection on $L \otimes E_0$. After scaling the sections of $(\det A_j)^{\star} \otimes \Lambda^a E_j$ so as to have L^2 norm 1, the convergence of the connections implies that a subsequence of the rescaled sections converges weakly in L_1^2 (say) and strongly in L^2 to a holomorphic section of $L \otimes \Lambda^a E_0$ of norm 1. From the Cauchy integral formula

integrated over a poly-annulus P, a sequence of holomorphic functions converging in L^1 on P is converging uniformly on compact subsets inside P, so if each term in the sequence has a zero there, then so too does the limit. But the limiting section is a holomorphic section of a bundle of degree zero that admits a connection with $\widehat{F} = 0$, so it is either identically zero or nowhere zero. $\qquad\square$

The conclusion of Lemma 4.1 can be strengthened to give a perturbed version; integrability of d_0 is again not required.

Proposition 4.5. *Suppose d_0 is a connection on E_h with $\widehat{F}(d_0) = 0$. There is a constant $\epsilon = \epsilon(d_0) > 0$ such that any connection $d_a = d_0 + a$ with $a = a' + a''$, $\bar{\partial}_0^* a'' = 0$ and $\|a\|_{L_1^p} < \epsilon$ has the property that any section $s \in A^{0,0}(E_h)$ with $\bar{\partial}_0 s + a'' s = 0$ must in fact satisfy $\bar{\partial}_0 s = 0 = a'' s$.*

Proof. Suppose $\bar{\partial}_0 s + a'' s = 0$ for some section $s \in A^{0,0}(E_h)$, and write $s = s_0 + s_1$ where $\bar{\partial}_0 s_0 = 0$ and s_1 is orthogonal in L^2 to the kernel of $\bar{\partial}_0$. By Lemma 4.1, $d_0 s_0 = 0$. Then

$$\bar{\partial}_0 s_1 + a'' s_1 + a'' s_0 = 0,$$

so after applying $\bar{\partial}_0^* = -i\, \Lambda \partial_0$ it follows that

$$0 = \triangle_0'' s_1 + i\, \Lambda(a'' \wedge \partial_0 s_1) = \triangle_0'' s_1 + \bar{\partial}_0^*(a'' s_1).$$

Using the L^2 inner product, this implies that

$$\|\bar{\partial}_0 s_1\|^2 = -\langle \bar{\partial}_0 s_1, a'' s_1 \rangle \le \sup |a''|\, \|\bar{\partial}_0 s_1\|\, \|s_1\|,$$

and therefore $\|\bar{\partial}_0 s_1\| \le \sup |a''|\, \|s_1\|$. Since $p > 2n$, there is a constant C such that $\sup |a''| \le C\|a''\|_{L_1^p}$, and since s_1 is orthogonal to $\ker \bar{\partial}_0$, there is a constant $c = c(d_0) > 0$ independent of s such that $c\, \|s_1\| \le \|\bar{\partial}_0 s_1\|$. So if $C\|a''\|_{L_1^p} < c$, then s_1 must be 0, giving $s = s_0 \in \ker \bar{\partial}_0$, with $0 = \bar{\partial}_0 s + a'' s = \bar{\partial}_0 s_0 + a'' s_0 = a'' s_0$. $\qquad\square$

For notational convenience, the group $Aut\, E_0$ consisting of the $\bar{\partial}_0$-closed elements of $\mathcal{G}$ will be denoted by Γ henceforth. An equivalence relation $\sim$ is defined on $S = \Psi^{-1}(0)$ by $s' \sim s''$ if $s'' = \gamma \cdot s'$ for some $\gamma \in \Gamma$. In this sense there exists a (pointwise) quotient $\Psi^{-1}(0)/\Gamma$ without assuming a genuine group action $\Gamma \times S \to S$, which does not exist in general (because of points being pushed outside the domain of Ψ by some elements of Γ).

Corollary 4.6. *Let d_0 be a Hermite-Einstein connection defining a holomorphic structure E_0, and let Ψ be the function of (2.4), defined in a neighbourhood of zero in $H^{0,1} = H^{0,1}(E_h, \bar{\partial}_0)$ with values in $H^{0,2}$. Let $\mathcal{F} := \{d_0 + a \mid F^{0,2}(d_0 + a) = 0\}$, equipped with the L_1^p topology. If Ψ is restricted to a sufficiently small neighbourhood of zero, then with respect to the quotient topologies, the natural map $\Psi^{-1}(0)/\Gamma \to \mathcal{F}/\mathcal{G}$ is a homeomorphism onto a neighbourhood of $[d_0]$.*

Proof. The continuity of the map is clear. If $\alpha_0 \in H^{0,1}$, ellipticity of the $\bar{\partial}_0$-Laplacian implies that $\|\alpha_0\|_{L_1^p} \le C\|\alpha_0\|_{L^2}$ for some constant $C = C(d_0)$. Hence if $\|\alpha_0\|_{L^2}$ is sufficiently small, Proposition 2.6 yields a unique $\beta_0 \in A^{0,2}(\text{End}\, E_h)$ orthogonal to $\ker \bar{\partial}_0^*$ with $\|\beta_0\|_{L_2^p} \le C\|\alpha_0\|_{L^2}^2$ such that $a_0'' := \alpha + \bar{\partial}_0^* \beta_0$ satisfies $\bar{\partial}_0^*(\bar{\partial}_0 a_0'' + a_0'' \wedge a_0'') = 0$. If $a = a' + a''$ is such that $\|a_0 - a\|_{L_1^p}$ is so small that Lemma 2.5 applies to $d_0 + a$, then there is a unique $g \in \mathcal{G}$ of the form $g = \exp(\varphi)$ with $\varphi \in (\ker \bar{\partial}_0)^\perp$ and $\|\varphi\|_{L_2^p} \le C\|\bar{\partial}_0^* a''\|_{L^p}$ such that $d_0 + a_1 := g \cdot (d_0 + a)$ satisfies $\bar{\partial}_0^* a_1'' = 0$. Since $p > 2n$, the Sobolev embedding theorem gives uniform C^1 estimates on g and g^{-1}, implying that

$$\|a'' - a_1''\|_{L_1^p} \le C\|\bar{\partial}_0^* a''\|_{L^p} = C\|\bar{\partial}_0^*(a'' - a_0'')\|_{L^p}$$

for some new constant $C = C(d_0)$. So if a is sufficiently close to a_0 in L_1^p, Proposition 2.6 now gives a uniquely determined $\alpha_1 \in H^{0,1}$ and $\beta_1 \in (\ker \bar{\partial}_0^*)^\perp \subseteq A^{0,2}(\text{End}\, E_h)$ such that $a_1'' = \alpha_1 + \bar{\partial}_0^* \beta_1$, with $\bar{\partial}_0^*(\bar{\partial}_0 a_1'' + a_1'' \wedge a_1'') = 0$ and with $\|\beta_1\|_{L_2^p} \le C\|\alpha_1\|^2$. Thus for some new constant $C = C(d_0)$,

$$\|\alpha_1 - \alpha_0\|_{L^2} \le \|a_1'' - a_0''\|_{L^2} \le \|a_1'' - a''\|_{L_1^p} + \|a'' - a_0''\|_{L_1^p} \le C\|a'' - a_0''\|_{L_1^p}.$$

In summary, given $\alpha_0 \in H^{0,1}$ sufficiently close to 0, for each $a \in A^{0,1}(\text{End } E_h)$ sufficiently close to a_0 in L_1^p there exists $g \in \mathcal{G}$ and $\alpha_1 \in H^{0,1}$ such that $d_0 + a$ is of the form $d_0 + a = g^{-1} \cdot (d_0 + a_1)$ for $a_1'' = \alpha_1 + \bar{\partial}_0^* \beta_1$ with $\|\alpha_1 - \alpha_0\|_{L^2}$ uniformly bounded by a multiple of $\|a - a_0\|_{L_1^p}$. If $\mathcal{A}$ denotes $A^{0,1}(\text{End } E_h)$ equipped with the L_1^p norm, this implies that the natural map $[\alpha] \mapsto [\alpha + \bar{\partial}_0^* \beta]$ from a neighbourhood of $[0] \in H^{0,1}/\Gamma$ to $\mathcal{A}/\mathcal{G}$ is an open mapping, and therefore so too is the natural map $\Psi^{-1}(0)/\Gamma \to \mathcal{F}/\mathcal{G}$.

To see that the mapping is 1–1, suppose that $d_0 + a_0$, $d_0 + a_1$ are (integrable) connections with $\bar{\partial}_0^* a_0'' = 0 = \bar{\partial}_0^* a_1''$ that define isomorphic holomorphic structures; that is, there exists $g \in \mathcal{G}$ with $d_0 + a_1 = g \cdot (d_0 + a_0)$. If $\|a_0\|_{L_1^p} + \|a_1\|_{L_2^p} < \epsilon$ with ϵ as in Proposition 4.5, apply that proposition to the connection on $\text{End } E_h = \text{Hom}(E_h, E_h)$ defined by $d_0 + a_0$ on one side and $d_0 + a_1$ on the other, with s being the section $g \in \text{Hom}(E_h, E_h)$. It follows from that result that $d_0 g = 0$ (so $g \in \Gamma$) and that $a_1 g = g a_0$. Writing $a_j'' = \alpha_j + \bar{\partial}_0^* \beta_j$ with β_j determined by Proposition 2.6, orthogonality of the decomposition implies that $\alpha_1 g = g \alpha_0$; that is, α_0, $\alpha_1 \in H^{0,1}$ represent the same point in the quotient $H^{0,1}/\Gamma$. □

Note that again, the integrability of the connections $d_0 + a_j$ is not critical; the essential ingredient is the "quasi-integrability" condition implicit in the statement of Proposition 2.6.

The main application of Proposition 4.5 will be to the connections on $\text{End } E_h$ induced by connections d_0 on E_h with $\widehat{F}(d_0)$ a scalar multiple of the identity, as in following theorem. Here $[x, y] = xy - yx$ is the usual Lie bracket on endomorphisms, and the reader is alerted to the fact that the hypotheses on σ differ slightly in 1. and 2., although the conclusions are essentially the same.

Theorem 4.7. *Let d_0 be a connection on E_h with $i\widehat{F}(d_0) = \lambda 1$, and let $d_0 + a$ be a connection with $a = a' + a''$, where $a' = -(a'')^*$, $a'' = \alpha + \bar{\partial}_0^* \beta$ for some $\alpha \in A^{0,1}(\text{End } E_h)$ satisfying $\bar{\partial}_0 \alpha = 0 = \bar{\partial}_0^* \alpha$ and $\beta \in A^{0,2}(\text{End } E_h)$. There is a constant $\epsilon > 0$ depending only on d_0 with the property that if $\|a\|_{L_1^p} < \epsilon$ then the following hold for any endomorphism $\sigma \in A^{0,0}(\text{End } E_h)$:*

1. *If $\bar{\partial}_0 \sigma + [a'', \sigma] = 0$ then $d_0 \sigma = 0$ and $[\alpha, \sigma] = 0 = [\bar{\partial}_0^* \beta, \sigma]$. Furthermore, $[\beta, \sigma] = 0$ if $\beta \in (\ker \bar{\partial}_0^*)^\perp$.*
2. *If $\bar{\partial}_0 \sigma + [\alpha, \sigma] = 0$ and $\beta \in (\ker \bar{\partial}_0^*)^\perp$ and $\bar{\partial}_0^*(\bar{\partial}_0 a'' + a'' \wedge a'') = 0$ then $d_0 \sigma = 0$ and $[\alpha, \sigma] = 0 = [\beta, \sigma]$.*
3. *If $\bar{\partial}_0 \sigma + [a'', \sigma] = 0$ and $i\Lambda(\alpha \wedge \alpha^* + \alpha^* \wedge \alpha)$ is orthogonal to $\ker \bar{\partial}_0$ then $d_0 \sigma = 0$ and $[\alpha, \sigma]$, $[\bar{\partial}_0^* \beta, \sigma]$ and $[\alpha, \sigma^*]$ all vanish. Furthermore, $[\beta, \sigma] = 0$ if $\beta \in (\ker \bar{\partial}_0^*)^\perp$.*
4. *If $\bar{\partial}_0 \sigma + [a'', \sigma] = 0$ and $\bar{\partial}_0^*(\bar{\partial}_0 a'' + a'' \wedge a'') = 0$ and $i\Lambda(\alpha \wedge \alpha^* + \alpha^* \wedge \alpha) \in (\ker \bar{\partial}_0)^\perp$ and $\beta \in (\ker \bar{\partial}_0^*)^\perp$ then $d_0 \sigma = 0$ and $[\alpha, \sigma]$, $[\beta, \sigma]$, $[\alpha, \sigma^*]$ and $[\beta, \sigma^*]$ all vanish.*

Proof. **1.** The central curvature of the connection on $\text{End } E_h$ induced by d_0 vanishes, so from Corollary 4.2 and Proposition 4.5,
$$d_0 \sigma = 0 = [a'', \sigma] = [\alpha, \sigma] + [\bar{\partial}_0^* \beta, \sigma] = [\alpha, \sigma] + \bar{\partial}_0^* [\beta, \sigma] .$$
Applying $\bar{\partial}_0$ gives $\bar{\partial}_0 \bar{\partial}_0^* [\beta, \sigma] = 0$, from which it follows that $\bar{\partial}_0^* [\beta, \sigma] = 0 = [\alpha, \sigma]$. If β is orthogonal to $\ker \bar{\partial}_0^*$, so too is $[\beta, \sigma]$ by virtue of identity $\langle \psi, [\beta, \sigma] \rangle = \langle [\psi, \sigma^*], \beta \rangle$, and if $\psi \in \ker \bar{\partial}_0^*$ then so too is $[\psi, \sigma^*]$ since $d_0 \sigma^* = 0$.

2. If $\tau := \bar{\partial}_0 a'' + a'' \wedge a''$, then
$$\tau = \bar{\partial}_0 \bar{\partial}_0^* \beta + \alpha \wedge \bar{\partial}_0^* \beta + \bar{\partial}_0^* \beta \wedge \alpha + \alpha \wedge \alpha + \bar{\partial}_0^* \beta \wedge \bar{\partial}_0^* \beta .$$
Setting $\beta = 0$ in 1. it follows from that case that $d_0 \sigma = 0 = [\alpha, \sigma]$. Since σ commutes with α, it also commutes with $\alpha \wedge \alpha$, implying that
$$[\sigma, \tau] = \bar{\partial}_0 \bar{\partial}_0^* [\sigma, \beta] + \alpha \wedge \bar{\partial}_0^* [\sigma, \beta] + \bar{\partial}_0^* [\sigma, \beta] \wedge \alpha + \bar{\partial}_0^* [\sigma, \beta] \wedge \bar{\partial}_0^* \beta + \bar{\partial}_0^* \beta \wedge \bar{\partial}_0^* [\sigma, \beta] . \tag{4.1}$$
As in the proof of 1., because β is orthogonal to $\ker \bar{\partial}_0^*$, so too is $[\sigma, \beta]$. On the other hand, since $\bar{\partial}_0^* \tau = 0$, so too is $\bar{\partial}_0^* [\sigma, \tau] = 0$. Thus $[\sigma, \beta]$ is orthogonal to $[\sigma, \tau]$. Taking the inner product on both sides of (4.1) with $[\sigma, \beta]$, rearranging terms and estimating, it follows that
$$\left\| \bar{\partial}_0^* [\sigma, \beta] \right\|^2 \leq C \left(\sup |\alpha| + \sup |\bar{\partial}_0^* \beta| \right) \left\| \bar{\partial}_0^* [\sigma, \beta] \right\| \, \| [\sigma, \beta] \| .$$

Since $[\sigma, \beta]$ is orthogonal to $\ker \vec{\bar\partial}_0^{\,\star}$, there is a constant $c > 0$ depending only on d_0 such that $\left\| \vec{\bar\partial}_0^{\,\star}[\sigma, \beta] \right\|^2 \geq c\|[\sigma, \beta]\|^2$. Since α is $\bar\partial_0$-harmonic, $\sup |\alpha|$ is bounded by a constant multiple of $\|\alpha\|$, and from Proposition 2.6 it can be assumed that $\sup |\beta|$ is bounded by a constant multiple of $\|\alpha\|^2$. It follows that there is a new constant ϵ_1 depending only on d_0 for which, if $\|\alpha\| < \epsilon_1$, then necessarily $[\sigma, \beta] = 0$.

3. By 1., $d_0\sigma = 0$ and $[\alpha, \sigma] = 0 = [\vec{\bar\partial}_0^{\,\star}\beta, \sigma]$. Since $\partial_0\sigma = 0$ it follows that $\bar\partial_0\sigma^\star = 0$. Using the fact that σ commutes with α, a short calculation gives

$$\operatorname{tr}\left((\sigma^\star\alpha - \alpha\sigma^\star)^\star \wedge (\sigma^\star\alpha - \alpha\sigma^\star) \right) = \operatorname{tr}\left((\sigma^\star\sigma - \sigma\sigma^\star)(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star) \right).$$

After applying $\omega^{n-1} \wedge$ to both sides and integrating over X, the fact that $(\sigma^\star\sigma - \sigma\sigma^\star)$ lies in $\ker \bar\partial_0$ together with the fact that $\Lambda(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star)$ is orthogonal to $\ker \bar\partial_0$ imply that $\left\| [\sigma^\star, \alpha] \right\|_{L^2}^2 = 0$.

4. The vanishing of $d_0\sigma$, $[\alpha, \sigma]$ and $[\beta, \sigma]$ follows from 1. Then from 3. it follows that $[\alpha, \sigma^\star] = 0$, so applying 2. to $\sigma^\star$ gives $[\beta, \sigma^\star] = 0$. $\qquad\square$

Remark.

Given $a'' \in A^{0,1}(\operatorname{End} E_h)$ with $\|a''\|_{L_1^p}$ small, Lemma 2.5 yields a uniquely determined $g \in \mathcal{G}$ near 1 such that $\bar\partial_0^{\,\star}(g \cdot a'') = 0$, where $g \cdot a'' := ga''g^{-1} - \bar\partial_0 g g^{-1}$. Also, Proposition 2.6 yields a uniquely determined $\beta \in A^{0,2}(\operatorname{End} E_h)$ for which $\tau := a'' + \bar\partial_0^{\,\star}\beta$ satisfies $\bar\partial_0\tau + \tau \wedge \tau \in \ker \bar\partial_0^{\,\star}$. The two operations do not commute in general, but if $a_0'' \in A^{0,1}(\operatorname{End} E_h)$ satisfies $\bar\partial_0^{\,\star}a_0'' = 0$ and $\bar\partial_0^{\,\star}(\bar\partial_0 a'' + a_0'' \wedge a_0'') = 0$, then for $\beta \in A^{0,2}(\operatorname{End} E_h)$ and g in $\mathcal{G}$ with both β and $g - 1$ sufficiently close to zero in L_2^p, the element a_1'' of $A^{0,1}(\operatorname{End} E_h)$ obtained by applying the operation of Lemma 2.5 to $g \cdot a_0'' + \bar\partial_0^{\,\star}\beta$ followed by the operation of Proposition 2.6 to the result of that has the form $a_1'' = \gamma a_0'' \gamma^{-1}$ for some automorphism $\gamma \in \Gamma$ near 1, where Γ here and subsequently denotes the $\bar\partial_0$-closed elements of $\mathcal{G}$. All of this follows using the analysis of Theorem 4.7.

