

Complex Manifolds

Research Article

Indranil Biswas*

Criterion for connections on principal bundles over a pointed Riemann surface

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Abstract: We investigate connections, and more generally logarithmic connections, on holomorphic principal bundles over a compact connected Riemann surface.

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1 Introduction

Let X be a compact connected Riemann surface and E a holomorphic vector bundle on X . It is a natural question to ask for a criterion that would ensure that E admits a holomorphic connection. Note that a holomorphic connection on E is same as a flat connection on E such that the locally defined flat sections are holomorphic. Answering the question, a well-known theorem of Atiyah and Weil says that E admits a holomorphic connection if and only if the degree of every indecomposable component of E is zero [16], [2]. One can ask the same question in broader set-ups. We give some such examples.

Let $0 = E_0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E$ be a filtration of E by holomorphic sub-bundles. One can pose the following questions:

1. What is a criterion for E to admit a holomorphic connection $\mathcal{D}$ such that each sub-bundle E_i is preserved by $\mathcal{D}$?
2. What is a criterion for E to admit a holomorphic connection $\mathcal{D}$ such that $\mathcal{D}(E_i) \subset E_{i+1} \otimes K_X$ for all $1 \leq i \leq k-1$? (The holomorphic cotangent bundle of X is denoted by K_X .)

If E admits a holomorphic connection $\mathcal{D}$ such that each sub-bundle E_i is preserved by $\mathcal{D}$, then $\mathcal{D}$ induces a holomorphic connection on each E_i/E_{i-1} , $1 \leq i \leq k$. A standard guess is that this necessary condition is also sufficient. However there are examples showing that this necessary condition is not sufficient (see [7]).

If the vector bundle E is semi-stable of degree zero, then from the criterion of Atiyah and Weil it follows that E admits a holomorphic connection. A theorem of Simpson gives a much finer statement: The vector bundle E has a *canonical* holomorphic connection [15, p. 36, Lemma 3.5]. One of the properties of this canonical connection is that if each E_i is of degree zero, then the filtration of E is preserved by the connection.

If E admits a holomorphic connection $\mathcal{D}$ such that $\mathcal{D}(E_i) \subset E_{i+1} \otimes K_X$ for all $1 \leq i \leq k-1$, then consider the associated graded vector bundle

$$\widetilde{E} := \bigoplus_{i=1}^k E_i/E_{i-1} \longrightarrow X.$$

*Corresponding Author: Indranil Biswas: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India, E-mail: indranil@math.tifr.res.in

The second fundamental form of $\mathcal{D}$ for the filtration of E produces $\mathcal{O}_X$ -linear homomorphisms

$$\theta_i : E_i/E_{i-1} \longrightarrow (E_{i+1}/E_i) \otimes K_X, \quad 1 \leq i \leq k-1.$$

Hence $\theta := \bigoplus_{i=1}^{k-1} \theta_i$ is a Higgs field on $\widetilde{E}$. This Higgs field θ takes the i -th graded piece $E_i/E_{i-1} \subset \widetilde{E}$ to $(E_{i+1}/E_i) \otimes K_X$.

A natural guess for the second question is the following: The filtered vector bundle E admits a holomorphic connection $\mathcal{D}$ such that $\mathcal{D}(E_i) \subset E_{i+1} \otimes K_X$ for all $1 \leq i \leq k-1$ if and only if $\widetilde{E}$ admits a Higgs field θ satisfying the following two conditions:

- $(\widetilde{E}, \theta)$ is semi-stable, and
- $\theta(E_i/E_{i-1}) \subset (E_{i+1}/E_i) \otimes K_X$ for all $1 \leq i \leq k-1$.

A holomorphic vector bundle of rank r is a special case of a holomorphic principal bundle where the structure group is $\mathrm{GL}(r, \mathbb{C})$. All the above questions generalize to the more general context of holomorphic principal bundles on X .

In another direction, the holomorphic connections generalize to logarithmic connections. So the general set-up is logarithmic connections on holomorphic principal bundles over X . Our aim is to describe the questions in this general set-up and explain some results on these questions.

2 Atiyah bundle and holomorphic connection

Let X be a compact connected Riemann surface. The holomorphic cotangent bundle of X will be denoted by K_X . Fix a finite subset

$$D := \{x_1, \dots, x_n\} \subset X.$$

The reduced effective divisor $x_1 + \dots + x_n$ will also be denoted by D . We allow n to be zero, so D may be the zero divisor.

The holomorphic tangent bundle of a complex manifold Y will be denoted by TY .

Let H be a connected affine algebraic group defined over $\mathbb{C}$. The Lie algebra of H will be denoted by $\mathfrak{h}$. Let

$$p : E_H \longrightarrow X \tag{1}$$

be a holomorphic principal H -bundle on X . This means that E_H is a complex manifold equipped with a holomorphic action

$$p' : E_H \times H \longrightarrow E_H$$

of H , and p is a holomorphic surjective submersion, such that

- $p \circ p' = p \circ p_E$, where $p_E : E_H \times H \longrightarrow E_H$ is the natural projection, and
- the map $p_E \times p' : E_H \times H \longrightarrow E_H \times_X E_H$ is an isomorphism; note that the first condition ensures that the image of $p_E \times p'$ is contained in $E_H \times_X E_H$.

Let

$$dp : TE_H \longrightarrow p^*TX \tag{2}$$

be the differential of the map p in (1). This homomorphism dp is surjective because p is a submersion. The kernel of dp is identified with the trivial vector bundle $E_H \times \mathfrak{h}$ using the action of H on E_H . Consider the action of H on TE_H given by the action of H on E_H . It preserves the sub-bundle $\ker(dp)$. Define the quotient

$$\mathrm{ad}(E_H) := \ker(dp)/H \longrightarrow X.$$

This $\mathrm{ad}(E_H)$ is a holomorphic vector bundle over X . In fact, it is identified with the vector bundle $E_H \times^H \mathfrak{h}$ associated to E_H for the adjoint action of H on $\mathfrak{h}$; this identification is given by the above identification of $\ker(dp)$ with $E_H \times \mathfrak{h}$. This vector bundle $\mathrm{ad}(E_H)$ is known as the adjoint vector bundle for E_H . Since the adjoint action of

H on $\mathfrak{h}$ preserves its Lie algebra structure, for any $x \in X$, the fiber $\text{ad}(E_H)_x$ is a Lie algebra isomorphic to $\mathfrak{h}$. In fact, $\text{ad}(E_H)_x$ is identified with $\mathfrak{h}$ uniquely up to a conjugation.

The direct image p_*TE_H is equipped with an action of H given by the action of H on TE_H . Note that p_*TE_H is a locally free quasi-coherent sheaf on X . Its H -invariant part

$$(p_*TE_H)^H \subset p_*TE_H$$

is a locally free coherent sheaf on X . The corresponding holomorphic vector bundle is denoted by $\text{At}(E_H)$; it is known as the Atiyah bundle for E_H [2]. It is straight-forward check that the quotient

$$(TE_H)/H \longrightarrow X$$

is identified with $\text{At}(E_H)$. Consider the short exact sequence of holomorphic vector bundles on E_H

$$0 \longrightarrow \text{kernel}(dp) \longrightarrow TE_H \xrightarrow{dp} p^*TX \longrightarrow 0. \quad (3)$$

Taking its quotient by H , we get the following short exact sequence of vector bundles on X

$$0 \longrightarrow \text{ad}(E_H) \longrightarrow \text{At}(E_H) \xrightarrow{d'p} TX \longrightarrow 0, \quad (4)$$

where $d'p$ is constructed from dp ; this is known as the Atiyah exact sequence for E_H .

A holomorphic connection on E_H is a holomorphic homomorphism

$$\theta : TX \longrightarrow \text{At}(E_H)$$

such that $d'p \circ \theta = \text{Id}_{TX}$, where $d'p$ is the homomorphism in (4). Giving such a homomorphism θ is equivalent to giving a homomorphism $\varpi : \text{At}(E_H) \longrightarrow \text{ad}(E_H)$ such that the composition

$$\text{ad}(E_H) \hookrightarrow \text{At}(E_H) \xrightarrow{\varpi} \text{ad}(E_H)$$

is the identity map of $\text{ad}(E_H)$.

