DE GRUYTER Complex Manifolds 2021; 8:403–414

Research Article Open Access

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Partially integrable almost CR structures

https://doi.org/10.1515/coma-2020-0124
Received June 28, 2021; accepted November 8, 2021

Abstract: Let (M, D) be a compact contact manifold with $dim_{\mathbf{R}} M = 2n - 1 \geq 5$. This means that: M is a C^∞ differential manifold with $dim_{\mathbf{R}} M = 2n - 1 \geq 5$. And D is a subbundle of the tangent bundle TM which satisfying ; there is a real one form θ such that $D = \{X : X \in TM, \theta(X) = 0\}$, and $\theta \wedge \bigwedge^{n-1}(d\theta) \neq 0$ at every point of p of M. Especially, we assume that our D admits almost CR structure,(M, S). In this paper, inspired by the work of Matsumoto([M]), we study the difference of partially integrable almost CR structures from actual CR structures. And we discuss partially integrable almost CR structures from the point of view of the deformation theory of CR structures ([A1],[AGL]).

Keywords: CR structure, partially integrable, contact structure, stein filling

MSC: 32V05, 32G07

1 Introduction

The purpose of this paper is to study a partially integrable almost CR structures from the point of view of the deformation theory of CR structures, developed in [A1,AGL]. Let (M, θ) be a contact structure where M is a C^∞ manifold with odd dimension and θ is a contact form. Recently, some people are studying more special kind of contact manifolds, that is to say, a partially integrable almost CR structure. Matsmoto pointed out that a partially integrable almost CR structure is related with the deformation complex of CR structures. Here, along this line, we discuss a partially integrable almost CR structure. We recall its definition. Let (M, D) be a compact contact manifold with $dim_{\mathbf{R}} M = 2n - 1 \geq 5$. This means that: M is a C^∞ differential manifold with $dim_{\mathbf{R}} M = 2n - 1 \geq 5$. And D is a subbundle of the tangent bundle TM which satisfying that: there is a real one form θ such that $D = \{X : X \in TM, \theta(X) = 0\}$, and $\theta \wedge \bigwedge^{n-1}(d\theta) \neq 0$ for any point of $p \in M$. Especially, we assume that our D admits almost CR structure and satisfies a week integrability condition.The purpose of this paper is to discuss partially integrable almost CR structures from the point of view of the deformation complex of CR structures(see[AGL]), which is successful. Because of the positivity of the Levi form, we can construct E^j_S for a partially integrable almost CR structure by the same way in [A1],[A2]. For a given partially integrable almost CR structure $\{(M, S), \theta\}$, we see that: $(M, {}^\phi S)$ (see definition [3.1] in this paper) is actual CR structure iff ϕ is a solution of a non-linear partial differential equation. And for this equation, if $H_S(obstr) = 0$ holds, our equation has an approximate solution (for the notation, see Sect.8). For higher order approximation, by now, we don't have a particular procedure. Because the convergence of our formal sequence is a very difficult problem. And some topologist says that there might be a counter example, that is to say, the existence of a partially integrable almost CR structure which does not come from a CR structure. However we dare to conjecture

Conjecture If $H_S(obstr) = 0$, then we have an actual solution.

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2 CR structures

In order to explain the partially integrable almost CR structures, which is strongly pseudo convex, we remember the notion of CR structures. Let M be a compact real $2n-1$ dimensional C^∞ manifold. Let ${}^0T''$ be a $n-1$ dimensional complex subbundle of the complexfied tangent bundle of M. The pair $(M, {}^0T'')$ is a CR structure if the following condition holds.

$$(1)\ {}^0T'' \cap {}^0T' = 0 \tag{2.1}$$

$$(2)\ [\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T'') \tag{2.2}$$

Here ${}^0T'$ means the complex conjugate of ${}^0T''$. We assume that our CR structure is strongly pseudo convex. Now we set a supplement real vector ζ, satisfying :for every point $p \in M$,

$$(1)\zeta_p \notin {}^0T''_p + {}^0T'_p \tag{2.3}$$

$$(2)[\zeta, \Gamma(M, {}^0T')] \subset \Gamma(M, {}^0T') + \Gamma(M, {}^0T'') \tag{2.4}$$

Because of strongly pseudo convexity, this is always possible. And set

$$T' = {}^0T' + \mathbf{C} \otimes \zeta$$

And we fix C^∞ vector bundle decomposition of the complexfied tangent bundle by

$$\mathbf{C} \otimes TM = {}^0T'' + T'$$

We introduce $\bar{\partial}_{T'}$ operator.

$$\bar{\partial}_{T'} : \Gamma(M, T') \to \Gamma(M, T' \otimes ({}^0T'')^*)$$

For $u \in \Gamma(M, T')$, $\bar{\partial}_{T'} u(X) = [X, u]_{T'}$. By the standard exterior derivative, we have $\bar{\partial}^{(i)}_{T'}$. Here we mention

$$\bar{\partial}^{(1)}_{T'} u(X, Y) = [X, u(Y)]_{T'} - [Y, u(X)]_{T'} - u([X, Y]).$$

And the Kohn-Rossi cohomology

$$H^{(i)} = \frac{Ker\bar{\partial}^{(i)}_{T'}}{Im\bar{\partial}^{i-1}_{T'}}$$

is introduced.

