Complex Manifolds 2019; 6:366–376 DE GRUYTER

Research Article Open Access

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On the Kähler-likeness on almost Hermitian manifolds

https://doi.org/10.1515/coma-2019-0020
Received May 20, 2019; accepted September 3, 2019

Abstract: We define a Kähler-like almost Hermitian metric. We will prove that on a compact Kähler-like almost Hermitian manifold (M^{2n}, J, g), if it admits a positive $\partial\bar{\partial}$-closed $(n-2, n-2)$-form, then g is a quasi-Kähler metric.

Keywords: almost Hermitian manifolds, Kähler-like metrics, Chern connection, quasi-Kähler metrics

MSC: 53C15 (primary); 53C55 (secondary)

1 Introduction

The geometry of almost Hermitian manifolds has been studied extensively in last years such as in [3], [10], [11] and [23]. In this paper, we will define a Kähler-like almost Hermitian metric. The aim of this manuscript is to investigate what conditions are needed for such metrics to be quasi-Kähler. In Hermitin case, Yang and Zheng examined the Hermitian curvature tensors of Hermitian metrics, as the curvature tensors satisfies all the symmetry conditions of the curvature tensor of a Kähler metric in [22]. They called these metrics Kähler-like. When a manifold is compact, these metrics are more special than balanced metrics since such metrics are always balanced, that is, $d(\omega^{n-1}) = 0$, where ω is the fundamental 2-form associated with a Hermitian metric and n is the complex dimension of the manifold. This fact has attracted attention in the reserch of non-Kähler Calabi-Yau manifolds. Their definitions are as follows. Given a Hermitian manifold (M^n, J, g), there are two canonical connections associated with g, the Chern connection ∇ and the Levi-Civita connection D. Denote R and R^L the curvature tensor of these two connections respectively. Notice that in this whole paper, in the almost Hermitian case M^{2n} indicates that $2n = \dim_{\mathbb{R}} M$, in the Hermitian case M^n means that $n = \dim_{\mathbb{C}} M$.

Definition 1.1. (Kähler-like and G-Kähler-like [22]) A Hermitian metric g will be called Kähler-like, if $R_{X\bar{Y}Z\bar{W}} = R_{Z\bar{Y}X\bar{W}}$ holds for any type $(1,0)$ tangent vectors X, Y, Z and W. Similarly, if $R^L_{XY\bar{Z}\bar{W}} = R^L_{XYZ\bar{W}} = 0$ for any type $(1,0)$ tangent vectors X, Y, Z and W, we will say that g is Gray-Kähler-like, or G-Kähler-like for short.

The G-Kähler-like condition was firstly introduced by Gray in [8]. Yang and Zheng showed that when $R = R^L$, then g is Kähler in [22, Theorem 1.1], and they also showed that when the Hermitian manifold is compact, either condition, the Kähler-likeness or the G-Kähler-likeness, would imply that the metric is balanced.

Proposition 1.1. ([22, Theorem 1.3]) Let (M^n, J, g) be a compact Hermitian manifold. If it is either Kähler-like or G-Kähler-like, then it must be balanced.

In this sence, the Kähler-likeness is more special than being balanced for compact Hermitian manifolds. Note that Vaisman has showed that any compact G-Kähler-like Hermitian surface is Kähler in [19].

Yang and Zheng have also shown that the folloing result in [22].

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Proposition 1.2. ([22, Theorem 3.1]) Let (M^n, J, g) be a Hermitian manifold that is Kähler-like. If M^n is compact and admits a positive, $\partial\bar{\partial}$-closed $(n-2, n-2)$-form χ, then g is Kähler. In particular, if M^n is compact, Kähler-like, and $\partial\bar{\partial}(\omega^{n-2}) = 0$, then g is Kähler. When $n = 2$, compactness implies that any Kähler-like metric is Kähler.

In this paper, we generalize the result of Yang and Zheng in Proposition 1.2 from the category of Hermitian manifolds to the category of almost Hermitian manifolds. We now extend their studies to almost Hermitian geometry. Let (M, J) be an almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1, 0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1, 1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric.

We define a Kähler-like almost Hermitian metric in the following as in [22, Definition (Kähler-like and G-Kähler-like)].

Definition 1.2. Let (M^{2n}, J, g) be an almost Hermitian manifold and let R^∇ be the curvature tensor with respect to the Chern connection ∇ associated with g. An almost Hermitian metric g will be called Kähler-like, if $R^\nabla_{X\bar{Y}Z\bar{W}} = R^\nabla_{Z\bar{Y}X\bar{W}}$ holds for any type $(1, 0)$-tangent vectors X, Y, Z and W. When the almost Hermitian metric g is Kähler-like, the triple (M^{2n}, J, g) will be called a Kähler-like almost Hermitian manifold.

When g is Kähler-like, by taking complex conjugations, we see that R is also symmetric with respect to its second and fourth positions, thus obeying all the symmetries of the curvature tensor of a Kähler metric.

A quasi-Kähler structure is an almost Hermitian structure whose real $(1, 1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$ (cf. [5], [8], [18]). It is important for us to study quasi-Kähler manifolds since they include the classes of almost Kähler manifolds and nearly Kähler manifolds. An almost Kähler or quasi-Kähler manifold with J integrable is a Kähler manifold. We get a result that a metric is actually quasi-Kähler under the same assumption as in Proposition 1.2 on an almost Hermitian manifold.

Theorem 1.1. Let (M^{2n}, J, g) be a compact Kähler-like almost Hermitian manifold with $n \geq 2$. If M^{2n} admits a positive $\partial\bar{\partial}$-closed $(n-2, n-2)$-form χ, then g is quasi-Kähler. In particular, if M^{2n} is compact, Kähler-like, and $\partial\bar{\partial}(\omega^{n-2}) = 0$, then g is quasi-Kähler. When $n = 2$, compactness implies that any Kähler-like metric is almost Kähler.

Note that in dimension 4, every quasi-Kähler manifold is almost Kähler. In general, there are known examples of quasi-Kähler manifolds which are not almost Kähler. In particular, if a compact real 6-dimensional almost Hermitian manifold M^6 admits a Kähler-like metric that is non-quasi-Kähler, then M^6 cannot have any almost pluriclosed metric.