Consider the smooth divisor $\widehat{D} := p^{-1}(D) \subset E_H$. We have the logarithmic tangent bundle

$$TE_H(-\log \widehat{D}) \subset TE_H,$$

whose derivation action on the structure sheaf $\mathcal{O}_{E_H}$ preserves $\mathcal{O}_{E_H}(-\widehat{D}) \subset \mathcal{O}_{E_H}$. The action of H on TE_H clearly preserves the sub-sheaf $TE_H(-\log \widehat{D})$. Now define

$$\text{At}(E_H, D) := TE_H(-\log \widehat{D})/H \longrightarrow X$$

We note that $\text{At}(E_H, D)$ coincides with the H -invariant direct image

$$p_*(TE_H(-\log \widehat{D}))^H \subset p_*(TE_H(-\log \widehat{D})).$$

The inclusion of $TE_H(-\log \widehat{D})$ in TE_H produces an inclusion of $\text{At}(E_H, D)$ in $\text{At}(E_H)$. Let

$$\sigma : \text{At}(E_H, D) \longrightarrow TX$$

be the restriction of the homomorphism $d'p$ in (4). It is straight-forward to check that the image of σ coincides with the logarithmic tangent bundle

$$TX(-D) := TX \otimes \mathcal{O}_X(-D) \subset TX.$$

We note that $TX(-D)$ coincides with the sub-sheaf of TX whose derivation action on the structure sheaf $\mathcal{O}_X$ preserves $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. Therefore, from (4) we get a short exact sequence of vector bundles on X

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{i_0} \text{At}(E_H, D) \xrightarrow{\sigma} TX(-D) \longrightarrow 0; \quad (5)$$

this is called the *logarithmic Atiyah exact sequence* for E_H .

From (3) we have the short exact sequence

$$0 \longrightarrow \ker(dp) \longrightarrow TE_H(-\log \widehat{D}) \xrightarrow{dp} p^*(TX(-D)) \longrightarrow 0.$$

Its quotient by H coincides with the above logarithmic Atiyah exact sequence for E_H . Indeed, this follows immediately from the fact that

$$\text{At}(E_H, D) = (d'p)^{-1}(TX(-D)),$$

where $d'p$ is the projection in (4).

A logarithmic connection on E_H singular over D is a holomorphic homomorphism

$$\theta : TX(-D) \longrightarrow \text{At}(E_H, D) \quad (6)$$

such that $\sigma \circ \theta = \text{Id}_{TX(-D)}$, where σ is the homomorphism in (5). Note that giving such a homomorphism θ is equivalent to giving a homomorphism $\varpi : \text{At}(E_H, D) \longrightarrow \text{ad}(E_H)$ such that $\varpi \circ i_0 = \text{Id}_{\text{ad}(E_H)}$, where i_0 is the injective homomorphism in (5).

So, when D is the zero divisor, a logarithmic connection on E_H is a holomorphic connection on E_H .

2.1 The special case of vector bundles

Set $H = \text{GL}(r, \mathbb{C})$. Given a holomorphic principal $\text{GL}(r, \mathbb{C})$ -bundle $E_{\text{GL}(r, \mathbb{C})}$ on X , let

$$E := E_{\text{GL}(r, \mathbb{C})} \times^{\text{GL}(r, \mathbb{C})} \mathbb{C}^r \longrightarrow X$$

be the holomorphic vector bundle associated to the standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. Let $\text{Diff}^i(E, E)$ be the holomorphic vector bundle on X whose holomorphic sections over any open subset $U \subset X$ are the holomorphic differential operators $E|_U \longrightarrow E|_U$ of order at most i . So we have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \text{End}(E) = \text{Diff}^0(E, E) \longrightarrow \text{Diff}^1(E, E) \xrightarrow{\sigma_0} \text{End}(E) \otimes TX \longrightarrow 0.$$

where σ_0 is the symbol homomorphism. The vector bundle $\text{End}(E)$ is the adjoint bundle $\text{ad}(E_{\text{GL}(r, \mathbb{C})})$. Now define the sub-sheaf

$$\text{At}(E)(-\log D) := \sigma_0^{-1}(\text{Id}_E \otimes_{\mathbb{C}} TX(-D)) \subset \text{Diff}^1(E, E).$$

It coincides with the vector bundle $\text{At}(E_{\text{GL}(r, \mathbb{C})}, D)$ constructed earlier.

Therefore, a logarithmic connection on $E_{\text{GL}(r, \mathbb{C})}$ singular over D is a first order holomorphic differential operator

$$\mathcal{D} : E \longrightarrow K_X \otimes \mathcal{O}_X(D)$$

satisfying the Leibniz identity which says that

$$\mathcal{D}(fs) = f\mathcal{D}(s) + s \otimes (df), \quad (7)$$

where s is any locally defined holomorphic section of E and f is any locally defined holomorphic function on X [10].

2.2 Residue

First we define residue for logarithmic connections on vector bundle considered in Section 2.1.

Take any point $x \in D$. Take a holomorphic coordinate z on an open subset $U \subset X$ containing x with $z(x) = 0$. Assume that $U \cap D = \{x\}$. Then $\frac{dz}{z}$ is a nowhere vanishing holomorphic section of $K_X \otimes \mathcal{O}_X(D)$ over U . Let $e_x \in (K_X \otimes \mathcal{O}_X(D))_x$ be the evaluation of this holomorphic section at the point x . It is straight-forward to check that this nonzero element e_x is independent of the choice of (U, z) . Therefore, the fiber $(K_X \otimes \mathcal{O}_X(D))_x$ is identified with $\mathbb{C}$ by sending any $c \in \mathbb{C}$ to $c \cdot e_x \in (K_X \otimes \mathcal{O}_X(D))_x$ (see [12, p. 146]).

Let $\mathcal{D}$ be a logarithmic connection singular over D on a holomorphic vector bundle E . Consider the composition

$$E \xrightarrow{\mathcal{D}} E \otimes K_X \otimes \mathcal{O}_X(D) \longrightarrow E_x \otimes (K_X \otimes \mathcal{O}_X(D))_x = E_x.$$

The Leibniz identity in (7) implies that this composition is $\mathcal{O}_X$ -linear; hence it defines an element of $\text{End}(E_x) = \text{End}(E)_x$. This element of $\text{End}(E_x)$, which we will denote by $\text{Res}(\mathcal{D}, x)$, is called the *residue* of $\mathcal{D}$ at x [10, p. 53].

Now we return to the more general case holomorphic principal bundles on X .

From (4) and (5) we have the commutative diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{i_0} & \text{At}(E_H, D) & \xrightarrow{\sigma} & \text{TX}(-D) \longrightarrow 0 \\ & & \parallel & & \downarrow j & & \downarrow \iota \\ 0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{i} & \text{At}(E_H) & \xrightarrow{d'p} & \text{TX} \longrightarrow 0 \end{array} \quad (8)$$

on X . From the commutativity of the diagram it follows that for any point $x \in X$, we have

$$d'p(x) \circ j(x) = \iota(x) \circ \sigma(x) : \text{At}(E_H, D)_x \longrightarrow (\text{TX})_x = T_x X.$$

Note that $\iota(x) = 0$ if $x \in D$, therefore $d'p(x) \circ j(x) = 0$ for all $x \in D$. Consequently, for every $x \in D$ there is a homomorphism

$$R_x : \text{At}(E_H, D)_x \longrightarrow \text{ad}(E_H)_x \quad (9)$$

uniquely defined by the identity $i(x) \circ R_x(v) = j(x)(v)$ for all $v \in \text{At}(E_H, D)_x$. Note that

$$R_x \circ i_0(x) = \text{Id}_{\text{ad}(E_H)_x},$$

where i_0 is the homomorphism in (8), which follows from the commutativity of (8). Therefore, from (5) we have

$$\text{At}(E_H, D)_x = \text{ad}(E_H)_x \oplus \text{kernel}(R_x) = \text{ad}(E_H)_x \oplus \text{TX}(-D)_x; \quad (10)$$

note that the composition $\text{kernel}(R_x) \hookrightarrow \text{At}(E_H, D)_x \xrightarrow{\sigma(x)} \text{TX}(-D)_x$ is an isomorphism.

