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Jie Liu, Wenhao Ou, and Xiaokui Yang*

Strictly nef vector bundles and characterizations of $\mathbb{P}^n$

Abstract: In this note, we give a brief exposition on the differences and similarities between strictly nef and ample vector bundles, with particular focus on the circle of problems surrounding the geometry of projective manifolds with strictly nef bundles.

1 Introduction

Let X be a complex projective manifold. A line bundle L over X is said to be *strictly nef* if

$$L \cdot C > 0$$

for each irreducible curve $C \subset X$. This notion is also called "numerically positive" in literatures (e.g. [25]). The Nakai-Moishezon-Kleiman criterion asserts that L is ample if and only if

$$L^{\dim Y} \cdot Y > 0$$

for every positive-dimensional irreducible subvariety Y in X. Hence, ample line bundles are strictly nef. In 1960s, Mumford constructed a number of strictly nef but non-ample line bundles over ruled surfaces (e.g. [25]), and they are tautological line bundles of stable vector bundles of degree zero over smooth curves of genus $g \geq 2$. By using the terminology of Hartshorne ([24]), a vector bundle $E \to X$ is called strictly nef (resp. ample) if its tautological line bundle $\mathcal{O}_E(1)$ is strictly nef (resp. ample). One can see immediately that the strictly nef vector bundles constructed by Mumford are actually *Hermitian-flat*. Therefore, some functorial properties for ample bundles ([24]) are not valid for strictly nef bundles. In this note, we give a brief exposition on the differences and similarities between strict nefness and ampleness, and survey some recent progress on understanding the geometry of projective manifolds endowed with some strictly nef bundles.

Starting in the mid 1960's, several mathematicians–notably Grauert, Griffiths and Hartshorne ([21–24])–undertook the task of generalizing to vector bundles the theory of positivity for line bundles. One of the goals was to extend to the higher rank setting as many as possible of the beautiful cohomological and topological properties enjoyed by ample divisors. In the past half-century, a number of fundamental results have been established. For this rich topic, we refer to the books [31, 32] of Lazarsfeld and the references therein.

Jie Liu: Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China, E-mail: jliu@amss.ac.cn

Wenhao Ou: Wenhao Ou, Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China, E-mail: wenhaoou@amss.ac.cn

*Corresponding Author: Xiaokui Yang: Xiaokui Yang, Department of Mathematics and Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China, E-mail: xkyang@mail.tsinghua.edu.cn

1.A Abstract strictly nef vector bundles.

Let's recall a criterion for strictly nef vector bundles (see [35, Proposition 2.1]) which is analogous to the Barton-Kleiman criterion for nef vector bundles (e.g. [32, Proposition 6.1.18]). This criterion will be used frequently in the sequel.

Proposition 1.1. *Let E be a vector bundle over a projective manifold X. Then the following conditions are equivalent.*

(1) E is strictly nef.
(2) For any smooth projective curve C with a finite morphism $v : C \to X$, and for any line bundle quotient $v^\star(E) \to L$, one has

$$\deg L > 0.$$

Recall that for an ample line bundle L, one has the Kodaira vanishing theorem

$$H^i(X, L^\star) = 0 \ \text{ for } \ i < \dim X.$$

For an ample vector bundle E with rank $r \geq 2$, one can only deduce

$$H^0(X, E^\star) = 0$$

and the higher cohomology groups $H^i(X, E^\star)$ ($i \geq 1$) may not vanish. By using Proposition 1.1, we obtain a similar vanishing theorem for strictly nef vector bundles.

Theorem 1.2. *Let E be a strictly nef vector bundle over a projective manifold X. Then*

$$H^0(X, E^\star) = 0.$$

It is worth pointing out that for a strictly nef vector bundle E with rank $r \geq 2$, the cohomology group $H^0(X, \mathrm{Sym}^k E^\star)$ may not vanish for $k \geq 2$, which is significantly different from properties of ample vector bundles. Indeed, strict nefness is not closed under tensor product, symmetric product or exterior product of vector bundles. This will be discussed in Section 2.

