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On the Calabi-Yau equation in the Kodaira-Thurston manifold

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Abstract: We review some previous results about the Calabi-Yau equation on the Kodaira-Thurston manifold equipped with an invariant almost-Kähler structure and assuming the volume form T^2-invariant. In particular, we observe that under some restrictions the problem is reduced to a Monge-Ampère equation by using the ansatz $\tilde{\omega} = \Omega - dJdu + da$, where u is a T^2-invariant function and a is a 1-form depending on u. Furthermore, we extend our analysis to non-invariant almost-complex structures by considering some basic cases and we finally take into account a generalization to higher dimensions.

MSC: 32Q25, 32Q60, 35J60

1 Introduction

The *Calabi-Yau problem* in 4-dimensional almost-Kähler manifolds is a PDEs system arising from the generalization of the classical Calabi-Yau theorem to the non-Kähler setting.

The Calabi-Yau theorem [14] states that on a compact Kähler manifold (X, J, Ω) for every smooth function $F\colon X \to \mathbb{R}$ such that

$$\int_X e^F\, \Omega^n = \int_X \Omega^n \tag{1.1}$$

there always exists a unique Kähler form $\tilde{\omega}$ on (X, J) satisfying

$$[\tilde{\omega}] = [\Omega]\,, \quad \tilde{\omega}^n = e^F\, \Omega^n\,. \tag{1.2}$$

An analogue problem still makes sense in the almost-Kähler case, when J is merely an almost-complex structure and Ω is a J-compatible symplectic form. It turns out that in this more general context, the PDEs system arising from (1.2) is overdetermined for $n \geq 3$, while it is elliptic in dimension 4 (see [3]). Consequently, the Calabi-Yau problem is mainly studied in 4-dimensional almost-Kähler manifolds (see [1, 2, 11–13, 15] and the references therein).

The study of the problem is strongly motivated by a project of Donaldson involving compact symplectic 4-manifolds (see [3]). The project is based on a conjecture stated in [3] and partially confirmed by Taubes in [10].

In [15] Weinkove attacked the problem by introducing a *symplectic potential*. Indeed, given two almost-Kähler forms Ω and $\tilde{\omega}$ on a compact almost-complex manifold (X, J) satisfying $[\Omega] = [\tilde{\omega}]$ there always exists a function u, called the *symplectic potential*, such that

$$(\tilde{\omega} - \Omega) \wedge \tilde{\omega} = -dJdu \wedge \tilde{\omega}\,.$$

In terms of u one can always write

$$\tilde{\omega} = \Omega - dJdu + da\,,$$

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where a is a 1-form which can be assumed co-closed with respect to the co-differential induced by $\tilde{\omega}$ (in this way a is unique up addiction of *harmonic* 1-forms).

Weinkove proved that in order to show the solvability of the Calabi-Yau problem (1.2) it's enough to provide an a priori estimate on the C^0-norm of the almost-Kähler potential (see theorem 1 in [15]); that can be always done if the L^1-norm of the Nijenhuis tensor of J is small enough (see theorem 2 in [15]).

In [13] Tosatti and Weinkove studied the Calabi-Yau problem on the Kodaira-Thurston manifold (M, Ω_0, J_0) showing that under the assumption on the initial datum F to be invariant by the action of a 2-dimensional torus the problem has a unique solution. The Kodaira-Thurston manifold M is a 4-dimensional 2-step nilmanifold carrying a natural almost-Kähler structure and it can be viewed as a torus bundle over a torus (more precisely M is an S^1-bundle over a 3-dimensional torus).

In [4] it is observed that if F is T^2-invariant, then (1.2) on the Kodaira-Thrurston manifold M can be rewritten in terms of the Monge-Ampère equation

$$(1 + u_{xx})(1 + u_{yy}) - u_{xy}^2 = \mathrm{e}^F$$

on the 2-dimensional torus $\mathbb{T}^2_{xy}$ and the Tosatti-Weinkove result in [13] can be alternatively obtained by applying a result of Y.Y. Li in [8]. A similar approach was then adopted in [1, 4] in order to study the Calabi-Yau problem in every 4-dimensional torus bundle over a torus equipped with an invariant almost-Kähler structure. In this more general case the equation writes in terms of a "modified" Monge-Ampère equation which is still solvable. Furthermore, in [2] it is studied the equation on the Kodaira-Thurston manifold when F is S^1-invariant (instead of T^2-invariant as in the previous papers). It turns out that in this last case the Calabi-Yau problem writes as a PDE on the 3-dimensional torus $\mathbb{T}^3_{xyt}$ which is not of Monge-Ampère type anymore.