We saw that for any $x \in D$ the fiber $\text{TX}(-D)_x$ is identified with $\mathbb{C}$. Therefore, from (10) we have

$$\text{At}(E_H, D)_x = \text{ad}(E_H)_x \oplus \mathbb{C} \quad (11)$$

for all $x \in D$.

To give an alternative description of R_x and the decomposition in (11), consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{ad}(E_H) \otimes \mathcal{O}_X(-D))_x & \xrightarrow{c_0} & (\text{At}(E_H) \otimes \mathcal{O}_X(-D))_x & \longrightarrow & (\text{TX}(-D))_x \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \parallel \\ 0 & \longrightarrow & \text{ad}(E_H)_x & \longrightarrow & \text{At}(E_H, D)_x & \longrightarrow & (\text{TX}(-D))_x \longrightarrow 0 \end{array}$$

where a is the zero homomorphism because $x \in D$; the top exact sequence is the restriction to x of the exact sequence in (4) tensored with $\mathcal{O}_X(-D)$, while the bottom one is the restriction of the exact sequence in (5) to x . Now from the snake lemma (see [14, p. 158, Lemma 9.1]) it follows that the kernel of the homomorphism b coincides with the image of c_0 . Hence the image of the fiber $(\text{At}(E_H) \otimes \mathcal{O}_X(-D))_x$ in $\text{At}(E_H, D)_x$ is identified with the quotient line $\text{TX}(-D)_x = \mathbb{C}$ of $(\text{At}(E_H) \otimes \mathcal{O}_X(-D))_x$. Now the image of the fiber $(\text{At}(E_H) \otimes \mathcal{O}_X(-D))_x$ in $\text{At}(E_H, D)_x$ and the image of $\text{ad}(E_H)_x$ in $\text{At}(E_H, D)_x$ together give a decomposition of $\text{At}(E_H, D)_x$. This decomposition coincides with the one in (11).

For a logarithmic connection $\theta : \text{TX}(-D) \longrightarrow \text{At}(E_H, D)$ as in (6), and any $x \in D$, define

$$\text{Res}(\theta, x) := R_x(\theta(1)) \in \text{ad}(E_H)_x, \quad (12)$$

where R_x is the homomorphism constructed in (9); in the above definition 1 is considered as an element of $\text{TX}(-D)_x$ using the identification of $\mathbb{C}$ with $\text{TX}(-D)_x$ mentioned earlier.

The element $\text{Res}(\theta, x)$ in (12) is called the *residue*, at x , of the logarithmic connection θ .

2.3 Homomorphisms of group

Let M be a complex affine algebraic group and

$$\rho : H \longrightarrow M$$

a holomorphic homomorphism. As before, E_H is a holomorphic principal H -bundle on X . Let

$$E_M := E_H \times^\rho M \longrightarrow X$$

be the holomorphic principal M -bundle on X obtained by extending the structure group of E_H using the above homomorphism ρ . So E_M is the quotient of $E_H \times M$ obtained by identifying (y, m) and $(yh^{-1}, \rho(h)m)$, where y, m and h run over E_H, M and H respectively. Therefore, we have a morphism

$$\widehat{\rho} : E_H \longrightarrow E_M, \quad y \longmapsto \widehat{(y, e_M)},$$

where $\widehat{(y, e_M)}$ is the equivalence class of (y, e_M) with e_M being the identity element of the group M . The homomorphism of Lie algebras

$$d\rho : \mathfrak{h} \longrightarrow \mathfrak{m} := \text{Lie}(M)$$

associated to ρ produces a homomorphism of vector bundles

$$\alpha : \text{ad}(E_H) \longrightarrow \text{ad}(E_M). \quad (13)$$

This α sends the equivalence class of $(z, v) \in E_H \times \mathfrak{h}$ to the equivalence class of $(\widehat{\rho}, d\rho(v))$. The maps $\widehat{\rho}$ and $d\rho$ together produce a homomorphism of vector bundles

$$\widetilde{A} : \text{At}(E_H) \longrightarrow \text{At}(E_M),$$

because $\widehat{\rho}$ is H -equivariant for the action of H on E_M given by the action of M on E_M and the homomorphism ρ . This map $\widetilde{A}$ produces a homomorphism

$$A : \text{At}(E_H, D) \longrightarrow \text{At}(E_M, D), \quad (14)$$

which fits in the following commutative diagram of homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{i_0} & \text{At}(E_H, D) & \xrightarrow{\sigma} & \text{TX}(-D) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow A & & \parallel \\ 0 & \longrightarrow & \text{ad}(E_M) & \longrightarrow & \text{At}(E_M, D) & \longrightarrow & \text{TX}(-D) \longrightarrow 0 \end{array} \quad (15)$$

where the top exact sequence is the one in (5) and the bottom one is the corresponding sequence for E_M .

If $\theta : \text{TX}(-D) \longrightarrow \text{At}(E_H, D)$ is a logarithmic connection on E_H as in (6), then

$$A \circ \theta : \text{TX}(-D) \longrightarrow \text{At}(E_M, D) \quad (16)$$

is a logarithmic connection on E_M singular over D . From the definition of residue in (12) it follows immediately that

$$\alpha(\text{Res}(\theta, x)) = \text{Res}(A \circ \theta, x) \quad (17)$$

for all $x \in D$, where α is the homomorphism in (13). This proves the following:

Lemma 2.1 ([8]). *With the above notation, if E_H admits a logarithmic connection θ singular over D with residue $w_x \in \text{ad}(E_H)_x$ at each $x \in D$, then E_M admits a logarithmic connection $\theta' = A \circ \theta$ singular over D with residue $\alpha(w_x)$ at each $x \in D$.*

3 Criterion for a holomorphic connection

Let G be a connected reductive affine algebraic group defined over the field of complex numbers. A Zariski closed connected subgroup $P \subset G$ is called a parabolic subgroup of G if G/P is a projective variety [9, 11.2], [13]. In particular, G itself is a parabolic subgroup of G . The unipotent radical of a parabolic subgroup $P \subset G$ will be denoted by $R_u(P)$; it is the unique maximal normal unipotent subgroup of P . The quotient group $P/R_u(P)$ is called the *Levi quotient* of P . A *Levi factor* of P is a Zariski closed connected subgroup $L \subset P$ such that the composition $L \hookrightarrow P \rightarrow P/R_u(P)$ is an isomorphism [13, p. 184]. We note that P admits Levi factors, and any two Levi factors of P are conjugate by an element of $R_u(P)$ [13, § 30.2, p. 185, Theorem]. A subgroup $L' \subset P$ is a Levi factor of P if and only if L' contains a maximal torus of P with L' being a maximal reductive subgroup of P .

Given a holomorphic principal G -bundle E_G over X and a complex Lie subgroup $H \subset G$, a holomorphic reduction of structure group of E_G to H is given by a holomorphic section of the holomorphic fiber bundle E_G/H over X . Let

$$q_H : E_G \longrightarrow E_G/H$$

be the quotient map. If $v : X \rightarrow E_G/H$ is a holomorphic section of the fiber bundle E_G/H over X , then note that $q_H^{-1}(v(X)) \subset E_G$ is a holomorphic principal H -bundle on X . If E_H is a holomorphic principal H -bundle on X , and χ is a holomorphic character of H , then the associated holomorphic line bundle $E_H(\chi) = (E_H \times \mathbb{C})/H \rightarrow X$ is the quotient of $E_H \times \mathbb{C}$, where two points $(z_1, c_1), (z_2, c_2) \in E_H \times \mathbb{C}$ are identified if there is an element $h \in H$ such that

- $z_2 = z_1 h$, and
- $c_2 = \frac{c_1}{\chi(h)}$.

