Complex Manifolds

Research Article

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Examples of solvmanifolds without LCK structures

Abstract: The purpose in this paper is to construct solvmanifolds without LCK structures such that the complex structure is left-invariant.

Introduction

Let G be a simply-connected solvable Lie group. A discrete co-compact subgroup Γ of G is said to be a *lattice* in G. We call the compact manifold $\Gamma\backslash G$ a *solvmanifold*. If G is a nilpotent Lie group, we call the compact manifold $\Gamma\backslash G$ a *nilmanifold*. It is well known that a Kähler solvmanifold is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus ([2], [8]).

Let (M, g, J) be a $2n$-dimensional compact Hermitian manifold. We denote by Ω the fundamental 2-form, that is, the 2-form defined by $\Omega(X, Y) = g(X, JY)$. A Hermitian manifold (M, g, J) is said to be *locally conformal Kähler (LCK)* if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The closed 1-form ω is called the *Lee form*. Note that if $\omega = df$, then $(M, e^{-f}g, J)$ is Kähler. A LCK manifold (M, g, J) is said to be *a Vaisman manifold* if the Lee form ω is parallel with respect to the Levi-Civita connection ∇ of the metric g. In this paper, we consider LCK structures and Vaisman structures on solvmanifolds with a left-invariant complex structure.

The main non-Kähler examples of LCK manifolds are Hopf manifold [24], Inoue surfaces [22], Kodaira-Thurston manifold [5] and Oeljeklaus-Toma manifold [15] (cf. [6]). Note that Hopf manifold and Kodaira-Thurston manifold are Vaisman manifolds, and Inoue surfaces, Kodaira-Thurston manifold and Oeljeklaus-Toma manifold have a structure of a solvmanifold. The author of [18] proved that, in the case of a nilmanifold $\Gamma\backslash G$ with a left-invariant complex structure J, $(\Gamma\backslash G, J)$ has a LCK structure if and only if $G = \mathbb{R} \times H(n)$, where $H(n)$ is a $(2n + 1)$-dimensional Heisenberg Lie group. Thus, a LCK structure on a nilmanifold with a left-invariant complex structure is a Vaisman structure. Bazzoni [3] proved that a Vaisman nilmanifold is a quotient of $G = \mathbb{R} \times H(n)$ even if J is not left-invariant. Kasuya [12] proved that Oeljeklaus-Toma manifold has no Vaisman structures. On the other hand, not much is known about LCK structures and Vaisman structures on general solvmanifolds.

In this paper, we consider a $(2n + 2)$-dimensional solvable Lie group G_n given by

$$G_n = \left\{ \left(t, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, z \right) : t, x_i, y_i, z \in \mathbb{R}, i = 1, \cdots, n \right\},$$

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where the structure group on G_n is defined by

$$\left(t, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, z\right) \cdot \left(t', \begin{pmatrix} x_i' \\ y_i' \end{pmatrix}, z'\right) = \left(t + t', \begin{pmatrix} e^{a_i t} x_i' + x_i \\ e^{-a_i t} y_i' + y_i \end{pmatrix}, z' + \frac{1}{2} \sum_{i=1}^{n} (-e^{a_i t} y_i x_i' + e^{-a_i t} x_i y_i') + z\right)$$

and $a_i \in \mathbb{Z} - \{0\}$. In section 1, we construct a lattice Γ_n in G_n (cf. [17]).

In the case of $n = 1$, the 4-dimensional solvmanifold $\Gamma_1 \backslash G_1$ has a LCK structure with non-parallel Lee form [1], and it is biholomorphic to Inoue surface S^+ [10]. Belgun [4] considered the universal covering space of a Vaisman surface and proved that Inoue surface S^+ has no Vaisman structures. In section 3, we reconsider LCK structures and Vaisman structures on Inoue surface S^+ of the form $\Gamma \backslash G$, and have the same result by a different proof:

Main Theorem 1. *The solvmanifold $\Gamma_1 \backslash G_1$ has no Vaisman structures.*

Moreover, we determine a LCK structure on the solvmanifold $\Gamma_1 \backslash G_1$.