The content of Theorem 4.7 implies and is implied by a corresponding result for connections on bundles of degree zero. For the purposes of transparency of the proof, the theorem was presented in terms of endomorphisms, but the alternative result is the following:

Corollary 4.8. *Let d_0 be a connection on E_h with $\widehat{F}(d_0) = 0$, and let $d_0 + a$ be a connection with $a = a' + a''$, $\bar\partial_0^{\,\star}a'' = 0$, $a'' = \alpha + \bar\partial_0^{\,\star}\beta$ for some $\alpha \in A^{0,1}(\operatorname{End} E_h)$ satisfying $\bar\partial_0\alpha = 0 = \bar\partial_0^{\,\star}\alpha$ and some $\beta \in A^{0,2}(\operatorname{End} E_h)$. There is a constant $\epsilon > 0$ depending only on d_0 with the property that if $\|a\|_{L_1^p} < \epsilon$ then the following hold for any section s in $A^{0,0}(E_h)$:*

1. *If $\bar\partial_0 s + a''s = 0$ then $d_0 s = 0$ and $\alpha s = 0 = (\bar\partial_0^{\,\star}\beta)s$. Furthermore $\beta s = 0$ if $\beta \in (\ker \bar\partial_0^{\,\star})^\perp$.*
2. *If $\bar\partial_0 s + \alpha s = 0$ and $\beta \in (\ker \bar\partial_0^{\,\star})^\perp$ and $\bar\partial_0^{\,\star}(\bar\partial_0 a'' + a'' \wedge a'') = 0$ then $d_0 s = 0$ and $\alpha s = 0 = \beta s$;*
3. *If $\bar\partial_0 s + a''s = 0$ and $i\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha) \in (\ker \bar\partial_0)^\perp$ then $d_0 s = 0$ and αs, $(\bar\partial_0^{\,\star}\beta)s$ and $\alpha^\star s$ are all zero. Furthermore, $\beta s = 0$ if $\beta \in (\ker \bar\partial_0^{\,\star})^\perp$.*
4. *If $\bar\partial_0 s + a''s = 0$ and $\bar\partial_0^{\,\star}(\bar\partial_0 a'' + a'' \wedge a'') = 0$ and $i\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha) \in (\ker \bar\partial_0)^\perp$ and $\beta \in (\ker \bar\partial_0^{\,\star})^\perp$ then $d_0 s = 0$ and αs, βs, $\alpha^\star s$ and $\beta^\star s$ are all zero. Hence $d_0 s = 0 = d_a s$.*

Proof. Let **1** be the trivial line bundle equipped with its standard flat connection. Now apply Theorem 4.7 to the endomorphism $\sigma = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}$ of $E_h \oplus \mathbf{1}$ using the direct sum connections on this bundle. $\qquad\square$

Remarks.

1. When the connection $d_0 + a$ of Theorem 4.7 is integrable, the second statement of the theorem implies that if the holomorphic structure E defined by $d_0 + a$ is not simple then the isotropy subgroup of Γ at α has dimension greater than 1; (since Γ is acting by conjugation, the constant multiples of the identity are always in the isotropy subgroup). The converse of this follows from the third statement of that theorem. It follows that if the (ineffectivity) kernel I of the representation of $\Gamma = Aut\, E_0$ on $H^1(X, End\, E_0)$ has dimension greater than

1 then no holomorphic structure defined by an integrable connection near d_0 can be stable. In this case, by writing E_0 as a direct sum of stable factors, it is easily seen that there are non-zero polystable bundles A_0 and B_0 of the same slope with $E_0 = A_0 \oplus B_0$ such that every small deformation of E_0 is given by small deformations of A_0 and B_0; the space of small deformations of E_0 splits as a product. (Of course, it can still be the case that E_0 has no stable deformations even if $I = \mathbb{C}^\star \cdot 1$, for example if A_0 and B_0 are stable and $H^1(X, Hom(A_0, B_0)) = 0 \neq H^1(X, Hom(B_0, A_0))$.)

2. As will be discussed in the next section, the condition that $i\Lambda(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star)$ should be orthogonal to $\ker \bar{\partial}_0$ implies that the form α is an element of $H^{0,1}$ that is polystable with respect to the action of Γ in the sense of geometric invariant theory, and in that context, the last statement of the theorem can also be interpreted as a holomorphic condition on E determined by an algebraic condition on α. This is a manifestation of one half of the "local" Hitchin-Kobayashi correspondence that is at the heart of this paper, made precise in Theorem 5.4 below. The other half, or "converse" of this is the focus of attention in §6, §7 and §8.

The proof of the last statement of Theorem 1 can now be completed:

Corollary 4.9. *For α, $\alpha' \in \Psi^{-1}(0)$, $E_\alpha \simeq E_{\alpha'}$ if and only if α and α' lie in the same orbit under Γ.*

Proof. The "only if" part of the statement has been proved in the last paragraph of the proof of Corollary 4.6. For the converse, given $\gamma \in \Gamma$ such that $\alpha' = \gamma \alpha \gamma^{-1}$, take the connection on $Hom(E_h, E_h)$ induced by $d_0 + \alpha - \alpha^\star$ and $d_0 + \alpha' - \alpha'^\star$ and apply 2. of Corollary 4.8, substituting γ for s (and $Hom(E_h, E_h)$ for E_h). $\square$

5 Γ-stability and ω-stability.

For the convenience of the reader this section commences with a summary of notions and facts from geometric invariant theory as far as these will be used here. The primary references include [29], [31], [20] and [34].

Recall that when a reductive Lie group G acts linearly on a finite-dimensional complex vector space V, a point $v \in V \setminus \{0\}$ is *unstable* for the action if $0 \in \overline{G \cdot v}$, is *semistable* if $0 \notin \overline{G \cdot v}$, is *polystable* if v is semistable and $G \cdot v$ is closed, and is *stable* if v is polystable and the isotropy subgroup of v is finite.

Fixing a positive hermitian form on V, in the closure of each orbit there is a point of smallest norm, unique up to the action of the compact subgroup of G preserving the hermitian form. For each $v \in V$, the derivative at $1 \in G$ of the function $G \ni g \mapsto \|g \cdot v\|^2$ gives a function $\mu : V \to \mathfrak{g}^\star$, the moment map, and its zeros on the set of semistable points are precisely the points of smallest norm in the closed orbits.

The Hilbert-Mumford criterion states that a point $v \in V$ is stable if and only if it is stable for every 1-parameter subgroup in G. A 1-parameter subgroup is given by a homomorphism $\chi : \mathbb{C}^\star \to G$, giving a representation of $\mathbb{C}^\star$ on V. The irreducible representations of $\mathbb{C}^\star$ are all 1-dimensional, so V splits as a direct sum of 1-dimensional subspaces V_j on each of which $\mathbb{C}^\star$ acts with a given weight $w_j \in \mathbb{Z}$: $\mathbb{C}^\star \times V_j \ni (t, v) \mapsto \chi(t) \cdot v = t^{w_j} v$. If v has 1-dimensional orbit under χ, it is clear that this orbit is closed if and only if $\mathbb{C}^\star \ni t \mapsto \chi(t) \cdot v \in V$ is proper, which in turn is equivalent to the condition that the maximum weight $w^\sharp$ and minimum weight $w^\flat$ of χ on the non-zero components of v in its decomposition into irreducibles should differ in sign. Since $t \mapsto \chi(t^{-1})$ is another 1-parameter subgroup of G for which the maximum and minimum such weights are respectively $-w^\flat$ and $-w^\sharp$, the criterion reduces to the condition that v is a stable point for the action of G if and only if $w^\flat$ is negative for every 1-parameter subgroup of G. In practice, this is the condition that $\lim_{t \to 0} \left(\log \|\chi(t) \cdot v\| / \log |t| \right) < 0$ for every 1-parameter subgroup χ of G, which is an analogue of the numerical condition in the definition of stability for a holomorphic vector bundle on a compact Kähler manifold.

Consider now the situation discussed in the previous section: (X, ω) is a compact Kähler n-manifold and E_h is a complex vector bundle over X equipped with a fixed hermitian structure. The group $\mathcal{G}$ of complex automorphisms of E_h acts on the affine space $\mathcal{A}$ of hermitian connections on E_h, preserving the subspace of integrable such connections. The group $\mathcal{G}$ is the complexification of the group $\mathcal{U}$ of unitary gauge transformations.

A connection $d_0 \in \mathcal{A}$ that is integrable and has curvature $F(d_0)$ satisfying $i\widehat{F}(d_0) = \lambda\,1$ is a minimum of the Yang-Mills functional. By Lemma 4.1, the group $\Gamma \subset \mathcal{G}$ of complex gauge transformations fixing $\bar\partial_0$ is the same group as the group that fixes d_0; these are the holomorphic automorphisms of the holomorphic structure E_0 defined by d_0. The group $\Gamma = Aut(E_0)$ acts on the space $H^1(X, End\,E_0)$ of infinitesimal deformations of E_0 by conjugation, and since each element of Γ is covariantly constant with respect to d_0, this action preserves the harmonic subspaces $H^{0,q} = H^{0,q}(\bar\partial_0, End\,E_h)$. From Corollary 4.3 the function Ψ of (2.4) is equivariant with respect to the action of Γ on $H^{0,1}$ (at least, near 0), from which it follows that $\Psi^{-1}(0) \subseteq H^{0,1}$ is invariant under Γ. The linearisation of the action of Γ on $H^{0,q}$ at $1 \in \Gamma$ is given by the Lie bracket, $H^{0,0} \times H^{0,q} \ni (\varphi, \tau) \mapsto [\tau, \varphi]$. The group Γ is the complexification of the subgroup $U(\Gamma)$ of d_0-closed unitary automorphisms of E_h, so is a reductive Lie group.

Lemma 5.1. *A form* $\alpha \in H^{0,1}$ *is of minimal norm in its orbit under* Γ *if and only if* $i\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha) \in A^{0,0}(End\,E_h)$ *is orthogonal to* $\ker\bar\partial_0$.

Proof. Given self-adjoint $\delta \in H^{0,0}$, let $\gamma_t := e^{t\delta}$ for $t \in \mathbb{R}$. Then with $\alpha_t := \gamma_t \alpha \gamma_t^{-1}$, differentiation with respect to t gives $\dot\alpha_t = [\delta, \alpha_t] = e^{t\delta}[\delta, \alpha]e^{-t\delta}$, using the fact that δ commutes with $e^{t\delta}$. Consequently

$$\frac{d}{dt}\|\alpha_t\|^2 = 2\,\mathrm{Re}\,\langle \alpha_t, [\delta, \alpha_t]\rangle = 2\,\mathrm{Re}\,\langle e^{2t\delta}\alpha e^{-2t\delta}, [\delta, \alpha]\rangle$$

so

$$\frac{d^2}{dt^2}\|\alpha_t\|^2 = 4\,\mathrm{Re}\,\langle [\delta, \alpha_t], [\delta, \alpha_t]\rangle \geq 0\,.$$

Thus any critical point of $t \mapsto \|\alpha_t\|^2$ is a minimum, and since $\mathrm{Re}\,\langle \alpha_t, [\delta, \alpha_t]\rangle = -\langle \delta, i\Lambda(\alpha_t \wedge \alpha_t^\star + \alpha_t^\star \wedge \alpha_t)\rangle$ (using self-adjointness of δ), the result follows. $\qquad\square$

The assignment

$$H^{0,1} \ni \alpha \mapsto m(\alpha) := \Pi^{0,0}\,i\,\Lambda\big(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha\big) \in H^{0,0} \tag{5.1}$$

maps elements of the hermitian vector space $H^{0,1}$ into the space of trace-free self-adjoint elements in $H^{0,0}$, the latter being i times the Lie algebra of the group $SU(\Gamma)$ of d_0-closed unitary automorphisms of E_h with unit determinant, which is canonically identified with its real dual. In view of this lemma, it is natural to presume that m is a moment map for the action of $SU(\Gamma)$ on $H^{0,1}$, where the latter is equipped with the symplectic form

$$H^{0,1} \times H^{0,1} \ni (\alpha, \beta) \mapsto \underline\omega(\alpha, \beta) := i\big(\langle \beta, \alpha\rangle - \langle \alpha, \beta\rangle\big) \in \mathbb{R}$$
$$= \int_X \mathrm{tr}\,\Lambda\big(\alpha^\star \wedge \beta - \beta^\star \wedge \alpha\big)\,dV\,,$$

(with the complex structure $J(\alpha) = i\,\alpha$). Using the definition in [31], to prove that this is the case, it must be shown that m is equivariant with respect to the action of $SU(\Gamma)$ on $H^{0,1}$ and that $dm_\delta(\dot\alpha) = i\,\underline\omega(X_\delta, \dot\alpha)$ for each skew-adjoint $\delta \in H^{0,0}$ and each $\dot\alpha \in H^{0,1}$, where X_δ is the vector field on $H^{0,1}$ determined by δ, this having the value $[\delta, \alpha]$ at $\alpha \in H^{0,1}$. (The factor of i in front of $\underline\omega$ comes from (5.1), included to make $m(\alpha)$ self-adjoint.)

Lemma 5.2. *Suppose that* $\int_X \omega^n = 1$. *Then for any* $u \in H^{0,0}$ *and* $v \in A^{0,0}(End\,E_h)$ *in* L^2, $\mathrm{tr}\,(u^\star \Pi^{0,0}v) = \int_X \mathrm{tr}\,(u^\star v)\,dV$.

Proof. If $\varphi, \psi \in H^{0,0}$ are arbitrary, they are both covariantly constant and therefore so too is $\varphi^\star\psi$. Hence the trace of this endomorphism is constant, which implies that $\mathrm{tr}\,(\varphi^\star\psi) = \int_X \mathrm{tr}\,(\varphi^\star\psi)\,dV = \langle \varphi, \psi\rangle$. Consequently, if $e_1, \ldots, e_m$ is an L^2-orthonormal basis for $H^{0,0}$, the endomorphisms $e_1, \ldots, e_m$ are pointwise orthonormal. Writing $u = \sum_j \langle e_j, u\rangle\,e_j$ and $\Pi^{0,0}v = \sum_k \langle e_k, v\rangle\,e_k$, it follows that

$$\mathrm{tr}\,(u^\star \Pi^{0,0}v) = \sum_j \langle u, e_j\rangle\,\langle e_j, v\rangle = \sum_j \langle u, e_j\rangle \int_X \mathrm{tr}\,(e_j^\star v)\,dV$$

$$= \int_X \sum_j \operatorname{tr}(u^\star e_j) \operatorname{tr}(e_j^\star v) \, dV = \int_X \operatorname{tr}(u^\star v) \, dV \, .$$

$\square$

It follows from this lemma that if $v \in A^{0,0}(\operatorname{End} E_h)$ is in L^2 and $u \in SU(\Gamma)$, then $\Pi^{0,0}(uvu^{-1}) = u(\Pi^{0,0}v)u^{-1}$, which implies that the map m of (5.1) is indeed equivariant with respect to the action of $SU(\Gamma)$. Furthermore, thinking of α as depending differentiably on a parameter $t \in \mathbb{R}$ and differentiating at $t = 0$, for $\dot{\alpha}_0 := d\alpha/dt|_{t=0}$ it follows

$$\frac{d}{dt}m(\alpha)\Big|_{t=0} = \Pi^{0,0} i\Lambda\left(\dot{\alpha}_0 \wedge \alpha^\star + \alpha \wedge \dot{\alpha}_0^\star + \dot{\alpha}_0^\star \wedge \alpha + \alpha^\star \wedge \dot{\alpha}_0\right) \, .$$

Hence for $\delta \in H^{0,0}$, using Lemma 5.2 it follows that

$$\frac{d}{dt}\operatorname{tr}\left(m(\alpha)\delta\right)\Big|_{t=0} = \int_X \operatorname{tr}\left(\delta \, i\Lambda(\dot{\alpha} \wedge \alpha^\star + \alpha \wedge \dot{\alpha}^\star + \dot{\alpha}^\star \wedge \alpha + \alpha^\star \wedge \dot{\alpha})\right)$$

$$= \langle [\alpha, \delta^\star], \dot{\alpha}_0 \rangle + \langle \dot{\alpha}_0, [\alpha, \delta] \rangle$$

$$= i \, \underline{\omega}(\dot{\alpha}_0, [\delta, \alpha]) \quad \text{if } \delta = -\delta^\star.$$

Thus:

Corollary 5.3. *The assignment $H^{0,1} \ni \alpha \mapsto m(\alpha) = \Pi^{0,0} i\Lambda\left(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha\right) \in H^{0,0}$ is the moment map for the action of $SU(\Gamma)$ on $H^{0,1}$.*

$\square$

Combining Lemma 5.1 with the results of the previous sections gives some of the main results of this article:

Theorem 5.4. *Let d_0 be an integrable connection on E_h with $i\widehat{F}(d_0) = \lambda \mathbf{1}$, and let Ψ be the holomorphic function (2.4) defined in a neighbourhood of zero in $H^{0,1}$ with values in $H^{0,2}$. Let $\alpha \in \Psi^{-1}(0)$ be a non-zero form, and let E_α be the corresponding holomorphic structure. Then α is polystable with respect to the action of $\Gamma = \ker \overline{\partial}_0 \subset \mathcal{G}$ if E_α is polystable. Moreover, when E_α is polystable, α is stable with respect to the action of Γ if and only if E_α is stable.*

Proof. The proof is by induction on the rank r of E_h, the case $r = 1$ being elementary.

Suppose first that E_α is stable but α is not Γ-polystable. Then the orbit of α under Γ is not closed, and indeed, the infimum of $\|\gamma \cdot \alpha\|^2$ over $\gamma \in \Gamma$ is not attained in that orbit. Let $\alpha_0 \in H^{0,1}$ be a point of smallest norm in the closure of the orbit of α under Γ, unique up to conjugation by unitary elements of Γ. So there is a sequence $\gamma_j \in \Gamma$ with $\det \gamma_j = 1$ for all j such that $\|\gamma_j \cdot \alpha\|^2$ is decreasing to $\|\alpha_0\|^2$ but $\|\gamma_j\|$ is not bounded. Let $\beta \in A^{0,2}(\operatorname{End} E_h)$ be the form orthogonal to $\ker \overline{\partial}_0^\star$ determined by Proposition 2.6 such that $\overline{\partial}_0 + \alpha + \overline{\partial}_0^\star \beta$ is an integrable semi-connection defining the holomorphic structure E_α that is L_1^p-near to E_0, and let $a'' = \alpha + \overline{\partial}_0^\star \beta$. If $\alpha_j := \gamma_j \cdot \alpha$ then $\|\alpha_j\|$ is decreasing. From Proposition 2.6 it follows that $\|\beta_j\|_{L_2^p}$ is uniformly bounded, where $\beta_j := \gamma_j \cdot \beta$. If $a_j'' := \alpha_j + \overline{\partial}_0^\star \beta_j = \gamma_j \cdot a''$, it follows that $\|a_j''\|_{L_1^p}$ is uniformly bounded, so after passing to a subsequence if necessary, the forms a_j'' converge weakly in L_1^p and strongly in C^0 to a limit a_0''. The limiting connection $d_0 + a_0$ is integrable, defining a holomorphic structure E_{α_0} on E_h. After rescaling γ_j to $\tilde{\gamma}_j$ with $\|\tilde{\gamma}_j\| = 1$, a subsequence of the automorphisms $\tilde{\gamma}_j$ converges to a non-zero limit $\tilde{\gamma}_0$ with $\|\tilde{\gamma}_0\| = 1$ and $\det \tilde{\gamma}_0 = 0$, this defining a holomorphic map from the holomorphic structure E_α defined by $d_0 + a$ to the holomorphic structure defined by $d_0 + a_0$. Since the latter can be assumed to be semistable (by Proposition 3.2), there is a non-zero holomorphic map from E_α to a semistable bundle of the same degree and rank that is not an isomorphism, and this contradicts the assumption that E_α is stable. Therefore α is Γ-polystable. If α is polystable but not stable with respect to the action of Γ, the isotropy subgroup $\Gamma_\alpha \subset \Gamma$ of α has dimension greater than one. Then it follows from 2. of Theorem 4.7 that E_α is polystable but not stable, a contradiction.