The following theorem is proved in [3] (see [3, Theorem 4.1]).

Theorem 3.1 ([3]). *A holomorphic principal G -bundle E_G over X admits a holomorphic connection if and only if for every triple of the form (H, E_H, λ) , where*

1. *H is a Levi factor of some parabolic subgroup of G ,*
 2. *$E_H \subset E_G$ is a holomorphic reduction of structure group to H , and*
 3. *λ is a holomorphic character of H ,*
- the degree of the associated line bundle $E_H(\lambda) = (E_H \times \mathbb{C})/H$ over X is zero.*

Note that setting $G = \mathrm{GL}(r, \mathbb{C})$ in Theorem 3.1 we get back the theorem of Atiyah and Weil which says that a holomorphic vector bundle E of rank r on X admits a holomorphic connection if and only if the degree of every direct summand of E is zero.

We will now describe a sketch of the proof of Theorem 3.1.

Let $p : E_G \rightarrow X$ be a holomorphic G -bundle over X equipped with a holomorphic connection $\mathcal{D}$. Take any triple (H, E_H, λ) as in Theorem 3.1. We will first show that the connection $\mathcal{D}$ produces a holomorphic connection on the holomorphic principal H -bundle E_H .

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. The group H has adjoint actions on both $\mathfrak{h}$ and $\mathfrak{g}$. The natural inclusion

$$\iota_0 : \mathfrak{h} \hookrightarrow \mathfrak{g}$$

is a homomorphism of H -modules. Since the group H is reductive, any exact sequence of H -modules splits. In particular, there is a homomorphism of H -modules

$$\psi : \mathfrak{g} \rightarrow \mathfrak{h}$$

such that the composition

$$\mathfrak{h} \xrightarrow{\iota_0} \mathfrak{g} \xrightarrow{\psi} \mathfrak{h}$$

is the identity map of $\mathfrak{h}$.

Since a holomorphic connection on E_G is given by a holomorphic splitting of the Atiyah exact sequence for E_G , a holomorphic connection on E_G produces a $\mathfrak{g}$ -valued holomorphic 1-form ω on the total space of E_G satisfying the following two conditions:

- ω is G -equivariant (G acts on $\mathfrak{g}$ by inner automorphism), and
- the restriction of ω to any fiber of E_G gives the isomorphism of $\ker(dp)$ with the trivial vector bundle with fiber $\mathfrak{g}$.

Conversely, any $\mathfrak{g}$ -valued holomorphic 1-form on the total space of E_G satisfying the above two conditions defines a holomorphic connection on E_G .

Let ω be the $\mathfrak{g}$ -valued holomorphic 1-form on E_G associated to the given connection $\mathcal{D}$. Using the chosen splitting homomorphism

$$\psi : \mathfrak{g} \longrightarrow \mathfrak{h},$$

the connection form ω on E_G defines a $\mathfrak{h}$ -valued holomorphic one-form $\omega' := \psi \circ \omega$ on E_G . The restriction of ω' to the complex sub-manifold $E_H \subset E_G$ satisfies the two conditions needed for a holomorphic $\mathfrak{h}$ -valued 1-form on E_H to define a holomorphic connection on E_H .

Therefore, E_H admits a holomorphic connection. A holomorphic connection on E_H induces a holomorphic connection on the associated line bundle $E_H(\lambda)$. Any line bundle admitting a holomorphic connection must be of degree zero [2]. Consequently, we conclude that if E_G admits a holomorphic connection then the degree of $E_H(\lambda)$ is zero.

To prove the converse statement, let E_G be a holomorphic principal G -bundle over X such that

$$\text{degree}(E_H(\lambda)) = 0 \quad (18)$$

for all triples (H, E_H, λ) of the above type in the statement of Theorem 3.1. We need to show that the Atiyah exact sequence for E_G splits holomorphically.

As the first step, in [3] the following is proved: To prove that the Atiyah exact sequence for E_G splits holomorphically, it is enough to prove that the Atiyah exact sequence for E_G splits holomorphically under the assumption that E_G does not admit any holomorphic reduction of structure group to a Levi factor of some proper parabolic subgroup of G . In view of this result, we assume that E_G does not admit any holomorphic reduction of structure group to a Levi factor of some proper parabolic subgroup of G .

The G -module $\mathfrak{g}$ is self-dual, because $\mathfrak{g}$ admits a G -invariant non-degenerate symmetric bilinear form. Consequently, we have

$$\text{ad}(E_G) = \text{ad}(E_G)^*. \quad (19)$$

As before, let K_X denote the holomorphic cotangent bundle of X . The obstruction for splitting of the Atiyah exact sequence for E_G is an element

$$\tau(E_G) \in H^1(X, K_X \otimes \text{ad}(E_G))$$

(see (4)). In view of (19), by Serre duality, we have

$$H^1(X, K_X \otimes \text{ad}(E_G)) = H^0(X, \text{ad}(E_G))^*.$$

This implies that

$$\tau(E_G) \in H^0(X, \text{ad}(E_G))^*. \quad (20)$$

Any homomorphic section f of $\text{ad}(E_G)$ has a Jordan decomposition

$$f = f_s + f_n,$$

where f_s is point-wise semi-simple and f_n is point-wise nilpotent. From the assumption that E_G does not admit any holomorphic reduction of structure group to a Levi factor of some proper parabolic subgroup of G , it follows that the semi-simple section f_s is given by some element of the center of $\mathfrak{g}$. Using this, from the assumption (18) on E_G it can be deduced that

$$\tau(E_G)(f_s) = 0,$$

where $\tau(E_G)$ is the element in (20).

The nilpotent section f_n of $\text{ad}(E_G)$ gives a holomorphic reduction of structure group $E_P \subset E_G$ of E_G to a proper parabolic subgroup P of G . This reduction E_P has the property that f_n lies in the image

$$H^0(X, R_n(E_P)) \hookrightarrow H^0(X, \text{ad}(E_G)),$$

where $R_n(E_P) \subset \text{ad}(E_P)$ is the nilpotent radical bundle of the adjoint vector bundle $\text{ad}(E_P)$ of E_P . Using this reduction it can be shown that $\tau(E_G)(f_n) = 0$.

Hence $\tau(E_G)(f) = 0$ for all $f \in H^0(X, \text{ad}(E_G))$, which implies that $\tau(E_G) = 0$. Therefore, the Atiyah exact sequence for E_G splits holomorphically, which means that E_G admits a holomorphic connection.

Note that the assumption that the group G is reductive played a crucial role in the above proof. Therefore, it is reasonable to ask the following question:

Question 3.2. *Let E_H be the holomorphic principal H -bundle on X , where H is a connected complex affine algebraic group. What is a criterion for E_H to admit a holomorphic connection? (The most interesting case is when P is a parabolic subgroup of a reductive group.)*

4 Connections preserving a section

As before, E_G is a holomorphic principal G -bundle over X , where G is a complex reductive group. We recall that for any point $x \in X$, the fiber $\text{ad}(E_G)_x$ is identified with $\mathfrak{g}$ uniquely up to a conjugation. Therefore, there is a natural map

$$\phi : \text{ad}(E_G) \longrightarrow \mathfrak{g}/G, \quad (21)$$

where $G/\mathfrak{g}$ is the space of all conjugacy classes of elements of $\mathfrak{g}$.

Note that a connection on E_G induces a connection of the adjoint bundle $\text{ad}(E_G)$.

Lemma 4.1. *Take a holomorphic connection $\mathfrak{D}$ on E_G , and let $\beta \in H^0(X, \text{ad}(E_G))$ be a flat section with respect to $\mathfrak{D}$. Then the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, where $x \in X$, is independent of x .*

Proof. Using $\mathfrak{D}$, we may holomorphically trivialize E_G on any connected simply connected open subset of X . With respect to such a trivialization, the section β is a constant one because it is flat with respect to $\mathfrak{D}$. This immediately implies that $\phi \circ \beta(x) \in \mathfrak{g}/G$ is independent of $x \in X$. $\square$

A nilpotent element v of the Lie algebra of a complex semi-simple group H is called *regular nilpotent* if the dimension of the centralizer of v in H coincides with the rank of H [13, p. 53].