The question how the geometry of compact almost Kähler manifolds can force the integrability of an almost complex structure has been investigated by many reserchers such as [6], [12] , [13], [14] and [15]. A well-known conjecture of Goldberg states that compact Einstein almost Kähler manifolds are necessarily Kähler. This conjecture is still open, but there are some partial results. Sekigawa has proven that the Goldberg conjecture is true if the Riemannian scalar curvature is non-negative in [16]. Especially in dimension 4, some other results have been shown under some conditions. Likewise, we would like to consider how the geometry of compact Kähler-like almost Hermitian 4-manifolds can force the integrability of of an almost complex structure below.

In the following a few cases, on a $2n$-dimensional almost Hermitian manifold with an almost Hermitian structure (J, g), we define the curvature and the Ricci tensor with respect to the Levi-Civita connection D in the following way for tangent vectors X, Y, Z and W:

$$R^L(X, Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z, \quad R^L(X, Y, Z, W) = g(R^L(X, Y)Z, W),$$

$$\rho(X, Y) = \mathrm{tr}(Z \mapsto R^L(Z, X)Y).$$

An almost Hermitian manifold (M, J, g) satisfying that (cf. [8], [9])

(1) $R^L(X, Y, Z, W) = R^L(X, Y, JZ, JW)$ for all vector fields X, Y, Z, W is called an AH_1-manifold;
(2) $R^L(X, Y, Z, W) = R^L(X, Y, JZ, JW) + R^L(X, JY, Z, JW) + R^L(JX, Y, Z, JW)$ for all vector fields X, Y, Z, W is called an AH_2-manifold;
(3) $R^L(X, Y, Z, W) = R^L(JX, JY, JZ, JW)$ for all vector fields X, Y, Z, W is called an AH_3-manifold (or RK-manifold (cf. [14], [15]).

Then we have $AH_1 \subset AH_2 \subset AH_3$. Note that if an AH_1-manifold is almost Kähler, then it is Kähler (cf. [8, Theorem 5.1]), which tells us that by combining with Theorem 1.1, we get that a 4-dimensional compact Kähler-like AH_1-manifold is Kähler. Actually, we can obtain a stronger result by using the following result states that there are no compact examples of strictly almost Kähler 4-manifolds satisfying the curvature condition (3).

Proposition 1.3. (cf. [2, Theorem 2]) A 4-dimensional compact almost Kähler AH_3-manifold is Kähler.

Combining Theorem 1.1 and Proposition 1.3, we get the following corollary.

Corollary 1.1. A 4-dimensional compact Kähler-like AH_3-manifold is Kähler.

We introduce the following result.

Proposition 1.4. (cf. [21, Theorem 4.4]) Let (M, J, g) be an almost Kähler manifold with the fundamental 2-form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$. If $(DJ)D = 0$, then (g, J, ω) is a Kähler structure on M, where D is the Levi-Civita connection with resprct to g.

By combining Theorem 1.1 and Proposition 1.4, we obtain the following corollary.

Corollary 1.2. Let (M^4, J, g) be a 4-dimensional compact Kähler-like almost Hermitian manifold with the fundamental 2-form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$. If $(DJ)D = 0$, then (J, g, ω) is a Kähler structure on M, where D is the Levi-Civita connection with respect to g.

Notice that with the definition of *static* in [17, Definition 9.1] and a corollary which states that 4-dimensional compact static almost Kähler manifold is Kähler-Einstein [17, Corollary 9.5], we obtain the result that a 4-dimensional compact static Kähler-like manifold is Kähler-Einstein. Here, we introduce the following results for 4-dimentional almost Kähler manifolds in [6], [12].

Proposition 1.5. (cf. [6, Theorem 2]) A 4-dimensional compact almost Kähler manifold (M^4, J, g) with J-invariant and nonnegative definite Ricci tensor ρ is Kähler.

By applying Theorem 1.1 and Proposition 1.5, we then obtain the following corollary.

Corollary 1.3. A 4-dimensional compact Kähler-like almost Hermitian manifold (M^4, J, g) with J-invariant and nonnegative definite Ricci tensor ρ is Kähler.

For $X \in T^{1,0}M$, the holomorphic sectional curvature is defined by

$$H(X) = \frac{R^\nabla(X, \bar{X}, X, \bar{X})}{g(X, \bar{X})g(X, \bar{X})},$$

where ∇ is the Chern connection associated to an almost Hermitian metric g and R^∇ is the curvature with respect to ∇. The holomorphic sectional curvature is constant at a point $p \in M$ if $H(X)$ is a constant $k(p)$ for all $X \in T_p^{1,0}M$. Note that if the constant k is the same at every point $p \in M$, we say that it is a *globally* constant, and if H is constant at each point of M, we say it is *pointwise* constant.

Proposition 1.6. (cf. [12, Theorem 1.1]) A 4-dimensional closed almost Kähler manifold of *globally* constant holomorphic sectional curvature $k \geq 0$ is Kähler-Einstein.

By applying Theorem 1.1 and Proposition 1.6, we then get the following result.

Corollary 1.4. A 4-dimensional closed Kähler-like almost Hermitian manifold of *globally* constant holomorphic sectional curvature $k \geq 0$ is Kähler-Einstein.

Notice that according to [12, Theorem 1.2], 4-dimensional closed Kähler-like almost Hermitian manifold of *pointwise* constant holomorphic sectional curvature $k < 0$ with the J-invariant Ricci tensor ρ is Kähler-Einstein.

We say that a metric g is an almost pluriclosed metric if g is an almost Hermitian metric whose associated real $(1,1)$-form ω satisfies $\partial\bar{\partial}\omega = 0$ (cf. [10, Definition 1.1]). From Theorem 1.1, we may say that if a compact Kähler-like almost Hermitian manifold M^6 admits an almost pluriclosed metric g, then g is actually a quasi-Kähler metric and (M^6, g) is a quasi-Kähler manifold.