Suppose now that E_α is polystable but not stable. Then E_α splits into a direct sum of stable subbundles, each of the same slope. In terms of connections, the bundle-with-connection (E_h, d_0+a) has a unitary splitting

into a direct sum of irreducible unitary bundles-with-connection. Orthogonal projection onto any of these subbundles (followed by inclusion) is a holomorphic endomorphism of E_α, and by Theorem 4.7, such an endomorphism is in fact covariantly constant with respect to d_0 and also commutes with α and β. Thus the bundle-with-connection (E_h, d_0) has a unitary splitting into a direct sum of subbundles-with-connection, each of which defines a polystable subbundle of E_0 of the same slope. If $(E_h, d_0) = \bigoplus_j (B_j, d_{0,j})$ is this last splitting, then $i\widehat{F}(d_{0,j}) = \lambda 1$ and for some skew-adjoint $a_j \in A^1(\mathrm{End}\, B_j)$, $(E_h, d_0 + a) = \bigoplus_j (B_j, d_{0,j} + a_j)$ corresponds to the splitting of E_α into stable components. The compatibility of the splittings implies that each a_j is of the form $a_j = a_j' + a_j''$ with $a_j'' = \alpha_j + \overline{\partial}_{0,j}^* \beta_j$ where α_j is a $\overline{\partial}_{0,j}$-harmonic $(0, 1)$-form with coefficients in $\mathrm{End}\, B_j$. By the inductive hypothesis, each α_j is stable with respect to the action of Γ_j, the group of $\overline{\partial}_{0,j}$-closed automorphisms of B_j. Hence there exists such an automorphism ρ_j such that $\rho_j \cdot \alpha_j =: \hat{\alpha}_j$ is a zero of the moment map, which means that $\Pi_j^{0,0} i\Lambda(\hat{\alpha}_j \wedge \hat{\alpha}_j^* + \hat{\alpha}_j^* \wedge \hat{\alpha}_j) = 0$, where $\Pi_j^{0,0}$ is the orthogonal projection onto $\ker \overline{\partial}_{0,j}$. But this implies that there is an automorphism $\sigma \in \Gamma$ such that $\hat{\alpha} := \sigma \cdot \alpha$ satisfies $\Pi^{0,0} i\Lambda(\hat{\alpha} \wedge \hat{\alpha}^* + \hat{\alpha}^* \wedge \hat{\alpha}) = 0$. Therefore α is polystable.

The last statement of the proposition follows immediately from Theorem 4.7. □

Remark.

If $\alpha \in H^{0,1}$ satisfies $\Pi^{0,0} i\Lambda(\alpha \wedge \alpha^* + \alpha^* \wedge \alpha) = 0$ and if $\Gamma_\alpha \subseteq \Gamma$ is the isotropy subgroup of α, then it follows from 2. of Theorem 4.7 that γ commutes with $\beta(t\alpha)$ for t sufficiently small, where $\beta(-)$ is the function defined implicitly in Proposition 2.6. Then by 3. of Theorem 4.7 it follows that γ^* also commutes with α, and indeed, it also commutes with $\beta(t\alpha)$. It then follows from Proposition 1.56 of [18] that Γ_α is itself a complex reductive group.

The group Γ_α acts fibrewise on E_h, splitting each fibre into a direct sum of irreducible Γ_α-invariant subspaces. These subspaces together form subbundles, namely the intersections of the eigen-bundles associated with the elements of Γ_α. The fact that Γ_α is closed under adjoints implies that the splitting of E_h into Γ_α-irreducible subbundles is a *unitary* splitting. For $|t|$ sufficiently small and $|s| \leq 1$, the connections $d_{t,s} = d_0 + a_{t,s}$ given by $a_{t,s}'' = t\alpha + s\overline{\partial}_0^*(t\alpha)$ preserve these splittings, and restrict to irreducible unitary connections on each of the Γ_α-irreducible subbundles.

As mentioned in the second of the Remarks towards the end of §4, Theorem 5.4 is one half of a local version of the Hitchin-Kobayashi correspondence: ω-(poly)stability of E_α implies Γ-(poly)stability of α, where the latter term means polystable with respect to the action of Γ in the sense of geometric invariant theory. The more difficult task is to establish the other half; that is, the converse, and this effectively involves solving differential equations. This will be the subject of the next three sections.

6 Connections with constant central curvature

As in previous sections, let (X, ω) be a compact Kähler n-manifold and let E_h be a fixed complex r-bundle equipped with a fixed hermitian metric; all conventions and notations from previous sections are also retained. In this section, the study of §2 into an L_1^p neighbourhood of a given (hermitian) connection will be continued but the focus is now is on the central component $\widehat{F} = \Lambda F$ of the curvature F rather than the $(0, 2)$-component. As previously, $\mathcal{G}$ is the group of complex automorphisms of E_h, with $\mathcal{U} \subset \mathcal{G}$ the subgroup preserving the given hermitian metric. Let $\mathcal{A}$ denote the space of hermitian connections $d = \partial + \overline{\partial}$ on E_h, so the action of $\mathcal{G}$ on $\mathcal{A}$ is given by

$$\mathcal{G} \times \mathcal{A} \ni (g, d) \mapsto g \cdot d := g^{*-1} \circ \partial \circ g^* + g \circ \overline{\partial} \circ g^{-1} = d + g^{*-1}\partial g^* - \overline{\partial}gg^{-1} \in \mathcal{A}. \tag{6.1}$$

Unless otherwise stated, elements of $\mathcal{G}$ are assumed to lie in L_2^p and elements of $\mathcal{A}$ to lie in L_1^p. Projection to the central component of the curvature defines a function Φ on $\mathcal{G} \times \mathcal{A}$ with values in the space of self-adjoint endomorphisms of E_h lying in L^p given by

$$\Phi: \mathcal{G} \times \mathcal{A} \to A^{0,0}(\mathrm{End}\, E_h), \quad \Phi(g, d) := i\Lambda F(g \cdot d), \tag{6.2}$$

and it is the properties of this function and its derivatives with respect to each of its arguments on which the analysis concentrates in this section.

Let $d_0 = \partial_0 + \bar{\partial}_0$ be a connection on E_h, and let $a = a' + a''$ be an element of $A^1(\text{End } E_h)$ with $a' = -(a'')^*$. If $d_a = d_0 + a$, the curvature of this connection is

$$F(d_a) = F(d_0) + d_0 a + a \wedge a \ . \tag{6.3}$$

It follows that if $a = a(t)$ depends differentiably on the real parameter t, then

$$\frac{d}{dt}\big[F(d_a)\big] = d_0 \dot{a} + \dot{a} \wedge a + a \wedge \dot{a} = d_a \dot{a} \ , \tag{6.4}$$

where $\cdot$ denotes differentiation with respect to t.

The action of $\mathcal{G}$ on $\mathcal{A}$ has the explicit form

$$g \cdot d_a = d_0 + \big(g^{*-1} a' g^* + g^{*-1} \partial_0 g^*\big) + \big(ga'' g^{-1} - \bar{\partial}_0 g g^{-1}\big) \ , \quad g \in \mathcal{G} \ . \tag{6.5}$$

If $g = g(t)$ also depends differentiably on t, then by direct calculation from (6.5), it follows that

$$\frac{d}{dt}\big[g \cdot d_a\big] = \big(g^{*-1} \dot{a}' g^* + \partial_{g \cdot d_a}(g^{*-1} \dot{g}^*)\big) + \big(g\dot{a}'' g^{-1} - \bar{\partial}_{g \cdot d_a}(\dot{g} g^{-1})\big) \ . \tag{6.6}$$

Hence, from (6.4) and (6.6), it follows that

$$\frac{d}{dt}\big[F(g \cdot d_a)\big] = d_{g \cdot d_a}\big(g^{*-1} \dot{a}' g^* + g\dot{a}'' g^{-1} + \partial_{g \cdot d_a}(g^{*-1} \dot{g}^*) - \bar{\partial}_{g \cdot d_a}(\dot{g} g^{-1})\big) \ . \tag{6.7}$$

Applying $i\Lambda$ to both sides and recalling that $\partial^* = i\Lambda \bar{\partial}$ and $\bar{\partial}^* = -i\Lambda\partial$ on 1-forms,

$$\frac{d}{dt}\big[i\widehat{F}(g \cdot d_a)\big] = i\Lambda \bar{\partial}_{g \cdot d_a}(g^{*-1} \dot{a}' g^*) + i\Lambda \partial_{g \cdot d_a}(g\dot{a}'' g^{-1}) + i\Lambda \bar{\partial}_{g \cdot d_a} \partial_{g \cdot d_a}(g^{*-1} \dot{g}^*) - i\Lambda \partial_{g \cdot d_a} \bar{\partial}_{g \cdot d_a}(\dot{g} g^{-1})$$

$$= \partial^*_{g \cdot d_a}(g^{*-1} \dot{a}' g^*) - \bar{\partial}^*_{g \cdot d_a}(g\dot{a}'' g^{-1}) + \partial^*_{g \cdot d_a} \partial_{g \cdot d_a}(g^{*-1} \dot{g}^*) + \bar{\partial}^*_{g \cdot d_a} \bar{\partial}_{g \cdot d_a}(\dot{g} g^{-1}) \ . \tag{6.8}$$

Since $\Lambda(\bar{\partial}\partial + \partial\bar{\partial}) = \widehat{F}$, writing $\sigma := \dot{g} g^{-1}$ and decomposing into self-adjoint and skew-adjoint components gives the following conclusion, which will be used frequently in this section:

Lemma 6.1. *Let* $a = a(t) \in A^1(\text{End } E_h)$ *be a differentiable* 1-*real parameter family of skew-adjoint forms, and let* $g = g(t)$ *be a differentiable* 1-*real parameter family of complex automorphisms of* E_h. *If* $d_0 = \partial_0 + \bar{\partial}_0$ *is a connection on* E_h *and* $d_a = d_0 + a$, *then*

$$\frac{d}{dt}\big[i\widehat{F}(g \cdot d_a)\big] = \partial^*_{g \cdot d_a}(g^{*-1} \dot{a}' g^*) - \bar{\partial}^*_{g \cdot d_a}(g\dot{a}'' g^{-1}) + \triangle_{g \cdot d_a} \sigma_+ - [i\widehat{F}(d_{g \cdot d_a}), \sigma_-] \ , \tag{6.9}$$

where $\sigma := \dot{g} g^{-1}$ *and* $\sigma_\pm := \frac{1}{2}(\sigma \pm \sigma^*)$. $\qquad\qquad \square$

Taking $a \in A^1(\text{End } E_h)$ to be independent of t, it follows from (6.9) that the linearisation of the map $\mathcal{G} \ni g \mapsto i\widehat{F}(g \cdot d_a) = \Phi(g, d_a)$ at a connection $g_0 \cdot d$ is

$$(D_1 \Phi)_{(g_0, d_a)}(\sigma) = \triangle_{g_0 \cdot d_a} \sigma_+ - [i\widehat{F}(g_0 \cdot d_a), \sigma_-], \quad \sigma \in A^{0,0}(\text{End } E_h) \ .$$

In particular, if $g_0 = 1$ and $a = 0$, the linearisation at $(1, d_0) \in \mathcal{G} \times \mathcal{A}$ of the action of $\mathcal{G}$ on the space of L_1^p (hermitian) connections is an isomorphism from the space of L_2^p self-adjoint sections of $\text{End } E_h$ that are orthogonal to the d_0-closed sections to the space of self-adjoint L^p sections of $\text{End } E_h$ that are again orthogonal to the kernel of d_0.

From now on, let d_0 be a connection with $i\widehat{F}(d_0) = \lambda 1$, so by Corollary 4.2, $\ker d_0 = \ker \bar{\partial}_0$. In general, the map $\mathcal{G} \times A^1(\text{End } E_h) \ni (g, a) \mapsto \Phi(g, d_a) = i\widehat{F}(g \cdot d_a)$ takes values in the self-adjoint endomorphisms of E_h, but it does not map into the space orthogonal to $\ker \bar{\partial}_0$. However, if $\Pi^{0,0}$ is the L^2 projection of $A^{0,0}(\text{End } E_h)$

onto $\ker \bar{\partial}_0$ and $\Pi_{\perp}^{0,0} = 1 - \Pi^{0,0}$ is the projection onto the orthogonal complement, then for any skew-adjoint L_1^p section $a \in A^1(\text{End}\, E_h)$, the composition

$$A^{0,0}(\text{End}\, E_h) \ni \varphi \mapsto \Pi_{\perp}^{0,0}\, i\,\widehat{F}\big(\exp(\varphi)\cdot(d_0 + a)\big) \in A^0(\text{End}\, E_h)$$

maps the Banach space of self-adjoint L_2^p sections in $A^{0,0}(\text{End}\, E_h)$ orthogonal to $\ker \bar{\partial}_0$ into the Banach space of such sections lying in L^p, and when $a = 0$, this map has the same linearisation at 0 as the earlier map. The implicit function theorem for Banach spaces then yields:

Proposition 6.2. *There exists $\epsilon > 0$ depending only on d_0 with the property that for each skew-adjoint section $a \in A^1(\text{End}\, E_h)$ satisfying $\|a\|_{L_1^p} < \epsilon$ there is a unique self-adjoint $\varphi \in (\ker \bar{\partial}_0)^{\perp} \subset A^{0,0}(\text{End}\, E_h)$ for which $\Pi_{\perp}^{0,0}\, i\widehat{F}\big(\exp(\varphi)\cdot(d_0+a)\big) = 0$. Furthermore, there is a constant C depending only on d_0 such that φ satisfies $\|\varphi\|_{L_2^p} \le C \left\| \Pi_{\perp}^{0,0}\, \Lambda(d_0 a + a \wedge a) \right\|_{L^p}$.* $\square$

The power series expansion in φ at $\varphi = 0$ of $i\,\widehat{F}(\exp(\varphi)\cdot d_a)$ is, to first order,

$$\begin{aligned}
i\widehat{F}\big(\exp(\varphi)\cdot(d_0+a)\big) &= i\widehat{F}(d_0) + i\Lambda(d_0 a + a \wedge a) - [i\widehat{F}(d_0) + i\Lambda(d_0 a + a \wedge a), \varphi_-] \\
&\quad + \triangle_0 \varphi_+ + 2i\Lambda\big(a'' \wedge \partial_0\varphi_+ + \partial_0\varphi_+ \wedge a'' - a' \wedge \bar{\partial}_0\varphi_- - \bar{\partial}_0\varphi_+ \wedge a'\big) + [d_0^\star a, \varphi_+] \\
&\quad + i\Lambda\big([a', \varphi_+] \wedge a'' + a'' \wedge [a', \varphi_+] - a' \wedge [a'', \varphi_+] - [a'', \varphi_+] \wedge a'\big) + R_2(\varphi), \quad (6.10)
\end{aligned}$$

where the term $R_2(\varphi)$ involves products of φ and its first and second derivatives with respect to d_a with at least two such factors, but where the second-order derivatives appear linearly and the first-order derivatives appear at most quadratically. Consequently, if a satisfies the hypotheses of Proposition 6.2 and if $\varphi \in A^{0,0}(\text{End}\, E_h)$ satisfies the inequality in the statement of that proposition, then $\|R_2(\varphi)\|_{L^p} \le C \|\Lambda(d_0 a + a \wedge a)\|_{L^p}^2$ for some constant $C = C(d_0)$.

If φ is self-adjoint and if $\bar{\partial}_0^\star a'' = 0$, the formula (6.10) simplifies somewhat. In this case, it is useful to project both sides into $\ker \bar{\partial}_0$ and $(\ker \bar{\partial}_0)^{\perp}$, respectively giving

$$\begin{aligned}
\Pi^{0,0} i\widehat{F}\big(\exp(\varphi)\cdot d_a\big) &= \lambda\, 1 + i\,\Pi^{0,0}\, \Lambda\big(a' \wedge a'' + a'' \wedge a'\big) &(a) \quad (6.11) \\
&\quad + i\,\Pi^{0,0}\Lambda\big([a', \varphi] \wedge a'' + a'' \wedge [a', \varphi] - [a'', \varphi] \wedge a' - a' \wedge [a'', \varphi]\big) + \Pi^{0,0}R_2(\varphi),
\end{aligned}$$

$$\begin{aligned}
\Pi_{\perp}^{0,0} i\widehat{F}\big(\exp(\varphi)\cdot d_a\big) &= \Pi_{\perp}^{0,0}\, i\,\Lambda\big(a' \wedge a'' + a'' \wedge a'\big) &(b) \\
&\quad + \triangle_0 \varphi + 2i\Lambda\big(a'' \wedge \partial_0\varphi + \partial_0\varphi \wedge a'' - a' \wedge \bar{\partial}_0\varphi - \bar{\partial}_0\varphi \wedge a'\big) \\
&\quad + \Pi_{\perp}^{0,0} i\Lambda\big([a', \varphi] \wedge a'' + a'' \wedge [a', \varphi] - [a'', \varphi] \wedge a' - a' \wedge [a'', \varphi]\big) + \Pi_{\perp}^{0,0} R_2(\varphi).
\end{aligned}$$

If $\varphi = \varphi(a)$ is the endomorphism of Proposition 6.2, the left-hand side of (6.11)(b) vanishes, giving the equation that effectively determines $\varphi(a)$. Since $\Lambda d_0 a = 0$ now, the uniform estimate on φ provided by that proposition then implies $\|\varphi(a)\|_{L_2^p} \le C\|a\|_{L^{2p}}^2$, so the remainder term $R_2(\varphi)$ appearing in (6.11) is uniformly bounded by a constant multiple of $\|a\|_{L^{2p}}^4$, something that is also true of the other terms on the second line of (6.11)(a). If $a'' = \alpha + \bar{\partial}_0^\star\beta$ for $\bar{\partial}_0$-harmonic $\alpha \in A^{0,1}(\text{End}\, E_h)$ and $\beta \in (\ker \bar{\partial}_0)^{\perp} \subseteq A^{0,2}(\text{End}\, E_h)$, the (negative of the) terms involving $a' \wedge a'' + a'' \wedge a'$ in (6.11) expand to

$$\begin{aligned}
i\,\Lambda(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star) + i\,\Lambda(\partial_0^\star\beta^\star \wedge \bar{\partial}_0^\star\beta + \bar{\partial}_0^\star\beta \wedge d_0^\star\beta^\star) \\
+ i\,\Lambda(\alpha^\star \wedge \bar{\partial}_0^\star\beta + \partial_0^\star\beta^\star \wedge \alpha + \alpha \wedge \partial_0^\star\beta^\star + \bar{\partial}_0^\star\beta \wedge \alpha^\star). \quad (6.12)
\end{aligned}$$

If $\sigma \in A^{0,0}(\text{End}\, E_h)$ is $\bar{\partial}_0$-closed, then $\partial_0\sigma = 0$ by Corollary 4.2, and

$$\big\langle \sigma, i\,\Lambda(\alpha^\star \wedge \bar{\partial}_0^\star\beta + \partial_0^\star\beta^\star \wedge \alpha + \alpha \wedge \partial_0^\star\beta^\star + \bar{\partial}_0^\star\beta \wedge \alpha^\star)\big\rangle = \big\langle [\alpha, \sigma], \bar{\partial}_0^\star\beta\big\rangle + \big\langle [\bar{\partial}_0^\star\beta, \sigma]), \alpha\big\rangle. \quad (6.13)$$

The first term on the right vanishes because $\bar{\partial}_0[\alpha, \sigma] = 0$, and the second term vanishes because $[\bar{\partial}_0^\star\beta, \sigma] = \bar{\partial}_0^\star[\beta, \sigma]$; consequently the projection of the term in the second line of (6.12) on $\ker \bar{\partial}_0$ is zero. To summarise the discussion so far:

Lemma 6.3. *Suppose that $a = a' + a''$ satisfies the hypotheses of* Proposition 6.2 *and $a'' = \alpha + \bar{\partial}_0^\star \beta$ where α is $\bar{\partial}_0$-harmonic. If $\varphi = \varphi(a)$ is the endomorphism of that proposition, then*

$$i\widehat{F}\big(\exp(\varphi)\cdot(d_0 + a)\big) - \lambda\,1 = -i\,\Pi^{0,0}\,\Lambda(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star) - i\,\Pi^{0,0}\,\Lambda(\partial_0^\star \beta^\star \wedge \bar{\partial}_0^\star \beta + \bar{\partial}_0^\star \beta \wedge \partial_0^\star \beta^\star)$$
$$+ i\,\Pi^{0,0}\Lambda\big([a',\varphi]\wedge a'' + a'' \wedge [a',\varphi] - [a'',\varphi]\wedge a' - a' \wedge [a'',\varphi]\big) + R(a) \tag{6.14}$$

where $\|R(a)\|_{L^p} \leq C\|a\|_{L_1^{2p}}^4$ for some constant $C = C(d_0)$. $\qquad\square$

The term $\Pi^{0,0}\Lambda(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star)$ is $O(\|\alpha\|^2)$ in general, whereas if β is as in Proposition 2.6, all the other terms on the right of (6.14) are $O(\|\alpha\|^4)$. But if $\Pi^{0,0}\Lambda(\alpha^\star \wedge \alpha + \alpha \wedge \alpha^\star)$ vanishes (as considered in §5) then the whole of the right-hand side of (6.14) is $O(\|\alpha\|^4)$ and the connection $\exp(\varphi(a)) \cdot (d_0 + a)$ is very close to having central curvature equal to $-i\,\lambda\,1$. Given that $\varphi(a)$ is orthogonal to $\ker\bar{\partial}_0$, it can be hoped that a small perturbation by an element of $\ker\bar{\partial}_0$ will yield a connection with $i\widehat{F} \equiv \lambda\,1$. For $\gamma \in \Gamma$,

$$\gamma \cdot (d_0 + a) = d_0 + \gamma \cdot a := d_0 + \gamma^{\star-1}a'\gamma^\star + \gamma a''\gamma^{-1},$$

so since $\exp(\varphi + \delta)$ is close to $\exp(\varphi)\exp(\delta)$ for small $\varphi \in (\ker\bar{\partial}_0)^\perp$ and small $\delta \in \ker\bar{\partial}_0$, an alternative is to perturb α by conjugation with an element of Γ close to 1. Such an argument will involve an application of the inverse function theorem in finite dimensions, for which purpose the variation in $\widehat{F}$ as α is varied in this way must be determined.