Proposition 4.2. *Take any $\beta \in H^0(X, \text{ad}(E_G))$. Assume that*

- E_G admits a holomorphic connection,
- the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x , where ϕ is defined in (21), and
- for every adjoint type simple quotient H of G , the section of the adjoint bundle $\text{ad}(E_H)$ given by β , where $E_H := E_G \times^G H$ is the holomorphic principal H -bundle over X associated to E_G , has the property that it is either zero or it is regular nilpotent at some point of X .

Then the principal G -bundle E_G admits a holomorphic connection for which the section β is flat.

(From Theorem 3.1 we know when E_G admits a holomorphic connection.)

Proof. Let $Z := G/[G, G]$ be the abelian quotient of G . It is a product of copies of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. There are quotients $H_1, \dots, H_\ell$ of G such that

1. each H_i is simple of adjoint type (the center is trivial), and

2. the natural homomorphism

$$\varphi : G \longrightarrow Z \times \prod_{i=1}^{\ell} H_i \quad (22)$$

is surjective, and the kernel of φ is a finite group.

In fact, $\prod_{i=1}^{\ell} H_i = G/Z_G$, where Z_G is the center of G ; note that the center of the semi-simple group G/Z_G is trivial, and hence it is a product of simple groups of adjoint type. Let

$$E_Z := E_G \times^G Z \quad \text{and} \quad E_{H_i} := E_G \times^G H_i, \quad i \in [1, \ell],$$

be the holomorphic principal Z -bundle and principal H_i -bundle associated to E_G for the quotient Z and H_i respectively. Let $\text{ad}(E_Z)$ and $\text{ad}(E_{H_i})$ be the adjoint vector bundles for E_Z and E_{H_i} respectively. Since the homomorphism φ in (22) induces an isomorphism of Lie algebras, we have

$$\text{ad}(E_G) = \text{ad}(E_Z) \oplus \left(\bigoplus_{i=1}^{\ell} \text{ad}(E_{H_i}) \right). \quad (23)$$

Let β_Z (respectively, β_i) be the holomorphic section of $\text{ad}(E_Z)$ (respectively, $\text{ad}(E_{H_i})$) given by β using the decomposition in (23), so

$$\beta = \beta_Z + \sum_{i=1}^{\ell} \beta_i.$$

Since the conjugacy class of $\beta(x)$ is independent of $x \in X$ (it is the second condition in the proposition), we conclude that the conjugacy class of $\beta_i(x)$ is also independent of $x \in X$ for each i .

A holomorphic connection on E_G induces a holomorphic connection on E_Z . Since Z is abelian, the vector bundle $\text{ad}(E_Z)$ is the trivial holomorphic vector bundle $X \times \text{Lie}(Z)$. Given any connection on the principal Z -bundle E_Z , the induced connection on $\text{ad}(E_Z)$ coincides with the trivial connection on the trivial vector bundle $X \times \text{Lie}(Z)$. In particular, the section β_Z is a constant one, and it is flat with respect to the any connection on $\text{ad}(E_Z)$ induced by a connection on the principal Z -bundle E_Z .

Now take any $i \in [1, \ell]$. A holomorphic connection on E_G induces a holomorphic connection on E_{H_i} . If the section β_i is zero at some point, then β_i is identically zero because the conjugacy class of $\beta_i(x)$ is independent of x . Hence, in that case β_i is flat with respect to any connection on $\text{ad}(E_{H_i})$. Therefore, assume that β_i is not zero at all points of X .

By the third assumption in the proposition, the section β_i is regularly nilpotent over some point of X . Since the conjugacy class of $\beta_i(x)$, $x \in X$, is independent of x , we conclude that β_i is regular nilpotent over every point of X . We will show that the holomorphic principal H_i -bundle E_{H_i} is semi-stable.

For each point $x \in X$, from the fact that $\beta_i(x)$ is regular nilpotent we conclude that there is a unique Borel sub-algebra $\widetilde{\mathfrak{b}}_x$ of $\text{ad}(E_{H_i})_x$ such that $\beta_i(x) \in \widetilde{\mathfrak{b}}_x$ [13, p. 187, Theorem 30.4(b)]. Let

$$\widetilde{\mathfrak{b}} \subset \text{ad}(E_{H_i})$$

be the Borel sub-algebra bundle such that for every point $x \in X$, the fiber $(\widetilde{\mathfrak{b}})_x$ is $\widetilde{\mathfrak{b}}_x$. Fix a Borel subgroup $B \subset H_i$. Using the above Borel sub-algebra bundle $\widetilde{\mathfrak{b}}$, we will construct a holomorphic reduction of structure group of E_{H_i} to the subgroup $B \subset H_i$.

Let $\mathfrak{b}$ be the Lie algebra of B . The Lie algebra of H_i will be denoted by $\mathfrak{h}_i$. We recall that $\text{ad}(E_{H_i})$ is the quotient of $E_{H_i} \times \mathfrak{h}_i$ where two points (z_1, v_1) and (z_2, v_2) of $E_{H_i} \times \mathfrak{h}_i$ are identified if there is an element $h \in H_i$ such that $z_2 = z_1 h$ and $v_2 = \text{Ad}(h)(v_1)$, where $\text{Ad}(h)$ is the automorphism of $\mathfrak{h}_i$ corresponding to the automorphism $y \mapsto h^{-1} y h$ of H_i . For any point $x \in X$, let $E_{B,x} \subset (E_{H_i})_x$ be the subset consisting of all $z \in (E_{H_i})_x$ such that for all $v \in \mathfrak{b}$, the image of (z, v) in $\text{ad}(E_{H_i})_x$ lies in the subspace $\widetilde{\mathfrak{b}}_x$. Since any two Borel sub-algebras of $\mathfrak{h}_i$ are conjugate, it follows immediately that $E_{B,x}$ is nonempty. The normalizer of $\mathfrak{b}$ in H_i coincides with B . From this it follows that

- $E_{B,x}$ is preserved by the action of B on $(E_{H_i})_x$, and
- the action of B on $E_{B,x}$ is transitive.

It is now evident that $E_{B,x}$ is a complex sub-manifold of $(E_{H_i})_x$.

Let

$$E_B \subset E_{H_i}$$

be the complex sub-manifold such that $E_B \cap (E_{H_i})_x = E_{B,x}$ for every $x \in X$. That E_B is a complex sub-manifold follows from that facts that

- $E_{B,x}$ is a complex sub-manifold of $(E_{H_i})_x$ for all $x \in X$, and
- the section β_i is holomorphic.

From the above properties of $E_{B,x}$ it follows immediately that E_B is a holomorphic reduction of structure group of the principal H_i -bundle E_{H_i} to the subgroup B .

Now consider the adjoint action of B on $\mathfrak{b}_1 := \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. Let

$$E_B(\mathfrak{b}_1) := E_B \times^B \mathfrak{b}_1 \longrightarrow X$$

be the holomorphic vector bundle associated to E_B for the B -module $\mathfrak{b}_1$. Since β_i is everywhere regular nilpotent, it follows that the vector bundle $E_B(\mathfrak{b}_1)$ is trivial. Consequently, for any character χ of B which is a nonnegative integral combination of simple roots, the line bundle $E_B(\chi) \longrightarrow X$ associated to E_B for the character χ is trivial [1, p. 708, Theorem 5]. Therefore, for any character χ of B , the line bundle $E_B(\chi)$ associated to E_B for χ is trivial.