It is well-known that vector bundles over $\mathbb{P}^1$ split into direct sums of line bundles. By using Proposition 1.1 again, one deduces that strictly nef vector bundles over $\mathbb{P}^1$ are ample. In [35, Theorem 3.1], the following result is obtained.

Corollary 1.3. *If E is a strictly nef vector bundle over an elliptic curve C, then E is ample.*

As we mentioned before, over smooth curves of genus $g \geq 2$, there are strictly nef vector bundles which are Hermitian-flat. There also exist strictly nef but non-ample bundles on some rational surfaces ([13, 30]). It is still a challenge to investigate strictly nef vector bundles over higher dimensional projective manifolds. We propose the following conjecture, which is also the first step to understand such bundles.

Conjecture 1.4. *Let E be a strictly nef vector bundle over a projective manifold X. If $-K_X$ is nef, then $\det E$ is ample.*

Although this conjecture is shown to be a consequence of the "*generalized abundance conjecture*" (e.g. [33, 34]), we still expect some other straightforward solutions. Indeed, we get a partial answer to it.

Theorem 1.5. *Let E be a strictly nef vector bundle over a projective manifold X. If $-K_X$ is nef and big, then $\det E$ is ample.*

We refer to [6, 26] for more details on positivity of equivariant vector bundles.

1.B The geometry of projective manifolds endow with strictly nef bundles.

Since the seminal works of Mori and Siu-Yau ([42], [49]) on characterizations of projective spaces, it becomes apparent that the positivity of the tangent bundle of a complex projective manifold carries important geometric information. In the past decades, many remarkable generalizations have been established, as, for instance, Mok's uniformization theorem on compact Kähler manifolds with semipositive holomorphic bisectional curvature ([41]) and fundamental works of Campana, Demailly, Peternell and Schneider ([7], [16], [46]) on the structure of projective manifolds with nef tangent bundles. For this comprehensive topic, we refer to [7, 10–12, 12, 16, 17, 35, 41, 42, 44, 46, 49, 57] and the references therein.

As we pointed out before, strict nefness is a notion of positivity weaker than ampleness. Even though there are significant differences between them, we still expect that strict nefness and ampleness could play similar roles in many situations. The following result, which extends Mori's Theorem, is obtained in [35, Theorem 1.4].

Theorem 1.6. *Let X be a projective manifold. If T_X is strictly nef, then X is isomorphic to a projective space.*

Therefore, T_X is ample if and only if it is strictly nef. However, this is not valid for cotangent bundles. Indeed, Shepherd-Barron proved in [48] that there exists a projective surface whose cotangent bundle is strictly nef but not ample (see e.g. Example 4.2).

Let us consider manifolds with strictly nef canonical or anti-canonical bundles. Campana and Peternell proposed in [7, Problem 11.4] the following conjecture, which is still a major problem along this line.

Conjecture 1.7. *Let X be a projective manifold. If K_X^{-1} is strictly nef, then X is Fano.*

This conjecture has been verified for projective manifolds of dimension 2 in [40] and dimension 3 in [47] (see also [52] and the references therein). Recently, some progress has been achieved in [35, Theorem 1.2].

Theorem 1.8. *If K_X^{-1} is strictly nef, then X is rationally connected.*

Indeed, we show in [39] that if (X, Δ) is a projective simple normal crossing pair and $-(K_X + \Delta)$ is strictly nef, then X is rationally connected.

The following dual version of Conjecture 1.7 is actually a consequence of the abundance conjecture.

Conjecture 1.9. *If K_X is strictly nef, then K_X is ample.*

As analogous to the Fujita conjecture, Serrano proposed in [47] the following conjecture, which is a generalization of Conjecture 1.7.

Conjecture 1.10. *Let X be a projective manifold. If L is a strictly nef line bundle, then $K_X \otimes L^{\otimes m}$ is ample for $m \geq \dim X + 2$.*

This conjecture has been solved for projective surfaces in [47]. For the progress on projective threefolds and higher dimensional manifolds, we refer to [9, Theorem 0.4] and the references therein.