Example

Let (V, o) be an isolated singularity in a complex euclidean space $\mathbf{C}^\mathbf{N}$. Set $M = V \cap S^{2N-1}_\epsilon$ and ${}^0T'' = \mathbf{C} \otimes TM \cap T''(\mathbf{C}^\mathbf{N})$.

3 The deformation theory of CR structures

For $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$, We set a new almost CR structure, determined by;

$${}^\phi T'' = \{X' : X' = X + \phi(X), X \in {}^0T''\}. \tag{3.1}$$

While, $(M, {}^\phi T'')$ may not be integrable.

Theorem 3.1. *For $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$, $(M, {}^\phi T'')$ is a CR structure, iff ϕ satisfies*

$$\bar{\partial}^{(1)}_{T'} \phi + R_2(\phi) + R_3(\phi) = 0.$$

If $dim_R M = 2n-1 \geqq 7$, and our M is compact strongly pseudo convex, there is the Neumann operator for $\bar{\partial}_{T'}$. However, this Neumann operator is only sub-elliptic(elliptic in ${}^0T' + {}^0T''$, but for the direction ζ, not elliptic). In spired with these results, we reformulate our deformation theory inside ${}^0T' + {}^0T''$.

We briefly remember the deformation theory of CR structures(in [A1], [AGL]). For $i \geqq 1$, we set

For $i \geqq 1$,

$$E^i = \{u : u \in \Gamma(M,{}^0 T' \otimes \bigwedge^i ({}^0 T'')^*), (\overline{\partial}_{T'}^{(i)} u)_{\mathbf{C}\otimes\zeta} = 0\}.$$

Here $(\overline{\partial}_{T'}^{(i)} u)_{\mathbf{C}\otimes\zeta} = 0$ means the $\mathbf{C} \otimes \zeta$ part of $\overline{\partial}_{T'}^{(i)} u$ according to (2.5), (2.6).

For an example of partialloy integrable almost CR strutures, we see E^1. $(\overline{\partial}_{T'}^{(1)} u)_{\mathbf{C}\otimes\zeta} = 0$ means that: for $X, Y \in {}^0 T''$,

$$(\overline{\partial}_{T'}^{(1)} u(X, Y))_{\mathbf{C}\otimes\zeta} = 0.$$

While

$$\overline{\partial}_{T'}^{(1)} u(X, Y) = [X, u(Y)]_{T'} - [Y, u(X)]_{T'} - u([X, Y])$$

So this becomes

$$[X, u(Y)]_{\mathbf{C}\otimes\zeta} - [Y, u(X)]_{\mathbf{C}\otimes\zeta} = 0. \tag{3.2}$$

As for D, we determine $X_g \in \Gamma(M, \mathbf{C})$ by Here $(\overline{\partial}_{T'}^{(i)} u)_{\mathbf{C}\otimes\zeta} = 0$ means the $\mathbf{C} \otimes \zeta$ part of $\overline{\partial}_{T'}^{(i)} u$ according to (2.5), (2.6). As for $calD$, we determine the Hamiltonian vector field with respect to the Leviform, $X_g \in \Gamma(M, \mathbf{C})$ by

$$[X_g, Y]_{\mathbf{C}\otimes\zeta} = (Yg)\zeta,$$

where $[X_g, Y]_{\mathbf{C}\otimes\zeta}$ means the $\mathbf{C} \otimes \zeta$ part of $[X_g, Y]$, and set $Z_g = X_g + g\zeta$. $\mathcal{H} = \{Z_g : g \in \Gamma(M, \mathbf{C})\}$. By these preparations, set

$$\mathcal{D}g = \overline{\partial}_{T'_S} Z_g.$$

Then, by the definition of Z_g, $\overline{\partial}_{T'_S} Z_g \in \Gamma(M, E^1)$. Our structure is integrable. So, $\overline{\partial}_{T'}^{(2)} \overline{\partial}_{T'}^{(1)} = 0$. Hence

Proposition 3.2. *(see Proposition 2.1in [A1])*

$$\textit{For } u \in \Gamma(M, E^1),\ \overline{\partial}_{T'}^{(1)} u \in \Gamma(M, E^2).$$

More precisely,we have our deformation complex, which is sub-ellptic(furthermore, the Kohn-Rossi cohomology can be recovered)(see [A1]).

<u>Our deformation complex</u>

$$\Gamma(M, \mathbf{C}) \xrightarrow{\mathcal{D}} \Gamma(M, E^1) \xrightarrow{\overline{\partial}_{T'}^{(1)}} \Gamma(M, E^2) \ldots$$

4 Partially integrable almost CR structures

We start with the definition of contact structures. Let M be a C^∞ manifold with $dim_{\mathbf{R}} M = 2n - 1$. Let θ be a real one form on M, satisfying

$$\theta \wedge \bigwedge^{n-1} (d\theta) \neq 0 \text{ for any point of } M$$

The pair (M, θ) is called a contact structure or a contact manifold. Recently, several mathematician consider more special kind of a contact manifold, a partially integrable almost CR structure.

Definition 4.1. *If our contact structure* (M, θ) *admits almost CR structure* (M, S)*, where* S *is the subbundle of the complexfied tangent bundle satisfying:*

$$(1)\ \theta\,|_S = 0, \tag{4.1}$$

$$(2)\ S \cap \overline{S} = 0,\ dim_{\mathbf{C}} \frac{C \otimes TM}{S + \overline{S}} = 1, \tag{4.2}$$

$$(3)\ [\Gamma(M, S), \Gamma(M, S)] \subset \Gamma(M, S + \overline{S}), \tag{4.3}$$

the tripple (M, S, θ) *is called a partially integrable almost CR structure.*

For this almost CR structure, by the same as in the case CR structure, we can introduce strongly pseudo convexity.