In the case of $n \geq 2$, we prove

Main Theorem 2. *In the case of $n \geq 2$, let J_n be a left-invariant complex structure on the solvmanifold $\Gamma_n \backslash G_n$. Then the solvmanifold $(\Gamma_n \backslash G_n, J_n)$ has no LCK structures.*

A non-degenerate 2-form Ω is said to be *locally conformal symplectic (LCS)* if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. Note that the solvmanifold $\Gamma_n \backslash G_n$ has LCS structures.

1 Construction of solvmanifolds

In this section, we construct a lattice Γ_n in G_n (cf. [17]).

Let B be a unimodular matrix with distinct positive eigenvalues λ, λ^{-1} given by

$$B = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

where $k > 2$. The characteristic polynomial of B is given by $f(x) = x^2 - kx + 1$. Let $P = \begin{pmatrix} 1 & \lambda \\ 1 & \lambda^{-1} \end{pmatrix} \in GL(2, \mathbb{R})$. Then we see that $|P| = (\lambda^{-1} - \lambda)$ and $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

We define a diffeomorphism $\varphi : \mathbb{R}^{2n+2} \to G_n$ by

$$\varphi(A) = \left((\log \lambda)t, 2P\begin{pmatrix} x_i \\ y_i \end{pmatrix}, |P|z\right)$$

for $A = (t, x_i, y_i, z) \in \mathbb{R}^{2n+2}$. Define $\Gamma_n = \varphi(\mathbb{Z}^{2n+2})$. We prove that Γ_n is a lattice in G_n. It is obvious that Γ_n is discrete and $\varphi(U)$ is compact, where $U = \{(t, x_i, y_i, z) : 0 \leq t, x_i, y_i, z \leq 1\}$. Thus, it is sufficient to prove that Γ_n is a subgroup of G_n.

Let $\gamma, \gamma' \in \Gamma_n$ such that $\gamma = \varphi((s, u_i, v_i, w)), \gamma' = \varphi((s', u_i', v_i', w'))$. Put

$$s(\gamma) = (\log \lambda)s, \quad \begin{pmatrix} u_i(\gamma) \\ v_i(\gamma) \end{pmatrix} = 2P\begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad w(\gamma) = |P|w.$$

Then we get $s(\gamma\gamma') = \log \lambda(s + s')$.

For each i, we see that

$$\begin{pmatrix} u_i(\gamma\gamma') \\ v_i(\gamma\gamma') \end{pmatrix} = \begin{pmatrix} e^{a_i(\log \lambda)s} u_i(\gamma') + u_i(\gamma) \\ e^{-a_i(\log \lambda)s} v_i(\gamma') + v_i(\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{a_i s} & 0 \\ 0 & \lambda^{-a_i s} \end{pmatrix} \begin{pmatrix} \boldsymbol{u_i}(\gamma') \\ \boldsymbol{v_i}(\gamma') \end{pmatrix} + \begin{pmatrix} \boldsymbol{u_i}(\gamma) \\ \boldsymbol{v_i}(\gamma) \end{pmatrix}$$

$$= PB^{a_i s}P^{-1}2P\begin{pmatrix} u_i' \\ v_i' \end{pmatrix} + 2P\begin{pmatrix} u_i \\ v_i \end{pmatrix} = 2P\left\{ B^{a_i s}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix} + \begin{pmatrix} u_i \\ v_i \end{pmatrix} \right\}.$$

Moreover, since

$$-e^{a_i(\log\lambda)s}\boldsymbol{v_i}(\gamma)\boldsymbol{u_i}(\gamma') + e^{-a_i(\log\lambda)s}\boldsymbol{u_i}(\gamma)\boldsymbol{v_i}(\gamma')$$

$$= -\lambda^{a_i s}\boldsymbol{v_i}(\gamma)\boldsymbol{u_i}(\gamma') + \lambda^{-a_i s}\boldsymbol{u_i}(\gamma)\boldsymbol{v_i}(\gamma')$$

$$= (-\boldsymbol{v_i}(\gamma), \boldsymbol{u_i}(\gamma))\begin{pmatrix} \lambda^{a_i s}\boldsymbol{u_i}(\gamma') \\ \lambda^{-a_i s}\boldsymbol{v_i}(\gamma') \end{pmatrix}$$