It is well-known that if the complex structure is integrable, then the $(2,0)$-part of the curvature tensor for the Chern connection vanishes. Generally, the converse is not true, but if some curvature conditions are assumed, then ChengJie has showed that the answer becomes affirmative in [3]. The Ricci curvature is said to be quasi-positive if it is nonnegative everywhere and strictly positive in any direction at (at least) one point. Note that a compact Riemannian manifold of quasi-positive Ricci curvature admits metric of strictly positive Ricci curvature (cf. [24]). In the following cases, we define the curvature and the Ricci tensor with respect to the Chern connection, see Section 2.2.

Proposition 1.7. (cf. [3, Theorem 1.2]) Let (M^{2n}, J, g) be a compact quasi-Kähler manifold with quasi-positive second Ricci curvature and parallel $(2,0)$-part of the curvature tensor. Then, the manifold must be Kähler.

By combining Theorem 1.1 and Proposition 1.7, and that we have the first Ricci curvature coincide with the second Ricci curvature under the assumption of the Kähler-likeness, we have the following corollary.

Corollary 1.5. Let (M^{2n}, J, g) be a compact Kähler-like almost Hermitian manifold with quasi-positive second Ricci curvature and parallel $(2,0)$-part of the curvature tensor and $n \geq 2$. If M^{2n} admits a positive $\partial\bar{\partial}$-closed $(n-2, n-2)$-form, then the manifold must be Kähler.

This paper is organized as follows: in section 2, we recall some basic definitions and computations. In the last section, we prepare some lemmas for the torsion and the curvature, and then by applying these results, we will prove the main theorem. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2 Preliminaries

2.1 The Nijenhuis tensor of the almost complex structure

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies $N(X, Y) = -N(Y, X)$, $N(JX, Y) = -JN(X, Y)$, $N(X, JY) = -JN(X, Y)$, $N(JX, JY) = -N(X, Y)$. For any $(1,0)$-vector fields W and V, $N(V, W) = -[V, W]^{(0,1)}$, $N(V, \bar{W}) = N(\bar{V}, W) = 0$ and $N(\bar{V}, \bar{W}) = -[\bar{V}, \bar{W}]^{(1,0)}$ since we have $4N(V, W) = -2([V, W] + \sqrt{-1}J[V, W])$, $4N(\bar{V}, \bar{W}) = -2([\bar{V}, \bar{W}] - \sqrt{-1}J[\bar{V}, \bar{W}])$. An almost complex structure J is called integrable if $N = 0$ everywhere on M. Giving a complex structure to a differentiable manifold M is equivalent to giving an integrable almost complex structure to M. Let (M, J) be an almost complex manifold. A Riemannian metric g on M is called

J-invariant if J is compatible with g, i.e., for any $X, Y \in \Gamma(TM)$, $g(X, Y) = g(JX, JY)$. In this case, the pair (J, g) is called an almost Hermitian structure. The fundamental 2-form ω associated to a J-invariant Riemannian metric g, i.e., an almost Hermitian metric, is determined by, for $X, Y \in \Gamma(TM)$, $\omega(X, Y) = g(JX, Y)$. Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y, X) = g(JY, X) = g(J^2 Y, JX) = -g(JX, Y) = -\omega(X, Y)$$

and $\omega \in \Gamma(\bigwedge^2 T^\star M)$. We will also refer to the associated real fundamental $(1, 1)$-form ω as an almost Hermitian metric. The form ω is related to the volume form dV_g by $n!dV_g = \omega^n$. Let a local $(1, 0)$-frame $\{Z_r\}$ on (M, J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, i.e., $\zeta^i(Z_j) = \delta^i_j$ for $i, j = 1, \dots, n$. Since g is almost Hermitian, its components satsfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$.

With using these local frame $\{Z_r\}$ and coframe $\{\zeta^r\}$, we have

$$N(Z_{\bar{i}}, Z_{\bar{j}}) = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} =: N^k_{\bar{i}\bar{j}} Z_k, \quad N(Z_i, Z_j) = -[Z_i, Z_j]^{(0,1)} = \overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}},$$

and

$$N = \frac{1}{2}\overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}} \otimes (\zeta^i \wedge \zeta^j) + \frac{1}{2} N^k_{\bar{i}\bar{j}} Z_k \otimes (\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}).$$

We write $T^{\mathbb{R}}M$ for the real tangent space of M. Then its complexified tangent space is given by $T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$. By extending J $\mathbb{C}$-linearly and g, ω $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent space $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \,|\, X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \,|\, X \in TM\}.$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q} M$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p, q)-forms, where $\Lambda^{p,q} M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$,

$$\Lambda^{1,0}M = \{\alpha + \sqrt{-1}J\alpha \,|\, \alpha \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\alpha - \sqrt{-1}J\alpha \,|\, \alpha \in \Lambda^1 M\}$$

and $\Lambda^1 M$ denotes the dual of TM.

Let (M^{2n}, J, g) be an almost Hermitian manifold. An affine connection D on TM is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (cf. [4], [7], [20], [23]).

Lemma 2.1. Let (M, J, g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. Then for any given vector valued $(1, 1)$-form $\Theta = (\Theta^i)_{1\le i\le n}$, there exists a unique almost Hermitian connection D on (M, J, g) such that the $(1, 1)$-part of the torsion is equal to the given Θ.

If the $(1, 1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connction is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇.