With 'sk' denoting skew-adjoint and 'sa' denoting self-adjoint, consider first the function G defined on $A^{0,0}(\text{End }E_h) \times A^1_{sk}(\text{End }E_h)$ with values in $A^0_{sa}(\text{End }E_h)$ defined by

$$G(\psi, a) := i\widehat{F}(\exp(\psi) \cdot (d_0 + a)) \qquad (= \Phi(\exp(\psi), d_0 + a) \text{ for } \Phi \text{ as in (6.2)}).$$

If $G_0 := \Pi^{0,0}G$ and $G_1 := \Pi^{0,0}_\perp G$, the conclusion of Proposition 6.2 is that for $a \in A^1_{sk}(\text{End }E_h)$ with $\|a\|_{L_1^p} < \epsilon$, $G_1(\varphi(a), a) \equiv 0$, where $a \mapsto \varphi(a)$ is the function specified in that proposition, the existence of which is guaranteed by the implicit function theorem.

If a moves in a differentiable 1-parameter family $a(t)$, then it follows that

$$0 \equiv \frac{d}{dt}\Big[G_1(\varphi(a(t)), a(t))\Big] = \big((D_1 G_1)_{(\varphi(a),a)} \circ (D\varphi)_a\big)(\dot{a}) + (D_2 G_1)_{(\varphi(a),a)}(\dot{a}),$$

where $D_1 G_1, D_2 G_1$ are respectively the partial derivatives of G_1 with respect to its first and second arguments and $\dot{a}$ denotes the derivative with respect to t as before. The implicit function theorem implies that if (ψ, a) is sufficiently close to $(0, 0)$, then $(D_1 G_1)_{(\psi,a)}$ is an isomorphism from the space of self-adjoint elements of $A^{0,0}(\text{End }E_h)$ orthogonal to $\ker\bar{\partial}_0$ lying in L_2^p to the space of self-adjoint elements of $A^{0,0}(\text{End }E_h)$ orthogonal to $\ker\bar{\partial}_0$ lying in L^p, so

$$(D\varphi)_a(\dot{a}) = -\big((D_1 G_1)^{-1}_{(\varphi(a),a)} \circ (D_2 G_1)_{(\varphi(a),a)}\big)(\dot{a}).$$

The variation in $G(\varphi(a), a)$ at a is therefore given by

$$\frac{d}{dt}\Big[G(\varphi(a), a)\Big] = \big((D_1 G)_{(\varphi(a),a)} \circ (D\varphi)_a\big)(\dot{a}) + (D_2 G)_{(\varphi(a),a)}(\dot{a}) \tag{6.15}$$
$$= -\big((D_1 G_0)_{(\varphi(a),a)} \circ (D_1 G_1)^{-1}_{(\varphi(a),a)} \circ (D_2 G_1)_{(\varphi(a),a)}\big)(\dot{a}) + (D_2 G_0)_{(\varphi(a),a)}(\dot{a}).$$

The partial derivatives appearing here are determined by the formula (6.9): setting $d_b := \exp(\varphi(a)) \cdot (d_0 + a)$ and substituting $g = e^{\varphi(a)}$ into (6.9) gives

$$(D_2 G)_{(\varphi(a),a)}(\dot{a}) = \partial_b^\star(e^{-\varphi(a)}\dot{a}' e^{\varphi(a)}) - \bar{\partial}_b^\star(e^{\varphi(a)}\dot{a}'' e^{-\varphi(a)}) \tag{6.16}$$
$$= e^{\varphi(a)}\partial_a^\star\big(e^{-2\varphi(a)}\dot{a}' e^{2\varphi(a)}\big)e^{-\varphi(a)} - e^{-\varphi(a)}\bar{\partial}_a^\star\big(e^{2\varphi(a)}\dot{a}'' e^{-2\varphi(a)}\big)e^{\varphi(a)},$$

where $\dot{a} = \dot{a}' + \dot{a}''$, with the derivatives $D_2 G_0$ and $D_2 G_1$ obtained by projecting into $\ker\bar{\partial}_0$ and its orthogonal complement respectively. Similarly, the partial derivative of G with respect to its first variable ψ at $(\varphi(a), a)$ is

obtained from (6.9) by substituting $\dot{a} = 0$ and $\sigma = (de^\psi/dt)\,e^{-\psi}\big|_{\psi=\varphi(a)}$ for a 1-parameter family $\psi(t)$ into that formula, so

$$(D_1 G)_{(\varphi,a)}(\sigma) = \bar{\partial}_b^\star \bar{\partial}_b \sigma + \partial_b^\star \partial_b \sigma^\star = \triangle_b \,\sigma_+ - [i\widehat{F}(d_b),\,\sigma_-]\,, \tag{6.17}$$

again with the derivatives of G_0 and G_1 obtained by taking L^2 projection into $\ker \bar{\partial}_0$ and its orthogonal complement. Note that σ is not self-adjoint in general, but satisfies $\sigma^\star = e^{-\varphi(a)}\sigma e^{\varphi(a)}$. Thus the second term on the right of (6.17) is not zero in general, unless $i\widehat{F}(d_b) = \lambda\,1$.

Lemma 6.4. *Under the hypotheses of* Proposition 6.2, *suppose in addition that* $\bar{\partial}_0^\star a'' = 0$. *Then there is a constant* $C = C(d_0)$ *such that, for skew-adjoint* $\dot{a} \in A^1(\mathrm{End}\,E_\mathrm{h})$,

$$\left\|\left((D_1 G_0)_{(\varphi(a),a)} \circ (D_1 G_1)^{-1}_{(\varphi(a),a)} \circ (D_2 G_1)_{(\varphi(a),a)}\right)(\dot{a})\right\|_{L^2} \le C\,\|a\|_{L_1^p}^3\left(\|\bar{\partial}_0^\star \dot{a}\|_{L^p} + \|\dot{a}\|_{L^p}\right)\,. \tag{6.18}$$

Proof. The assumption that $\bar{\partial}_0^\star a'' = 0$ implies that $\Lambda d_0 a = 0$, so the bound of Proposition 6.2 implies that $\|\varphi\|_{L_2^p} \le C\|a\|_{L^{2p}}^2$ for some uniform constant C. By the Sobolev embedding theorem, there is a similar such bound on the C^1 norm of φ, so (given that $\|a\|_{L_1^p}$ is sufficiently small), an arbitrary endomorphism $\psi \in A^{0,0}(\mathrm{End}\,E_\mathrm{h})$ will satisfy a pointwise bound of the form $|\psi - e^\varphi \psi e^{-\varphi}| \le C\|a\|_{L^{2p}}^2\,|\psi|$, which can be seen by orthogonally diagonalising φ at the point in question. Since $(\bar{\partial}_0 e^\varphi)e^{-\varphi}$ is uniformly bounded in C^0 by a multiple of $\|a\|_{L^{2p}}^2$, it follows that $b'' = e^\varphi a'' e^{-\varphi} - (\bar{\partial}_0 e^\varphi)e^{-\varphi}$ also satisfies a pointwise bound of the form $|b'' - a''| \le C\|a\|_{L^{2p}}^2$, and therefore $(\bar{\partial}_a e^\varphi)e^{-\varphi} = a'' - b''$ satisfies this bound; similarly, $e^{-\varphi}\bar{\partial}_a e^\varphi$ also satisfies such a bound. Consequently, for any $\chi \in A^{0,1}(\mathrm{End}\,E_\mathrm{h})$, there is a uniform pointwise bound of the form

$$\left|\,e^{-\varphi}\bar{\partial}_a^\star\left(e^{2\varphi}\chi e^{-2\varphi}\right)e^\varphi - \bar{\partial}_a^\star \chi\,\right| \le C\|a\|_{L^{2p}}^2\left(|\bar{\partial}_a^\star \chi| + |\chi|\right)\,,$$

and since $\bar{\partial}_a^\star \chi = \bar{\partial}_0^\star \chi - i\Lambda(a' \wedge \chi + \chi \wedge a')$,

$$\left|\,e^{-\varphi}\bar{\partial}_a^\star\left(e^{2\varphi}\chi e^{-2\varphi}\right)e^\varphi - \bar{\partial}_a^\star \chi\,\right| \le C\|a\|_{L_1^p}^2\left(|\bar{\partial}_0^\star \chi| + |\chi|\right) \tag{6.19}$$

for some new uniform constant C. Taking $\chi = \dot{a}$ in (6.16), it follows that

$$\left|(D_2 G)_{(\varphi,a)}(\dot{a})\right| \le C\|a\|_{L_1^p}^2\left(|\bar{\partial}_0^\star \dot{a}| + |\dot{a}|\right)\quad \text{pointwise.} \tag{6.20}$$

It follows from this that the L^2 norm of $(D_2 G)_{(\varphi,a)}(\dot{a})$ is uniformly bounded above by a constant multiple of $\|a\|_{L_1^p}^2\left(\|\bar{\partial}_0^\star \dot{a}\|_{L^2} + \|\dot{a}\|_{L^2}\right)$, which implies the same such bound for its orthogonal projection onto $\ker \bar{\partial}_0$. Since $\ker \bar{\partial}_0$ is finite dimensional, the L^2 norm on this space is equivalent to any other, so it follows from (6.20) that there is uniform bound of the form

$$\left\|(D_2 G_1)_{(\varphi,a)}(\dot{a})\right\|_{L^p} \le C\|a\|_{L_1^p}^2\left(\|\bar{\partial}_0^\star \dot{a}\|_{L^p} + \|\dot{a}\|_{L^p}\right)\,. \tag{6.21}$$

From the proof of Proposition 6.2 using the implicit function theorem, the operator $(D_1 G_1)_{(\varphi,a)}$ is an isomorphism from the space of self-adjoint L_2^p sections of $A^{0,0}(\mathrm{End}\,E_\mathrm{h})$ orthogonal to $\ker \bar{\partial}_0$ to the same such space of L^p sections, so

$$\left\|(D_1 G_1)^{-1}_{(\varphi,a)}\left((D_2 G_1)_{(\varphi,a)}\right)(\dot{a})\right\|_{L_2^p} \le C\left\|(D_2 G_1)_{(\varphi,a)}(\dot{a})\right\|_{L^p} \tag{6.22}$$

for some new constant $C = C(d_0)$.

For a section $\sigma \in A^{0,0}(\mathrm{End}\,E_\mathrm{h})$, (6.17) gives

$$(D_1 G_0)_{(\varphi,a)}(\sigma) = \Pi^{0,0}\left(\bar{\partial}_b^\star \bar{\partial}_b \sigma + \partial_b^\star \partial_b \sigma^\star\right)\,,$$

where d_b is the connection $d_b = e^{\varphi(a)} \cdot d_a$. Since

$$\bar{\partial}_b^\star \bar{\partial}_b \sigma = \bar{\partial}_0^\star \bar{\partial}_b \sigma - i\Lambda(b' \wedge \bar{\partial}_b \sigma + \bar{\partial}_b \sigma \wedge b')$$

for which the first term is annihilated by $\Pi^{0,0}$, estimation of the second term gives

$$\left\|\Pi^{0,0}\bar{\partial}_b^\star \bar{\partial}_b \sigma\right\|_{L^2} \le Const.\left\|\Pi^{0,0}\bar{\partial}_b^\star \bar{\partial}_b \sigma\right\|_{L^1} \le C\left(\|b\|_{L^2}\|\sigma\|_{L_1^2} + \|b\|_{L^4}^2\|\sigma\|_{L^2}\right)\,.$$

The a priori L_1^p bound on a and the estimate on $\varphi(a)$ from Proposition 6.2 implies that $\|b\|_{L^{2p}}$ is uniformly bounded above by a multiple of $\|a\|_{L^{2p}}$, so $\|\Pi^{0,0}\,\bar{\partial}_b^\star\,\bar{\partial}_b\sigma\|_{L^2} \leq C\|a\|_{L^{2p}}\|\sigma\|_{L_1^2}$. Taking adjoints, this same estimate applies to $\Pi^{0,0}\partial_b^\star\partial_b\sigma^\star$ to give

$$\left\|(D_1 G_0)_{(\varphi(a),a)}(\sigma)\right\|_{L^2} \leq C\|a\|_{L^{2p}}\|\sigma\|_{L_1^2} \,.$$

Combining this with the estimates of (6.22) and (6.20) gives (6.18).   □

In accordance with the strategy outlined following the statement of Lemma 6.3, Lemma 6.4 provides sufficient information to analyse the variation of $i\widehat{F}(e^{\varphi(a)}\cdot d_a)$ as $a'' = \alpha + \bar{\partial}_0^\star\beta(a)$ is varied according to $H^{0,1} \ni \alpha \mapsto \gamma\alpha\gamma^{-1}$ for $\gamma \in \Gamma$, at least when the connections d_a are sufficiently near to d_0.

7 Γ-polystable locally implies ω-polystable

Retaining all of the notation and definitions of the previous section, suppose now that $\gamma_t \in \Gamma$ is a family depending differentiably on the real variable t, with $\gamma_0 = 1$ and $\dot{\gamma}_t|_{t=0} = \delta \in H^{0,0}$. Suppose that $a = a' + a''$ satisfies the hypotheses of Proposition 6.2 as well as $\bar{\partial}_0^\star a'' = 0$, and let $a_t := \gamma_t \cdot a = \gamma_t a'' \gamma_t^{-1} + \gamma_t^{\star-1} a' \gamma_t^\star$. Then at $t = 0$, $\dot{a}_0'' := \dot{a}_t''|_{t=0} = -[a'', \delta]$, so $\bar{\partial}_0^\star \dot{a}_0'' = 0$ and Lemma 6.4 gives an estimate of the contribution of the first term on the right of (6.15) to the variation in $i\widehat{F}$ in terms of $\|[a'', \delta]\|_{L^p}$. But since δ is d_0-closed, its C^0 norm is bounded by a uniform multiple of its L^2 norm so $\|[a'', \delta]\|_{L^p} \leq C\|a\|_{L^p}\|\delta\|$ for $C = C(d_0)$, and therefore that contribution is uniformly bounded above by a constant multiple of $\|a\|_{L_1^p}^4\|\delta\|$, (the fourth power coming from (6.18)). It follows that if $\|a\|_{L_1^p}$ is sufficiently small, the dominant term in the variation (6.15) of $i\widehat{F}$ is that coming from $(D_2 G_0)_{(\varphi_0, a)}([a'', \delta]_+)$, given by the projection of (6.16) onto $\ker \bar{\partial}_0$, provided that this is appropriately non-degenerate as a function of δ.

As in the proof of Lemma 6.4, (6.19) gives a bound

$$\left\| e^{-\varphi}\,\bar{\partial}_a^\star\big(e^{2\varphi}[a'',\delta]e^{-2\varphi}\big)e^\varphi - \bar{\partial}_a^\star[a'',\delta] \right\|_{L^2} \leq C\,\|a\|_{L_1^p}^2\,\|[a'',\delta]\| \,.$$

Noting that $\dot{a}'' = -[a'', \delta]$ and $\bar{\partial}_a\delta = [a'', \delta]$, it follows that

$$\begin{aligned}
\big\langle \delta, e^{-\varphi}\,\bar{\partial}_a^\star\big(e^{2\varphi}[a'',\delta]e^{-2\varphi}\big)e^\varphi \big\rangle &\geq \big\langle \delta, \bar{\partial}_a^\star[a'',\delta]\big\rangle - C\|a\|_{L_1^p}^2\,\|[a'',\delta]\|\,\|\delta\| \\
&= \big\|[a'',\delta]\big\|^2 - C\|a\|_{L_1^p}^2\,\|[a'',\delta]\|\,\|\delta\| \\
&\geq \frac{1}{2}\big\|[a'',\delta]\big\|^2 - C'\|a\|_{L_1^p}^4\,\|\delta\|^2 \,.
\end{aligned}$$

By taking adjoints and using $\dot{a}' = +[a', \delta^\star]$, the same estimates apply to the other term in (6.16) with δ replaced by $\delta^\star$. Then if δ is taken to be self-adjoint, by combining these estimates with those of Lemma 6.4 the following conclusion is reached:

Proposition 7.1. *Under the hypotheses of* Proposition 6.2*, assume in addition that* $\bar{\partial}_0^\star a'' = 0$*. Let* $\widehat{\Phi}$ *be the function defined in a neighbourhood of* $1 \in \Gamma$ *with values in* $H^{0,0}$ *given by* $\widehat{\Phi}(\gamma) := i\widehat{F}(\varphi(\gamma \cdot a) \cdot \gamma \cdot (d_0 + a))$*, where* $\varphi(-)$ *is the function of* Proposition 6.2*. Then there is a constant* $C = C(d_0) > 0$ *such that*

$$\big\|[a'',\delta]\big\|^2 \leq 2\big\langle \delta, (D\widehat{\Phi})(\delta)\big\rangle + C\|a\|_{L_1^p}^4\|\delta\|^2$$

for any self-adjoint $\delta \in H^{0,0}$.   □

With a'' and δ as in this proposition, write $a'' = \alpha + \bar{\partial}_0^\star\beta$ where $\alpha \in H^{0,1}$ and $\beta \in A^{0,2}(\mathrm{End}\,E_h)$. Given that $d_0\delta = 0$,

$$[a'', \delta] = [\alpha, \delta] + [\bar{\partial}_0^\star\beta, \delta] = [\alpha, \delta] + \bar{\partial}_0^\star[\beta, \delta] \,,$$

and since this is an orthogonal decomposition, it follows that

$$\left\| [a'', \delta] \right\|^2 = \left\| [\alpha, \delta] \right\|^2 + \left\| \bar{\partial}_0^\star [\beta, \delta] \right\|^2 .$$

Assuming that β satisfies the uniform bound of Proposition 2.6, from Proposition 6.2 it follows that there is a bound of the form $\|a\|_{L_1^p} \le C \|\alpha\|$ for some constant $C = C(d_0)$. Hence for some new constant $C = C(d_0)$, the bound of Lemma 6.4 implies a bound of the form

$$\left\| [\alpha, \delta] \right\|^2 + \left\| \bar{\partial}_0^\star [\beta, \delta] \right\|^2 \le 2 \left\langle \delta, (D\widehat{\Phi})(\delta) \right\rangle + C\|\alpha\|^4 \|\delta\|^2 \quad \text{for } \delta = \delta^\star \in H^{0,0}. \tag{7.1}$$

This estimate has been derived under the assumption that β satisfies the uniform bound given in Proposition 2.6. In particular, it applies if $\beta = \beta_0(\alpha)$ where $\beta_0(\alpha)$ is the unique element of $A^{0,2}(\text{End } E_h)$ specified in that result, but more generally, it also applies if $\beta = s\,\beta_0(\alpha)$ where $s \in [0, 1]$. This observation facilitates some homotopy arguments to follow.