In this paper we review some results in [4] showing that when the projection is Lagrangian, the reduction of the Calabi-Yau problem on the Kodaira-Thurston manifold to a scalar PDE can be obtained by setting

$$\tilde{\omega} = \Omega + d(-Ju + u\gamma_1 + u_y\gamma_2)$$

where γ_1 and γ_2 are suitable invariant forms depending on (Ω, J), u is in the same space of F and y is a coordinate on the base.

In section 3 we study the Calabi-Yau equation on (M, Ω_0) for S^1-invariant almost complex structures J compatible to Ω_0. Under some strong restrictions on J, the equation can be still reduced to a PDE in a single unknown function. In section 4 we prove the solvability of the arising equations in some special cases leaving the more general cases for an eventually future work.

In the last section we consider a generalization of the previous sections to 2-step nilmanifold in higher dimensions.

A remark on the notation. If P is an m-torus bundle over an n-torus, we denote by $\mathbb{T}^n$ the base of P and by T^m the principal fiber, in order to distinguish the base and the fibers.

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2 Calabi-Yau equations on the Kodaira-Thurston manifold

In this section we review some results in [1, 2, 4] about the Calabi-Yau equation on the Kodaira-Thurston manifold. The *Kodaira-Thurston manifold* is a compact 2-step nilmanifold M defined as the quotient $M = \Gamma\backslash G$, where G is the Lie group given by $\mathbb{R}^4$ in the variables (x_1, x_2, y_1, y_2) with the multiplication

$$(x_1, x_2, y_1, y_2) \cdot (x_1', x_2', y_1', y_2') = (x_1 + x_1', x_2 + x_2', y_1 + y_1', y_2 + y_2' + x_1 x_2')$$

and Γ is the co-compact lattice given by $\mathbb{Z}^4$ with the induced multiplication. Alternatively M can be defined as the product $M = \Gamma_0\backslash\mathrm{Nil}^3 \times S^1$, where Nil^3 is the 3-dimensional real Heisenberg group

$$\mathrm{Nil}^3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

and Γ_0 is the lattice in Nil^3 of matrices having integers entries. M has a natural structure of principal S^1-bundle over a 3-dimensional torus $\mathbb{T}^3$ induced by the map $[x_1, x_2, y_1, y_2] \mapsto [x_1, x_2, y_1]$ and it is parallelizable. A global co-frame on M is for instance given by

$$e^1 = dx_1, \quad e^2 = dx_2, \quad f^1 = dy_1, \quad f^2 = dy_2 - x_1 dx_2 .$$

For such co-frame we have

$$de^1 = de^2 = df^1 = 0, \quad df^2 = -e^1 \wedge e^2$$

and its dual basis is given by $\{\partial_{x_1}, \partial_{x_2} + x_1\partial_{y_2}, \partial_{y_1}, -\partial_{y_2}\}$. Furthermore, M has the "natural" almost-Kähler structure (Ω_0, J_0) given by the symplectic form

$$\Omega_0 = e^1 \wedge f^1 + e^2 \wedge f^2 \tag{2.1}$$

and the Riemannian metric

$$g_0 = e^1 \otimes e^1 + f^1 \otimes f^1 + e^2 \otimes e^2 + f^2 \otimes f^2 . \tag{2.2}$$

The following proposition is proved in [2]