It is also known that the existence of "positive" subsheaves of the tangent bundle can also characterize the ambient manifold. For instance, Andreatta and Wiśniewski established in [1, Theorem] that if the tangent bundle T_X of a projective manifold X contains a locally free ample subsheaf $\mathcal{F}$, then X is isomorphic to a projective space. When $\mathcal{F}$ is a line bundle, this result is proved by Wahl in [54], and in [8], Campana and Peternell established the cases rank($\mathcal{F}$) $\geqslant \dim X - 2$. It is also shown that the assumption on the local freeness can be dropped ([2, 37]). On the other hand, according to Mumford's construction (see Example 2.5), Andreatta-

Wiśniewski's result does not hold if the subsheaf $\mathcal{F}$ is assumed to be strictly nef. Indeed, we obtained in [38, Theorem 1.3] the following result.

Theorem 1.11. *Let X be a projective manifold. Assume that the tangent bundle T_X contains a locally free strictly nef subsheaf $\mathcal{F}$ of rank r. If X is not isomorphic to a projective space, then X admits a $\mathbb{P}^d$-bundle structure $\varphi \colon X \to T$ for some integer $d \geq r$ where T is a hyperbolic projective manifold of general type.*

Recall that T is said to be hyperbolic if every holomorphic map from $\mathbb{C}$ to it is constant. We expect a stronger geometric positivity on the cotangent bundle of the base T in Theorem 1.11. Moreover, we obtain in [38, Theorem 1.4] a characterization of projective spaces.

Theorem 1.12. *Let X be an n-dimensional complex projective manifold such that T_X contains a locally free strictly nef subsheaf $\mathcal{F}$. If $\pi_1(X)$ is virtually solvable, then X is isomorphic to $\mathbb{P}^n$, and $\mathcal{F}$ is isomorphic to either $T_{\mathbb{P}^n}$ or $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.*

There are many other characterizations of projective spaces for which we refer to [3, 4, 8, 18, 19, 27, 28, 36, 45, 51, 55] and the references therein.

2 Basic properties and examples

In this section, we investigate basic properties of strictly nef bundles and discuss some examples. As we mentioned before, Mumford constructed a strictly nef vector bundle which is not ample (see [25, Chapter I, Example 10.6]). We shall describe this example in details. Let E be a rank 2 vector bundle over a smooth curve C of genus $g \geq 2$, $X = \mathbb{P}(E)$ be the projectivized bundle and $\pi : \mathbb{P}(E) \to C$ be the projection. Let $\mathcal{O}_E(1)$ be the tautological line bundle of $\mathbb{P}(E)$ and D be the corresponding divisor over X.

Lemma 2.1. *[25, Chapter I, Proposition 10.2] For any $m > 0$, there is a one-to-one correspondence between*

(1) *effective curves Y in X, having no fibers as components, of degree m over C; and*
(2) *sub-line bundles L of $\mathrm{Sym}^m E$.*

Moreover, under this correspondence, one has

$$D \cdot Y = m \deg(E) - \deg(L). \tag{2.1}$$

For any effective curve Y in X, we denote by $m(Y)$ the degree of Y over C. Then there is an exact sequence

$$0 \to Pic(C) \xrightarrow{\pi^*} Pic(X) \xrightarrow{m} \mathbb{Z} \to 0.$$

It follows that the divisors on X, modulo numerical equivalence, form a free abelian group of rank 2, generated by D and F where F is any fiber of $\mathbb{P}(E)$.