$$= (-\boldsymbol{v_i}(\gamma), \boldsymbol{u_i}(\gamma))\begin{pmatrix} \lambda^{a_i s} & 0 \\ 0 & \lambda^{-a_i s} \end{pmatrix}\begin{pmatrix} \boldsymbol{u_i}(\gamma') \\ \boldsymbol{v_i}(\gamma') \end{pmatrix}$$

$$= 2(-u_i - \lambda^{-1}v_i, u_i + \lambda v_i)PB^{a_i s}P^{-1}2P\begin{pmatrix} u_i' \\ v_i' \end{pmatrix}$$

$$= 4\left\{ u_i(-1,1) + v_i(-\lambda^{-1}, \lambda) \right\}\begin{pmatrix} 1 & \lambda \\ 1 & \lambda^{-1} \end{pmatrix}B^{a_i s}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix}$$

$$= 4\left\{ u_i(0, |P|) + v_i(-|P|, 0) \right\}B^{a_i s}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix}$$

$$= 4|P|(-v_i, u_i)B^{a_i s}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix},$$

we have

$$\boldsymbol{w}(\gamma\gamma') = \boldsymbol{w}(\gamma') + \frac{1}{2}\sum_{i=1}^n (-e^{a_i(\log\lambda)s}\boldsymbol{v_i}(\gamma)\boldsymbol{u_i}(\gamma') + e^{-a_i(\log\lambda)s}\boldsymbol{u_i}(\gamma)\boldsymbol{v_i}(\gamma')) + \boldsymbol{w}(\gamma')$$

$$= |P|\,w' + \sum_{i=1}^n 2|P|(-v_i, u_i)B^{a_i s}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix} + |P|\,w$$

$$= |P|\left\{ w' + \sum_{i=1}^n 2(-v_i, u_i)B^{a_i s}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix} + w \right\}.$$

Then Γ_n is a subgroup of G_n. Thus we have

Proposition 1.1. *Γ_n is a lattice in G_n.*

2 LCK structures on solvmanifolds

In this section, we determine a form of the fundamental 2-form on a Vaisman manifold, and obtain a left-invariant LCK metric provided one begins with a left-invariant complex structure.

Let M be a manifold and $A^\star(M)$ be the de Rham complex of M with the exterior differential operator d. For a closed 1-form θ on M, we define the new differential operator d_θ from $A^p(M)$ to $A^{p+1}(M)$ by

$$d_\theta\alpha = \theta \wedge \alpha + d\alpha.$$

Since θ is closed, we see that $d_\theta^2 = 0$. Then, we can define the new cohomology group $H_\theta^\star(M)$ in a natural manner. A p-form α is called θ-*closed* if $d_\theta\alpha = 0$. It is called θ-*exact* if there exists a $(p-1)$-form β such that $\alpha = d_\theta\beta$. Similarly, we can define a new differential operator and the new cohomology group on a Lie algebra.

Let Ω be the fundamental 2-form of a LCK structure (g, J). There exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$, which implies that Ω is $-\omega$-closed. Moreover, we have

Proposition 2.1 ([13])**.** *Let (M, g) be a compact Riemannian manifold. If a closed 1-form θ is parallel with respect to the Levi-Civita connection ∇ of the metric g, then any θ-closed form is θ-exact.*

From Proposition 2.1, the fundamental 2-from Ω of a Vaisman manifold is $-\omega$-exact, that is, it is given by

$$\Omega = d_{-\omega}\eta = -\omega \wedge \eta + d\eta,$$

where η is a 1-form.

A solvable Lie group G is said to be *completely solvable* if a linear map $\mathrm{ad}(X)$ from $\mathfrak{g}$ to $\mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. We have

Theorem 2.2 ([9])**.** *If a completely solvable Lie group G admits a lattice Γ, then*

$$H_{DR}^p(\Gamma \backslash G) \cong H^p(\mathfrak{g})$$

for each p, where $\mathfrak{g}$ is the Lie algebra of G.