Now let ∇ be the Chern connection on M. We denote the structure coefficients of Lie bracket by

$$[Z_i, Z_j] =: B^r_{ij} Z_r + B^{\bar{r}}_{ij} Z_{\bar{r}} = B^r_{ij} Z_r - \overline{N^r_{\bar{i}\bar{j}}} Z_{\bar{r}}, , \quad [Z_i, Z_{\bar{j}}] =: B^r_{i\bar{j}} Z_r + B^{\bar{r}}_{i\bar{j}} Z_{\bar{r}},$$

$$[Z_{\bar{i}}, Z_{\bar{j}}] =: B^r_{\bar{i}\bar{j}} Z_r + B^{\bar{r}}_{\bar{i}\bar{j}} Z_{\bar{r}} = -N^r_{\bar{i}\bar{j}} Z_r + B^{\bar{r}}_{\bar{i}\bar{j}} Z_{\bar{r}},$$

where we used that $[Z_i, Z_j]^{(0,1)} = -\overline{N^r_{\bar{i}\bar{j}}} Z_{\bar{r}}$, $[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} = -N^r_{\bar{i}\bar{j}} Z_r$ and then $B^{\bar{r}}_{ij} = -\overline{N^r_{\bar{i}\bar{j}}}$, $B^r_{\bar{i}\bar{j}} = -N^r_{\bar{i}\bar{j}}$. Also we here note that for instance, $[Z_i, Z_{\bar{j}}] = [Z_i, Z_{\bar{j}}]^{(1,0)} + [Z_i, Z_{\bar{j}}]^{(0,1)}$, where

$$[Z_i, Z_{\bar{j}}]^{(1,0)} = \frac{1}{2}([Z_i, Z_{\bar{j}}] - \sqrt{-1}J[Z_i, Z_{\bar{j}}]), \quad [Z_i, Z_{\bar{j}}]^{(0,1)} = \frac{1}{2}([Z_i, Z_{\bar{j}}] + \sqrt{-1}J[Z_i, Z_{\bar{j}}]).$$

Notice that J is integrable if and only if the $B^{\bar{r}}_{ij}$'s vanish.

Note that for any p-form ψ, there holds that

$$d\psi(X_1,\dots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}X_i(\psi(X_1,\dots,\widehat{X}_i,\dots,X_{p+1})) + \sum_{i<j}(-1)^{i+j}\psi([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{p+1})$$

for any vector fields $X_1,\dots,X_{p+1}$ on M (cf. [23]). We directly compute that

$$d\zeta^s = -\frac{1}{2}B^s_{kl}\zeta^k\wedge\zeta^l - B^s_{k\bar{l}}\zeta^k\wedge\zeta^{\bar{l}} + \frac{1}{2}N^s_{\bar{k}\bar{l}}\zeta^{\bar{k}}\wedge\zeta^{\bar{l}}.$$

According to the direct computation above, we may split the exterior differential operator $d : \Lambda^pM\otimes_{\mathbb{R}}\mathbb{C}\to\Lambda^{p+1}M\otimes_{\mathbb{R}}\mathbb{C}$, into four components

$$d = A + \partial + \bar{\partial} + \bar{A}$$

with

$$\partial : \Lambda^{p,q}M\to\Lambda^{p+1,q}M,\quad \bar{\partial} : \Lambda^{p,q}M\to\Lambda^{p,q+1}M,$$
$$A : \Lambda^{p,q}M\to\Lambda^{p+2,q-1}M,\quad \bar{A} : \Lambda^{p,q}M\to\Lambda^{p-1,q+2}M.$$

In terms of these components, the condition $d^2 = 0$ can be written as

$$A^2 = 0,\quad \partial A + A\partial = 0,\ \bar{\partial}\bar{A} + \bar{A}\bar{\partial} = 0,\quad \bar{A}^2 = 0,$$

$$A\bar{\partial} + \partial^2 + \bar{\partial}A = 0,\quad A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0,\quad \partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0.$$

For any real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar{j}}\zeta^i\wedge\zeta^{\bar{j}}$, we have

$$\partial\eta = \frac{\sqrt{-1}}{2}\Big(Z_i(\eta_{j\bar{k}}) - Z_j(\eta_{i\bar{k}}) - B^s_{ij}\eta_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}\eta_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}\eta_{i\bar{s}}\Big)\zeta^i\wedge\zeta^j\wedge\zeta^{\bar{k}},$$

$$\bar{\partial}\eta = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(\eta_{k\bar{i}}) - Z_{\bar{i}}(\eta_{k\bar{j}}) - B^s_{k\bar{i}}\eta_{s\bar{j}} + B^s_{k\bar{j}}\eta_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}\eta_{k\bar{s}}\Big)\zeta^k\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}}.$$

From these computations above, we have

$$\partial\omega = \frac{\sqrt{-1}}{2}\Big(Z_i(g_{j\bar{k}}) - Z_j(g_{i\bar{k}}) - B^s_{ij}g_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}g_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}g_{i\bar{s}}\Big)\zeta^i\wedge\zeta^j\wedge\zeta^{\bar{k}}$$

and

$$\bar{\partial}\omega = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(g_{k\bar{i}}) - Z_{\bar{i}}(g_{k\bar{j}}) - B^s_{k\bar{i}}g_{s\bar{j}} + B^s_{k\bar{j}}g_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}g_{k\bar{s}}\Big)\zeta^k\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}}.$$

2.2 The torsion and the curvature on almost complex manifolds

Since the Chern connection ∇ preserves J, we are able to define the Christoffel symbols: for $i,j,r = 1,\dots,n$,

$$\nabla_iZ_j = \nabla_{Z_i}Z_j = \Gamma^r_{ij}Z_r,\quad \nabla_iZ_{\bar{j}} = \nabla_{Z_i}Z_{\bar{j}} = \Gamma^{\bar{r}}_{i\bar{j}}Z_{\bar{r}},$$

where

$$\Gamma^r_{ij} = g^{r\bar{s}}Z_i(g_{j\bar{s}}) - g^{r\bar{s}}g_{j\bar{l}}B^{\bar{l}}_{i\bar{s}},\quad \Gamma^p_{ip} = Z_i(\log\det g) - B^{\bar{s}}_{i\bar{s}}.$$