Lemma 2.2. *[25, Chapter I, Theorem 10.5] Let C be a smooth curve of genus $g \geq 2$.*

(1) *If E is a stable vector bundle, then every symmetric power $\mathrm{Sym}^k E$ is semi-stable.*
(2) *For any $r > 0$ and $d \in \mathbb{Z}$, there exists a stable vector bundle with rank r and degree d such that all symmetric powers $\mathrm{Sym}^k E$ are stable.*

Theorem 2.3. *[25, Chapter I, Example 10.6] Let E be a rank 2 vector bundle over a smooth curve of genus $g \geq 2$. If $\deg(E) = 0$ and all symmetric powers $\mathrm{Sym}^k E$ are stable, then E is a strictly nef vector bundle, i.e. $\mathcal{O}_E(1)$ is a strictly nef line bundle. Moreover E is not ample.*

Proof. Let Y be an arbitrary irreducible curve on X. If Y is a fiber, then $D \cdot Y = 1$. If Y is an irreducible curve of degree $m > 0$ over C, then by Lemma 2.1, Y is corresponding to a sub-line bundle L of $\mathrm{Sym}^m E$. Note that since $\mathrm{Sym}^m E$ are stable and of degree zero for all $m \geq 1$, we have

$$\deg(L) < \frac{\deg\left(\mathrm{Sym}^m E\right)}{\mathrm{rank}\left(\mathrm{Sym}^m E\right)} = 0.$$

Therefore, by formula (2.1)

$$D \cdot Y = m \deg(E) - \deg(L) = -\deg(L) > 0.$$

Hence, the line bundle $\mathcal{O}_E(1)$ of the divisor D is strictly nef, i.e. E is a strictly nef vector bundle. Since $\deg(E) = 0$, E cannot be ample. □

Lemma 2.4. *Let E be a vector bundle over a smooth curve C. If E is stable and $\deg(E) = 0$, then E admits a Hermitian-flat metric.*

Proof. Since E is stable over C, there exists a Hermitian-Einstein metric h on E (e.g. [53]), i.e. $g^{-1} \cdot R_{\alpha\bar{\beta}} = c \cdot h_{\alpha\bar{\beta}}$ for some constant c where g is a smooth metric on C. Since $\deg(E) = 0$, we deduce $c = 0$, i.e. (E, h) is Hermitian-flat. □

We summarize Mumford's example as following.

Example 2.5. Let C be a smooth curve of genus $g \geq 2$. There exists a rank 2 vector bundle $E \to C$ satisfying the following properties:

(1) $\deg(E) = 0$;
(2) $\mathrm{Sym}^k E$ are stable for all $k \geq 1$;
(3) E is strictly nef but not ample; E^* is strictly nef but not ample;
(4) E admits a Hermitian-flat metric.
(5) Let $X = \mathbb{P}(E)$, $\pi : X \to C$ be the projection and $\mathcal{O}_E(1)$ be the tautological line bundle. Then $T_{X/C} = K_{X/C}^{-1} \cong \mathcal{O}_E(2) \otimes \pi^* \det E^*$ is strictly nef.

Although the strict nefness is not closed under tensor products and wedge products, we still have the following properties by using the Barton-Kleiman type criterion (Proposition 1.1).

Proposition 2.6. *Let E and F be two vector bundles on a projective manifold X.*

(1) E is a strictly nef vector bundle if and only if for every smooth curve C and for any non-constant morphism $f : C \to X$, $f^* E$ is strictly nef.
(2) If E is strictly nef, then any non-zero quotient bundle Q of E is strictly nef.
(3) $E \oplus F$ is strictly nef if and only if both E and F are strictly nef.
(4) If the symmetric power $\mathrm{Sym}^k E$ is strictly nef for some $k \geq 1$, then E is strictly nef.
(5) Let $f : Y \to X$ be a finite morphism such that Y is a smooth projective variety. If E is strictly nef, then so is $f^* E$.
(6) Let $f : Y \to X$ be a surjective morphism such that Y is a smooth projective variety. If $f^* E$ is strictly nef, then E is strictly nef.

Example 2.7. Let E be the strictly nef vector bundle in Example 2.5.