The leading term on the right of (6.14) is the negative of $m(\alpha) = \Pi^{0,0} i\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha)$. Fixing α temporarily, this gives a map $\Gamma \to H^{0,0}$ give by $\Gamma \ni \gamma \mapsto m(\gamma\alpha\gamma^{-1})$, and the derivative of this map at $\gamma = 1$ is given by

$$\left(D_\gamma \left[m(\gamma\alpha\gamma^{-1}) \right]\right)\big|_{\gamma=1}(\delta) = \Pi^{0,0} i\Lambda \left([\delta, \alpha] \wedge \alpha^\star + \alpha \wedge [\alpha^\star, \delta^\star] + [\alpha^\star, \delta^\star] \wedge \alpha + \alpha^\star \wedge [\delta, \alpha] \right), \quad \delta \in H^{0,0}. \tag{7.2}$$

If $m_\alpha : H^{0,0} \to H^{0,0}$ is the $\mathbb{R}$-linear function on the right of (7.2) and $L_\alpha : H^{0,0} \to H^{0,1}$ is the $\mathbb{C}$-linear function $L_\alpha(\delta) := [\alpha, \delta]$, it is clear that the kernel of L_α is contained in that of m_α. In fact, by direct calculation, for $\delta \in H^{0,0}$ with $\delta = \delta_+ + \delta_-$ and $(\delta_\pm)^\star = \pm\delta_\pm$,

$$\text{Re} \left\langle \delta, m_\alpha(\delta) \right\rangle = \left\langle \delta_+, m_\alpha(\delta) \right\rangle = -2 \left\| [\alpha, \delta_+] \right\|^2 + \left\langle [\delta_-, \delta_+], m(\alpha) \right\rangle . \tag{7.3}$$

Thus when acting on the self-adjoint elements of $H^{0,0}$, m_α and L_α have the same kernel. Since m_α is itself self-adjoint as an $\mathbb{R}$-linear map $H^{0,0} \to H^{0,0}$, it follows that m_α maps the space of self-adjoint elements of $H^{0,0}$ that are orthogonal to $\ker L_\alpha$ isomorphically into the same space.

At this point, arguments are simplified if it is assumed that α is stable with respect to the action of Γ, not just polystable. Given that α is Γ-polystable, the assumption of stability is equivalent to the condition that $\ker L_\alpha = span\{1\}$. For $g \in \mathcal{G}$, (1.1) implies

$$\text{tr } i\widehat{F}(g \cdot (d_0 + a)) = \text{tr } i\widehat{F}(d_0 + a) + i\,\bar{\partial}\partial \log \det(g^\star g) = r\lambda + i\Lambda d \text{ tr } a + i\,\Lambda\,\bar{\partial}\partial \log \det(g^\star g),$$

so if $\bar{\partial}_0^\star a'' = 0$ then

$$\text{tr } i\widehat{F}(g \cdot (d_0 + a)) = r\lambda + \triangle' \log \det(g^\star g) .$$

If $g = \exp(\varphi)\gamma$ for self-adjoint $\varphi \in A^{0,0}(\text{End } E_h)$ orthogonal to $\ker \bar{\partial}_0$ and $\gamma \in \Gamma$, $\log \det(g^\star g) = 2 \text{ tr } \varphi + \log \det(\gamma\gamma^\star)$, so $\text{tr } \left(i\widehat{F}(g \cdot (d_0 + a)) - \lambda\,1 \right) = \triangle \text{tr } \varphi$. Hence for $\bar{\partial}_0^\star$-closed a'' and $g = \exp(\varphi)\gamma$, by taking the trace-free part of φ and rescaling γ to have determinant 1, it can be assumed without loss of generality that $\text{tr } i\widehat{F}(d_0 + a) = r\lambda = \text{tr } i\widehat{F}(g \cdot (d_0 + a))$, with $\det g \equiv 1 = \det \gamma$.

If V is a hermitian vector space, the real vector space $End^{sa}(V)$ of self-adjoint endomorphisms has a canonically induced orientation. This can be seen by induction on $\dim V$, given that the space of self-adjoint endomorphisms of $V \oplus \mathbb{C}$ is canonically isomorphic to $End^{sa}(V) \oplus V \oplus \mathbb{R}$. So the space of self-adjoint automorphisms of V, which is a real closed submanifold of $Aut(V)$, is orientable. Similarly, the space of trace-free self-adjoint endomorphisms of $V \oplus \mathbb{C}$ is canonically isomorphic to $End^{sa}(V) \oplus V$, so the space of self-adjoint automorphisms of determinant 1, which is a real closed submanifold of $Aut(V)$, is orientable. The function $\Gamma \ni \gamma \mapsto m(\gamma\alpha\gamma^{-1})$ restricted to the space Γ_0^{sa} of self-adjoint elements of Γ of determinant 1 maps Γ_0^{sa} into the space $(H^{0,0})_0^{sa}$ of trace-free self-adjoint elements of $H^{0,0}$; that is, into its tangent space at $1 \in \Gamma_0^{sa}$, and (given that α is Γ-stable) its derivative at 1 is an isomorphism between $T_1 \Gamma_0^{sa}$ and $(H^{0,0})_0^{sa}$.

Replacing α in (7.1) with $t\alpha$ for $t > 0$ sufficiently small (depending on α), it follows from that estimate that once t is sufficiently small, the kernel of the $\mathbb{R}$-linear function $(D\widehat{\Phi})_1$ acting on the trace-free self-adjoint elements of $H^{0,0}$ is zero, and this holds for any such t and any $s\beta(t\alpha)$ with $|s| \le 1$ where $\beta(\alpha) \in A^{0,2}(\text{End } E_h)$ is the section specified by Proposition 2.6.

Now view $\widehat{\Phi}$ in Proposition 7.1 as a function of $\gamma \in \Gamma$ and $\bar{\partial}_0^\star$-closed $a'' \in A^{0,1}(\text{End }E_h)$, taking values in the trace-free self-adjoint elements of $H^{0,0}$. When restricted to those $a'' \in A^{0,1}(\text{End }E_h)$ of the form $a'' = t\alpha + s\,\bar{\partial}_0^\star\beta_0(t\alpha)$ for $0 < t \le 1$ and $0 \le s \le 1$ and self-adjoint γ near 1 of determinant 1, there is an induced map $\Gamma_0^{\text{sa}} \to (H^{0,0})_0^{sa}$, for which, from (7.1), the derivative with respect to its first variable γ at 1 is injective once t is sufficiently small, where "sufficiently" depends upon α; more specifically, on the first non-zero eigenvalue of $L_\alpha^\star L_\alpha$.

Theorem 7.2. *Suppose $\alpha \in H^{0,1}$ satisfies $\Pi^{0,0}\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha) = 0$. Then for $t > 0$ sufficiently small (depending on α) and $s \in [0, 1]$ there exists self-adjoint $\delta \in \ker \bar{\partial}_0$ and self-adjoint $\varphi \in (\ker \bar{\partial}_0)^\perp$ (both depending on s and t) such that $i\widehat{F}\big(\exp(\varphi) \cdot \exp(\delta) \cdot (d_0 + a_{s,t}) \big) = \lambda \mathbb{1}$, where $a_{s,t} = a'_{s,t} + a''_{s,t}$, $a''_{s,t} = t\alpha + s\,\bar{\partial}_0^\star\beta(t\alpha)$, and $\beta(-) \in A^{0,2}(\text{End }E_h)$ is determined by Proposition 2.6. Furthermore, there is a constant $C = C(d_0, \alpha)$ such that $\|\delta\| \le Ct^2$ and $\|\varphi\|_{L_2^p} \le Ct^2$ for $t > 0$ sufficiently small (depending on α).*

Proof. The assumption on α implies that it is polystable with respect to the action of Γ. Assume initially that α is stable with respect to this action.

If $t > 0$ is sufficiently small and $s \in [0, 1]$, $(D_1\widehat{\Phi})_{(1, a''_{s,t})}$ gives an isomorphism between the tangent space to Γ_0^{sa} at 1 and $(H^{0,0})_0^{sa}$. The manifold Γ_0^{sa} is orientable, and the degree of $\widehat{\Phi}(-, a''_{s,t})$ at $\lambda \mathbb{1}$ is independent of s and sufficiently small t, and is therefore equal to the degree at $\lambda \mathbb{1}$ for such t and $s = 0$. If $\varphi_t := \varphi(a_{0,t})$, then by (6.14), Lemma 6.3 and the remarks following that lemma, for $\delta \in H^{0,0}$ near 0, there is a function $R_4 = R_4(t, \delta)$ with $\|R_4(t, \delta)\|_{L^p} \le Ct^4$ such that, for $\gamma_\delta := \exp(\delta)$,

$$
\begin{aligned}
\widehat{\Phi}(\gamma_\delta, a''_{0,t}) &= \Pi^{0,0}i\widehat{F}\big(\exp(\varphi(\gamma_\delta \cdot a_{0,t})) \cdot (d_0 + \gamma_\delta \cdot a_{0,t}) \big) \\
&= \lambda\mathbb{1} + \triangle_0\varphi(a_{0,t}) - i\,t^2\Pi^{0,0}\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha) + t^2\,m_\alpha(\delta) + R_4(t, \delta) \\
&= \lambda\mathbb{1} + t^2\,m_\alpha(\delta) + R_4(t, \delta) \,.
\end{aligned}
\tag{7.4}
$$

Since both $\widehat{\Phi} - \lambda\mathbb{1}$ and m_α take their values in the space of trace-free self-adjoint elements of $H^{0,0}$, so too does R_4. Since m_α is an isomorphism on this space and $R_4(t, \delta)/t^4$ is uniformly bounded as $t \to 0$, it follows that $\lambda\mathbb{1}$ is in the range of $\widehat{\Phi}(-, a''_{0,t})$ for t sufficiently small, and indeed that the degree of $\widehat{\Phi}(-, a''_{0,t})$ at $\lambda\mathbb{1}$ is precisely 1. Therefore, for $s \in [0, 1]$ and $t > 0$ sufficiently small (depending on α), there exists $\gamma \in \Gamma$ such that

$$
i\widehat{F}\big(\exp(\varphi(\gamma \cdot a_{s,t})) \cdot \gamma \cdot (d_0 + a_{s,t}) \big) = \lambda\mathbb{1} \,.
$$

The invertibility of m_α on the space $(H^{0,0})_0^{sa}$ of trace-free self-adjoint elements of $H^{0,0}$ implies that there is a solution $\gamma = \exp(\delta(t))$ to $i\widehat{F}\big(\exp(\gamma \cdot a_{s,t}) \cdot \gamma \cdot (d_0 + a_{s,t}) \big) = \lambda\mathbb{1}$ depending continuously on t, and from (7.4), the section $\delta(t)$ satisfies a bound of the form $\|\delta(t)\| \le Ct^2/c_\alpha$ where c_α is the lowest eigenvalue of $L_\alpha^\star L_\alpha$ acting on $(H^{0,0})_0^{sa}$ and $C = C(d_0, \alpha)$. Then uniform bounds on $\exp(\delta(t))$ give estimates on $\exp(\delta(t)) \cdot \alpha$, and together with the estimates of Proposition 6.2, a uniform L_2^p bound on φ of the form Ct^2/c_α follows for some new constant C depending on d_0 and α.

If now α is assumed to be polystable but not stable, then the isotropy subgroup Γ_α has dimension greater than one. Hence there are non-zero trace-free elements of $H^{0,0}$ commuting with α, and by 2. of Theorem 4.7, so too do their adjoints, as they all do with $\beta(t\alpha)$ for any (small) $t > 0$. Hence there are non-zero trace-free self-adjoint elements of $H^{0,0}$ commuting with α and $\beta(t\alpha)$ for any such t, and therefore there are non-zero trace-free self-adjoint endomorphisms of E_h that are covariantly constant with respect to $d_0 + a_{s,t}$ for all s, t. Any such endomorphism gives a unitary splitting of the bundle and connections, and these splittings are all compatible with one another, including the splitting of (E_h, d_0). With respect to such a splitting, the form α splits into a collection of endomorphism-valued $(0, 1)$-forms on X, each of which is $\bar{\partial}$-harmonic with respect to the induced connection. The "off-diagonal" components of α with respect to such a splitting are zero, since α commutes with the trace-free endomorphisms determining the splitting. Since α is of minimal norm in its orbit under γ, each of the "diagonal" components of α must be of minimal norm under the action of the subgroup of Γ that is the automorphism group of the corresponding component of (E_h, d_0). Hence each of these components defines an element of the corresponding space that is polystable with respect to the action of the

corresponding automorphism group, so it follows by induction on the rank r that for t sufficiently small, for each of these new bundles with connection, once $t > 0$ is sufficiently small there is a complex gauge transformation that gives a new connection with $i\widehat{F}$ a scalar multiple of 1, (the case $r = 1$ being elementary). Since the splitting of (E_h, d_0) is unitary and $i\widehat{F}(d_0) = \lambda\,1$, the relevant scalar in all cases is λ. For each summand, the estimates on the corresponding endomorphisms δ and φ imply an estimate of the required form on the direct sum connection, verifying the last statement of the theorem. □

A direct proof of Theorem 7.2 that does not use induction on rank appears to be possible, but raises a number of interesting representation-theoretic questions.

Corollary 7.3. *Under the hypotheses of* Theorem 7.2, *if $t > 0$ is sufficiently small and $d_0 + a_{s,t}$ is integrable, then the corresponding holomorphic bundle is polystable, and is stable only if α is stable with respect to the action of Γ.*

Proof. By Theorem 7.2 and the results of Kobayashi and Lübke, the holomorphic bundle defined by $d_0 + a_{s,t}$ is polystable. If it is not stable, then there exists a non-zero trace-free holomorphic endomorphism of this bundle. By Theorem 4.7, this endomorphism is covariantly constant with respect to d_0 and commutes with α, these facts contradicting the assumption that α is stable with respect to the action of Γ. □

Remarks.

1. From the viewpoint of deformation theory, an unobstructed infinitesimal deformation is (poly)stable with respect to the action of Γ if and only if there is a 1-complex parameter family of (poly)stable holomorphic structures whose tangent at E_0 is the given infinitesimal deformation. Of course, in the case that the latter is polystable but not stable, there may be families of bundles that are semistable but not polystable with that tangent vector, which will often be the case if $H^2(X, End\,E_0)$ vanishes. (Direct sums of non-isomorphic line bundles of degree zero on a torus provide an example when this is not the case.)

2. A relatively straightforward application of the continuity method applied to the assignment $t \mapsto g_t \in \mathcal{G}$ solving $i\widehat{F}(g_t \cdot (d_0 + a_{s,t}) = \lambda\,1$ gives a more quantitative version of Theorem 7.2, namely that the dependence of t on α stated in the theorem can be made explicit if the constant C there is replaced by by $C_0\|\alpha\|^2/c_\alpha$ where C_0 is a constant depending only on d_0 and where c_α is the first non-zero eigenvalue of $L_\alpha^\star L_\alpha \,:\, H^{0,0} \to H^{0,0}$. In the interests of brevity, an explicit proof will not be given.

The following proposition is the companion uniqueness result to Theorem 7.2 (existence). The proof of the first statement is based on the proof of Corollary 9 in [11]:

Proposition 7.4. *Suppose $\alpha \in H^{0,1}$, $a'' = \alpha + \bar{\partial}_0^\star\beta$, $a = -(a'')^\star + a''$, and $\|a\|_{L_1^p} < \epsilon$ where $\epsilon > 0$ is as in* Theorem 4.7. *If $i\widehat{F}(g_1 \cdot (d_0 + a)) = \lambda\,1 = i\widehat{F}(g_2 \cdot (d_0 + a))$, then $g_2 = u\,g_1\gamma$ for some $u \in \mathcal{U}$ and $\gamma \in \Gamma_\alpha$. Furthermore, there exists $g_0 \in \mathcal{G}$ with $i\widehat{F}(g_0 \cdot (d_0 + a)) = \lambda\,1$ satisfying the conditions that $g_0 = g_0^\star$ is positive, $\det g_0 \equiv 1$, and $\Pi^{0,0}(g_0^\star g_0) \in H^{0,0}$ is orthogonal to the space of trace-free elements of $\ker L_\alpha$, with these conditions determining g_0 uniquely up to conjugation by unitary elements of Γ_α.*

Proof. Let $d_a := d_0 + a$ and set $d_b := g_1 \cdot d_a$, so for $g := g_2 g_1^{-1}$ it follows that $g_2 \cdot d_a = g \cdot d_b$ with $i\widehat{F}(d_b) = \lambda\,1 = i\widehat{F}(g \cdot d_b)$. After a unitary change of gauge applied to $g_2 \cdot d_a$, it can be supposed that g is positive self-adjoint, with $g = \exp(v)$ for some self-adjoint v. If $y \in \mathbb{R}$ and with $d_y := \exp(y\,v) \cdot d_b$, by (6.9) the function $\mathbb{R} \ni y \mapsto \langle i\widehat{F}(\exp(y\,v) \cdot d_b) - \lambda\,1, v \rangle \in \mathbb{R}$ has derivative $\langle \triangle_y v, v \rangle = \|d_y v\|^2 \geq 0$ and is therefore a non-decreasing function on $\mathbb{R}$. Since it attains the value 0 at both $y = 0$ and $y = 1$, it must be constant on $[0, 1]$ with derivative identically 0. Hence $d_b v = 0$, which implies that $\bar{\partial}_a(g_1^{-1}g_2) = 0$. By 1. of Theorem 4.7, $\gamma := g_1^{-1}g_2$ is d_0-covariantly constant and commutes with α. Thus $ug_2 = g_1\gamma$ for some $u \in \mathcal{U}$ and $\gamma \in \Gamma_\alpha$.

To prove the second statement, note that Γ acts freely on $\mathcal{G}$ by right multiplication, as does the closed subgroup $\Sigma_\alpha \subset \Gamma$ of elements in Γ_α of unit determinant. Given a fixed $g \in \mathcal{G}$ there is a constant $c = c(g)$ such that $c\|\gamma\|^2 \leq \|g\gamma\|^2 \leq c^{-1}\|\gamma\|^2$, so there exists $\gamma_0 \in \Sigma_\alpha$ minimising $\|g\gamma\|^2$ over all $\gamma \in \Sigma_\alpha$. The Euler-Lagrange

equation for this functional on Σ_α is $\Pi(g_0^\star g_0) = 0$, where Π is L^2-orthogonal projection onto the space of trace-free elements in $\ker L_\alpha$, the Lie algebra of Σ_α.

Suppose now that g_1, $g_2 \in \mathcal{G}$ are as in the statement of the proposition, with $g_2 = ug_1\gamma$ for some $u \in \mathcal{U}$ and some $\gamma \in \Gamma_\alpha$. Suppose in addition that both g_1 and g_2 have unit determinant, are both positive and self-adjoint, and that $\Pi^{0,0}(g_j^\star g_j)$ is orthogonal to the trace-free elements of $\ker L_\alpha$ for $j = 1, 2$. Then for any trace-free $\phi \in \ker L_\alpha$, and using the fact that the trace of a covariantly constant endomorphism is constant,

$$0 = \langle g_2^\star g_2, \phi \rangle = \langle \gamma^\star g_1^\star g_1 \gamma, \phi \rangle = \langle g_1^\star g_1, \gamma \phi \gamma^\star \rangle$$

$$= \langle g_1^\star g_1, \frac{\operatorname{tr} \gamma \phi \gamma^\star}{r} 1 \rangle = \|g_1\|_{L^2}^2 \frac{\operatorname{tr} \gamma \phi \gamma^\star}{r} = \|g_1\|_{L^2}^2 \frac{\langle \gamma^\star \gamma, \phi \rangle}{r} \ .$$

Therefore $\gamma^\star \gamma$ is a multiple of 1, and this multiple must be 1 since $1 = \det u \, \det \gamma$. Thus $\gamma \in U(\Gamma_\alpha)$, the group of unitary elements in Γ commuting with α. Then since g_1 and g_2 are both self-adjoint, $g_2^2 = g_2^\star g_2 = \gamma^\star g_1^\star g_1 \gamma = (\gamma^{-1} g_1 \gamma)^2$, and positivity implies $g_2 = \gamma^{-1} g_1 \gamma$. From $g_2 = ug_1\gamma$, it then follows that $u = \gamma^{-1}$. $\square$

Theorem 7.2 gives a condition under which a connection near d_0 has a connection with central component of the curvature equal to a scalar multiple of the identity in its orbit under $\mathcal{G}$, but the deficiency of the result is that how near to d_0 the connection must be depends on the connection itself, dictated by the relative sizes of the eigenvalues of $L_\alpha^\star L_\alpha$. This issue is addressed in the next section.