Definition 4.2. *If our partially integrable almost CR structure* $\{(M, S), \theta\}$ *satisfies,*

$$d\theta : \overline{S} \times S \to \mathbf{C},\ \textit{is positive or negative definite}$$

defined by : $\sqrt{-1}d\theta(\overline{X}, Y)$ *for* $X, Y \in \Gamma(M, S)$*, then* $\{(M, S), \theta\}$ *is called strongly pseudo convex.*

Henceforth, we assume that our partially integrable almost CR structure is strongly pseudo convex. Hence, we can introduce a C^∞ vector bundle decomposition,

$$C \otimes TM = \overline{S} + \mathbf{C} \otimes \zeta + S, \tag{4.4}$$

where ζ is a C^∞ real vector field on M, determined by:

$$(1)\ \theta(\zeta) = 1 \tag{4.5}$$

$$(2)\ d\theta(X, \zeta) = 0 \text{ for } X \in S, \tag{4.6}$$

and $\mathbf{C} \otimes \zeta$ means the complex line bundle, generated by ζ.

<u>**Example**</u> As for partially integrable almost CR structures, we take a strongly pseudo convex CR structure $(M, {}^0T'')$. And for any $\phi \in \Gamma(M, E^1)$, $(M, {}^\phi T'')$ is the partially integrable almost CR structure. For any $X', Y' \in \Gamma(M, {}^0T'')$, $[X', Y'] \in {}^\phi T' + {}^\phi T''$ (actually ${}^0T' + {}^0T''$. In fact, $X' = X + \phi(X)$, $Y' = Y + \phi(Y)$, where $X, Y \in {}^0T''$,

$$[X', Y'] = [X + \phi(X), Y + \phi(Y)] \tag{4.7}$$

$$= [X, Y] + [X, \phi(Y)] - [Y, \phi(X)] + [\phi(X), \phi(Y)] \tag{4.8}$$

Since $(M, {}^0T'')$ is a CR structure, the $\mathbf{C} \otimes \zeta$ part of the first term and the fourth term vanish. And because of ϕ being of E^1, the $\mathbf{C} \otimes \zeta$ of (the second term + the third term) vanishes. So our $[X', Y']$ is of ${}^0T' + {}^0T'' = {}^\phi T' + {}^\phi T''$.

5 $\overline{\partial}_{T'_S}$ complex

For our partially integrable almost CR structure $\{(M, S), \theta\}$, we introduce $\overline{\partial}_{T'_S}$ - complex by the same way as the case the standard CR structure. Set

$$T'_S = \overline{S} + \mathbf{C} \otimes \zeta,$$

and consider the first order differential operator $\overline{\partial}_{T'_S}$,

$$\overline{\partial}_{T'_S} : \Gamma(M, T'_S) \to \Gamma(M, T'_S \otimes S^*),$$

defined by $\overline{\partial}_{T'_S}u(X) = [X, u]_{T'_S}$, where $u \in \Gamma(M, T'_S)$, $X \in \Gamma(M, S)$, and $[X, u]_{T'_S}$ means the projection of $[X, u]_{T'S}$ to $T'S$, with respect to (2.4). Then, by the standard way as for a scalar valued diferential form, we have a differential operator $\overline{\partial}^{(p)}_{T'_S}$: from $\Gamma(M, T'_S \otimes \bigwedge^p S^\star)$ to $\Gamma(M, T'_S \otimes \bigwedge^{p+1} S^\star)$.

We note that in this case, because of the lack of integrability, in general

$$\overline{\partial}^{(1)}_{T'_S}\overline{\partial}_{T'_S} \neq 0.$$

Even though this is not a diiferential complex, we have a pseudo-"Standard deformation complex"

$$0 \longrightarrow \Gamma(M, T'_S) \xrightarrow{\overline{\partial}_{T'_S}} \Gamma(M, T'_S \otimes S^\star) \xrightarrow{\overline{\partial}^{(1)}_{T'_S}} \Gamma(M, T'_S \otimes \bigwedge^2 S^\star)\ldots$$

And also like the case integrable CR structures, we can introduce a pseudo- Our deformation complex

$$\Gamma(M, \mathbf{C}) \xrightarrow{\mathcal{D}} \Gamma(M, E^1_S) \xrightarrow{\overline{\partial}^{(1)}_{T'}} \Gamma(M, E^2_S)\ldots$$

Here $\mathcal{D}$ is the second order partial differential operator. For the details, see $[AGL]$, $[A1]$. We briefly remember these. For $i \geqq 1$,

$$E^i_S = \{u : u \in \Gamma(M, \overline{S} \otimes \bigwedge^i S^\star), (\overline{\partial}^{(i)}_{T'_S}u)_{\mathbf{C}\otimes\zeta} = 0\}.$$

Here $(\overline{\partial}^{(i)}_{T'_S}u)_{\mathbf{C}\otimes\zeta}$ means the $\mathbf{C} \otimes \zeta$ part of $\overline{\partial}^{(i)}_{T'_S}u$ according to (4.4). As for $\mathcal{D}$, we determine $X_g \in \Gamma(M, \overline{S})$ by

$$[X_g, Y]_\zeta = (Yg)\zeta$$

annd set $Z_g = X_g + g\zeta$. $\mathcal{H} = \{Z_g : g \in \Gamma(M, \mathbf{C})\}$. By these preparations, set

$$\mathcal{D}g = \overline{\partial}_{T'_S}Z_g.$$

Then, by the definition of Z_g, $\overline{\partial}_{T'_S}Z_g \in \Gamma(M, E^1_S)$.