(1) $\Lambda^2 E = \det E$ is numerically trivial and it is not strictly nef;

(2) the tensor product $E \otimes E^{\star}$ is not strictly nef since $H^0(C, E \otimes E^{\star}) \cong \mathbb{C}$;

(3) $E \oplus E^{\star}$ is strictly nef, but $\mathrm{Sym}^2(E \oplus E^{\star})$ is not strictly nef.

The following result is well-known and it will be used frequently in the sequel.

Lemma 2.8. *If L is a strictly nef and semi-ample line bundle, then it is ample.*

Proof. Since L is semi-ample, mL is globally generated for some large m. Let $\phi_m : X \to Y$ be the morphism defined by $|mL|$. Since L is strictly nef, it is easy to see that ϕ_m is finite and so L is ample. $\qquad\square$

Corollary 2.9. *Let L be a strictly nef line bundle. Then $H^0(X, L^{\star}) = 0$. Moreover, for any coheret sheaf $\mathcal{F}$, there exists a postive integer $m_0 = m_0(L, \mathcal{F})$ such that for $m \geqslant m_0$*

$$H^0(X, (L^{\star})^{\otimes m} \otimes \mathcal{F}) = 0. \tag{2.2}$$

Proof. If $H^0(X, L^{\star}) \neq 0$, then $L^{\star}$ is effective. Since L is nef, we deduce that L is trivial and this is a contradiction. Hence $H^0(X, L^{\star}) = 0$ and $L^{\star}$ is not pseudo-effective. By [56, Corollary 1.6], L is $(\dim X - 1)$-ample, i.e. for any coherent sheaf $\mathcal{F}'$,

$$H^n(X, L^{\otimes m} \otimes \mathcal{F}') = 0$$

for $m \geq m_1(L, \mathcal{F}')$. By Serre duality, the vanishing (2.2) holds. $\qquad\square$

Recall that for an ample line bundle L, one has the Kodaira vanishing theorem. For an ample vector bundle E, one can only deduce $H^0(X, E^{\star}) = 0$ and the higher cohomology groups $H^i(X, E^{\star})$ ($i \geqslant 1$) may not vanish. For instance, when $X = \mathbb{P}^n$ with $n \geqslant 2$ and $E = T\mathbb{P}^n$, one has

$$H^1(X, E^{\star}) \cong H^{1,1}(X, \mathbb{C}) \cong \mathbb{C} \neq 0.$$

For strictly nef vector bundles, we have a similar vanishing theorem.

Theorem 2.10. *Let E be a strictly nef vector bundle over a projective manifold X. Then*

$$H^0(X, E^{\star}) = 0.$$

Proof. Suppose $\sigma \in H^0(X, E^{\star})$ is a nonzero section. Then by [16, Proposition 1.16], σ does not vanish anywhere. This section gives a trivial subbundle of $E^{\star}$ and so a trivial quotient bundle of the strictly nef vector bundle E. This contradicts to Proposition 2.6 (2). $\qquad\square$

Note that the vanishing in (2.2) does not hold for higher rank vector bundles. Indeed, let E be the strictly nef vector bundle in Example 2.5, $X = \mathbb{P}(E)$ and $\pi : X \to C$ be the projection. Let $F = K_C^{\otimes m}$ for some sufficiently large m. Then

$$H^0(C, \mathrm{Sym}^k E^{\star} \otimes F) \neq 0$$

for all $k > 0$. Since $\mathrm{Sym}^k E^{\star}$ is a direct summand of $(E^{\star})^{\otimes k_1}$ for some large k_1, we obtain the non-vanishing $H^0(C, (E^{\star})^{\otimes k_1} \otimes F) \neq 0$ for strictly nef vector bundle E.

Remark 2.11. For a strictly nef vector bundle E with rank $r \geqslant 2$, in general, $H^0(X, \mathrm{Sym}^k E^{\star}) = 0$ dos not hold for $k \geqslant 2$.