The torsion $T = (T^i)$ of the Chern connection ∇ is defined by

$$T^i := d\zeta^i - \zeta^p\wedge\gamma^i_p,\quad T^{\bar{i}} := d\zeta^{\bar{i}} - \zeta^{\bar{p}}\wedge\gamma^{\bar{i}}_{\bar{p}},$$

where $\gamma = (\gamma^i_j)$ is the connection form defined by $\gamma^i_j := \Gamma^i_{kj}\zeta^k + \Gamma^i_{\bar{k}j}\zeta^{\bar{k}}$. Since the torsion T of the Chern connection ∇ has no $(1,1)$-part;

$$0 = T^i_{k\bar{l}} = T^i(Z_k,Z_{\bar{l}}) = -\zeta^i([Z_k,Z_{\bar{l}}]) - (\Gamma^i_{sp}\zeta^p\wedge\zeta^s + \Gamma^i_{\bar{s}p}\zeta^p\wedge\zeta^{\bar{s}})(Z_k,Z_{\bar{l}}) = -B^i_{k\bar{l}} - \Gamma^i_{\bar{l}k}$$

$$0 = T^{\bar{i}}_{k\bar{l}} = T^{\bar{i}}(Z_k, Z_{\bar{l}}) = -\zeta^{\bar{i}}([Z_k, Z_{\bar{l}}]) - (\Gamma^{\bar{i}}_{s\bar{p}}\zeta^{\bar{p}} \wedge \zeta^s + \Gamma^{\bar{i}}_{\bar{s}\bar{p}}\zeta^{\bar{p}} \wedge \zeta^{\bar{s}})(Z_k, Z_{\bar{l}}) = -B^{\bar{i}}_{k\bar{l}} + \Gamma^{\bar{i}}_{k\bar{l}},$$

we have

$$\Gamma^{\bar{r}}_{i\bar{j}} = B^{\bar{r}}_{i\bar{j}}.$$

Here note that $B^{\bar{q}}_{j\bar{b}}$, $B^{q}_{\bar{j}b}$'s do not depend on g, which depend only on J since the mixed derivatives $\nabla_j Z_{\bar{b}}$ do not depend on g (cf. [20]).

The torsion T of ∇ has no $(1,1)$-part and the only non-vanishing components are as follows:

$$T^s_{ij} = T^s(Z_i, Z_j) = -\zeta^s([Z_i, Z_j]) - (\Gamma^i_{sp}\zeta^p \wedge \zeta^s + \Gamma^i_{\bar{s}p}\zeta^p \wedge \zeta^{\bar{s}})(Z_i, Z_j) = -B^s_{ij} - \Gamma^s_{ji} + \Gamma^s_{ij},$$

$$T^{\bar{s}}_{ij} = T^{\bar{s}}(Z_i, Z_j) = d\zeta^{\bar{s}}(Z_i, Z_j) = -\zeta^{\bar{s}}([Z_i, Z_j]) = -B^{\bar{s}}_{ij}$$

and on the other hand we have $d\zeta^{\bar{s}}(Z_i, Z_j) = N^{\bar{s}}_{ij}$, hence we obtain that $T^{\bar{s}}_{ij} = N^{\bar{s}}_{ij} = -B^{\bar{s}}_{ij}$. These computations tell us that T splits in $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M \otimes T^{1,0}M)$, a section of $\Lambda^{2,0}M \otimes T^{1,0}M$, and $T'' \in \Gamma(\Lambda^{2,0}M \otimes T^{0,1}M)$. The torsion $T = (T^i)$ can be split into $T = T^{(2,0)} + T^{(1,1)} + T^{(0,2)} = T^{(2,0)} + T^{(0,2)}$ since $T^{(1,1)} = 0$, where $T^{(2,0)} = \left(\frac{1}{2}T^i_{jk}\zeta^j \wedge \zeta^k\right)_{1\le i\le n}$, $T^{(0,2)} = \left(\frac{1}{2}N^i_{\bar{j}\bar{k}}\zeta^{\bar{j}} \wedge \zeta^{\bar{k}}\right)_{1\le i\le n}$, which tells us that $(0,2)$-part of the Chern connection is uniquely determined by the Nijenhuis tensor (cf. [22]). Let (M, J, g) be an almost Hermitian manifold. Let $\{Z_r\}$ be a local $(1,0)$-frame with respect to g and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}\zeta^r \wedge \zeta^{\bar{r}}$. With using these notations, we may write

$$(\natural)\quad \begin{aligned} d\omega &= \sqrt{-1}(d\zeta^r \wedge \zeta^{\bar{r}} - \zeta^r \wedge d\zeta^{\bar{r}}) \\ &= \sqrt{-1}\{(-\gamma^r_p \wedge \zeta^p + T^r) \wedge \zeta^{\bar{r}} - \zeta^r \wedge (-\gamma^{\bar{r}}_{\bar{p}} \wedge \zeta^{\bar{p}} + T^{\bar{r}})\} \\ &= \sqrt{-1}(-\gamma^r_p \wedge \zeta^p \wedge \zeta^{\bar{r}} + T^r \wedge \zeta^{\bar{r}} - \zeta^r \wedge \gamma^p_r \wedge \zeta^{\bar{p}} - \zeta^r \wedge T^{\bar{r}}) \\ &= \sqrt{-1}(T^r \wedge \zeta^{\bar{r}} - \zeta^r \wedge T^{\bar{r}} - \gamma^r_p \wedge \zeta^p \wedge \zeta^{\bar{r}} + \gamma^p_r \wedge \zeta^r \wedge \zeta^{\bar{p}}) \\ &= \sqrt{-1}(T^r \wedge \zeta^{\bar{r}} - \zeta^r \wedge T^{\bar{r}}) \\ &= \frac{\sqrt{-1}}{2}(N^k_{\bar{i}\bar{j}}\zeta^{\bar{k}} \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}} - \overline{N^k_{\bar{i}\bar{j}}}\zeta^k \wedge \zeta^i \wedge \zeta^j + T^k_{ij}\zeta^{\bar{k}} \wedge \zeta^i \wedge \zeta^j + T^{\bar{k}}_{\bar{j}\bar{i}}\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}}), \end{aligned}$$

where we used the skew-Hermitian property $\gamma^p_r + \gamma^{\bar{r}}_{\bar{p}} = 0$, which can be obtained with using $\nabla g = 0$ (cf. [18]). This expression $(\natural)$ implies that we have

$$\partial\omega = \frac{\sqrt{-1}}{2}T^k_{ij}\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}} = \sqrt{-1}(T^k)^{(2,0)} \wedge \zeta^{\bar{k}}$$

and

$$\bar{\partial}\omega = \frac{\sqrt{-1}}{2}T^{\bar{k}}_{\bar{j}\bar{i}}\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}} = -\sqrt{-1}(T^{\bar{k}})^{(0,2)} \wedge \zeta^k,$$

where we put

$$(T^k)^{(2,0)} := \frac{1}{2}T^k_{ij}\zeta^i \wedge \zeta^j, \quad (T^{\bar{k}})^{(0,2)} := \frac{1}{2}T^{\bar{k}}_{\bar{i}\bar{j}}\zeta^{\bar{i}} \wedge \zeta^{\bar{j}} = \overline{(T^k)^{(2,0)}}.$$

Note that T'' depends only on J and it can be regarded as the Nijenhuis tensor of J, that is, J is integrable if and only if T'' vanishes.