Proposition 2.1. *Let $u\colon M \to \mathbb{R}$ be an S^1-invariant function and*

$$\alpha := -J_0 du - ue^1.$$

Then

$$d\alpha \textit{ is of type } (1, 1)$$

and

$$(\Omega_0 + d\alpha)^2 = \left(\det(I + \mathcal{A}(u)) - u^2_{x_2 y_1}\right) \Omega_0^2 , \tag{2.3}$$

where I is the identity 2×2 matrix and

$$\mathcal{A}(u) = \begin{pmatrix} u_{x_1x_1} + u_{y_1y_1} + u_{y_1} & u_{x_1x_2} \\ u_{x_1x_2} & u_{x_2x_2} \end{pmatrix} . \tag{2.4}$$

Proof. Let $u\colon M \to \mathbb{R}$ be an S^1-invariant function. Then

$$du = u_{x_1} e^1 + u_{x_2} e^2 + u_{y_1} f^1$$

and

$$-J_0 du = u_{x_1} f^1 + u_{x_2} f^2 - u_{y_1} e^1$$

and

$$-dJ_0 du = \sum_{i,j=1}^{2} u_{x_i x_j} e^i \wedge f^j + u_{x_2 y_1} e^1 \wedge e^2 + u_{x_2 y_1} f^1 \wedge f^2 + u_{y_1 y_1} e^1 \wedge f^1 - u_{x_2} e^1 \wedge e^2 .$$

Therefore, if $\alpha = -J_0 du - ue^1$, we have

$$d\alpha = -dJ_0 du - du \wedge e^1 = \sum_{i,j=1}^{2} u_{x_i x_j} e^i \wedge f^j + u_{x_2 y_1} e^1 \wedge e^2 + u_{x_2 y_1} f^1 \wedge f^2 + u_{y_1 y_1} e^1 \wedge f^1 + u_{y_1} e^1 \wedge f^1$$

which is a form of type $(1, 1)$ with respect to J_0. Formula (2.3) follows from a straightforward computation. □

Proposition 2.1 is useful in the study of the Calabi-Yau problem on (M, Ω_0, J_0). Indeed, let $F\colon M \to \mathbb{R}$ be an S^1-invariant function satisfying $\int_M \mathrm{e}^F\, \Omega_0^2 = 1$ and consider the Calabi-Yau equation $(\Omega_0 + d\alpha)^2 = \mathrm{e}^F\, \Omega_0^2$ on (M, Ω_0, J_0). In view of proposition 2.1, we can study the Calabi-Yau problem by introducing the ansatz

$$\alpha = -J_0 du - ue^1$$

where u is an unknown S^1-invariant map. In this way the Calabi-Yau problem reduces to the single equation

$$\det(I + \mathcal{A}(u)) - u^2_{x_2 y_1} = \mathrm{e}^F\,, \tag{2.5}$$

on the 3-dimensional torus $\mathbb{T}^3_{x_1 x_2 y_1}$, where $\mathcal{A}(u)$ is given by (2.4). The main result in [2] is the following

Theorem 2.2. *Equation* (2.5) *has a solution for every S^1-invariant initial datum $F\colon M \to \mathbb{R}$ satisfying* $\int_M \mathrm{e}^F\, \Omega_0^2 = 1$. *Consequently the Calabi-Yau problem* $(\Omega_0 + d\alpha)^2 = \mathrm{e}^F\, \Omega_0^2$ *has a unique solution for every S^1-invariant function $F\colon M \to \mathbb{R}$.*

Special cases of equation (2.5) occur when we see M as a 2-torus bundle over a 2-dimensional torus and we assume F depending only on the coordinates of the base. Those cases correspond to assume F depending either on (x_1, x_2) or on (x_2, y_1) (the case $F = F(x_1, y_1)$ is equivalent to $F = F(x_2, y_1)$).

If $F = F(x_1, x_2)$, we can assume u depending only on (x_1, x_2) and (2.5) reduces to the Monge-Ampère type equation

$$(1 + u_{x_1x_1})(1 + u_{x_2x_2}) - u^2_{x_1x_2} = \mathrm{e}^F \tag{2.6}$$

on the 2-dimensional torus $\mathbb{T}^2_{x_1x_2}$. This equation has a solution in view of a theorem of Y.Y. Li (see [8]). Note that in this case the solution u to (2.6) is an almost-Kähler potential of $\tilde\omega = \Omega_0 + d\alpha$ with respect to Ω_0. Indeed,