8 The local Hitchin-Kobayashi correspondence

As stated at the end of the previous section, the objective of this section is to remove the dependency of Theorem 7.2 on α other than through $\|\alpha\|$. That is, retaining all of the notion of that section, the objective is to prove the following result:

Theorem 8.1. *Let d_0 be a connection on E_h with $i\widehat{F}(d_0) = \lambda 1$. Then there is a constant $\epsilon = \epsilon(d_0)$ with the following property: If $\alpha \in H^{0,1}$ is polystable with respect to the action of Γ and $\|\alpha\| < \epsilon$, and if $\beta \in A^{0,2}(\mathrm{End}\,\mathrm{E}_h)$ is as in* Proposition 2.6, *then there exists $g \in \mathcal{G}$ with $i\widehat{F}(g \cdot (d_0 + a)) = \lambda 1$, where $a = a' + a''$ for $a'' = \alpha + \bar{\partial}_0^\star \beta$.*

The approach to proving this result is to ensure that the analysis is performed in a sufficiently small neighbourhood of d_0 where the connections are well-approximated by their linearisations, which has the effect of reducing the problem to a finite-dimensional question that is naturally attacked using the methods of classical geometric invariant theory. Before commencing the proof of the theorem, there are several remarks and observations that simplify matters considerably.

First, consider the case in which the rank r of the bundle E_h is 1. Then α is a harmonic $(0, 1)$-form on X, β must be zero since $\alpha \wedge \alpha = 0$, and the connection $d_a = d_0 + (\alpha - \alpha^\star)$ has curvature $F(d_0)$, which already satisfies the condition $i\widehat{F} = \lambda$. Thus $g \equiv 1$ solves the equation. In the general case, if $i\widehat{F}(g \cdot (d_0 + a)) = \lambda 1$, then on taking the trace of both sides it follows that $i\Lambda \big(\operatorname{tr} (F(d_0) + d_0 a + a \wedge a) + \partial \bar{\partial} \log \det(g^\star g) \big) = r\lambda$, which implies that $i\Lambda \, \partial \bar{\partial} \log \det(g^\star g) \equiv 0$ and hence that $\det(g^\star g)$ is constant. After rescaling g by a constant, it can therefore be assumed that $|\det g| \equiv 1$.

Second, given that α is polystable with respect to the action of Γ, it may be assumed without loss of generality that α is of minimal norm in its orbit under Γ, and therefore $i\Lambda(\alpha \wedge \alpha^\star + \alpha^\star \wedge \alpha)$ is orthogonal to $\ker \bar{\partial}_0$, by Lemma 5.1.

Third, as it was for the proof of Theorem 7.2, if α is polystable but not stable, precisely the same argument using induction on r that was employed at the end of the proof of Theorem 7.2 reduces the problem to the case when α is stable with respect to the action of Γ. Given this, the uniqueness result Proposition 7.4 implies that the only freedom in choice of g is that of replacing g by ug for $u \in \mathcal{U}$.

Hitherto, little use has been made of unitary gauge freedom $\mathcal{U} \ni u \mapsto u \cdot d$ for a connection d, as this is subsumed into the complex gauge freedom $\mathcal{G} \ni g \mapsto g \cdot d$. But since the equation $i\widehat{F}(d) = \lambda 1$ is invariant

under unitary gauge transformations of the connection d, it is helpful to make use of the opportunity to place connections in good (unitary) gauges:

Lemma 8.2. *There are constants $\epsilon > 0$ and C depending only on d_0 with the property that if $d_0 + b$ is a connection with $\|b\|_{L_1^p} < \epsilon$ then there is a unique skew-adjoint section $\psi \in A^{0,0}(\mathrm{End}\,E_h)$ orthogonal to $\ker d_0$ for which $d_0^\star(e^\psi \cdot (d_0 + b) - d_0) = 0$, with $\|\psi\|_{L_2^p} \le C\|d_0^\star b\|_{L^p}$.*

Proof. The linearisation of the function $\mathcal{U} \ni u \mapsto d_0^\star(ubu^{-1} - d_0u\,u^{-1})$ at $u = 1$ and $b = 0$ is $A^{0,0}(\mathrm{End}\,E_h) \ni \sigma \mapsto \triangle_0\sigma$, which is an isomorphism from the space of skew-adjoint elements of $A^{0,0}(\mathrm{End}\,E_h)$ orthogonal to $\ker d_0$ lying in L_2^p to the same such space of elements lying in L^p. Since the original function takes values in the latter space, an application of the implicit function theorem implies that there is a number $\epsilon > 0$ such that the equation $d_0^\star(e^\psi b e^{-\psi} - (d_0 e^\psi)e^{-\psi}) = 0$ has a unique skew-adjoint solution $\psi \in (\ker d_0)^\perp \subseteq A^{0,0}(\mathrm{End}\,E_h)$ if $\|b\|_{L_1^p} < \epsilon$, and moreover $\|\psi\|_{L_2^p} \le C\|d_0^\star b\|_{L^p}$ for some $C = C(d_0)$. $\qquad\square$

Turning now to the proof of Theorem 8.1 and retaining most of the notation of the previous section, suppose that $\alpha \in H^{0,1}$ is stable with respect to the action of Γ and is of minimal norm in its orbit under this action, with $\|\alpha\| = 1$. For $t > 0$ sufficiently small that Proposition 2.6 is valid, let $\beta_t := \beta(t\alpha)$ and let $a_t = a_t' + a_t''$ for $a_t'' := t\alpha + \bar{\partial}_0^\star\beta_t$, with $d_t := d_0 + a_t$. Fix a number $\epsilon_0 \in (0, 1]$, the precise value of which will be fixed later, but for the moment should satisfy the condition that for any $t \in (0, \epsilon_0]$, a_t satisfies the hypotheses of Lemma 2.5, Proposition 2.6, Theorem 4.7, Corollary 4.8 and Lemma 8.2. Now let

$$S := \left\{ t_0 \in (0, \epsilon_0] \ \middle|\ \begin{array}{l} \textit{for every } t \in (0, t_0] \textit{ there exists } g \in \mathcal{G} \textit{ with } \left\|(\bar{\partial}_t g)g^{-1}\right\|_{L_1^p} < t \\ \textit{for which } g \cdot d_t =: d_0 + b_t \textit{ satisfies } d_0^\star b_t = 0 \ \textit{and} \ i\widehat{F}(d_0 + b_t) = \lambda\,1. \end{array} \right\}$$

By Theorem 7.2, for $t > 0$ sufficiently small (depending on α) there exist trace-free self-adjoint $\delta \in \ker \bar{\partial}_0$ and $\varphi \in (\ker \bar{\partial}_0)^\perp$ with $\|\delta\| + \|\varphi\|_{L_2^p} \le C_\alpha t^2$ such that $i\widehat{F}(g \cdot d_t) = \lambda\,1$ for $g = \exp(\varphi)\exp(\delta)$. Then $(\bar{\partial}_t g)g^{-1} = (\bar{\partial}_0 g)g^{-1} + a_t'' - ga_t''g^{-1} = (\bar{\partial}_0 e^\varphi)e^{-\varphi} + [a_t'', g]g^{-1}$. The first term is bounded in L_1^p by $C_\alpha t^2$, and since g^{-1} is uniformly bounded in C^0 whilst $\|g - 1\|_{L_1^p} \le C_\alpha t$, the bound $\|a_t''\| \le Ct$ implies that $\|(\bar{\partial}_t g)g^{-1}\|_{L_1^p} \le C_\alpha t^2$ for some new constant C_α. Since $\bar{\partial}_0^\star a_t'' = 0$, it follows easily that $\|\bar{\partial}_0^\star(-(\bar{\partial}_0 g)g^{-1} + ga_t''g^{-1})\|_{L^p} \le C_\alpha t^2$, so by Lemma 8.2, after a unitary gauge transformation $u \cdot g \cdot d_t = d_0 + b_t$ so that $d_0^\star b_t = 0$, the complex automorphism $\tilde{g} = ue^\varphi e^\delta \in \mathcal{G}$ satisfies the requirements for t to lie in S once $t > 0$ is sufficiently small. Thus S is not empty.

The fact that S is open (if ϵ_0 is sufficiently small) will be shown shortly, this being a straightforward consequence of the implicit function theorem. The proof that S is closed is more involved, involving a priori estimates on solutions.

To see that S is open, suppose that $t_0 \in (0, \epsilon_0) \cap S$, and let $g_0 \in \mathcal{G}$ satisfy $\|(\bar{\partial}_{t_0} g_0)g_0^{-1}\|_{L_1^p} < t_0$, $d_0^\star b_0 = 0$ for $d_0 + b_0 := g_0 \cdot d_{t_0}$ and $i\widehat{F}(d_{t_0}) = \lambda\,1$. The linearisation of the function $\mathcal{G} \ni g \mapsto i\widehat{F}(g \cdot d_{t_0})$ at $g_0 \in \mathcal{G}$ is $A^{0,0}(\mathrm{End}\,E_h) \ni \sigma \mapsto d_{b_0}^\star d_{b_0}\sigma_+ \in A^{0,0}(\mathrm{End}\,E_h)$. If σ is in the kernel of this map, then $\bar{\partial}_{b_0}\sigma_+ = 0$, so $(\bar{\partial}_0 + a_{t_0}'')(g_0^{-1}\sigma_+g_0) = 0$. Given that $\|a_{t_0}\|_{L_1^p} \le Ct_0$ and t_0 is sufficiently small, it follows from Theorem 4.7 that the endomorphism $g_0^{-1}\sigma_+g_0$ is covariantly constant with respect to d_0 and commutes with α as well as with β_{t_0}. Since α is Γ-stable, this implies that $g_0^{-1}\sigma_+g_0$ is a scalar multiple of the identity, and therefore so too is σ_+. Hence the kernel of $\triangle_{b_0} = d_{b_0}^\star d_{b_0}$ acting on the trace-free self-adjoint sections of $A^{0,0}(\mathrm{End}\,E_h)$ is zero, and so an application of the implicit function theorem implies that there is a small neighbourhood of t_0 in $(0, \epsilon_0)$ that lies in S, proving that S is open.

It remains to show that S is also closed. Suppose now that $(0, t_0) \subset S$, and for each $t \in (0, t_0)$, let $g_t \in \mathcal{G}$ satisfy $\det g_t \equiv 1$, $i\widehat{F}(g_t \cdot d_t) = \lambda\,1$ and $d_0^\star b_t = 0$ for $g_t \cdot d_t =: d_0 + b_t$, with $\|(\bar{\partial}_t g_t)g_t^{-1}\|_{L_1^p} < t$. Since

$$b_t'' = -(\bar{\partial}_0 g_t)g_t^{-1} + g_t a_t'' g_t^{-1} = -(\bar{\partial}_t g_t)g_t^{-1} + a_t'',$$

it follows that $\|b_t\|_{L_1^p} \le Ct$ for some constant $C = C(d_0)$, and therefore the preparatory results Lemma 2.5, Proposition 2.6, Theorem 4.7, and Lemma 8.2 apply to the connection $d_0 + b_t$ if ϵ_0 is sufficiently small.

The equation $i\widehat{F}(d_0 + b_t) = \lambda\,1 = i\widehat{F}(d_0)$ implies that $i\Lambda(d_0 b_t + b_t \wedge b_t) = 0$, which can be re-written as

$$\partial_0^\star b_t' - \bar{\partial}_0^\star b_t'' = -i\Lambda(b_t' \wedge b_t'' + b_t'' \wedge b_t')\,.$$

Since $0 = d_0^\star b_t = \partial_0^\star b_t' + \bar{\partial}_0^\star b_t''$, it follows that $2\,\bar{\partial}_0^\star b_t'' = i\Lambda(b_t' \wedge b_t'' + b_t'' \wedge b_t')$, implying that $\|\bar{\partial}_0^\star b_t''\|_{L^p} \leq Ct^2$ for some $C = C(d_0)$. Note that since $\bar{\partial}_0^\star a_t'' = 0$, this is a bound on the L^p norm of $\bar{\partial}_0^\star((\bar{\partial}_t g_t)g_t^{-1})$, and because of the uniform bounds on a_t, this can also be seen as a uniform bound on the L^p norm of $\bar{\partial}_t^\star((\bar{\partial}_t g_t)g_t^{-1})$.

By Lemma 2.5, there is a unique self-adjoint $\varphi_t \in A^{0,0}(\mathrm{End}\,E_h)$ orthogonal to $\ker \bar{\partial}_0$ such that

$$\bar{\partial}_0^\star\big(e^{-\varphi_t}\bar{\partial}_0 e^{\varphi_t} + e^{-\varphi_t}b_t'' e^{\varphi_t}\big) = 0 \quad \text{with} \quad \|\varphi_t\|_{L_2^p} \leq C\|\bar{\partial}_0^\star b_t''\|_{L^p} \leq Ct^2\,. \tag{8.1}$$

If $d_0 + c_t := \exp(-\varphi_t)\cdot(d_0 + b_t) = (\exp(-\varphi_t)g_t)\cdot(d_0 + a_t)$, then $\bar{\partial}_0^\star c_t'' = 0 = \bar{\partial}_0^\star a_t''$, so by the first statement of Corollary 4.8 (using the connection on $\mathrm{End}\,E_h = \mathrm{Hom}(E_h, E_h)$ induced by $d_0 + a_t$ and $d_0 + c_t$ respectively), $e^{-\varphi_t}g_t =: \gamma_t$ is d_0-covariantly constant and $c_t''\gamma_t = \gamma_t a_t''$. Note that since φ_0 must be trace-free (being orthogonal to 1) and $\det g_t = 1$, it follows that $\det \gamma_t = 1$. Since γ_t is invertible, $c_t'' = t\,\gamma_t \alpha \gamma_t^{-1} + \gamma_t\,\bar{\partial}_0^\star \beta_t \gamma_t^{-1}$, but since $t \leq \epsilon_0$ is sufficiently small, $\gamma_t\,\bar{\partial}_0^\star \beta_t \gamma_t^{-1} = \bar{\partial}_0^\star \beta(t\gamma_t \alpha \gamma_t^{-1})$. Since $\bar{\partial}_0 + c_t'' = \exp(-\varphi_t)\cdot(\bar{\partial}_0 + b_t'')$ and $\|\varphi_t\|_{L_2^p} \leq Ct^2$ whilst $\|b_t''\|_{L_1^p} \leq Ct$, it follows that $\|c_t''\|_{L_1^p} \leq Ct$ and in particular, $\|c_t''\|_{L^2} \leq Ct$. But since

$$\|c_t''\|_{L^2}^2 = t^2\big\|\gamma_t \alpha \gamma_t^{-1}\big\|_{L^2}^2 + \big\|\bar{\partial}_0^\star \beta(t\gamma_t \alpha \gamma_t^{-1})\big\|_{L^2}^2\,, \tag{8.2}$$

it follows that $\alpha_t := \gamma_t \alpha \gamma_t^{-1}$ is uniformly bounded in L^2 independently of t and α, and since α is stable with respect to the action of Γ, the assignment $\Gamma \ni \gamma \mapsto \|\gamma \alpha \gamma^{-1}\|$ is proper on the elements $\gamma \in \Gamma$ that have unit determinant; therefore γ_t converges to some $\gamma_0 \in \Gamma$ as $t \to t_0$. Since t_0 is sufficiently small that Proposition 2.6 applies, $\gamma_t \beta_t \gamma_t^{-1} = \beta(t\gamma_t \alpha \gamma_t^{-1})$ converges in L_2^p to $\beta(t_0 \gamma_0 \alpha \gamma_0^{-1})$ as $t \to t_0$.

The uniform L_2^p bounds on φ_t imply that these converge weakly in L_2^p and by the Sobolev embedding theorem, strongly in C^1 to $\varphi_0 \in (\ker \bar{\partial}_0)^\perp$, so the corresponding automorphisms $g_t = \exp(\varphi_t)\gamma_t$ have the same convergence. By ellipticity of the $\bar{\partial}_0$-Laplacians on $A^{0,1}(\mathrm{End}\,E_h)$ and $A^{0,2}(\mathrm{End}\,E_h)$, the function β of Proposition 2.6 depends smoothly on its argument, and therefore $\beta(t\gamma_t \alpha \gamma_t^{-1})$ converges smoothly to $\beta(t_0 \gamma_0 \alpha \gamma_0^{-1})$ as $t \to t_0$. Ellipticity of the equations $\partial_t^\star(h_t^{-1}\partial_t h_t) = \lambda\,1 - h_t^{-1}F(d_t)h_t$ for $h_t := g_t^\star g_t$ and smooth dependence of a_t on t then imply that the family (h_t) depends smoothly on t, and the uniqueness of the unitary gauge stated in Lemma 8.2 then gives smooth dependence of (g_t) on t, so g_t converges smoothly to some $g_0 \in \mathcal{G}$ as $t \to t_0$, with $d_0 + b_0 := g_0 \cdot (d_0 + a_{t_0})$ satisfying $i\widehat{F}(d_0 + b_0) = \lambda\,1$, $d_0^\star b_0 = 0$, and $\big\|(\bar{\partial}_{t_0}g_0)g_0^{-1}\big\|_{L_1^p} \leq t_0$. The proof that S is closed will be complete if it can be shown that this is a strict inequality, but ensuring this is the critical issue.

As above, $\alpha_t := \gamma_t \cdot \alpha = \gamma_t \alpha \gamma_t^{-1}$ is uniformly bounded in L^2 and hence in L_2^p independent of $t \in (0, \epsilon_0]$ and of α. Since $c_t'' = t\alpha_t + \bar{\partial}_0^\star \beta(t\alpha_t)$,

$$\|c_t'' - t\alpha_t\|_{L^2} \leq C\|\beta(t\alpha_t)\|_{L_1^p} \leq C\|t\alpha_t\|_{L^2}^2 \leq Ct^2$$

for some uniform constant C, using here Proposition 2.6. Again using the L_2^p bounds on φ_t, it then follows that

$$t^2\big\|i\Lambda\big(\alpha_t \wedge \alpha_t^\star + \alpha_t^\star \wedge \alpha_t\big)\big\| \leq \big\|i\Lambda\big(b_t'' \wedge b_t' + b_t' \wedge b_t''\big)\big\| + Ct^3 = Ct^3\,. \tag{8.3}$$

Thus, if

$$m(\sigma) := \Pi^{0,0}i\Lambda\big(\sigma \wedge \sigma^\star + \sigma^\star \wedge \sigma\big) \quad \text{for } \sigma \in H^{0,1}$$

is the moment map for the action of Γ as in §5, then $\|m(\alpha_t)\| \leq Ct$ for some constant C that is independent of α and t. In fact, it follows from (6.11)(a) that there is a better estimate, namely $\|m(\alpha_t)\| \leq Ct^2$ for some new constant C independent of α and t, provided that ϵ_0 is sufficiently small.

Note that the important relationship between $\|\sigma\|$ and $\|m(\sigma)\|$ for $\sigma \in H^{0,1}$ (or more precisely, the behaviour of the projectively-invariant function $\|m(\sigma)\|/\|\sigma\|^2$) has been studied earlier; cf. for example [31], [20], [30].