We give more examples of strictly nef vector bundle over higher dimensional projective manifolds (see [38, Section 5] for details). A line bundle L over a projective variety X of dimension n is called *k- strictly nef* if

$$L^{\dim Y} \cdot Y > 0$$

for every irreducible subvariety Y in X with $0 < \dim Y \leq k$. Hence, 1-strictly nef is exactly strictly nef, and an n-strictly nef line bundle is ample.

Theorem 2.12. *[50, Lemma 3.2 and Theorem 6.1] Let C be a smooth curve of genus $g \geqslant 2$. Then for any $r \geqslant 2$, there exists a Hermitian flat vector bundle E of rank r such that the tautological line bundle $\mathcal{O}_E(1)$ is $(r-1)$-strictly nef. In particular, E is strictly nef.*

Fix a smooth curve C of genus $g \geqslant 2$. Let $r \geqslant 2$ and E be a vector bundle of rank r as in Theorem 2.12.

Example 2.13. Let $X = \mathbb{P}(E)$. Then we have the following relative Euler exact sequence

$$0 \to \mathcal{O}_X \to p^\star E^\star \otimes \mathcal{O}_E(1) \to T_{X/C} \to 0,$$

where $p : X = \mathbb{P}(E) \to C$ is the natural projection. It is shown in [38, Example 5.9] that $p^\star E^\star \otimes \mathcal{O}_E(1)$ is strictly nef.

Example 2.14. We consider the following extension of vector bundles

$$0 \to Q \to G \to E^\star \to 0,$$

where Q is a nef vector bundle of positive rank. Since $E^\star$ is Hermitian flat, it is numerically flat. In particular, $E^\star$ is nef and so is G ([16, Proposition 1.15]). Let $X = \mathbb{P}(G)$ and $p : X = \mathbb{P}(G) \to C$ be the natural projection. Then we have the following relative Euler sequence

$$0 \to \mathcal{O}_X \to p^\star G^\star \otimes \mathcal{O}_G(1) \to T_{X/C} \to 0.$$

Since E is a subbundle of $G^\star$, it follows that $F := p^\star E \otimes \mathcal{O}_G(1)$ is a subbundle of $p^\star G^\star \otimes \mathcal{O}_G(1)$. We proved in [38, Example 5.10] that $p^\star E \otimes \mathcal{O}_G(1)$ is strictly nef and the restriction of F to fibers of p is isomorphic to $\mathcal{O}_{\mathbb{P}^d}(1)^{\oplus r}$. In particular, F is not a subbundle of $T_{X/C}$.

3 Strictly nef vector bundles

In this section, we consider strictly nef vector bundles over higher dimensional projective manifolds. As pointed out in the previous sections, the properties of strictly nef vector bundles are closely related to the geometry of ambient manifolds. When X is $\mathbb{P}^1$ or an elliptic curve, a strictly nef line bundle L over X is ample. It is also known that every strictly nef line bundle over an abelian variety is ample ([47, Proposition 1.4]), and Chaudhuri proved in [13] that every strictly nef homogeneous bundle on a complex flag variety is ample. The following conjecture is proposed for strictly nef vector bundles which is a consequence of Serrano's Conjecture 1.10.

Conjecture 3.1. *Let E be a strictly nef vector bundle over a projective manifold X. If $-K_X$ is nef, then $\det E$ is ample.*

Proposition 3.2. *Conjecture 1.10 implies Conjecture 3.1.*

Proof. Suppose Conjecture 1.10 is valid. Let $Y = \mathbb{P}(E)$, $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and $\pi : Y \to X$ be the projection. For large m, $K_Y \otimes L^{\otimes m}$ is ample. Since $-K_X$ is nef, $K_{Y/X} \otimes L^{\otimes m}$ is ample. We know $\det \pi_\star \left(K_{Y/X} \otimes \left(K_{Y/X} \otimes L^{\otimes m} \right) \right)$ is ample and so is $\det E$. □

A special case of Conjecture 3.1 is established and we expect a direct proof of it by adapting similar ideas.