$$\tilde\omega = (1 + u_{x_1x_1})e^1 \wedge f^1 + (1 + u_{x_2x_2})e^2 \wedge f^2 + u_{x_1x_2} e^1 \wedge f^2 + u_{x_1x_2} f^1 \wedge e^2$$

and

$$\tilde\omega - \Omega_0 = -dJ_0 du + da$$

where

$$a = -ue^1\,.$$

Hence $da = u_{x_2} e^1 \wedge e^2$ and

$$\tilde\omega \wedge da = 0$$

which implies

$$(\tilde\omega - \Omega_0) \wedge \tilde\omega = -dJ_0 du \wedge \tilde\omega\,.$$

If $F = F(x_2, y_1)$, we assume u depending only on (x_2, y_1) and (2.5) reduces to the “modified” Monge-Ampère equation

$$(1 + u_{y_1y_1} + u_{y_1})(1 + u_{x_2x_2}) - u^2_{x_2y_1} = \mathrm{e}^F \tag{2.7}$$

on the 2-dimensional torus $\mathbb{T}^2_{x_2y_1}$. The existence of a solution to this last equation was proved in [4]. Note that in this case

$$\tilde\omega = (1 + u_{y_1y_1} + u_{y_1})e^1 \wedge f^1 + (1 + u_{x_2x_2})e^2 \wedge f^2 + u_{x_2y_1} e^1 \wedge e^2 + u_{x_2y_1} f^1 \wedge f^2$$

and if u solves (2.7), then

$$d\alpha = -dJ_0 du + da\,,$$

where $da = -u_{x_2} e^1 \wedge e^2 - u_{y_1} e^1 \wedge f^1$. Therefore

$$da \wedge \tilde\omega \neq 0$$

and u is not an almost-Kähler potential.

Next, we take into account the Calabi-Yau problem on M viewed as a 2-torus bundle over a 2-torus equipped with an invariant Lagrangian almost-Kähler structure (Ω, J) and we assume F defined on the base. Here by *Lagrangian* we mean that the fibers of the fibration are Lagrangian submanifolds.

Proposition 2.3. *Let* (Ω, J) *be an* invariant *almost-Kähler structure on* M. *Then there exist real numbers* μ_1 *and* μ_2 *and an invariant* 1*-form* β *such that if* $u = u(x_1, x_2)$ *is a smooth function on* M, *then*

$$\alpha = -Jdu + \mu_1 u\, e^1 - \mu_2 u\, e^2 - u_y \beta$$

is such that $d\alpha$ *is of type* $(1,1)$. *Moreover*

$$(\Omega + d\alpha)^2 = \frac{1}{l_1 l_2}\left((l_1 + u_{x_1x_1})(l_2 + u_{x_2x_2}) - (u_{x_1x_2})^2\right)\Omega^2 . \tag{2.8}$$

where l_1 *and* l_2 *are positive real numbers.*

Proof. We set $x_1 = x$ and $x_2 = y$ in order to simplify the notation. We can find an invariant Hermitian coframe $\{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ on M such that

$$\Omega = \alpha^1 \wedge \beta^1 + \alpha^2 \wedge \beta^2$$

and

$$dx = A\alpha^1 , \quad dy = B\alpha^1 + C\alpha^2 .$$

Note that $dx \wedge dy = AC\alpha^1 \wedge \alpha^2$ and we can write

$$d\beta^1 = \lambda_1 dx \wedge dy , \quad d\beta^2 = \lambda_2 dx \wedge dy$$

for some λ_1, λ_2 in $\mathbb{R}$. Now

$$du = u_x dx + u_y dy = (Au_x + Bu_y)\alpha^1 + Cu_y\alpha^2$$

and

$$-Jdu = (Au_x + Bu_y)\beta^1 + Cu_y\beta^2$$

So

$$-\,dJdu = Au_{xx}dx \wedge \beta^1 + Au_{xy}dy \wedge \beta^1 + Cu_{xy}dx \wedge \beta^2 + Cu_{yy}dy \wedge \beta^2 + d(\gamma + Bu_y\beta^1)$$