Lemma 8.3. *For every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\gamma\alpha\gamma^{-1}\|^2 - 1 < \epsilon$ for every polystable $\alpha \in H^{0,1}$ with $m(\alpha) = 0$ and $\|\alpha\| = 1$ for which $\|m(\gamma\alpha\gamma^{-1})\| < \delta$.*

Assuming for the moment that this "uniform continuity" result holds, the proof of Theorem 8.1 is easily completed. As before,

$$\bar{\partial}_t g_t g_t^{-1} = \bar{\partial}_0 g_t g_t^{-1} + a_t'' - g_t a_t'' g_t^{-1} = (\bar{\partial}_0 e^{\varphi_t}) e^{-\varphi_t} + a_t'' - e^{\varphi_t} \gamma_t a_t'' \gamma_t^{-1} e^{-\varphi_t} ,$$

so with the earlier estimates on $\|\varphi_t\|_{L_2^p}$ and on $\|\beta\|_{L_2^p}$,

$$\|\bar{\partial}_t g_t g_t^{-1}\|_{L_1^p} \le t\|\alpha_t - \alpha\|_{L_1^p} + Ct^2 \le C't\|\alpha_t - \alpha\|_{L^2} + Ct^2 \tag{8.4}$$

for some uniform constants C and C'.

The unitary gauge transformations provided by Lemma 8.2 have been applied to the connections with central curvature $-i\lambda\,1$, but as yet, none has been applied to the connections d_t. This is now done by writing $\gamma_t = p_t u_t$ for some uniquely determined positive self-adjoint $p_t \in \Gamma$ and some unitary $u_t \in \Gamma$. The convergence of γ_t to γ_0 implies convergence of p_t and u_t to some positive p_0 and u_0 in Γ respectively. Note that all the estimates above apply equally with no change of constants when α is replaced by α_t since they depended on α only through $\|\alpha\|$. Since $\alpha_t' := u_t \alpha u_t^{-1}$ is of minimal norm in its orbit under Γ (i.e., that of α), it follows that

$$\left\|\alpha_t - \alpha_t'\right\|^2 = \|\alpha_t\|^2 - 2\,\mathrm{Re}\,\langle \alpha_t, \alpha_t'\rangle + \|\alpha_t'\|^2 = \|\alpha_t\|^2 - 2\,\langle p_t^{1/2} \cdot \alpha_t', p_t^{1/2} \cdot \alpha_t'\rangle + \|\alpha_t'\|^2 \le \|\alpha_t\|^2 - 1 .$$

From (8.4), $\|\bar{\partial}_t g_t g_t^{-1}\|_{L_1^p}$ will be less than $t/2$ if both $t \le t_0$ and t_0 is sufficiently small (depending only on d_0) and $\|\alpha_t - \alpha_t'\|$ is also sufficiently small. The latter condition will hold if $\|\alpha_t\|^2$ is sufficiently close to 1, and by Lemma 8.3, this in turn will hold if $\|m(\alpha_t)\|$ is sufficiently small. From (8.3), that last condition will be satisfied provided that t is sufficiently small, where "sufficiently small" is a condition that depends only on d_0, and not on α. Consequently, provided that ϵ_0 is chosen to be sufficiently small, the set S will be closed as well as open, and hence be equal to $(0, \epsilon_0]$, completing the proof of Theorem 8.1. $\qquad\square$

It remains to prove Lemma 8.3, which will be a consequence of the following:

Lemma 8.4. *Suppose $\alpha \in H^{0,1}\setminus\{0\}$ satisfies $m(\alpha) = 0$. Then for any $\gamma \in \Gamma$, $\left\|\gamma\alpha\gamma^{-1}\right\|^2 - \|\alpha\|^2 \le C\left\|m(\gamma\alpha\gamma^{-1})\right\|$ for some constant C depending only on d_0.*

Proof. The polystable holomorphic bundle E_0 splits as a direct sum $\bigoplus_{i=1}^m E_i$ of stable bundles all of the same slope. With respect to this splitting of E_0, a form $\tau \in H^{0,1}$ corresponds to an $m \times m$ matrix $[\tau_j^i]$ of $(0, 1)$-forms, with τ_j^i being $\bar{\partial}$-harmonic with respect to the induced Hermite-Einstein connection on $Hom(E_j, E_i)$. Then

$$\|\tau\|^2 = \sum_{i=1}^m \sum_{j=1}^m \|\tau_j^i\|^2 .$$

Moreover, $m(\tau) = \Pi^{0,0} i\Lambda(\tau \wedge \tau^\star + \tau^\star \wedge \tau)$ corresponds to an $m \times m$ matrix for which the i-th diagonal entry is

$$m(\tau)_i^i = \sqrt{-1} \sum_{j=1}^m \Pi_i^{0,0} \Lambda\left((\tau_j^i)^\star \wedge \tau_j^i + \tau_i^j \wedge (\tau_i^j)^\star\right) ,$$

where $\Pi_i^{0,0}$ is L^2-orthogonal projection onto the (i, i)-component (of $\ker \bar{\partial}_0$). Since E_i is stable for each i, $Aut(E_i) = \mathbb{C}^\star$ and so the projection $\Pi_i^{0,0}$ here is simply given by integrating the trace over X. Thus

$$m(\tau)_i^i = \sum_{j=1}^m \left(\|\tau_j^i\|^2 - \|\tau_i^j\|^2\right) .$$

(More precisely, $m(\tau)_i^i$ is the number on the right multiplied by the identity endomorphism of E_i, but this fact only changes estimates by combinatorial factors bounded by a combinatorial function of r.)

Suppose now that $\alpha \in H^{0,1}$ satisfies $m(\alpha) = 0$ and $\gamma \in \Gamma$. Using a Cartan decomposition of Γ into $\Gamma = U\,T\,U$ where T is a maximal complex torus and $U = U(\Gamma)$, it follows from the left and right unitary invariance of the norm and the unitary equivariance of m that γ may be assumed to lie in T; that is, $\gamma = diag(t_1, \ldots, t_m)$ for some $t_j \in \mathbb{C}^\star$.

Instead of working directly with α, it is more convenient to work with $\tau := \alpha\gamma^{-1}$, so $m(\tau\gamma) = 0$ and it must be shown that $\|\gamma\tau\|^2 - \|\tau\gamma\|^2 \le C\|m(\gamma\tau)\|$. This will follow if it can be shown that

$$\sum_{j=1}^{m} \left(|t_j|^2 \|\tau_j{}^i\|^2 - |t_i|^2 \|\tau_i{}^j\|^2 \right) = 0 \quad \text{for } i = 1, \ldots, m \text{ implies that} \tag{8.5}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \left(|t_i|^2 \|\tau_j{}^i\|^2 - |t_j|^2 \|\tau_j{}^i\|^2 \right) \le C \sum_{i=1}^{m} \left| \sum_{j=1}^{m} \left(|t_i|^2 \|\tau_j{}^i\|^2 - |t_j|^2 \|\tau_i{}^j\|^2 \right) \right| \tag{8.6}$$

for some constant C, using here the fact that the ℓ_1 and ℓ_2 norms on $H^{0,0}$ are equivalent in this representation.

Observe that

$$\left(|t_i|^2 - |t_j|^2 \right)\left(\|\tau_j{}^i\|^2 - \|\tau_i{}^j\|^2 \right) = \left(|t_i|^2 \|\tau_j{}^i\|^2 - |t_j|^2 \|\tau_j{}^i\|^2 \right)| + \left(|t_i|^2 \|\tau_i{}^j\|^2 - |t_j|^2 \|\tau_j{}^i\|^2 \right),$$

and by (8.5), for each fixed i the second term on the right sums to zero on application of $\sum_j$. Similarly,

$$\left(|t_i|^2 - |t_j|^2 \right)\left(\|\tau_j{}^i\|^2 + \|\tau_i{}^j\|^2 \right) = \left(|t_i|^2 \|\tau_j{}^i\|^2 - |t_j|^2 \|\tau_i{}^j\|^2 \right) + \left(|t_i|^2 \|\tau_i{}^j\|^2 - |t_j|^2 \|\tau_j{}^i\|^2 \right),$$

and again (8.5) implies that the second term on the right sums to zero on application of $\sum_j$. So (8.6) is equivalent to

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \left(|t_i|^2 - |t_j|^2 \right)\left(\|\tau_j{}^i\|^2 - \|\tau_i{}^j\|^2 \right) \le C \sum_{i=1}^{r} \left| \sum_{j=1}^{r} \left(|t_i|^2 - |t_j|^2 \right)\left(\|\tau_j{}^i\|^2 + \|\tau_i{}^j\|^2 \right) \right|.$$

After renumbering, it can be assumed that $|t_i| \ge |t_j|$ if $i < j$. The summand on the left is symmetric under interchange of i and j, so it can be written as $2\sum_{i=1}^{r}\sum_{j=i+1}^{r} \left(|t_i|^2 - |t_j|^2 \right)\left(\|\tau_j{}^i\|^2 - \|\tau_i{}^j\|^2 \right)$. Then the desired inequality will certainly follow if it can be shown that

$$\sum_{i=1}^{r}\sum_{j=i+1}^{r} \left| |t_i|^2 - |t_j|^2 \right| \left(\|\tau_j{}^i\|^2 + \|\tau_i{}^j\|^2 \right) \le C \sum_{i=1}^{r} \left| \sum_{j=1}^{r} \left(|t_i|^2 - |t_j|^2 \right)\left(\|\tau_j{}^i\|^2 + \|\tau_i{}^j\|^2 \right) \right|.$$

That this is true is a consequence of the following:

Lemma 8.5. *Let $S = (s_{ij})$ be a skew-symmetric $m \times m$ matrix with $s_{ij} \ge 0$ if $i < j$. Then*

$$\sum_{i=1}^{m}\sum_{j=i+1}^{m} s_{ij} \le 2^{m-1} \sum_{i=1}^{m} \left| \sum_{j=1}^{m} s_{ij} \right|.$$

Proof. It will be shown inductively that for $k \in \{1, \ldots, m\}$,

$$\sum_{i=1}^{k}\sum_{j=i+1}^{m} s_{ij} \le 2^{k-1} \sum_{i=1}^{k} \left| \sum_{j=1}^{m} s_{ij} \right|. \tag{8.7}$$

For $k = 1$, the inequality clearly holds since $s_{1j} \ge 0$ for every j. Suppose that the inequality has been shown to hold for $k = 1, \ldots, \ell - 1$. Then for $k = \ell$, the new term on the left is $L_\ell := \sum_{j=\ell+1}^{m} s_{\ell j}$, and on the right the new term is $R_\ell := \left| \sum_{j=1}^{m} s_{\ell j} \right|$. Let $A_\ell := -\sum_{j=1}^{\ell-1} s_{\ell j} \ge 0$ and $B_\ell := \sum_{j=\ell+1}^{m} s_{\ell j} \ge 0$, so $R_\ell = \left| B_\ell - A_\ell \right|$ and by inspection, $B_\ell = L_\ell$.

By skew-symmetry of S,

$$A_\ell = \sum_{j=1}^{\ell-1} s_{j\ell} = \sum_{i=1}^{ell-1} s_{i\ell} \le \sum_{i=1}^{\ell-1}\sum_{j=\ell}^{m} s_{ij} \le 2^{\ell-2} \sum_{i=1}^{\ell-1} \left| \sum_{j=1}^{m} s_{ij} \right|,$$

using the inductive hypothesis. Now if $B_\ell - A_\ell \leq 0$, then $L_\ell = B_\ell \leq A_\ell \leq 2^{\ell-2} \sum_{i=1}^{\ell-1} \left| \sum_{j=1}^{m} s_{ij} \right|$, so

$$\sum_{i=1}^{\ell} \sum_{j=i+1}^{m} s_{ij} = B_\ell + \sum_{i=1}^{\ell-1} \sum_{j=i+1}^{m} s_{ij} \leq A_\ell + \sum_{i=1}^{\ell-1} \sum_{j=i+1}^{m} s_{ij} \leq 2^{\ell-1} \sum_{i=1}^{\ell-1} \left| \sum_{j=1}^{m} s_{ij} \right| \leq 2^{\ell-1} \sum_{i=1}^{\ell} \left| \sum_{j=1}^{m} s_{ij} \right|.$$

On the other hand, if $B_\ell - A_\ell \geq 0$, then

$$B_\ell = (B_\ell - A_\ell) + A_\ell = \left| B_\ell - A_\ell \right| + A_\ell \leq \left| B_\ell - A_\ell \right| + 2^{\ell-2} \sum_{i=1}^{\ell-1} \left| \sum_{j=1}^{m} s_{ij} \right|,$$

so by the inductive hypothesis again,

$$\sum_{i=1}^{\ell} \sum_{j=i+1}^{m} s_{ij} = B_\ell + \sum_{i=1}^{\ell-1} \sum_{j=i+1}^{m} s_{ij} \leq \left| B_\ell - A_\ell \right| + 2^{\ell-1} \sum_{i=1}^{\ell-1} \left| \sum_{j=1}^{m} s_{ij} \right| \leq 2^{\ell-1} \sum_{i=1}^{\ell} \left| \sum_{j=1}^{m} s_{ij} \right|.$$

$\square$

This completes the proof of the lemma, and with it, the proof of Lemma 8.4 and hence of Lemma 8.3. $\square$

Remark.

There is an alternative approach to Lemma 8.3 in terms of general theory. Namely, for $\alpha_0 \in H^{0,1}$ one can study the downwards gradient flow for the function $\Gamma \ni \gamma \mapsto \|\gamma\alpha_0\gamma^{-1}\|^2$, for which the relevant ODE is $\dot{\gamma}_t\gamma_t^{-1} = [m(\alpha_t), \alpha_t]$ with $\gamma_0 = 1$, where $\alpha_t := \gamma_t\alpha_0\gamma_t^{-1}$. It is easily checked that the flow $t \mapsto \alpha_t$ is the same as the downwards gradient flow for $H^{0,1} \ni \alpha \mapsto \|m(\alpha)\|^2$. Modulo reparameterisation, the latter covers the downwards gradient flow for $\mathbb{P}(H^{0,1}) \ni [\alpha] \mapsto \|m(\alpha)\|^2/\|\alpha\|^4$, for which $\mathbb{P}(H^{0,1}) \ni [\alpha] \mapsto m(\alpha)/\|\alpha\|^2$ is the moment map for the action of Γ on $\mathbb{P}(H^{0,1})$ ([31]). An unpublished theorem of Duistermaat using the Łojasiewicz inequality presented in [25] shows that this flow defines a (strong) deformation retract of the set of Γ-polystable points onto the zero set of the moment map (analogous to the result of Neeman [30] in the algebraic setting), and Lemma 8.3 follows quite easily from this; see also §§3,4 of [9].

9 Non-stability

Theorem 8.1 is a version of the Hitchin-Kobayashi correspondence for bundles in an L_1^p neighbourhood of a polystable bundle, but it does not provide much detail in the case of connections and/or classes that are not polystable. Whereas non-zero elements of $H^{0,1}$ may be unstable with respect to the action of Γ—that is, zero is in the closures of their orbits, Proposition 3.2 states that there are no strictly unstable bundles near E_0, so the correspondence between the two different notions of stability is not perfect, (although this is more a distinction between (semi)stability in the affine versus projective settings). However, it is nevertheless true that the interrelation between the two notions goes further than just that described by Theorem 8.1, as will be seen in this section. All notation from earlier sections continues to be retained.

In general, if $\mathcal{E}$ is an arbitrary torsion-free semistable sheaf that is not stable, there is a non-zero subsheaf $\mathcal{S} \subset \mathcal{E}$ with $\mu(\mathcal{S}) = \mu(\mathcal{E})$ and with torsion-free quotient $\mathcal{Q} = \mathcal{E}/\mathcal{S}$ for which $\mu(\mathcal{Q}) = \mu(\mathcal{E})$. Both $\mathcal{S}$ and $\mathcal{Q}$ are necessarily semistable, and if $\mathcal{S}$ is of maximal rank, then $\mathcal{Q}$ is stable. Iterating this process yields a filtration of $\mathcal{E}$, $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_k = \mathcal{E}$ such that the successive quotients are all torsion-free and stable. Any such filtration is known as a *Seshadri filtration* or sometimes a Jordan-Hölder filtration, and although it is not unique, the graded object $Gr(\mathcal{E}) = \bigoplus_{j=1}^{k} (\mathcal{S}_j/\mathcal{S}_{j-1})$ is unique after passing to the double-dual.

In the current setting of holomorphic structures E near to E_0, Proposition 3.2 states that E is semistable whilst Proposition 4.4 states that any destabilising subsheaf of E is a subbundle. In this case therefore, there is a Seshadri filtration of E defined by subbundles, so the graded object $Gr(E)$ associated to E is a polystable holomorphic structure on E_h.

Recall from the proof of Lemma 3.1 that if A is a holomorphic subbundle of E with quotient B, then in a unitary frame for A and B, a hermitian connection d_E on E and its curvature F_E have the form

$$d_E = \begin{bmatrix} d_A & \beta \\ -\beta^\star & d_B \end{bmatrix} , \qquad F_E = \begin{bmatrix} F_A - \beta \wedge \beta^\star & d_{BA}\beta \\ -d_{AB}\beta^\star & F_B - \beta^\star \wedge \beta \end{bmatrix} ,$$

where now $\beta \in A^{0,1}(Hom(B, A))$ is a $\bar\partial$-closed form representing the extension $0 \to A \to E \to B \to 0$ and where d_A and d_B are the connections on A and B induced by the hermitian structure and d_E. If $\mathrm{rk}\, A = a$ and $\mathrm{rk}\, B = b$, and if $t > 0$, let $h_t = \begin{bmatrix} t^b & 0 \\ 0 & t^{-a} \end{bmatrix}$ so h_t is covariantly constant with respect to the direct sum connection $d_{A\oplus B}$ on $A \oplus B$, $\det h_t = 1$, and $h_t \cdot d_E$ has the same form as d_E with β replaced by $t^{a+b}\beta$. So as $t \to 0$, $h_t \cdot d_E \to d_{A\oplus B}$. Proceeding inductively, it follows easily that there exist $g_t \in \mathcal{G}$ such that $g_t \cdot (d_0 + a)$ converges in C^∞ to a Hermite-Einstein connection on $Gr(E)$. Note that if the homomorphic structure E is defined by $d_a = d_0 + a$, this argument shows that the holomorphic structure on $Gr(E)$ is defined by some connection $d_{\bar a} = d_0 + \bar a$ where $\|\bar a\|_{L_1^p} \le \|a\|_{L_1^p}$, so $d_{\bar a}$ is close to d_0 in L_1^p if d_a is.

By Proposition 4.5, every holomorphic endomorphism of $Gr(E)$ is also $\bar\partial_0$-closed, and is therefore covariantly constant with respect to d_0, by Corollary 4.2. Thus the automorphisms $g_t \in \mathcal{G}$ can even be taken to lie in Γ. The following result gives something of a converse to this observation, albeit in a rather special case. In its hypotheses, how small is "sufficiently small" is determined by the connection d_0, so that Corollary 2.4 is applicable.