Let $(M = \Gamma \backslash G, g, J)$ be a LCK completely solvable solvmanifold with a left-invariant complex structure J, and $\mathfrak{g}$ be the Lie algebra of G. From Theorem 2.2, there exists a left-invariant closed 1-form ω_0 such that $\omega - \omega_0 = df$. Then, for left-invariant vector fields X, Y, we define an inner product $\langle \ , \ \rangle$ by

$$\langle X, Y \rangle = \int\limits_M e^{-f} g(X, Y) d\mu,$$

where $d\mu$ is the volume element induced by a bi-invariant volume element on G. Since the complex structure J is left-invariant, we get a left-invariant Hermitian structure $(\langle \ , \ \rangle, J)$. Let Ω_0 be the fundamental 2-form of $(\langle \ , \ \rangle, J)$. We have

Proposition 2.3 ([4])**.** $d\Omega_0 = \omega_0 \wedge \Omega_0$.

Thus a LCK structure (g, J) with left-invariant J induces a left-invariant LCK structure $(\langle \ , \ \rangle, J)$ on $\mathfrak{g}$. We say that $(\mathfrak{g}, \langle \ , \ \rangle, J)$ is a *locally conformal Kähler (LCK)* solvable Lie algebra. Moreover, we have

Proposition 2.4 ([19])**.** *If Ω is $-\omega$-exact, then Ω_0 is also $-\omega_0$-exact, that is, it is given by*

$$\Omega_0 = d_{-\omega_0}\eta_0 = -\omega_0 \wedge \eta_0 + d\eta_0,$$

where η_0 is a 1-form on $\mathfrak{g}$.

Since the solvable Lie group G_n is completely solvable, we see that a LCK structure (g, J_n) on $\Gamma_n \backslash G_n$ induces a left-invariant LCK structure $(\langle \ , \ \rangle, J_n)$, where J_n is a left-invariant complex structure.

3 Proof of Main Theorem 1

In this section, we consider LCK structures in the case of $n = 1$ and prove Main Theorem 1.

In the case of $n = 1$, the Lie algebra $\mathfrak{g}_1$ of G_1 is given by

$$\mathfrak{g}_1 = \mathrm{span}\{A, X, Y, Z : [A, X] = a_1 X, [A, Y] = -a_1 Y, [X, Y] = Z\},$$

where $a_1 \neq 0$. Put $\widehat{A} = \dfrac{1}{a_1} A$. Then the Lie algebra $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_1'$ given by

$$\mathfrak{g}_1' = \mathrm{span}\{\widehat{A}, X, Y, Z : [\widehat{A}, X] = X, [\widehat{A}, Y] = -Y, [X, Y] = Z\}.$$

From now on, we write $\mathfrak{g}_1'$ as $\mathfrak{g}_1$ and $\widehat{A}$ as A, respectively. Let $\{\theta, \alpha, \beta, \delta\}$ be the dual basis of $\{A, X, Y, Z\}$:

$$d\theta = 0, \ d\alpha = -\theta \wedge \alpha, \ d\beta = \theta \wedge \beta, \ d\delta = -\alpha \wedge \beta.$$

Then we see that $H^1(\mathfrak{g}_1) = \mathrm{span}\{[\theta]\}$.

The 4-dimensional solvmanifold $\Gamma_1 \backslash G_1$ has a LCK structure with non-parallel Lee form as follows:

Example 3.1 ([1]). *Let $\langle \, , \, \rangle$ be a left-invariant metric such that $\{A, X, Y, Z\}$ is an orthonormal frame, and J be a left-invariant complex structure defined by*

$$JA = Y, JY = -A, JZ = X, JX = -Z.$$

Then $(\Gamma_1 \backslash G_1, \langle \, , \, \rangle, J)$ is a LCK manifold with the fundamental 2-form Ω given by

$$\Omega = -\theta \wedge \beta - \delta \wedge \alpha,$$

and Lee form $-\theta$. We easily see that

$$\nabla_Y \theta(Y) = -\theta(\nabla_Y Y) = -\langle A, \nabla_Y Y \rangle = -\langle [A, Y], Y \rangle = \langle Y, Y \rangle \neq 0.$$

Then Lee form $-\theta$ is not parallel.