Next, we see

Proposition 5.1.

For $u \in \Gamma(M, E^1_S)$, $\overline{\partial}^{(1)}_{T'_S}u \in \Gamma(M, E^2_S)$.

Proof. By the defintion,

$$\overline{\partial}^{(1)}_{T'_S}u \in \Gamma(M, \overline{S} \otimes \bigwedge^2 S^\star).$$

We have to show

$$(\overline{\partial}^{(2)}_{T'_S}\overline{\partial}^{(1)}_{T'_S}u)_{(C\otimes\zeta)\otimes\bigwedge^3 S^\star} = 0.$$

For $X, Y, Z \in \Gamma(M, S)$, we compute

$$(\overline{\partial}^{(2)}_{T'_S}\overline{\partial}^{(1)}_{T'_S}u)(X, Y, Z)$$

Henceforth, we abbreviate $\overline{\partial}$ for $\overline{\partial}^{(i)}_{T'_S}$.

By the definition,

$$(\overline{\partial}\overline{\partial}u)(X, Y, Z) = [X, \overline{\partial}u(Y, Z)]_{T'_S} - [Y, \overline{\partial}u(X, Z)]_{T'_S} + [Z, \overline{\partial}u(X, Y)]_{T'_S}$$
$$- \overline{\partial}u([X, Y]_S, Z) + \overline{\partial}u([X, Z]_S, Y) - \overline{\partial}u([Y, Z]_S, X)$$

While

$$[X, \overline{\partial} u(Y, Z)]_{T'_S} = [X, [Y, u(Z)]_{T'_S} - [Z, u(Y)]_{T'_S} - u([Y, Z]_S)]_{T'_S}$$

$$[Y, \overline{\partial} u(X, Z)]_{T'_S} = [Y, [X, u(Z)]_{T'_S} - [Z, u(X)]_{T'_S} - u([X, Z]_S)]_{T'_S}$$

$$[Z, \overline{\partial} u(X, Y)]_{T'_S} = [Z, [X, u(Y)]_{T'_S} - [Y, u(X)]_{T'_S} - u([X, Y]_S)]_{T'_S}$$

Combined these,

$$\begin{aligned}(\overline{\partial}\overline{\partial} u)(X, Y, Z) = &\{[X, [Y, u(Z)]_{T'_S} - [Z, u(Y)]_{T'_S} - u([Y, Z]_S)]_{T'_S}\}\\ &- \{[Y, [X, u(Z)]_{T'_S} - [Z, u(X)]_{T'_S} - u([X, Z]_S)]_{T'_S}\}\\ &+ \{[Z, [X, u(Y)]_{T'_S} - [Y, u(X)]_{T'_S} - u([X, Y]_S)]_{T'_S}\}\\ &- \{[[X, Y]_S, u(Z)]_{T'_S} - [Z, u([X, Y]_S)]_{T'_S} - u([[X, Y]_S, Z]_S)\}\\ &+ \{[[X, Z]_S, u(Y)]_{T'_S} - [Y, u([X, Z]_S)]_{T'_S} - u([[X, Z]_S, Y]_S)\}\\ &- \{[[Y, Z]_S, u(X)]_{T'_S} - [X, u([Y, Z]_S)]_{T'_S} - u([[Y, Z]_S, X]_S)\}\end{aligned} \tag{5.1}$$

We see that the $\mathbf{C} \otimes \zeta$ part of $(\overline{\partial}\overline{\partial} u)(X, Y, Z)$ vanishes. First, our u takes its value in $\overline{S}$ (by the definition of u). Hence it is enough to show thatthe $\mathbf{C} \otimes \zeta$ part of

$$\begin{aligned}\{[X, [Y, u(Z)]_{T'_S} - [Z, u(Y)]_{T'_S} - u([Y, Z]_S)]_{T'_S}\} &- \{[Y, [X, u(Z)]_{T'_S} - [Z, u(X)]_{T'_S} - u([X, Z]_S)]_{T'_S}\}\\ &+ \{[Z, [X, u(Y)]_{T'_S} - [Y, u(X)]_{T'_S} - u([X, Y]_S)]_{T'_S}\}\\ &- \{[[X, Y]_S, u(Z)]_{T'_S} - [Z, u([X, Y]_S)]_{T'_S}\}\\ &+ \{[[X, Z]_S, u(Y)]_{T'_S} - [Y, u([X, Z]_S)]_{T'_S}\}\\ &- \{[[Y, Z]_S, u(X)]_{T'_S} - [X, u([Y, Z]_S)]_{T'_S}\}\end{aligned} \tag{5.2}$$

vanishes. This becomes

$$\begin{aligned}\{[X, [Y, u(Z)]_{T'_S} - [Z, u(Y)]_{T'_S}]_{T'_S}\} &- \{[Y, [X, u(Z)]_{T'_S} - [Z, u(X)]_{T'_S}]_{T'_S}\}\\ &+ \{[Z, [X, u(Y)]_{T'_S} - [Y, u(X)]_{T'_S}]_{T'_S}\}\\ &- \{[[X, Y]_S, u(Z)]_{T'_S}]_{T'_S}\}\\ &+ \{[[X, Z]_S, u(Y)]_{T'_S}]_{T'_S}\}\\ &- \{[[Y, Z]_S, u(X)]_{T'_S}]_{T'_S}\}\end{aligned} \tag{5.3}$$