Lemma 9.1. *Let $d_0 + a$ be an integrable connection on E_h with $\|a\|_{L_1^p}$ sufficiently small, and suppose that $a'' = \alpha + \bar\partial_0^\star\beta$ for some $\alpha \in H^{0,1}$ and $\beta \in A^{0,2}(\mathrm{End}\, E_h)$. Then the following are equivalent:*

1. *For any $\epsilon > 0$ there exists $\gamma \in \Gamma$ such that $\|\gamma\alpha\gamma^{-1}\| < \epsilon$;*
2. *For any $\epsilon > 0$ there exists $g \in \mathcal{G}$ such that $\|g \cdot (d_0 + a) - d_0\|_{L_1^p} < \epsilon$.*

Proof. The implication 1. $\Rightarrow$ 2. follows immediately from Corollary 2.4. For the converse, assume $\epsilon > 0$ is smaller than the number specified in Lemma 2.5 and let $g \in \mathcal{G}$ be an automorphism such that $\|g \cdot (d_0 + a) - d_0\|_{L_1^p} < \epsilon$. Applying Lemma 2.5 to the semi-connection $g \cdot (\bar\partial_0 + a'')$ yields a unique $\varphi \in A^{0,0}(\mathrm{End}\, E_h)$ orthogonal to $\ker \bar\partial_0$ such that $d_0 + \tilde a := \exp(\varphi) \cdot g \cdot (d_0 + a)$ satisfies $\bar\partial_0^\star\tilde a'' = 0$, with $\|\tilde a''\|$ bounded by a fixed multiple of ϵ. Applying Proposition 4.5 to the connection on $\mathrm{Hom}(E_h, E_h)$ induced by $d_0 + \tilde a$ and $d_0 + a$ and the section $\exp(\varphi)g$ of this bundle, it follows that if ϵ is sufficiently small then $\exp(\varphi)g =: \gamma$ must be $\bar\partial_0$-closed with $a''\gamma = \gamma\tilde a''$. Then if $\tilde a'' = \tilde\alpha + \bar\partial_0^\star\tilde\beta$, orthogonality of the decompositions gives $\gamma^{-1}\alpha\gamma = \tilde\alpha$, and $\|\tilde\alpha\|$ is bounded by a fixed multiple of ϵ since $\|\tilde a''\|_{L_1^p}$ is. $\qquad\square$

Consider now an integrable connection $d_0 + a$, with $\|a\|_{L_1^p}$ assumed to be appropriately small and with $a'' = \alpha + \bar\partial_0^\star\beta$ for some $\alpha \in H^{0,1}$ and some $\beta \in A^{0,2}(\mathrm{End}\, E_h)$ orthogonal to the kernel of $\bar\partial_0^\star$. Under the action of Γ on $H^{0,1}$, there is a point $\bar\alpha \in H^{0,1}$ of smallest norm in the closure of the orbit of α unique up to conjugation by unitary elements in Γ, and this is a Γ-polystable point (if not zero). Since $\bar\alpha$ is in the closure of the orbit of α and each element near 0 in this orbit lies in the analytic set $\Psi^{-1}(0)$, there is a unique section $\bar\beta \in A^{0,2}(E_h)$ such that $\bar\partial_0 + \bar a'' := \bar\partial_0 + \bar\alpha + \bar\partial_0^\star\bar\beta$ is integrable, so by Theorem 8.1 the corresponding holomorphic bundle $\overline E$ is polystable. The following is the main result of this section, this being Theorem 4 of the introduction:

Theorem 9.2. *With the preceding definitions, let E be the holomorphic structure defined by $d_0 + a$. Then $\overline E \simeq Gr(E)$.*

The proof, which is principally by induction on the rank r of E_h (with the initial case $r = 1$ being self-evident) proceeds in several stages, corresponding to three cases: 1. *That $\alpha = 0$;*   2. *That α is not zero and is not Γ-semistable;* and   3. *That α is Γ-semistable.* The first is the totally degenerate case for which $\alpha = 0$:

Proposition 9.3. *Let $d_0 + a$ be an integrable connection on E_h with $\bar{\partial}_0^* a'' = 0$, and let E be the corresponding holomorphic structure. If $\|a\|_{L_1^p}$ is sufficiently small then $\Pi^{0,1} a'' = 0$ if and only if $E \simeq E_0$.*

Proof. If $\Pi^{0,1} a'' = 0$, then it follows from Corollary 2.4 that $a'' = 0$, and therefore $a = 0$. Conversely, if $E \simeq E_0$, then there exists $g \in \mathcal{G}$ such that $g \cdot d_0 = d_0 + a$, or equivalently, $\bar{\partial}_0 g + a'' g = 0$. Applying Proposition 4.5 to the connection on $Hom(E_h, E_h)$ induced by d_0 and $d_0 + a$, it follows that $d_0 g = 0 = a'' g$, so $a'' = 0$. □

Before moving on to the other two stages of the proof of Theorem 9.2, consider first some general features applicable in all cases. Let $d_0 + a$ be as above with $a'' = \alpha + \bar{\partial}_0^* \beta$. Choose a sequence (γ_j) in Γ with $\det \gamma_j = 1$ for every j such that $\|\gamma_j \alpha \gamma_j^{-1}\|^2 \to \inf_{\gamma \in \Gamma} \|\gamma \alpha \gamma^{-1}\|^2$ as $j \to \infty$, so after passing to a subsequence if necessary, it can be assumed that $\alpha_j := \gamma_j \alpha \gamma_j^{-1}$ converges to some $\bar{\alpha} \in H^{0,1}$.

If $\beta_j := \gamma_j \beta \gamma_j^{-1}$ and $a_j'' := \alpha_j + \bar{\partial}_0^* \beta_j$, then $d_0 + a_j = \gamma_j \cdot (d_0 + a)$ is an integrable connection defining a holomorphic structure isomorphic to E, with $\bar{\partial}_0^* a_j'' = 0$. By Corollary 2.4, $\|a_j''\|_{L_1^p}$ is uniformly bounded independent of j, so after passing to another subsequence if necessary, the connections $d_0 + a_j$ can be assumed to converge weakly in L_1^p and strongly in C^0 (say) to a limiting connection $d_0 + \bar{a}$, with $a \in L_1^p$. Elliptic regularity combined with integrability of the connection together with the equation $\bar{\partial}_0^* \bar{a}'' = 0$ imply that $\bar{a}$ is in fact smooth. Indeed, using the analysis of §1, the forms β_j can be assumed to be converging in L_2^p to a limit in $A^{0,2}(End E_h)$ that is orthogonal to $\ker \bar{\partial}_0^*$, and by the uniqueness statement of Proposition 2.6, this limit must be the form $\bar{\beta}$ mentioned earlier, with $\bar{a}'' = \bar{\alpha} + \bar{\partial}_0^* \bar{\beta}$.

Since $\det \gamma_j = 1$ for every j, it follows that if $\|\gamma_j\|$ is uniformly bounded then a subsequence can be found converging to some $\gamma_0 \in \Gamma$, and then $d_0 + \bar{a} = \gamma_0 \cdot (d_0 + a)$. This is the case considered in the previous section when $\alpha \in H^{0,1}$ is a Γ-polystable point. So it may be supposed that $\|\gamma_j\|$ is not uniformly bounded, and after replacing γ_j by $\gamma_j/\|\gamma_j\|$, these may be assumed to converge to some $\gamma_0 \in H^{0,0}$ with $\|\gamma_0\| = 1$ and $\det \gamma_0 = 0$. It may also be assumed without loss of generality that γ_j is self-adjoint and positive for each j, so γ_0 is also self-adjoint and non-negative.

The equation $\gamma_j \cdot (d_0 + a) = d_0 + a_j$ is equivalent to $\bar{\partial}_j \gamma_j = 0$ where d_j is the connection on $Hom(E_h, E_h)$ induced by $d_0 + a$ and $d_0 + a_j$, these connections converging to the connection on this bundle induced by $d_0 + a$ and $d_0 + \bar{a}$. So γ_0 defines a non-zero holomorphic map from E to $\bar{E}$, this map having determinant 0. Since γ_0 must be of constant rank on X, its kernel K is a holomorphic subbundle of E, necessarily of the same slope as that of E_h (because E and $\bar{E}$ are semistable of the same slope). Thus E may be expressed as an extension by holomorphic semistable bundles $0 \to K \to E \to Q \to 0$, where $Q := E/K$.

Consider now Case 2. of Theorem 9.2, namely when α is non-zero and is not Γ-semistable. By definition, zero is in the closure of the orbit of α under Γ, so $\bar{\alpha} = 0$ and therefore $\bar{E} = E_0$ by Proposition 9.3. For notational convenience, set $E_1 := K = \ker \gamma_0$ and $E_2 := Q = E/K$. Since γ_0 is self-adjoint, E_2 can be identified with $E_1^\perp \subset E_h$ as a unitary bundle. The holomorphic structures on E_1 and E_2 are those induced from E as holomorphic sub- and quotient bundles. But since $E_1 = \ker \gamma_0$ and γ_0 is a d_0-closed self-adjoint endomorphism of the holomorphic bundle E_0, both E_1 and E_2 have hermitian connections induced from d_0, with respect to which the connections are Hermite-Einstein with the same Einstein constant as E_0. These connections will be denoted by $d_{0,1}, d_{0,2}$ respectively, so $d_0 = d_{0,1} \oplus d_{0,2}$ using self-evident notation.

The limit γ_0 of the (rescaled) automorphisms γ_j is d_0-closed and satisfies $\gamma_0 \alpha = 0 = \gamma_0 \beta$ and also $\gamma_0 a'' = 0$, so in terms of the splitting $E_h = E_1 \oplus E_2$,

$$\gamma_0 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{\gamma} \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ 0 & 0 \end{bmatrix}, \quad a'' = \begin{bmatrix} a_{11}'' & a_{12}'' \\ 0 & 0 \end{bmatrix}, \tag{9.1}$$

where $\bar{\gamma} = \bar{\gamma}^*$ has non-zero determinant. Since γ_0 is d_0-closed, the connection on E_2 induced by the connection $d_0 + a$ (i.e., as a quotient of E) is the same as the connection on this bundle induced by d_0 (i.e., as a subbundle of E_0), so the holomorphic bundle E_2 is isomorphic to a direct sum of stable summands of E_0.

The connection on E_1 induced by $d_0 + a$ is identified with $d_{0,1} + a_{11}$, with $a_{11} = \alpha_{11} + \bar{\partial}_{0,1}^* \beta_{11}$. By the inductive hypothesis (of Theorem 9.2), under the action of $\Gamma_1 = Aut(E_1(d_{0,1}))$, there are connections in the

orbit of $d_{0,1} + a_{11}$ that are arbitrarily close in L_1^p to the Hermite-Einstein connection d_G on $Gr(E_1)$. Then using automorphisms of the form h_t as described earlier, the off-diagonal term a_{12}'' in (9.1) can be made arbitrarily small whilst leaving the diagonal terms fixed, from which it follows that there exist $g_t \in \mathcal{G}$ for $t > 0$ such that $g_t \cdot (d_0 + a) \to (d_G \oplus d_{0,2})$ as $t \to 0$.

Thus the two Hermite-Einstein connections d_0 and $d_{G \oplus E_2}$ lie in the closure of the orbit of $d_0 + a$ under $\mathcal{G}$, but up to unitary isomorphism, there is at most one such connection since the space of Yang-Mills connections modulo unitary gauge is Hausdorff, by the reasoning of §6 of [1]. So $Gr(E_1) \oplus E_2 \simeq E_0$, which implies that $Gr(E) \simeq E_0$.

It remains to complete the proof of Theorem 9.2 in Case 3., this being when $\alpha \in H^{0,1}$ is Γ-semistable but not Γ-polystable. With the same objects as earlier, this is the case that $\bar{\alpha} \neq 0$, so $\overline{E}$ is not isomorphic to E_0 (by Proposition 9.3), but $\overline{E}$ is polystable, by Theorem 8.1.

By Theorem 4.7, every endomorphism of E_h that is holomorphic with respect to $\overline{\partial}_0 + \bar{a}''$ is in fact covariantly constant with respect to d_0 and commutes with $\bar{a}$, so $\overline{\Gamma} := Aut(\overline{E})$ is the subgroup of $\Gamma = Aut(E_0)$ commuting with $\bar{a}$. The connection $d_0 + \bar{a}$ defines an ω-polystable point, and $\gamma_j \cdot (d_0 + a) \to d_0 + \bar{a}$. From Lemma 9.1, once $d_0 + a$ has been placed in the good complex gauge $d_0 + \tilde{a}$ of Lemma 2.5 with respect to the Hermite-Einstein connection $\bar{d}$ inducing $\overline{E}$, there are automorphisms $\bar{\gamma}_j$ that are $\bar{d}$-closed such that $\bar{\gamma}_j \cdot (d_0 + \tilde{a}) \to \bar{d}$. But now Case 2. applies with $\overline{E}$ replacing E_0, for which the conclusion is that the bundle $Gr(E)$ is isomorphic to $\overline{E}$, as desired. Consequently, the proof of Theorem 9.2 (i.e., Theorem 4 of the introduction) is complete.   □

Conclusion

We end this paper with several concluding remarks.

1. The results presented here appear to be of some significance even in the case of compact Riemann surfaces. When the degree and the rank of E_h are coprime, the moduli space of stable holomorphic structures on E_h is a smooth compact manifold, of considerable interest in its own right, not least because this space carries a natural hyper-Kähler metric of Weil-Petersson type. When the rank and degree of E_h are not coprime, the stable bundles are naturally compactified by adding the polystable bundles, and the results here provide a description of neighborhoods of boundary points.

The case of compact Riemann surfaces is also helpful for obtaining a better understanding of several of the results presented here. There are no integrability conditions to be considered, and the only singularities occurring in moduli spaces result from quotient singularities which can be viewed in the light of isotropy for the action of Γ on classes in $H^{0,1}$. The cases of genus 0, 1 and 2 for E_0 being the trivial bundle of rank 2, or the direct sum of a non-trivial line bundle of degree 0 with the trivial line bundle all provide considerable insight.

2. In the case $n = 2$, relatively explicit examples of moduli spaces can be computed, particularly when X is a ruled surface and even more particularly when $X = \mathbb{P}_1 \times \mathbb{P}_1$. Using monads, moduli spaces of 2-bundles have been computed explicitly in [4], including an explicit description of the space of deformations of the bundle $\mathcal{O}(1, -1) \oplus \mathcal{O}(-1, 1)$, which again illustrates many of the results here; cf. the Remark following Proposition 3.2. Monads also feature in Donaldson's paper [10], which presents another illustration of the interrelation between the notions of stability in Kähler geometry and geometric invariant theory, one that is not independent of the results in this paper.

3. Because the results here focus on a *neighbourhood* of a fixed polystable bundle, it is reasonable to expect that they will hold mutatis mutandis on arbitrary compact complex manifolds equipped with Gauduchon metrics. However, in light of the Remark following the proof of Proposition 3.2, there may be some unforeseen subtleties. For the sake of brevity and simplicity, we have considered only the Kähler case.

4. It is evident from the analysis that the assumption of integrability for connections is not nearly as important as might be expected, as the (0, 2) and (2, 0) components of the curvature are well-controlled by Proposition 2.6, given that the calculations are local to d_0. This highlights the interesting class of solutions d

of the Yang-Mills equations on a compact Kähler manifold for which $\partial F^{0,2}(d) = 0 = \overline{\partial}\widehat{F}(d)$ (which includes some *self*-dual solutions on compact surfaces), these bearing some formal similarities to solutions of the Seiberg-Witten equations.

5. At its heart, the proof of Proposition 3.2 is a manifestation of a very coarse compactness property of stable bundles, a desirable property used to great effect in gauge theory. Moduli spaces of stable holomorphic bundles on a Kähler surface can fail to be compact in two ways, one reflecting the degeneration from stable to polystable and the other in terms of the concentration of curvature of Hermite-Einstein connections. The former is the subject of this paper, whereas the latter is considered in [6]. Although the failure of moduli spaces of stable bundles on compact Kähler surfaces to be compact can be controlled to some extent as described in that reference, in higher dimensions there is less control on the degeneration and one is forced to consider compactifications in terms of sheaves ([2]). The Bogomolov inequality $(c_2 - (r-1)c_1^2/2r) \cdot \omega^{n-1} \geq 0$ for semistable sheaves and bundles does not provide sufficient control on subbundles in dimensions greater than 2.

6. As alluded to in the introduction, there are profound relationships between the theory of stable holomorphic vector bundles on compact Kähler manifolds and the theory of constant scalar curvature Kähler metrics, these relationships mediated by geometric invariant theory. In that the former theory is a quasi-linear analogue of the latter (in the sense of partial differential equations), it can be hoped that the results here may provide useful directions for the further investigation of moduli of compact complex manifolds and their geometries.

7. To conclude on an even more speculative note, in view of the critical importance of Yang-Mills theory and of representation theory in contemporary physics, it might also be hoped that our results may provide deeper insight into the nature of elementary particles and their interactions.

Conflict of interest: The authors state no conflict of interest.

References

[1] M. F. Atiyah, N. J. Hitchin and I. M. Singer: *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A, **362** (1978), 425–461.

[2] S. Bando and Y.-T. Siu: *Stable sheaves and Einstein-Hermitian metrics.* In: *Geometry and analysis on complex manifolds*, (World Sci. Publ., River Edge, NJ, 1994), 39–50.

[3] T. A. Brönnle: *Deformation constructions of extremal metrics*. Ph.D. thesis, Imperial College, London, UK, 2011.

[4] N. P. Buchdahl: *Stable 2-bundles on Hirzebruch surfaces*, Math. Z. **194** (1987), 143–152.

[5] N. P. Buchdahl: *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces*, Math. Ann. **280** (1988), 625–648.

[6] N. P. Buchdahl: *Sequences of stable bundles over compact complex surfaces*, J. Geom. Anal. **9** (1999), 391–428

[7] N. P. Buchdahl and G. Schumacher: *An analytic application of Geometric Invariant Theory*, J. Geom. Phys. **165** (2021), 104237.

[8] N. P. Buchdahl and G. Schumacher: *An analytic application of Geometric Invariant Theory II: Coarse moduli spaces.* J. Geom. Phys. **175** (2022), 104467.

[9] X. Chen and S. Sun: *Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics*, Ann. of Math. **180** (2014), 407–454.

[10] S. K. Donaldson: *Instantons and geometric invariant theory*, Comm. Math. Phys. **93** (1984), 453–460.

[11] S. K. Donaldson: *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), 1–26.

[12] S. K. Donaldson and P. B. Kronheimer: *The geometry of four-manifolds*, Oxford University Press, New York, 1990.

[13] Y. Fan: *Construction of the moduli space of Higgs bundles using analytic methods*, `arXiv:2004.07182v2` (2020).

[14] O. Forster and K. Knorr: *Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen*, Math. Ann. **209** (1974), 291–346

[15] A. Fujiki and G. Schumacher: *The moduli space of Hermite-Einstein bundles on a compact Kähler manifold*, Proc. Japan Acad. Ser. A Math. Sci. **63** (1987), 69–72.

[16] M. Garcia-Fernandez and C. Tipler: *Deformation of complex structures and the coupled Kähler-Yang-Mills equations*, J. London Math. Soc. **89** (2014), 779–796.

[17] A. Jacob: *Existence of approximate Hermitian-Einstein structures on semi-stable bundles*, Asian J. Math. **18** (2014), 859–883.

[18] A. W. Knapp: *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[19] G. Kempf and L. Ness: *Lengths of vectors in representation spaces*, Lecture Notes in Mathematics Vol. 732, Springer, Berlin-Heidelberg-New York, 1978, 233–244.

[20] F. C. Kirwan: *Cohomology of quotients in symplectic and algebraic geometry.* Princeton University Press, Princeton, NJ, 1984.

[21] S. Kobayashi: *First Chern class and holomorphic tensor fields*, Nagoya Math. J. **77** (1980), 5–11.

[22] S. Kobayashi: *Curvature and stability of vector bundles*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), 158–162.

[23] S. Kobayashi: *Differential geometry of complex vector bundles*, Princeton University Press, Princeton, NJ 1987.

[24] M. Kuranishi: *On the locally complete families of complex analytic structures*, Ann. of Math. **75** (1962), 536–577.

[25] E. Lerman: *Gradient flow of the norm squared of a moment map*, Enseign. Math. **51** (2005), 117–127.

[26] M. Lübke: *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math. **42** (1983), 245–257.

[27] D. Luna: *Slices étales*, Bull. Soc. Math. France **33** (1973), 81–105.

[28] K. Miyajima: *Kuranishi family of vector bundles and algebraic description of the moduli space of Einstein-Hermitian connections*, Publ. Res. Inst. Math. Sci. 25 (1989), 301–320.

[29] D. Mumford, J. Fogarty and F. Kirwan: *Geometric invariant theory (3rd ed.)*, Springer, Berlin, 1994.

[30] A. Neeman: *The topology of quotient varieties*, Ann. of Math. **122** (1985), 419–459.

[31] L. Ness: *A stratification of the null cone via the moment map. With an appendix by David Mumford.* Amer. J. Math. **106** (1984), 1281–1329.

[32] G. Székelyhidi: *The Kähler-Ricci flow and K-polystability*, Am. J. Math. **132** (2010), 1077–1090.

[33] F. Takemoto: *Stable vector bundles on algebraic surfaces*, Nagoya Math. J. **47** (1972), 29–48.

[34] R. P. Thomas: *Notes on GIT and symplectic reduction for bundles and varieties.* In: *Surveys in differential geometry 10*, Int. Press, Somerville, MA (2006), 221–273.

[35] K. K. Uhlenbeck and S.-T. Yau: *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39** (1986), S257–S293.