Two complex structures J_1 and J_2 on a Lie algebra $\mathfrak{g}$ are *equivalent* if there exists $F \in \mathrm{Aut}(\mathfrak{g})$ such that $F \circ J_2 = J_1 \circ F$. If J_1 and J_2 are equivalent, then a LCK metric g_2 with a complex structure J_2 is one-to-one correspondence with a LCK metric g_1 with a complex structure J_1, by $g_2 = F^* g_1$ [23].

Proof of Main Theorem 1. A complex structure J on a 4-dimensional solvmanifold is left-invariant [7]. Then, a LCK structure (g, J) on $\Gamma_1 \backslash G_1$ induces a left-invariant LCK structure $(\langle \, , \, \rangle, J)$ as in section 2. Particularly, we assume that the LCK structure (g, J) is a Vaisman structure. From Proposition 2.1 and Proposition 2.4, the fundamental 2-form Ω of $(\langle \, , \, \rangle, J)$ is $-k\theta$-exact ($k \in \mathbb{R}$).

A complex structure J on $\mathfrak{g}_1$ is equivalent to a complex structure J_q such that

$$J_q A = Y + qZ, J_q Y = -A - qX, J_q Z = X, J_q X = -Z,$$

where $q = 0$ or 1 [16]. Thus, the LCK structure $(\langle \, , \, \rangle, J)$ induces the LCK structure $(\langle \, , \, \rangle_q, J_q)$ with the fundamental 2-form Ω_q given by

$$\Omega_q = d_{-k\theta} \eta = -k\theta \wedge \eta + d\eta,$$

because there exists $F \in \mathrm{Aut}(\mathfrak{g}_1)$ such that $\Omega_q = F^* \Omega$. However, we easily see that

$$\langle Z, Z \rangle = d_{-k\theta} \eta(J_q Z, Z) = (-k\theta \wedge \eta + d\eta)(X, Z) = 0$$

because $X, Z \in [\mathfrak{g}_1, \mathfrak{g}_1]$ and Z is in the center of $\mathfrak{g}_1$. This is a contradiction. □

We give some remarks:

Remark 3.2. *Kasuya [12] proved that, in the case of $G = \mathbb{R}^n \ltimes_\varphi \mathbb{R}^m$, a solvmanifold $\Gamma \backslash G$ with some conditions has no Vaisman structures. Thus, Inoue surface S^0 and Oeljeklaus-Toma manifold have no Vaisman structures. On the other hand, the solvable Lie group G_1 is the form of $\mathbb{R} \ltimes_\varphi H(1)$, where $H(1)$ is a 3-dimensional Heisenberg Lie group.*

Remark 3.3. *It is known that the first Betti number of a Vaisman manifold is odd ([11], [24]). From Theorem 2.2 [9], we see that $\dim H^1_{DR}(\Gamma_1 \backslash G_1) = \dim H^1(\mathfrak{g}_1) = 1$.*

Next, we consider LCK structures on the solvable Lie algebra $\mathfrak{g}_1$. From the argument in [14], the new cohomology group $H^2_{k\theta}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ is given by

$$H^2_{-\theta}(\mathfrak{g}_1) = \mathrm{span}\{[\delta \wedge \beta]_{-\theta}, [\theta \wedge \beta]_{-\theta}\},$$
$$H^2_{\theta}(\mathfrak{g}_1) = \mathrm{span}\{[\delta \wedge \alpha]_{\theta}, [\theta \wedge \alpha]_{\theta}\},$$
$$H^2_{k\theta}(\mathfrak{g}_1) = \{0\} \text{ for } k \neq \pm 1,$$

where $[\quad]_{\pm\theta}$ is a cohomology class of a new cohomology group $H^2_{\pm\theta}(\mathfrak{g}_1)$ defined by $d_{\pm\theta}$, respectively. From the above argument, we see that $\mathfrak{g}_1$ has no LCK structures such that the fundamental 2-form is exact.

A $-\theta$-closed 2-form Ω^- is given by

$$\Omega^- = a^- \delta \wedge \beta + b^- \theta \wedge \beta + d_{-\theta} \eta^-.$$

Then we see that

$$\Omega^-(J_q Z, Z) = \Omega^-(X, Z) = (-\theta \wedge \eta^- + d\eta^-)(X, Z) = 0$$

because $X, Z \in [\mathfrak{g}_1, \mathfrak{g}_1]$ and Z is in the center of $\mathfrak{g}_1$. Thus, (Ω^-, J_q) is not a LCK structure for any $q = 0$ or 1.