While because of partially integrability, for example,

$$\begin{aligned}[X, [Y, u(Z)]_{T'_S}]_{\mathbf{C}\otimes\zeta} &= [X, [Y, u(Z)]_{T'_S} + [Y, u(Z)]_S]_{\mathbf{C}\otimes\zeta}\\ &= [X, [Y, u(Z)]]_{\mathbf{C}\otimes\zeta}\end{aligned} \tag{5.4}$$

And so on.

Hence, by the Jacobi identity, the $\mathbf{C} \otimes \zeta$ part of (5.1) vanishes.

□

Then, like the case integrable almost CR structures, we have a " differential complex" (in our case, $\bar{\partial}^{(1)}_{T'_S} \mathcal{D} \neq 0.$).

Even though "our differential complex" is not an actual differential complex, we have the harmonic decomposition on $\Gamma(M, E^1_S)$ by the standard functional analysis argument.

Theorem 5.2. *Assume that $\dim_{\mathbf{R}} M = 2n - 1 \geqq 5$ and (M, S) is a compact strongly pseudo convex partially integrable CR structure. Then, on $\Gamma(M, E^1_S)$, we have the harmonic type operator H_S and the Neumann type operator N_S which satisfies:*

$$(1)\ H_S N_S = N_S H_S = 0, \tag{5.5}$$

$$(2)\ \textit{for}\ \phi \in \Gamma(M, E^1_S),\ \ \phi = \Box_S N_S \phi + H_S \phi, \tag{5.6}$$

where $\Box_S = \mathcal{D}\mathcal{D}^\star + \bar{\partial}^{(1)\star}_{T'_S} \bar{\partial}^{(1)}_{T'_S}$.

Theorem 5.3. *Assume that $\dim_{\mathbf{R}} M = 2n - 1 \geqq 7$ and (M, S) is a compact strongly pseudo convex partially integrable CR structure. Then, on $\Gamma(M, E^2_S)$, we have the harmonic type operator H_S and the Neumann type operator N_S which satisfies:*

$$(1)\ H_S N_S = N_S H_S = 0, \tag{5.7}$$

$$(2)\ \textit{for}\ \phi \in \Gamma(M, E^2_S),\ \ \phi = \Box_S N_S \phi + H_S \phi, \tag{5.8}$$

where $\Box_S = \bar{\partial}^{(1)}_{T'_S} \bar{\partial}^{(1)\star}_{T'_S} + \bar{\partial}^{(2)\star}_{T'_S} \bar{\partial}^{(2)}_{T'_S}$.

6 Obstruction

We set the element $obstr \in \Gamma(M, \bar{S} \otimes \bigwedge^2 S^\star)$ by

$$obstr(X, Y) = [X, Y]_{\bar{S}},\ \ X, Y \in \Gamma(M, S).$$

Theorem 6.1. *$obstr$ is of $\Gamma(M, E^2_S)$ and $\bar{\partial}^{(2)}_{T'_S} obstr = 0$.*

Proof. By the definition of the partially integrable almost CR structure $\{(M, S), \theta\}$,

$$obstr \in \Gamma(M, \bar{S} \otimes \bigwedge^2 S^\star).$$

We have to show $(\bar{\partial}^{(2)}_S obstr)_{\mathbf{C}\otimes\zeta} = 0$. While, in this case, we can show $\bar{\partial}^{(2)}_{T'_S} obstr = 0$. In fact, for $X, Y, Z \in \Gamma(M, S)$,

$$\begin{aligned}\bar{\partial}^{(2)}_{T'_S} obstr(X, Y, Z) = {} & [X, obstr(Y, Z)]_{T'_S} - [Y, obstr(X, Z)]_{T'_S} + [Z, obstr(X, Y)]_{T'_S} \\ & - obstr([X, Y]_S, Z) + obstr([X, Z]_S, Y) - obstr([Y, Z]_S, X)\end{aligned}$$

So

$$\overline{\partial}^{(2)}_{T'_S} obstr(X,Y,Z) = [X, obstr(Y,Z)]_{T'_S} - [Y, obstr(X,Z)]_{T'_S} + [Z, obstr(X,Y)]_{T'_S}$$
$$- [[X,Y]_S, Z]_{\overline{S}} + [[X,Z]_S, Y]_{\overline{S}} - [[Y,Z]_S, X]_{\overline{S}}$$

As our almost CR structure is partially integrable, this becomes

$$[X,[Y,Z]_{\overline{S}}]_{T'_S} - [Y,[X,Z]_{\overline{S}}]_{T'_S} + [Z,[X,Y]_{\overline{S}}]_{T'_S}$$
$$- [[X,Y]_S, Z]_{\overline{S}} + [[X,Z]_S, Y]_{-}[[Y,Z]_S, X]_{\overline{S}} \tag{6.1}$$

So

$$[X,[Y,Z]_{S+\overline{S}}]_{T'_S} - [Y,[X,Z]_{S+\overline{S}}]_{T'_S} + [Z,[X,Y]_{S+\overline{S}}]_{T'_S}$$

Again, by the partial integrablity,

$$[Y,Z]_{S+\overline{S}} = [Y,Z],\ [X,Z]_{S+\overline{S}} = [X,Z],\ [X,Y]_{S+\overline{S}} = [X,Y].$$

Hence by the Jacobi identity, the above becomes 0.