We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2 M \otimes TM$, $\Omega \in \Gamma(\Lambda^2 M \otimes TM)$ and Ω splits in

$$\Omega = \Omega^{(2,0)} + \Omega^{(1,1)} + \Omega^{(0,2)} = H + R + \bar{H},$$

with

$$\Omega^{(2,0)} = \left(\frac{1}{2}H_{kli}{}^j\zeta^k \wedge \zeta^l\right), \quad \Omega^{(1,1)} = \left(R_{k\bar{l}i}{}^j\zeta^k \wedge \zeta^{\bar{l}}\right), \quad \Omega^{(0,2)} = \left(\frac{1}{2}H_{\bar{k}\bar{l}i}{}^j\zeta^{\bar{k}} \wedge \zeta^{\bar{l}}\right),$$

where $R \in \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M)$, $H \in \Gamma(\Lambda^{2,0}M \otimes \Lambda^{1,1}M)$. The curvature form can be written by $\Omega^i_j = d\gamma^i_j + \gamma^i_s \wedge \gamma^s_j$. Then the Chern-Ricci form is $(\sqrt{-1}\Omega^i_i) \in 2\pi c_1(M, J) \in H^2(M, \mathbb{R})$, where $c_1(M, J)$ is the first Chern class of (M, J).

In terms of Z_r's, we have

$$R_{i\bar{j}k}{}^{r} = \Omega_k^r(Z_i, Z_{\bar{j}}) = Z_i(\Gamma_{\bar{j}k}^r) - Z_{\bar{j}}(\Gamma_{ik}^r) + \Gamma_{is}^r\Gamma_{\bar{j}k}^s - \Gamma_{\bar{j}s}^r\Gamma_{ik}^s - B_{i\bar{j}}^s\Gamma_{sk}^r + B_{\bar{j}i}^{\bar{s}}\Gamma_{\bar{s}k}^r,$$

$$H_{ijk}{}^{r} = \Omega_k^r(Z_i, Z_j) = Z_i(\Gamma_{jk}^r) - Z_j(\Gamma_{ik}^r) + \Gamma_{is}^r\Gamma_{jk}^s - \Gamma_{js}^r\Gamma_{ik}^s - B_{ij}^s\Gamma_{sk}^r - B_{ij}^{\bar{s}}\Gamma_{\bar{s}k}^r,$$

$$H_{\bar{i}\bar{j}k}{}^{r} = \Omega_k^r(Z_{\bar{i}}, Z_{\bar{j}}) = Z_{\bar{i}}(\Gamma_{\bar{j}k}^r) - Z_{\bar{j}}(\Gamma_{\bar{i}k}^r) + \Gamma_{\bar{i}s}^r\Gamma_{\bar{j}k}^s - \Gamma_{\bar{j}s}^r\Gamma_{\bar{i}k}^s - B_{\bar{i}\bar{j}}^s\Gamma_{sk}^r - B_{\bar{i}\bar{j}}^{\bar{s}}\Gamma_{\bar{s}k}^r,$$

and we deduce that with using $\Gamma_{kp}^p = Z_k(\log\det g) - B_{k\bar{p}}^{\bar{p}}$,

$$P_{i\bar{j}} = R_{i\bar{j}r}{}^{r} = -(Z_iZ_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\log\det g) + Z_{\bar{j}}(B_{i\bar{r}}^{\bar{r}}) + Z_i(B_{\bar{j}r}^r) + B_{i\bar{j}}^sB_{s\bar{r}}^{\bar{r}} - B_{i\bar{j}}^{\bar{s}}B_{\bar{s}r}^r$$

$$R_{ij} = H_{ijr}{}^{r} = [Z_i, Z_j]^{(0,1)}(\log\det g) - Z_i(B_{j\bar{r}}^{\bar{r}}) + Z_j(B_{i\bar{r}}^{\bar{r}}) + B_{ij}^sB_{s\bar{r}}^{\bar{r}} + \overline{N_{\bar{i}\bar{j}}^{s}}B_{\bar{s}r}^r$$

and

$$R_{\bar{i}\bar{j}} = H_{\bar{i}\bar{j}r}{}^{r} = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)}(\log\det g) + Z_{\bar{i}}(B_{\bar{j}r}^r) - Z_{\bar{j}}(B_{\bar{i}r}^r) - N_{\bar{i}\bar{j}}^sB_{s\bar{r}}^{\bar{r}} - B_{\bar{i}\bar{j}}^{\bar{s}}B_{\bar{s}r}^r.$$

Note that by computing with using a local g-unitary $(1,0)$-frame $\{Z_r\}$, we obtain the following formula (cf. [11, Lemma 2.3]):

$$R_{i\bar{j}k\bar{l}} = g(\nabla_i\nabla_{\bar{j}}Z_k - \nabla_{\bar{j}}\nabla_iZ_k - \nabla_{[Z_i,Z_{\bar{j}}]}Z_k, Z_{\bar{l}}).$$

The Chern-Ricci form $\mathrm{Ric}(\omega)$ is defined by

$$\mathrm{Ric}(\omega) := \frac{\sqrt{-1}}{2}R_{kl}\zeta^k \wedge \zeta^l + \sqrt{-1}P_{k\bar{l}}\zeta^k \wedge \zeta^{\bar{l}} + \frac{\sqrt{-1}}{2}R_{\bar{k}\bar{l}}\zeta^{\bar{k}} \wedge \zeta^{\bar{l}}.$$