Theorem 3.3. *Let E be a strictly nef vector bundle over a projective manifold X. If $-K_X$ is nef and big, then $\det E$ is ample.*

Proof. If E is a strictly nef line bundle, then $E - K_X$ is nef and big. By Kawamata-Reid-Shokurov base point free theorem, E is semi-ample. Thanks to Lemma 2.8, E is ample. If E has rank $r \geqslant 2$, we consider the projective

bundle $Y = \mathbb{P}(E)$. Let $\mathcal{O}_E(1)$ be the tautological line bundle of the projection $\pi : Y \to X$. By the projection formula, we have

$$-K_Y = \mathcal{O}_E(r) \otimes \pi^\star(-K_X) \otimes \pi^\star(\det E^\star).$$

For any $m > 0$, the line bundle $L = \mathcal{O}_E(m) \otimes \pi^\star(\det E)$ is strictly nef. Since

$$L - K_Y = \mathcal{O}_E(m + r) \otimes \pi^\star(-K_X),$$

we deduce $L - K_Y$ is strictly nef. On the other hand, $L - K_Y$ is big. Indeed, since both $\mathcal{O}_E(1)$ and $-K_X$ are nef, the top intersection number

$$
\begin{aligned}
(L - K_Y)^{n+r-1} &= (\mathcal{O}_E(m + r) \otimes \pi^\star(-K_X))^{n+r-1} \\
&\geq \left(\mathcal{O}_E(m + r)\right)^{r-1} \cdot \left(\pi^\star(-K_X)\right)^n > 0.
\end{aligned}
$$

Therefore, by the base point free theorem again, L is semi-ample and so L is ample. By the positivity of direct image sheaves ([43]), we deduce that $\pi_\star(K_{Y/X} \otimes L^{\otimes k})$ is ample for k large enough. By using the projection formula, one can see that the ample vector bundle $\pi_\star(K_{Y/X} \otimes L^{\otimes k})$ is of the form $\mathrm{Sym}^{k_0} E \otimes (\det E)^{\otimes k_1}$ where k_0 and k_1 are some positive integers. In particular, $\det E$ is ample. $\qquad\square$

The following result is proved in [35, Section 3].

Proposition 3.4. *Let E be a strictly nef vector bundle over a projective manifold X. If either of the following holds*

(1) the Kodaira dimension $\kappa(X)$ satisfies $0 \leqslant \kappa(X) < \dim X$,
(2) $-K_X$ is pseudo-effective,

then $\det E$ is not numerically trivial.

4 Geometry of projective manifolds with strictly nef bundles

In this section, we describe the geometry related to strictly nef and ample bundles. As analogous to classical results of Mori ([42]), Cho-Miyaoka-Shepherd-Barron ([14]) and Dedieu-Hoering ([15]), we obtained characterizations of $\mathbb{P}^n$ and quadrics ([35, Theorem 1.3 and Theorem 1.5]).

Theorem 4.1. *Let X be a projective manifold of dimension n.*

(1) If T_X is strictly nef, then X is isomorphic to a projective space.
(2) If $n \geq 3$ and $\bigwedge^2 T_X$ is strictly nef, then X is isomorphic to $\mathbb{P}^n$ or a quadric $\mathbb{Q}^n$.

Gachet studied in [20] the case when $\Lambda^3 T_X$ is strictly nef.

We have already established in general that the tangent bundle T_X is strictly nef if and only if it is ample. However, the same does not hold for cotangent bundles.

Example 4.2. Let X be a bidisk quotient, $\Delta \times \Delta/\Gamma$, with Γ an irreducible torsion-free cocompact lattice. Let $E = T_X^\star$ and L be its tautological line bundle. It is proved in [48] that L is strictly nef and big, but it is not semi-ample.

We propose the following problem which is analogous to the classical result of Kobayashi that projective manifolds with ample cotangent bundle are hyperbolic.

Problem 4.3. *Let X be a projective manifold. If $T_X^\star$ is strictly nef, is X hyperbolic?*

Let us consider manifolds with strictly nef canonical or anti-canonical bundles by recalling the Conjecture 1.7 of Campana and Peternell.