□

7 P.d.e and a variation formula

We would like to find a $\psi \in \Gamma(M, \overline{S} \otimes S^*)$ satisfying: $(M, {}^{\psi}S)$ is integrable. Here

$${}^{\psi}S = \{X' : X' = X + \psi(X), X \in S\}.$$

This becomes solving a partial differential equation. We see this. ${}^{\psi}S$ is integrable iff

$$\text{for any } X', Y' \in {}^{\psi}S,\ [X', Y'] \in {}^{\psi}S.$$

This means that for any $X, Y \in S$, there is a Z of S,

$$[X + \psi(X), Y + \psi(Y)] = Z + \psi(Z).$$

We can solve Z. That is to say, $Z = [X + \psi(X), Y + \psi(Y)]_S$, where $[X + \psi(X), Y + \psi(Y)]_S$ means the projection of $[X + \psi(X), Y + \psi(Y)]$ to S with respect to (2.4). Hence ${}^{\psi}S$ is integrable iff for any $X, Y \in S$,

$$[X + \psi(X), Y + \psi(Y)] = [X + \psi(X), Y + \psi(Y)]_S + \psi([X + \psi(X), Y + \psi(Y)]_S).$$

We write this. For any $X, Y \in S$,

$$[X + \psi(X), Y + \psi(Y)]_{T'_S} - \psi([X + \psi(X), Y + \psi(Y)]_S) = 0.$$

Here $[X + \psi(X), Y + \psi(Y)]_{T'_S}$ means the T'_S part of $[X + \psi(X), Y + \psi(Y)]$ with respect to (4.4). The 0-the order part with respect to ψ is

$$[X,Y]_{T'_S} = [X,Y]_{\overline{S}},\ \text{which is introduced in Sect.6 as } obstr.$$

The 1-st order part is

$$[X, \psi(Y)]_{T'_S} - [Y, \psi(X)]_{T'_S} - \psi([X,Y]_S).$$

The 2nd order part is

$$[\psi(X), \psi(Y)]_{T'_S} - \psi([X, \psi(Y)]_S + [\psi(X), Y]_S),$$

and the 3rd order part is

$$-\psi([\psi(X), \psi(Y)]_S)$$

Both terms are of $\Gamma(M, E_S^2)$ and include first order derivative with respect to S and $\overline{S}$(we note that the they are only $S\psi$, $\overline{S}\psi$, and $\zeta\psi$ term doesn't appear. Hence it is rather good to handle). Therefore we have

Proposition 7.1. *$(M, {}^{\psi}S)$ is integrable iff $\psi \in \Gamma(M, E_S^1)$ satisfies*

$$obstr + \overline{\partial}_{T_S'}^{(1)}\psi + R(\psi) = 0. \tag{7.1}$$

We see our partial differencial equation more. For $\psi \in \Gamma(M, E_S^1)$, we consider the modified almost partially integrable CR structure $(M, {}^{\psi}S)$ and set the C^∞ vector bundle decomposion

$$\mathbf{C} \otimes TM = {}^{\psi}S + \overline{{}^{\psi}S} + \mathbf{C} \otimes \zeta$$

Then, consider $obstr(\psi)(X, Y) = [X+\psi(X), Y+\psi(Y)]_{\overline{\psi S}}$, where $X, Y \in \Gamma(M, S)$. Here $[X+\psi(X), Y+\psi(Y)]_{\overline{\psi S}}$ means the projection of $[X + \psi(X), Y + \psi(Y)]$ according to the C^∞ vector bundle decomposition.

Obviously, if $obstr(\psi)$ vanishes, our modified alomost partial CR structure is actual CR structure. By these consideration, we propose to study

$$obstr(\psi) = obstr + \overline{\partial}_{T_S'}^{(1)}\psi + R(\psi).$$

8 An approximate solution

In Section 7, we propose a partial differential equation

$$obstr + \overline{\partial}_{T_S'}^{(1)}\psi + R(\psi) = 0,$$

for $\psi \in \Gamma(M, E_S^1)$. As an approximate solutiuon, we adapt $\psi^{(1)} = -\overline{\partial}_{T_S'}^{(1)\star} N_S obstr$. Then,

$$obstr(\psi^{(1)}) = -\overline{\partial}_{T_S'}^{(1)\star}\overline{\partial}_{T_S'}^{(1)} N_S obstr + R(-\overline{\partial}_{T_S'}^{(1)\star} N_S obstr). \tag{8.1}$$