It is a closed real 2-form. If J is integrable, it is a closed real $(1,1)$-form. If furthermore, J is integrable and $d\omega = 0$, then the Chern-Ricci form coincides with the Ricci form defined by the Levi-Civita connection of ω. We denote by S one of the Ricci-type curvatures of the Chern curvature, which is called the first Ricci curvature and with an arbitrary $(1,0)$-frame $\{Z_r\}$ with respect to g, is locally given by $S_{i\bar{j}} = g^{k\bar{l}}\Omega_{k\bar{l}i\bar{j}}$. The curvature P is one of the Ricci-type curvatures of the Chern curvature, which is called the second Ricci curvature of the almost Hermitian metric g, and is locally given by $P_{i\bar{j}} = g^{k\bar{l}}\Omega_{i\bar{j}k\bar{l}}$. In the Kähler-like case, since the curvature tensor R then satisfies all the symmetries of the curvature tensor of a Kähler metric, the first and second Ricci curvature coincides with each other; $S_{i\bar{j}} = P_{i\bar{j}}$, and then we call it the Ricci curvature.

3 Proof of Theorem 1.1

Let (M^{2n}, J, g) be an almost Hermitian manifold. Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to such a frame, we have $g_{i\bar{j}} = \delta_{ij}$, $Z_k(g_{i\bar{j}}) = 0$. By using a local g-unitary $(1,0)$-frame, the Christoffel symbols satisfy $\Gamma_{ij}^k = -\Gamma_{i\bar{k}}^{\bar{j}} = -B_{i\bar{k}}^{\bar{j}}$ (cf. [10, Lemma 2.2]), and the components of the torsion can be written as $T_{ij}^k = -B_{i\bar{k}}^{\bar{j}} + B_{j\bar{k}}^{\bar{i}} - B_{ij}^k$ and the components of w can be written as $w_j = -B_{jr}^r - B_{j\bar{r}}^{\bar{r}} + B_{r\bar{r}}^{\bar{j}}$.

In order to prove Theorem 1.1, we prepare the following lemmas. The first one below can be obtained easily to see the expression of (♮) in section 2.

Lemma 3.1. (cf. [18, Lemma 2.4]) An almost Hermitian manifold (M^{2n}, J, g) is quasi-Kähler if and only if $T_{ij}^k = 0$ for all i, j and k when a local unitary $(1,0)$-frame is fixed.

We can have the following result as in [22, Lemma 5].

Lemma 3.2. Given an almost Hermitian manifold (M^{2n}, J, g), g is Kähler-like if and only if $\Omega^{(1,1)} \wedge \zeta = 0$.

Proof. Assume that g is Kähler-like. Then since we have $R_{k\bar{l}i}{}^{j} = R_{i\bar{l}k}{}^{j}$, we obtain

$$\begin{aligned} \Omega^{(1,1)} \wedge \zeta &= R_{k\bar{l}i}{}^{j}\zeta^k \wedge \zeta^{\bar{l}} \wedge \zeta^i \\ &= R_{i\bar{l}k}{}^{j}\zeta^k \wedge \zeta^{\bar{l}} \wedge \zeta^i \\ &= -R_{i\bar{l}k}{}^{j}\zeta^i \wedge \zeta^{\bar{l}} \wedge \zeta^k \\ &= -R_{k\bar{l}i}{}^{j}\zeta^k \wedge \zeta^{\bar{l}} \wedge \zeta^i. \end{aligned}$$

Hence we have $\Omega^{(1,1)} \wedge \zeta = 0$. Conversely, if $\Omega^{(1,1)} \wedge \zeta = R_{k\bar{l}i}{}^{j}\zeta^k \wedge \zeta^{\bar{l}} \wedge \zeta^i = 0$, then we have

$$\begin{aligned} 0 &= R_{k\bar{l}i}{}^{j}\zeta^k \wedge \zeta^{\bar{l}} \wedge \zeta^i \\ &= \frac{1}{2}(R_{k\bar{l}i}{}^{j} - R_{i\bar{l}k}{}^{j})\zeta^k \wedge \zeta^{\bar{l}} \wedge \zeta^i, \end{aligned}$$

which tells us that $R_{k\bar{l}i}{}^{j} = R_{i\bar{l}k}{}^{j}$ and g is Kähler-like. □

Now we give a proof of Theorem 1.1.

Proof. (Theorem 1.1) Suppose that (M^{2n}, J, g) be a compact Kähler-like almost Hermitian manifold with $n \geq 2$. Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}\zeta^r \wedge \zeta^{\bar{r}}$. As we see in section 2, we then have $\partial\omega = \sqrt{-1}(T^k)^{(2,0)} \wedge \zeta^{\bar{k}}$, where $(T^k)^{(2,0)} = \frac{1}{2}T^k_{ij}\zeta^i \wedge \zeta^j$. We compute by using that $d\zeta^i = -\gamma^i_p \wedge \zeta^p + T^i$, $d\gamma^i_j = -\gamma^i_s \wedge \gamma^s_j + \Omega^i_j$, then we have

$$\begin{aligned} dT^i &= -d(\zeta^p \wedge \gamma^i_p) \\ &= -d\zeta^p \wedge \gamma^i_p + \zeta^p \wedge d\gamma^i_p \\ &= -(T^p - \gamma^p_s \wedge \zeta^s) \wedge \gamma^i_p + \zeta^p \wedge (\Omega^i_p - \gamma^i_s \wedge \gamma^s_p) \\ &= -T^p \wedge \gamma^i_p + \zeta^p \wedge \Omega^i_p + \gamma^p_s \wedge \zeta^s \wedge \gamma^i_p - \zeta^p \wedge \gamma^i_s \wedge \gamma^s_p \\ &= -T^p \wedge \gamma^i_p + \zeta^p \wedge \Omega^i_p, \end{aligned}$$

where at the fifth line, we used that

$$\gamma^p_s \wedge \zeta^s \wedge \gamma^i_p - \zeta^p \wedge \gamma^i_s \wedge \gamma^s_p = \gamma^p_s \wedge \zeta^s \wedge \gamma^i_p - \zeta^s \wedge \gamma^i_p \wedge \gamma^p_s = \gamma^p_s \wedge \zeta^s \wedge \gamma^i_p - \gamma^p_s \wedge \zeta^s \wedge \gamma^i_p = 0.$$