Let d be the complex dimension of $\mathfrak{h}_i$. Consider the adjoint action on B on $\mathfrak{h}_i$. Note that $\text{ad}(E_{H_i})$ is identified with the vector bundle associated to the principal B -bundle E_B for this B -module $\mathfrak{h}_i$. Since B is solvable, there is a filtration of B -modules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = \mathfrak{h}_i \quad (24)$$

such that $\dim V_j = j$ for all $j \in [1, d]$. Let

$$E_B(V_j) := E_B \times^B V_j \longrightarrow X$$

be the vector bundle associated to E_B for the B -module V_j . Since (24) is a filtration of B -submodules of $\mathfrak{h}_i$, we have a filtration of sub-bundles

$$0 = E_B(V_0) \subset E_B(V_1) \subset \cdots \subset E_B(V_{d-1}) \subset E_B(V_d) = E_B(\mathfrak{h}_i) = \text{ad}(E_{H_i}). \quad (25)$$

For every $j \in [1, d]$, the quotient bundle $E_B(V_j)/E_B(V_{j-1})$ is identified with the vector bundle

$$E_B(V_j/V_{j-1}) := E_B \times^B (V_j/V_{j-1}) \longrightarrow X$$

associated to E_B for the B -module V_j/V_{j-1} .

We noted above that the line bundles associated to E_B for the characters of B are trivial. Therefore, each $E_B(V_j/V_{j-1})$ is a trivial line bundle. Consequently, (25) is filtration of holomorphic sub-bundles of $\text{ad}(E_{H_i})$ such that each successive quotient is a trivial line bundle. This immediately implies that the vector bundle $\text{ad}(E_{H_i})$ is semi-stable. Hence the holomorphic principal H_i -bundle E_{H_i} is semi-stable [1, p. 698, Lemma 3].

Since H_i is simple, and the principal H_i -bundle E_{H_i} is semi-stable, there is a natural holomorphic connection on E_{H_i} [5, p. 20, Theorem 1.1] (set the Higgs field in [5, Theorem 1.1] to be zero). Let $\mathfrak{D}^{H_i}$ denote this connection. The vector bundle $\text{ad}(E_{H_i})$ being semi-stable of degree zero has a natural holomorphic connection [15, p. 36, Lemma 3.5]. See also [5, p. 20, Theorem 1.1]. (In both [15, Lemma 3.5] and [5, Theorem 1.1] set the Higgs field to be zero.) Let $\mathfrak{D}^{\text{ad}}$ denote this holomorphic connection on $\text{ad}(E_{H_i})$. This connection $\mathfrak{D}^{\text{ad}}$ coincides with the one induced by $\mathfrak{D}^{H_i}$ (see the construction of the connection in [5]).

Any holomorphic section of $\text{ad}(E_{H_i})$ is flat with respect to $\mathfrak{D}^{\text{ad}}$. To see this, let

$$\phi_i : \mathcal{O}_X \longrightarrow \text{ad}(E_{H_i})$$

be the homomorphism given by a nonzero holomorphic section of $\text{ad}(E_{H_i})$. Since $\text{image}(\phi_i)$ is a semi-stable sub-bundle of $\text{ad}(E_{H_i})$ of degree zero,

- the connection $\mathfrak{D}^{\text{ad}}$ preserves $\text{image}(\phi_i)$, and

- the restriction of $\mathfrak{D}^{\text{ad}}$ to $\text{image}(\phi_i)$ coincides with the canonical connection of $\text{image}(\phi_i)$ [15, p. 36, Lemma 3.5].

The canonical connection on the trivial holomorphic line bundle $\text{image}(\phi_i)$ is the trivial connection (the monodromy is trivial).

In particular, the connection on $\text{ad}(E_{H_i})$ induced by $\mathfrak{D}^{H_i}$ has the property that the section β_i is flat with respect to it.

Since the homomorphism of Lie algebras corresponding to φ (in (22)) is an isomorphism, if we have holomorphic connections on E_Z and E_{H_i} , $[1, \ell]$, then we get a holomorphic connection on E_G . Indeed, simply pullback the connection form using the map

$$E_G \longrightarrow E_Z \times_X E_{H_1} \times_X \cdots \times_X E_{H_\ell}.$$

The connection on E_G given by the connections on E_Z and E_{H_i} , $[1, \ell]$, constructed above satisfies the condition that β is flat with respect to it. This completes the proof of the proposition. $\square$

Lemma 4.3. *Take any semi-simple section $\beta_s \in H^0(X, \text{ad}(E_G))$. Then β_s produces a holomorphic reduction of structure group of E_G to a Levi factor of a parabolic subgroup of G .*

Proof. We will first show that the conjugacy class of $\phi \circ \beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x , where ϕ is defined in (21).

Take any complex G -module V . Let $E_V := E_G \times^G V$ be the holomorphic vector bundle over X associated to E_G for the G -module V . The section β_s produces a holomorphic section

$$\beta_V \in H^0(X, \text{End}(E_V)).$$

The X is compact and connected, the coefficients of the characteristic polynomial of $\beta_V(x)$ are constants over X . This implies that the eigenvalues of $\beta_V(x)$, along with their multiplicities, are independent of x . Since β_s is semi-simple, from this it follows that the conjugacy class of $\phi \circ \beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x .

Fix an element

$$v_0 \in \mathfrak{g}$$

such that the image of v_0 in $\mathfrak{g}/G$ coincides with $\phi \circ \beta_s(x)$. Let $\mathbb{L} \subset G$ be the centralizer of v_0 . It is known that $\mathbb{L}$ is a Levi subgroup of some parabolic subgroup of G [11, p. 26, Proposition 1.22] (note that $\mathbb{L}$ is the centralizer of the torus in G generated by v_0). In particular, $\mathbb{L}$ is connected and reductive.

For any point $x \in X$, let $F_x \subset (E_G)_x$ be the complex sub-manifold consisting of all points z such that the image of (z, v_0) in $\text{ad}(E_G)_x$ coincides with $\beta_s(x)$. (Recall that $\text{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$.) Let

$$F_{\mathbb{L}} \subset E_G$$

be the complex sub-manifold such that $F_{\mathbb{L}} \cap (E_G)_x = F_x$ for all $x \in X$. It is straightforward to check that $F_{\mathbb{L}}$ is a holomorphic reduction of structure group of the principal G -bundle E_G to the subgroup $\mathbb{L}$. $\square$

From the Jordan decomposition of a complex reductive Lie algebra we know that for any holomorphic section θ of $\text{ad}(E_G)$, there is a naturally associated semi-simple (respectively, nilpotent) section θ_s (respectively, θ_n) such that $\theta = \theta_s + \theta_n$.

Take any $\beta \in H^0(X, \text{ad}(E_G))$. Let

$$\beta = \beta_s + \beta_n$$

be the Jordan decomposition. Assume that the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x , where ϕ is defined in (21). This implies that $\phi \circ \beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is also independent of x . Let $(\mathbb{L}, F_{\mathbb{L}})$ be the principal bundle constructed in Lemma 4.3 from β_s . Let H be an adjoint type simple quotient of $\mathbb{L}$. Let

$$E_H := F_{\mathbb{L}} \times^{\mathbb{L}} H \longrightarrow X$$

be the holomorphic principal H -bundle associated to $F_{\mathbb{L}}$.

Since $[\beta_s, \beta_n] = 0$, from the construction of $F_{\mathbb{L}}$ it follows that

$$\beta_n \in H^0(X, \operatorname{ad}(F_{\mathbb{L}})) \subset H^0(X, \operatorname{ad}(E_G)).$$

Therefore, using the natural projection $\operatorname{ad}(F_{\mathbb{L}}) \rightarrow \operatorname{ad}(E_H)$, given by the projection $\operatorname{Lie}(\mathbb{L}) \rightarrow \operatorname{Lie}(H)$, the above section β_n produces a holomorphic section of $\operatorname{ad}(E_H)$. Let

$$\widetilde{\beta}_n \in H^0(X, \operatorname{ad}(E_H)) \quad (26)$$

be the section constructed from β_n .

Theorem 4.4. Take any $\beta \in H^0(X, \operatorname{ad}(E_G))$. Let $\beta = \beta_s + \beta_n$ be the Jordan decomposition. Assume that

- E_G admits a holomorphic connection,
- the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x , where ϕ is defined in (21), and
- for every adjoint type simple quotient H of $\mathbb{L}$, the section $\widetilde{\beta}_n$ in (26) of has the property that it is either zero or it is regular nilpotent at some point of X .

Then the principal G -bundle E_G admits a holomorphic connection for which β is flat.