We see that our $\psi^{(1)}$ is a first approximation for the proposed equation. As for our norm, we use $\| \|'_{(m)}$ (m is an integer, satisfying: $m \geq n$), which is successfully, introduced in [A1]. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a local finite coordinate covering of M. Let $\{\rho_\lambda\}_{\lambda\in\Lambda}$ be a partition of unity subordinate to this covering. For a C^∞ function f on M, we set

$$\|f\|'_{(m)} = \sum_{\lambda\in\Lambda}\sum_{k=1}^{n-1}\|\rho_\lambda(e_k f)\|_{(m)} + \sum_{\lambda\in\Lambda}\sum_{k=1}^{n-1}\|\rho_\lambda(\overline{e}_k f)\|_{(m)} + \|f\|_{(m)},$$

where $\| \|$ means the Sobolev m norm. Henceforth we omit ρ_λ. By using these, we can introduce $\| \|'_{(m)}$ norm on $\Gamma(M, \overline{S} \otimes \wedge^2 S^\star)$. If $p = 2$, $dim_{\mathbf{R}}M = 2n - 1 \geqq 7$, because of strongly pseudo convexity, our norm is equivalent

$$\|\overline{\partial}_{T_S'}^{(2)} u\|_{(m)} + \|\overline{\partial}_{T_S'}^{(1)\star} u\|_{(m)} + \|u\|_{(m)}$$

for $u \in \Gamma(M, \overline{S} \otimes \wedge^2 S^\star)$.

Henceforth, we abbreviate several subscripts. For example, $\overline{\partial} = \overline{\partial}_{T_S'}$, $N = N_S$,..

$\underline{(A1)}$

$$\|XNu\|'_{(m)} \leqq c_m\|u\|_{(m)}$$

for $u \in \Gamma(M, \overline{S} \otimes \wedge^2 S^\star)$ and c_m is a constant, which doesn't depend on u. Here $X = e_1, .., e_{n-1}, \overline{e}_1, .., \overline{e}_{n-1}$ (X is a $S + \overline{S}$ - valued partial differential operator). We note that $\overline{\partial}^\star$ is the formal adjoint of $\overline{\partial}$ with respect to the

Levi metric(therefore this ia a first differential operator). While the actual adjoint with respect to $\|\|_{(m)}$ norm must be

$$\overline{\partial}^{\star} + 0\text{-th order operator}$$

by the functional analysis method(obviously, this may not be a differential operator). But in our problem(for estimates), these term can be absorbed in a lower term(negligible term). So, in this paper, we treat $\overline{\partial}^{\star}$ as the adjoint operator. By the subellptic estimate

$$\|X_i X_j v\|_{(m)} \leqq c_m(\|\Box v\|_{(m)} + \|v\|_{(m)})$$

for $v \in \Gamma(M, \overline{S} \otimes \wedge^2 S^{\star})$ and c_m is a constant which doesn't depend on v. And $X_i, X_j = e_1, .., e_{n-1}, \overline{e}_1, .., \overline{e}_{n-1}$. In this inequality, set $v := Nu$. Then, we have our inequality. And also

<u>$(A2)$</u>

$$\|NYu\|'_{(m)} \leqq c_m\|u\|_{(m)}$$

for $u \in \Gamma(M, \overline{S} \otimes \wedge^2 S^{\star})$ and c_m is a constant, which doesn't depend on u. Here $Y = e_1, .., e_{n-1}, \overline{e}_1, .., \overline{e}_{n-1}$. By the above fact,

$$\|NYu\|'^2_{(m)} \simeq (\overline{\partial}NYu, \overline{\partial}NYu)_{(m)} + (\overline{\partial}^{\star}Yu, \overline{\partial}^{\star}Yu)_{(m)} + (u, u)_{(m)},$$

where $(,)_{(m)}$ means the Sobolev m inner product. Then,

$$(\overline{\partial}NYu, \overline{\partial}NYu)_{(m)} \sim (\overline{\partial}^{\star}\overline{\partial}NYYu, NYu)_{(m)}$$

We note that $\sim$ means modulo "negligible terms". And also

$$(\overline{\partial}^{\star}NYu, \overline{\partial}^{\star}NYu)_{(m)} \sim (\overline{\partial}\overline{\partial}^{\star}NYYu, NYu)_{(m)}$$

Hence

$$\|NYu\|'^2_{E} : E(m) \simeq (\Box Nu, NYu)_{(m)} + \|u\|^2_{(m)}$$

As $\Box Nu = u$,

$$\|NYu\|'^2_{(m)} < (\text{large constant})\|u\|^2_{(m)} + (\text{small constant})\|NYu\|^2_{(m)}.$$

Hence we have $(A2)$. By the similar method, we have

<u>$(A3)$</u>

$$\|X_i X_j X_k Nu\|_{(m)} \leqq c_m\|u\|'_{(m)}$$

Here $X_i, X_j, X_k = e_1, .., e_{(n-1)}, \overline{e}_1, ..., \overline{e}_{(n-1)}$.

And

<u>$(B1)$</u> By the Sobolev lemma,

$$\|fg\|'_{(m)} \leqq c_m\|f\|'_{(m)}\|g\|'_{(m)}$$

(by $m \geqq n$)

8.1 First term

We see the first term of (8.1). That is to say, $-\overline{\partial}^{\star}\overline{\partial}Nobstr$. First, we note that: even though $\Box N = N\Box$, we can't expect $\overline{\partial}^{\star}\overline{\partial}N = N\overline{\partial}^{\star}\overline{\partial}$. However,

Proposition 8.1. *For $u \in \Gamma(M, E^2_S)$,*

$$\|(\overline{\partial}^{\star}\overline{\partial}N - N\overline{\partial}^{\star}\overline{\partial})u\|'_{(m)} \leqq c_m\|obstr\|'_{(m)}\|u\|_{(m)}.$$

where c_m is a constant which doesn't depend on u.