A θ-closed 2-form Ω^+ is given by

$$\Omega^+ = a^+ \delta \wedge \alpha + b^+ \theta \wedge \alpha + d_\theta \eta^+.$$

If (Ω^+, J_q) is a LCK structure, then we see that

$$\Omega^+(J_q Z, Z) = \Omega^+(X, Z) = (a^+ \delta \wedge \alpha)(X, Z) = -a^+ \neq 0.$$

Moreover, we get

$$\Omega^+(X, Y) = \Omega^+(J_q X, J_q Y) = \Omega^+(-Z, -A - qX)$$
$$d\eta^+(X, Y) = (a^+ \delta \wedge \alpha + \theta \wedge \eta^+)(-Z, -A - qX)$$
$$-\eta^+([X, Y]) = -\eta^+(Z) = -\eta^+(Z) + a^+ q$$
$$0 = a^+ q,$$

that is, $q = 0$.

Let (Ω^+, J_0) be a LCK structure with Lee form θ as above. Then we see that

$$\Omega^+(A, X) = \Omega^+(J_0 A, J_0 X) = \Omega^+(Y, -Z)$$
$$(b^+ \theta \wedge \alpha + d_\theta \eta^+)(A, X) = 0$$
$$b^+ + \eta^+(X) - \eta^+(X) = 0,$$

that is, $b^+ = 0$. Then, we see that $[\Omega^+]_\theta \in \mathrm{span}\{[\delta \wedge \alpha]_\theta\}$. This is Example 3.1. Thus, we have

Proposition 3.4. *$(\Gamma_1 \backslash G_1, J_q)$ has a LCK structure Ω if and only if $q = 0$, that is, Inoue surface S^+. Moreover, a LCK structure (Ω, J_0) has a Lee form θ and $[\Omega]_\theta \in \mathrm{span}\{[\delta \wedge \alpha]_\theta\}$.*

4 Proof of Main Theorem 2

In this section, we consider LCK structures in the case of $n \geq 2$ and prove Main Theorem 2.

In the case of $n \geq 2$, the Lie algebra $\mathfrak{g}_n$ of G_n is given by

$$\mathfrak{g}_n = \mathrm{span}\{A, X_i, Y_i, Z : [A, X_i] = a_i X_i, [A, Y_i] = -a_i Y_i, [X_i, Y_i] = Z\},$$

where $a_i \neq 0$. Let $\{\theta, \alpha_i, \beta_i, \delta\}$ be the dual basis of $\{A, X_i, Y_i, Z\}$:

$$d\theta = 0, \quad d\alpha_i = -a_i \theta \wedge \alpha_i, \quad d\beta_i = a_i \theta \wedge \beta_i, \quad d\delta = -\sum_{i=1}^n \alpha_i \wedge \beta_i.$$

Then we see that $H^1(\mathfrak{g}_n) = \mathrm{span}\{[\theta]\}$.

We have

Lemma 4.1. *Let Ω_0 be a $-k\theta$-closed 2-form on $\mathfrak{g}_n$ ($k \in \mathbb{R}$). Then $\Omega_0(Z, X_i) = \Omega_0(Z, Y_i) = 0$ for each i.*

Proof. For each i, since $n \geq 2$, we can take j such that $j \neq i$. Then we get

$$\begin{aligned}
\Omega_0(Z, X_i) &= \Omega_0([X_j, Y_j], X_i) \\
&= -d\Omega_0(X_j, Y_j, X_i) + \Omega_0([X_j, X_i], Y_j) - \Omega_0([Y_j, X_i], X_j) \\
&= -k\theta \wedge \Omega_0(X_j, Y_j, X_i) = 0.
\end{aligned}$$

Similarly, we get $\Omega_0(Z, Y_i) = 0$. □

Proof of Main Theorem 2. Let J_n be a left-invariant complex structure on $\Gamma_n \backslash G_n$. We assume that the solvmanifold $(\Gamma_n \backslash G_n, J_n)$ has a LCK metric g. From section 2, a LCK structure (g, J_n) induces a left-invariant LCK structure $(\langle \ , \ \rangle, J_n)$. Let Ω_0 be the fundamental 2-form of $(\langle \ , \ \rangle, J_n)$ and $-k\theta$ be Lee form. □