Proof. Note that

$$\beta_s \in H^0(X, \operatorname{ad}(F_{\mathbb{L}})) \subset H^0(X, \operatorname{ad}(E_G)).$$

In fact, for each point $x \in X$, the element $\beta_s(x) \in \operatorname{ad}(F_{\mathbb{L}})_x$ is in the center of $\operatorname{ad}(F_{\mathbb{L}})_x$. Consider the abelian quotient

$$Z_{\mathbb{L}} = \mathbb{L}/[\mathbb{L}, \mathbb{L}].$$

Let $F_{Z_{\mathbb{L}}}$ be the holomorphic principal $Z_{\mathbb{L}}$ -bundle over X obtained by extending the structure group of $F_{\mathbb{L}}$ using the quotient map $\mathbb{L} \rightarrow Z_{\mathbb{L}}$. The adjoint bundle $\operatorname{ad}(F_{Z_{\mathbb{L}}})$ is a direct summand of $\operatorname{ad}(F_{\mathbb{L}})$. In fact, for each $x \in X$, the subspace $\operatorname{ad}(F_{Z_{\mathbb{L}}})_x \subset \operatorname{ad}(F_{\mathbb{L}})_x$ is the center of the Lie algebra $\operatorname{ad}(F_{\mathbb{L}})_x$.

A holomorphic connection on $F_{\mathbb{L}}$ induces a holomorphic connection on E_G . We can now apply Proposition 4.2 to $F_{\mathbb{L}}$ to complete the proof of the theorem. But for that we need to show that $F_{\mathbb{L}}$ admits a holomorphic connection.

Let $\mathfrak{l}$ be the Lie algebra of $\mathbb{L}$. Consider the inclusion of $\mathbb{L}$ -modules $\mathfrak{l} \hookrightarrow \mathfrak{g}$ given by the inclusion of $\mathbb{L}$ in G . Since $\mathbb{L}$ is reductive, there is a sub $\mathbb{L}$ -module $S \subset \mathfrak{g}$ such that the natural homomorphism

$$\mathfrak{l} \oplus S \rightarrow \mathfrak{g}$$

is an isomorphism (so S is a complement of $\mathfrak{l}$). Let

$$p : \mathfrak{g} \rightarrow \mathfrak{l} \quad (27)$$

be the projection given by the above decomposition of $\mathfrak{g}$.

Let D be a holomorphic connection on E_G . So D is a holomorphic 1-form on the total space of E_G with values in the Lie algebra $\mathfrak{g}$. Let D' be the restriction of this 1-form to the complex sub-manifold $F_{\mathbb{L}} \subset E_G$. Consider the $\mathfrak{l}$ -valued 1-form $p \circ D'$ on $E_{\mathbb{L}}$, where p is the projection in (27). This $\mathfrak{l}$ -valued 1-form on $F_{\mathbb{L}}$ defines a holomorphic connection of the principal $\mathbb{L}$ -bundle $F_{\mathbb{L}}$. Now Proposition 4.2 completes the proof of the theorem. $\square$

5 Logarithmic connections with prescribed singularity

This section is an exposition of [8].

Fix a holomorphic principal H -bundle E_H on X , and fix elements

$$w_x \in \operatorname{ad}(E_H)_x$$

for all $x \in D$. Consider the decomposition of $\operatorname{At}(E_H, D)_x$ in (11). For any $x \in D$, let

$$\ell_x := \mathbb{C} \cdot (w_x, 1) \subset \operatorname{ad}(E_H)_x \oplus \mathbb{C} = \operatorname{At}(E_H, D)_x$$

be the line in the fiber $\text{At}(E_H, D)_x$. Let

$$\mathcal{A} \subset \text{At}(E_H, D)$$

be the sub-sheaf that fits in the short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \text{At}(E_H, D) \longrightarrow \bigoplus_{x \in D} \text{At}(E_H, D)_x / \ell_x \longrightarrow 0. \quad (28)$$

Note that the composition

$$\text{ad}(E_H)_x \xrightarrow{i_0(x)} \text{At}(E_H, D)_x \longrightarrow \text{At}(E_H, D)_x / \ell_x$$

is injective, hence it is an isomorphism, where i_0 is the homomorphism in (15); this composition will be denoted by ϕ_x . Therefore, from (5) and (28) we have a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \text{ad}(E_H) \otimes \mathcal{O}_X(-D) & \longrightarrow & \mathcal{A} & \xrightarrow{\sigma_1} & \text{TX}(-D) & \longrightarrow 0 \\ & \downarrow & & \downarrow \nu & & \downarrow \text{id} & \\ 0 \longrightarrow & \text{ad}(E_H) & \xrightarrow{i_0} & \text{At}(E_H, D) & \xrightarrow{\sigma} & \text{TX}(-D) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \bigoplus_{x \in D} \text{ad}(E_H)_x & \xrightarrow{\bigoplus_{x \in D} \phi_x} & \bigoplus_{x \in D} \text{At}(E_H, D)_x / \ell_x & \longrightarrow & 0 & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \quad (29)$$

where all the rows and columns are exact; the restriction of σ to the sub-sheaf $\mathcal{A}$ is denoted by σ_1 .

Lemma 5.1 ([8]). *Consider the space of all logarithmic connections on E_H singular over D such that the residue over every $x \in D$ is w_x . It is in bijection with the space of all holomorphic splittings of the short exact sequence of vector bundles*

$$0 \longrightarrow \text{ad}(E_H) \otimes \mathcal{O}_X(-D) \longrightarrow \mathcal{A} \xrightarrow{\sigma_1} \text{TX}(-D) \longrightarrow 0$$

on X in (29).

5.1 T -rigid elements of adjoint bundle

As before, H is a complex affine algebraic group and $p : E_H \longrightarrow X$ a holomorphic principal H -bundle on X . An automorphism of E_H is a holomorphic map $F : E_H \longrightarrow E_H$ such that

- $p \circ F = p$, and
- $F(zh) = F(z)h$ for all $z \in E_H$ and $h \in H$.

Let $\text{Aut}(E_H)$ be the group of all automorphisms of E_H . We will show that $\text{Aut}(E_H)$ is a complex affine algebraic group.

First consider the case of $H = \text{GL}(r, \mathbb{C})$. For a holomorphic principal $\text{GL}(r, \mathbb{C})$ -bundle E_{GL} on X , let $E := E_{\text{GL}} \times^{\text{GL}(r, \mathbb{C})} \mathbb{C}^r$ be the holomorphic vector bundle of rank r on X associated to E_{GL} for the standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. Then $\text{Aut}(E_{\text{GL}})$ is identified with the group of all holomorphic automorphisms $\text{Aut}(E)$ of the vector bundle E over the identity map of X . Note that $\text{Aut}(E)$ is the Zariski open subset of the complex affine space $H^0(X, \text{End}(E))$ consisting of all global endomorphisms f of E such that $\det(f(x_0)) \neq 0$ for a fixed point $x_0 \in X$; since $x \mapsto \det(f(x))$ is a holomorphic function on X , it is in fact a constant function. Therefore, $\text{Aut}(E_{\text{GL}})$ is an affine algebraic variety over $\mathbb{C}$.

For a general H , fix an algebraic embedding $\rho : H \hookrightarrow \mathrm{GL}(r, \mathbb{C})$ for some r . For a holomorphic principal H -bundle E_H on X , let $E_{\mathrm{GL}} := E_H \times^{\rho} \mathrm{GL}(r, \mathbb{C})$ be the holomorphic principal $\mathrm{GL}(r, \mathbb{C})$ -bundle on X obtained by extending the structure group of E_H using ρ . The injective homomorphism ρ produces an injective homomorphism

$$\rho' : \mathrm{Aut}(E_H) \longrightarrow \mathrm{Aut}(E_{\mathrm{GL}}).$$

The image of ρ' is Zariski closed in the algebraic group $\mathrm{Aut}(E_{\mathrm{GL}})$. Hence ρ' produces the structure of a complex affine algebraic group on $\mathrm{Aut}(E_H)$. This structure of a complex algebraic group is independent of the choices of r, ρ . Therefore, $\mathrm{Aut}(E_H)$ is an affine algebraic group. Note that $\mathrm{Aut}(E_H)$ need not be connected, although the automorphism group of a holomorphic vector bundle is always connected (as it is a Zariski open subset of a complex affine space).