Then a direct calculation shows that since we have $dT^k = -T^p \wedge \gamma^k_p + \zeta^p \wedge \Omega^k_p$, we obtain

$$\begin{aligned} & d\partial\omega = d(\sqrt{-1}(T^k)^{(2,0)} \wedge \zeta^{\bar{k}}) \\ = \ & \sqrt{-1}dT^k \wedge \zeta^{\bar{k}} + \sqrt{-1}(T^k)^{(2,0)} \wedge d\zeta^{\bar{k}} \\ = \ & \sqrt{-1}(-T^p \wedge \gamma^k_p + \zeta^p \wedge \Omega^k_p) \wedge \zeta^{\bar{k}} + \sqrt{-1}(T^k)^{(2,0)} \wedge (T^{\bar{k}} - \gamma^{\bar{k}}_{\bar{s}} \wedge \zeta^{\bar{s}}) \\ = \ & \sqrt{-1}\zeta^p \wedge \Omega^k_p \wedge \zeta^{\bar{k}} + \sqrt{-1}(T^k)^{(2,0)} \wedge T^{\bar{k}} - \sqrt{-1}T^p \wedge \gamma^k_p \wedge \zeta^{\bar{k}} - \sqrt{-1}(T^k)^{(2,0)} \wedge \gamma^{\bar{k}}_{\bar{s}} \wedge \zeta^{\bar{s}} \\ = \ & \sqrt{-1}\Omega^k_p \wedge \zeta^p \wedge \zeta^{\bar{k}} + \sqrt{-1}(T^k)^{(2,0)} \wedge T^{\bar{k}} - \sqrt{-1}T^p \wedge \gamma^k_p \wedge \zeta^{\bar{k}} + \sqrt{-1}(T^k)^{(2,0)} \wedge \gamma^s_k \wedge \zeta^{\bar{s}}, \end{aligned}$$

where we used the skew-Hermitian property $\gamma^p_k + \gamma^{\bar{k}}_{\bar{p}} = 0$ at the fifth line. From the Kähler-likeness, as in the proof of Lemma 3.2, we have

$$(\Omega^k_p \wedge \zeta^p \wedge \zeta^{\bar{k}})^{(2,2)} = R_{i\bar{j}p}{}^{k}\zeta^i \wedge \zeta^{\bar{j}} \wedge \zeta^p \wedge \zeta^{\bar{k}} = 0.$$

We also have that

$$\begin{aligned} & -\sqrt{-1}(T^p \wedge \gamma^k_p \wedge \zeta^{\bar{k}})^{(2,2)} + \sqrt{-1}((T^k)^{(2,0)} \wedge \gamma^s_k \wedge \zeta^{\bar{s}})^{(2,2)} \\ = \ & \frac{\sqrt{-1}}{2}(-T^p_{ij}\Gamma^k_{\bar{s}p}\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{s}} \wedge \zeta^{\bar{k}} + T^k_{ij}\Gamma^s_{\bar{r}k}\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{r}} \wedge \zeta^{\bar{s}}) = 0. \end{aligned}$$

By combining these, we then obtain that

$$\sqrt{-1}(d\partial\omega)^{(2,2)} = -((T^k)^{(2,0)} \wedge T^{\bar{k}})^{(2,2)} = -(T^k)^{(2,0)} \wedge (T^{\bar{k}})^{(0,2)} = -(T^k)^{(2,0)} \wedge \overline{(T^k)^{(2,0)}}.$$

Here, note that the exterior differential operator d is splitted into four components; $d = A + \partial + \bar{\partial} + \bar{A}$. We have that $\partial\omega \wedge A\chi = 0$, $\partial\omega \wedge \bar{A}\chi = 0$ and $\partial\omega \wedge \partial\chi = 0$. Hence we have $\partial\omega \wedge d\chi = \partial\omega \wedge \bar{\partial}\chi$. Also we have $A(\omega \wedge \bar{\partial}\chi) = 0$, $\bar{A}(\omega \wedge \bar{\partial}\chi) = 0$ and $\bar{\partial}(\omega \wedge \bar{\partial}\chi) = 0$, which tells us that $d(\omega \wedge \bar{\partial}\chi) = \partial(\omega \wedge \bar{\partial}\chi)$. Since M is supposed to be compact, by integrating over M and applying the Stokes Theorem, we have

$$0 = \int_M d(\partial\omega \wedge \chi) = \int_M (d\partial\omega)^{(2,2)} \wedge \chi - \int_M \partial\omega \wedge d\chi = \int_M (d\partial\omega)^{(2,2)} \wedge \chi - \int_M \partial\omega \wedge \bar{\partial}\chi,$$

$$0 = \int_M d(\omega \wedge \bar{\partial}\chi) = \int_M \partial(\omega \wedge \bar{\partial}\chi) = \int_M \partial\omega \wedge \bar{\partial}\chi + \int_M \omega \wedge \partial\bar{\partial}\chi,$$

where note that $d\partial\omega \wedge \chi = (d\partial\omega)^{(2,2)} \wedge \chi$ since χ is $(n-2, n-2)$-form. Hence we have

$$\int_M \sqrt{-1}(d\partial\omega)^{(2,2)} \wedge \chi = \int_M \sqrt{-1}\partial\omega \wedge \bar{\partial}\chi = -\int_M \sqrt{-1}\omega \wedge \partial\bar{\partial}\chi.$$

By combining these with the assumption that χ is a positive $\partial\bar{\partial}$-closed $(n-2, n-2)$-form, we obtain that

$$\int_M (T^k)^{(2,0)} \wedge \overline{(T^k)^{(2,0)}} \wedge \chi = \int_M \sqrt{-1}\omega \wedge \partial\bar{\partial}\chi = 0,$$

where note that $(T^k)^{(2,0)} \wedge \overline{(T^k)^{(2,0)}}$ is a global, nonnegative torsion $(2, 2)$-form on M. Therefore, we have $T^k_{ij} = 0$ for all i, j and k, which implies that g is quasi-Kähler from Lemma 3.1. □

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