Let γ_0 be the isomorphism from $\mathfrak{g}_n^*$ to $\mathfrak{g}_n$ induced by $\langle \ , \ \rangle$. Since $[\mathfrak{g}_n, \mathfrak{g}_n] = \mathrm{span}\{X_i, Y_i, Z\}$ and θ is closed, we see that $[\mathfrak{g}_n, \mathfrak{g}_n]^\perp = \mathrm{span}\{\gamma_0(\theta)\}$. From Lemma 4.1, we get

Corollary 4.2. $J \circ \gamma_0(\theta) \in span\{Z\}$.

Proof. From Lemma 4.1, we see that

$$\langle JZ, X_i \rangle = -\Omega_0(Z, X_i) = 0 \text{ and } \langle JZ, Y_i \rangle = -\Omega_0(Z, Y_i) = 0$$

for each i. Since $[\mathfrak{g}_n, \mathfrak{g}_n] = \mathrm{span}\{X_i, Y_i, Z\}$, we have $JZ \in [\mathfrak{g}_n, \mathfrak{g}_n]^\perp = \mathrm{span}\{\gamma_0(\theta)\}$, that is, $J \circ \gamma_0(\theta) \in \mathrm{span}\{Z\}$.
□

Let $(\mathfrak{g}, \langle \ , \ \rangle, J)$ be a LCK solvable Lie algebra as in section 2, and ω_0 be the Lee form. In the previous papers [20], [21], we have

Theorem 4.3 ([20]). *If* $\langle [\gamma_0(\omega_0), J \circ \gamma_0(\omega_0)], J \circ \gamma_0(\omega_0) \rangle = 0$, *then* $(\langle \ , \ \rangle, J)$ *is a Vaisman structure.*

Theorem 4.4 ([21]). *If* $\mathfrak{g}$ *is completely solvable and* $(\langle \ , \ \rangle, J)$ *is a Vaisman structure, then* $\mathfrak{g} = \mathbb{R} \times \mathfrak{h}(n)$, *where* $\mathfrak{h}(n)$ *is a* $(2n+1)$*-dimensional Heisenberg Lie algebra.*

Since Z is in the center of $\mathfrak{g}_n$, from Corollary 4.2, we see that $(\mathfrak{g}_n, \langle \ , \ \rangle, J_n)$ satisfies the assumption on Theorem 4.3. Then a LCK structure $(\langle \ , \ \rangle, J_n)$ on $\mathfrak{g}_n$ is a Vaisman structure. However, $\mathfrak{g}_n$ is completely solvable and it is the form of $\mathbb{R} \ltimes_\varphi \mathfrak{h}(n)$, and the action φ is non-trivial. This is contradiction to Theorem 4.4. Thus, $(\mathfrak{g}_n, J_n)$ has no LCK structures.

We give some remarks:

Remark 4.5. *A non-degenerate* $-k\theta$*-closed 2-form is said to be locally conformal symplectic (LCS). The solvmanifold* $\Gamma_n \backslash G_n$ *has LCS structures* Ω_1, Ω_2 *defined as follows:*

$$\Omega_1 = d_{-\theta}\delta = -\theta \wedge \delta - \sum_{i=1}^{n} \alpha_i \wedge \beta_i,$$

$$\Omega_2 = \alpha_k \wedge \alpha_l + d_{(a_k + a_l)\theta}\delta = \alpha_k \wedge \alpha_l + (a_k + a_l)\theta \wedge \delta - (a_k + a_l)\sum_{i=1}^{n} \alpha_i \wedge \beta_i,$$

where $a_k + a_l \neq 0$. *Note that* Ω_1 *is exact and* Ω_2 *is non-exact.*

Remark 4.6. *If* $a_i = 0$ *for each* i, *then the solvmanifold* $\Gamma_n \backslash G_n$ *is a nilmanifold* $S^1 \times \Gamma \backslash H(n)$, *where* $H(n)$ *is a* $(2n+1)$*-dimensional Heisenberg Lie group and* Γ *is its lattice. A nilmanifold* $S^1 \times \Gamma \backslash H(n)$ *has a Vaisman structure [5].*

Remark 4.7. *From Theorem 4.4, we can easily prove Main Theorem 1.*

References

[1] L. C. de Andrés, L. A. Cordero, M. Fernández and J. J. Mencía : Examples of four dimensional locally conformal Kähler solvmanifolds, Geom. Dedicata **29** (1989), 227-232.