Conjecture 4.4. *Let X be a projective manifold. If K_X^{-1} is strictly nef, then X is Fano.*

Recently, some evidences are established in [35, Theorem 1.2].

Theorem 4.5. *Let X be a projective manifold of dimension n and $1 \le r \le n$. If $\Lambda^r T_X$ is strictly nef, then X is rationally connected. In particular, if K_X^{-1} is strictly nef, then X is rationally connected.*

Let $f : X \to Y$ be a smooth surjective morphism between two projective manifolds. It is well-known that if K_X^{-1} is ample, then so is K_Y^{-1} ([29], see also [5] for semi-ampleness). It is natural to propose the following conjecture.

Conjecture 4.6. *Let $f : X \to Y$ be a smooth surjective morphism between two projective manifolds. If K_X^{-1} is strictly nef, then so is K_Y^{-1}.*

Indeed, this conjecture can be regarded as a consequence of Conjecture 4.4. Thanks to Theorem 4.5, one obtains a partial answer to Conjecture 4.6.

Corollary 4.7. *Let $f : X \to Y$ be a smooth surjective morphism between two projective manifolds. If K_X^{-1} is strictly nef, then Y is rationally connected.*

Example 4.8. Let $f : X \to Y$ be a smooth surjective morphism between two projective manifolds. It is well-known that $K_{X/Y}^{-1}$ cannot be ample ([29]). However, it can be strictly nef.

We also propose the following general conjecture concerning strictly nef bundles.

Conjecture 4.9. *Let X be a projective manifold.*

(1) If $\Lambda^r T_X$ is strictly nef for some $r > 0$, then K_X^{-1} is ample;
(2) If $\Lambda^r T_X^\star$ is strictly nef for some $r > 0$, then K_X is ample.

The case hen $T_X^\star$ is strictly nef is of particular interest and it is also related to the Kobayashi-Lang conjecture on hyperbolicity.

Let us consider the geometry of projective manifolds whose tangent bundle contains a "positive" subsheaf. Recall that, Andreatta and Wiśniewski obtained in [1, Theorem] the following characterization of projective spaces.

Theorem 4.10. *Let X be an n-dimensional projective manifold. Assume that the tangent bundle T_X contains a locally free ample subsheaf $\mathcal{F}$ of rank r. Then $X \cong \mathbb{P}^n$ and either $\mathcal{F} \cong T_{\mathbb{P}^n}$ or $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.*

According to Example 2.5, this does not hold if the subsheaf $\mathcal{F}$ is assumed to be strictly nef. Indeed, we obtained in [38, Theorem 1.3] the following structure theorem for projective manifolds whose tangent bundle contains a strictly nef subsheaf.

Theorem 4.11. *Let X be a projective manifold. Assume that the tangent bundle T_X contains a locally free strictly nef subsheaf $\mathcal{F}$ of rank r. Then X admits a $\mathbb{P}^d$-bundle structure $\varphi : X \to T$ for some $d \ge r$. Moreover, if T is not a single point, then T is a hyperbolic projective manifold of general type.*

Actually, we obtained in [38, Theorem 8.1] a concrete description of the structure of the subsheaf $\mathcal{F}$ and it is exactly one of the following:

(1) $\mathcal{F} \cong T_{X/T}$ and X is isomorphic to a flat projective bundle over T;
(2) $\mathcal{F}$ is a numerically projectively flat vector bundle and its restriction on every fiber of φ is isomorphic to $\mathcal{O}_{\mathbb{P}^d}(1)^{\oplus r}$.

When dim $T > 0$, we established in [38, Corollary 1.5] the existence of non-zero symmetric differentials.

Corollary 4.12. *Let X be a projective manifold whose tangent bundle contains a locally free strictly nef subsheaf. If X is not isomorphic to a projective space, then X has a non-zero symmetric differential, i.e. $H^0(X, \mathrm{Sym}^i \Omega_X) \neq 0$ for some $i > 0$.*

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