The Lie algebra of $\mathrm{Aut}(E_H)$ is $H^0(X, \mathrm{ad}(E_H))$. The group $\mathrm{Aut}(E_H)$ acts on any fiber bundle associated to E_H . In particular, $\mathrm{Aut}(E_H)$ acts on the adjoint vector bundle $\mathrm{ad}(E_H)$. This action evidently preserves the Lie algebra structure on the fibers of $\mathrm{ad}(E_H)$.

Let $\mathrm{Aut}(E_H)^0 \subset \mathrm{Aut}(E_H)$ be the connected component containing the identity element. Fix a maximal torus

$$T \subset \mathrm{Aut}(E_H)^0.$$

An element $w \in \mathrm{ad}(E_H)_x$, where $x \in X$, will be called T -rigid if the action of T on $\mathrm{ad}(E_H)_x$ fixes w .

Consider the adjoint action of H on itself. Let

$$\mathrm{Ad}(E_H) := E_H \times^H H \longrightarrow X \quad (30)$$

be the associated holomorphic fiber bundle. Since this adjoint action preserves the group structure of H , the fibers of $\mathrm{Ad}(E_H)$ are complex algebraic groups isomorphic to H . More precisely, each fiber of $\mathrm{Ad}(E_H)$ is identified with H uniquely up to an inner automorphism of H . The corresponding Lie algebra bundle on X is $\mathrm{ad}(E_H)$.

The group $\mathrm{Aut}(E_H)$ is the space of all holomorphic sections of $\mathrm{Ad}(E_H)$. For any $x \in X$, the action of $\mathrm{Aut}(E_H)$ on the fiber $\mathrm{ad}(E_H)_x$ coincides with the one obtained via the composition

$$\mathrm{Aut}(E_H) \xrightarrow{\mathrm{ev}_x} \mathrm{Ad}(E_H)_x \xrightarrow{\mathrm{ad}} \mathrm{Aut}(\mathrm{ad}(E_H)_x),$$

where ev_x is the evaluation map that sends a section of $\mathrm{Ad}(E_H)$ to its evaluation at x , and ad is the adjoint action of the group $\mathrm{Ad}(E_H)_x$ on its Lie algebra $\mathrm{ad}(E_H)_x$.

Therefore, an element $w \in \mathrm{ad}(E_H)_x$ is T -rigid if and only if the adjoint action of $\mathrm{ev}_x(T) \subset \mathrm{Ad}(E_H)_x$ on $\mathrm{ad}(E_H)_x$ fixes w .

5.2 Logarithmic connections with T -rigid residue

As before, G is a connected reductive affine algebraic group defined over $\mathbb{C}$. Let E_G be a holomorphic principal G -bundle over X . Fix a maximal torus

$$T \subset \mathrm{Aut}(E_G)^0,$$

where $\mathrm{Aut}(E_G)^0$ as before is the connected component containing the identity element of the group of automorphisms of E_G .

We now recall some results from [4], [6].

As in (30), define the adjoint bundle $\mathrm{Ad}(E_G) = E_G \times^G G$. For any point $y \in X$, consider the evaluation homomorphism

$$\varphi_y : T \longrightarrow \mathrm{Ad}(E_G)_y, \quad s \longmapsto s(y).$$

Then φ_y is injective and its image is a torus in G [4, p. 230, Section 3]. Since G is identified with $\mathrm{Ad}(E_G)_y$ uniquely up to an inner automorphism, the image $\varphi_y(T)$ determines a conjugacy class of tori in G ; this conjugacy class is independent of the choice of y [4, p. 230, Section 3], [6, p. 63, Theorem 4.1]. Fix a torus

$$T_G \subset G \quad (31)$$

in this conjugacy class of tori. The centralizer

$$H := C_G(T_G) \subset G \quad (32)$$

of T_G in G is a Levi factor of a parabolic subgroup of G [4, p. 230, Section 3], [6, p. 63, Theorem 4.1]. The principal G -bundle E_G admits a holomorphic reduction of structure group

$$E_H \subset E_G \quad (33)$$

to the above subgroup H [4, p. 230, Theorem 3.2], [6, p. 63, Theorem 4.1]. Since T_G is in the center of H , the action of T_G on E_H commutes with the action of H , so $T_G \subset \text{Aut}^0(E_H)$. The image of T_G in $\text{Aut}^0(E_H)$ coincides with T . This reduction E_H is minimal in the sense that there is no Levi factor L of some parabolic subgroup of G such that

- $L \subsetneq H$, and
- E_G admits a holomorphic reduction of structure group to L .

(See [4, p. 230, Theorem 3.2].)

The above reduction E_H is unique in the following sense. Let L be a Levi factor of a parabolic subgroup of G and $E_L \subset E_G$ a holomorphic reduction of structure group to L satisfying the condition that E_G does not admit any holomorphic reduction of structure group to a Levi factor L' of some parabolic subgroup of G such that $L' \subsetneq L$. Then there is an automorphism $\varphi \in \text{Aut}(E_G)^0$ such that $E_L = \varphi(E_H)$ [6, p. 63, Theorem 4.1]. In particular, the subgroup $L \subset G$ is conjugate to H .

The Lie algebras of G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The inclusion of $\mathfrak{h}$ in $\mathfrak{g}$ and the reduction in (33) together produce an inclusion $\text{ad}(E_H) \hookrightarrow \text{ad}(E_G)$. This sub-bundle $\text{ad}(E_H)$ of $\text{ad}(E_G)$ coincides with the invariant sub-bundle $\text{ad}(E_G)^T$ for the action of T on $\text{ad}(E_G)$ [4, p. 230, Theorem 3.2], [6, p. 61, Proposition 3.3], in other words,

$$\text{ad}(E_H) = \text{ad}(E_G)^T \subset \text{ad}(E_G). \quad (34)$$

For every $x \in D$ fix a T -rigid element

$$w_x \in \text{ad}(E_G)_x \quad (35)$$

Since each w_x is T -rigid, from (34) we conclude that

$$w_x \in \text{ad}(E_H)_x \quad \forall x \in D. \quad (36)$$

So w_x determines a conjugacy class in $\mathfrak{h}$. For any character χ of H , the corresponding homomorphism of Lie algebras $d\chi : \mathfrak{h} \rightarrow \mathbb{C}$ factors through the conjugacy classes in $\mathfrak{h}$, because $\mathbb{C}$ is abelian. Therefore, we have $d\chi(w_x) \in \mathbb{C}$.

Theorem 5.2 ([8]). *There is a logarithmic connection on E_G singular over D , and with T -rigid residue w_x at every $x \in D$ (see (35)), if and only if*

$$\text{degree}(E_H(\chi)) + \sum_{x \in D} d\chi(w_x) = 0 \quad (37)$$

for every character χ of H , where $E_H(\chi)$ is the holomorphic line bundle on X associated to E_H for χ , and $d\chi$ is the homomorphism of Lie algebras corresponding to χ .

5.3 T -invariant logarithmic connections with given residue

The automorphism group $\text{Aut}(E_G)$ has a natural action on the space of all logarithmic connections on E_G singular over D . Given a maximal torus $T \subset \text{Aut}(E_G)^0$, by a T -invariant logarithmic connection we mean a logarithmic connection on E_G singular over D which is fixed by the action of T .

Theorem 5.3 ([8]). *Let E_G be a holomorphic principal G -bundle on X , where G is reductive. Fix $w_x \in \text{ad}(E_G)_x$ for each $x \in D$. Fix a maximal torus $T \subset \text{Aut}(E_G)^0$. The following two are equivalent:*

1. *There is a T -invariant logarithmic connection on E_G singular over D with residue w_x at every $x \in D$.*
2. *The element w_x is T -rigid for each $x \in D$, and (37) holds for every character χ of H .*

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