[2] D. Arapura and M. Nori: Solvable fundamental groups of algebraic varieties and Kähler manifolds, Compositio Math, **116** (1999), 173-188.

[3] G. Bazzoni: Vaisman nilmanifolds, Bull. Lond. Math. Soc. **49** (2017), no. 5, 824-830.

[4] F. A. Belgun: On the metric structure of non-Kähler complex surfaces, Math. Ann. **317** (2000), 1-40.

[5] L. A. Cordero, M. Fernández and M. de Léon: Compact locally conformal Kähler nilmanifolds, Geom. Dedicata **21** (1986), 187-192.

[6] S. Dragomir and L. Ornea: *Locally conformal Kähler geometry*, Birkhäuser (1998).

[7] K. Hasegawa: Complex and Kähler structures on compact solvmanifolds, J. Symplectic Geom. **3** (2005), 749-767.

[8] K. Hasegawa: A note on compact solvmanifolds with Kähler structures, Osaka J. Math. **43** (2006), no. 1, 131-135.

[9] A. Hattori : Spectral sequence in the de Rham cohomology of fibre bundles, J. Fac. Sci. Univ. Tokyo Sect. I . **8** (1960), 289-331.

[10] Y. Kamishima: Note on locally conformal Kähler surfaces, Geom. Dedicata **84** (2001), no. 1-3, 115-124.

[11] T. Kashiwada and S. Sato: On harmonic forms in compact locally conformal Kähler manifolds with the parallel Lee form, Ann. Fac. Sci. Univ. Nat. Zaire (Kinshasa) Sect. Math.-Phys. **6** (1980), 17-29

[12] H. Kasuya : Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds, Bull. Lond. Math. Soc. **45** (2013), no. 1, 15-26.

[13] M. de León, B. López, J. C. Marrero and E. Padrón: On the computation of the Lichnerowicz-Jacobi cohomology, J. Geom. Phys. **44** (2003), no. 4, 507-522.

[14] D. V. Millionschikov: Cohomology of solvable Lie algebras and solvmanifolds, Mat. Zametki **77** (2005), no. 1, 67-79,

[15] K. Oeljeklaus and M. Toma: Non-Kähler compact complex manifolds associated to number fields, Ann. Inst. Fourier (Grenoble) **55** (2005) 161-171.

[16] G. Ovando: Complex, symplectic and Kähler structures on four dimensional Lie groups, Rev. Un. Mat. Argentina **45** (2004), no. 2, 55-67.

[17] H. Sawai: A construction of lattices on certain solvable Lie groups, Topology Appl. **154** (2007), no. 18, 3125-3134.

[18] H. Sawai: Locally conformal Kähler structures on compact nilmanifolds with left-invariant complex structures, Geom. Dedicata **125** (2007), 93-101.

[19] H. Sawai: Locally conformal Kähler structures on compact solvmanifolds, Osaka J. Math. **49** (2012), no. 4, 1087-1102.

[20] H. Sawai: Vaisman structures on compact solvmanifolds, Geom. Dedicata **178** (2015), 389-404.

[21] H. Sawai: Structure theorem for Vaisman completely solvable solvmanifolds, J. Geom. Phys. **114** (2017), 581-586.

[22] F. Tricerri: Some examples of locally conformal Kähler manifolds, Rend. Sem. Math. Univ. Politec. Torino **40** (1982), no. 1, 81-92.

[23] L. Ugarte: Hermitian structures on six-dimensional nilmanifolds, Transform. Groups **12** (2007), no. 1, 175-202.

[24] I. Vaisman: Generalized Hopf manifolds, Geom. Dedicata **13** (1982), no. 3, 231-255.