Proof. As $u = \Box Nu$,

$$\begin{aligned}
\bar\partial^*\bar\partial Nu - N\bar\partial^*\bar\partial u &= \bar\partial^*\bar\partial N(\Box Nu) - N\bar\partial^*\bar\partial(\Box Nu) \\
&= \bar\partial^*\bar\partial(N\Box)Nu - N\bar\partial^*\bar\partial\Box Nu \\
&= \bar\partial^*\bar\partial Nu - N\bar\partial^*\bar\partial\Box Nu \\
&= \bar\partial^*\bar\partial Nu - (N\Box\bar\partial^*\bar\partial Nu + N[\bar\partial^*\bar\partial, \Box]Nu) \\
&= -N[\bar\partial^*\bar\partial, \Box]Nu \\
&= -N(\bar\partial^*\overline{\partial\partial\partial}^* - \overline{\partial\partial}^*\bar\partial^*\bar\partial)Nu
\end{aligned}$$

While, by the simple calculation,

Lemma 8.2. *For $\alpha \in \Gamma(M, E_S^1)$,*

$$\overline{\partial\partial}\alpha = (obstr)\alpha.$$

By taking the adjoint, for $\beta \in \Gamma(M.E_S^3)$,

$$\bar\partial^*\bar\partial^*\beta = \overline{(obstr)}\beta + (\textit{negligible term})$$

It might be better to comment about this lemma. For $u \in \Gamma(M, T'_S)$, $X, Y \in S$,

$$\overline{\partial\partial}u(X, Y) = [X, [Y, u]_{T'_S}]_{T'_S} - [Y, [X, u]_{T'_S}]_{T'_S} - [[X, Y]_S, u]_{T'_S}$$

By the Jacobi identity,

$$\begin{aligned}
\text{the above} &= [[X, Y]_{T'_S}, u]_{T'_S} - [X, [Y, u]_S]_{T'_S} + [Y, [X, u]_S]_{T'_S} \\
&= [obstr(X, Y), u]_{T'_S} - obstr(X, [Y, u]_S) + obstr(Y, [X, u]_S)
\end{aligned}$$

While even though α is a 1-form, we have a similar equality(a little bit complicated). Here, $\overline{\partial\partial}\alpha$ includes the first derivative($\bar S$ -direction) of $obstr$ and α both, but, we note that $obstr$ takes its value in $\bar S$. Hence,

$$N\bar\partial^*\overline{\partial\partial\partial}^*Nu = N\bar\partial^*(obstr)\bar\partial^*Nu$$

is estimated as follows.

$$\begin{aligned}
\| - N\bar\partial^*(obstr)(\bar\partial^*Nu)\|'_{(m)} &\leqq \|(obstr)(\bar\partial^*Nu)\|_{(m)} \text{ by (A2)} \\
&\leqq \|obstr\|'_{(m)}\|u\|_{(m)} \text{ by (B2)}
\end{aligned}$$

For $\overline{\partial\partial}^*\bar\partial^*\bar\partial Nu$, we have the same estimate. This proposition holds for general strongly pseudo convex almost CR structures(it is not necessary to assume the cohomological condition and even partial integrability). Namely, without any proof, the author mentions

$$\|(\bar\partial^*\bar\partial N - N\bar\partial^*\bar\partial)u\|'_{(m)} \leqq c_m\|obstr\|'_{(m)}\|u\|'_{(m)},$$

where c_m is a constant which doesn't depend on u.

Unfortunately, as far as I know, there is no reference. But these are a kind of well known results.

And in our case, because of partial integrability, $obstr$ takes its value in $\bar S$ and so we have a better estimate. □

Hence

Proposition 8.3. *For $u \in \Gamma(M, E_S^2)$, which satisfies $\bar\partial u = 0$,*

$$\|\bar\partial^*\bar\partial Nu\|'_{(m)} \leqq c_m\|obstr\|'_{(m)}\|u\|_{(m)},$$

where c_m is a constant which doesn't depend on u.

8.2 Second term

We discuss $R(-\overline{\partial}^{*}Nobstr)$. We estimate $R(-\overline{\partial}^{*}Nu)$ for $u \in \Gamma(M, \overline{S}\otimes\wedge^{2}S^{*})$. In Sect.7, we write down $R(v)$, where $v \in \Gamma(M, E_{S}^{1})$, quite precisely. And the crucial term of $R(v)$ is the second order term like

$$\sum(\text{some function})(X_i v_j)v_k$$

where X_i takes its value in $\overline{S}$. And v_i's are component of $v \in \Gamma(M, E_{S}^{1})$. (The third order term of $R(v)$ doesn't include any derivatives.)

For $u \in \Gamma(M, E_{S}^{2})$, we set

$$v = -\overline{\partial}Nu.$$

Then, by $\underline{(A1)}$ and $\underline{(A3)}$ with $\underline{(B1)}$,

$$\|R(-\overline{\partial}Nu)\|_{(m)}^{'} \leqq c_m(\|u\|_{(m)}^{'2} + \|u\|_{(m)}^{3}),$$

where c_m doesn't depend on u.

With these consideration(the first term and the secon term), we see that our first approximation satisfies

$$\|obstr(\psi^{(1)})\|_{(m)}^{'} \leqq c_m\|obstr\|_{(m)}^{'2},$$

if $\|obstr\|_{(m)}^{'}$ is sufficiently small.

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