where

$$\gamma = \lambda_1 Au\, dy - \lambda_2 Cu\, dx .$$

Hence

$$\begin{aligned}-\,dJdu = {} & A^2u_{xx}\alpha^1 \wedge \beta^1 + ABu_{xy}\alpha^1 \wedge \beta^1 + ACu_{xy}\alpha^2 \wedge \beta^1 + ACu_{xy}\alpha^1 \wedge \beta^2 \\ & + BCu_{yy}\alpha^1 \wedge \beta^2 + C^2u_{yy}\alpha^2 \wedge \beta^2 + d(Bu_y\alpha^2 + \gamma) .\end{aligned}$$

which implies that

$$\alpha = -Jdu - Bu_y\alpha^2 - \gamma$$

is such that $d\alpha$ is of type $(1,1)$.

Moreover,

$$(\Omega + d\alpha)^2 = \left((1 + A^2u_{xx})(1 + C^2u_{yy}) - (ACu_{xy})^2\right)\Omega^2 = \frac{1}{l_1 l_2}\left((l_1 + u_{xx})(l_2 + u_{yy}) - (u_{xy})^2\right)\Omega^2$$

where $l_1 = 1/A^2$ and $l_2 = 1/C^2$ and the claim follows. □

Proposition 2.4. *Let* (Ω, J) *be an* invariant *almost-Kähler structure on* M *which is* Lagrangian *with respect to the fibration* $[x_1, x_2, y_1, y_2] \mapsto [x_2, y_1]$. *There exist invariant* 1*-forms* γ^1, γ^2 *such that if* $u = u(x_2, y_1)$ *is a smooth function on* M, *then*

$$\alpha = -Jdu + u\gamma^1 + u_{y_1}\gamma^2$$

is such that $d\alpha$ *is of type* $(1,1)$. *Moreover*

$$(\Omega + d\alpha)^2 = \frac{1}{l_1 l_2}\left((l_1 + u_{x_2x_2})(l_2 + u_{y_1y_1} + m_1u_{x_2} + m_2u_{y_1}) - (u_{x_2y_1})^2\right)\Omega^2 \tag{2.9}$$

where $l_1, l_2, m_1, m_2 \in \mathbb{R}$ *and* $l_1, l_2 < 0$.

Proof. First of all we use that (Ω, J) is an invariant almost-Kähler structure on M which is Lagrangian with respect to $[x_1, x_2, y_1, y_2] \mapsto [x_2, y_1]$, then there exists an invariant Hermitian co-frame $\{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ on M such that

$$\Omega = \alpha^1 \wedge \beta^1 + \alpha^2 \wedge \beta^2$$

and

$$\alpha^2 \in \langle e^2 \rangle\,, \quad \beta^1 \in \langle e^2, f^1 \rangle\,, \quad \alpha^1 \in \langle e^1, e^2, f^1 \rangle$$

(see lemma 5.1 in [4]). In this way

$$dx_2 = A\alpha^2\,, \quad dy_1 = B\alpha^2 + C\beta^1\,, \quad d\beta^2 = \lambda\,\alpha^1 \wedge \beta^1 + \mu\alpha^2 \wedge \beta^1$$

for some $A, B, C, \lambda, \mu \in \mathbb{R}$. In order to semplify the notation we set $x_2 = x$ and $y_1 = y$. Then

$$du = u_x dx + u_y dy = Au_x\alpha^2 + u_y(B\alpha^2 + C\beta^1) = (Au_x + Bu_y)\alpha^2 + Cu_y\beta^1$$

and

$$-Jdu = -(Au_x + Bu_y)\beta^2 + Cu_y\alpha^1\,.$$

So

$$\begin{aligned} -\,dJdu = &-Au_{xx}dx \wedge \beta^2 - Au_{xy}dy \wedge \beta^2 + Cu_{xy}dx \wedge \alpha^1 \\ &+ Cu_{yy}dy \wedge \alpha^1 - (Au_x + Bu_y)\left(\lambda\,\alpha^1 \wedge \beta^1 + \mu\alpha^2 \wedge \beta^1\right) - d(Bu_y\beta^2) \end{aligned}$$

i.e.,

$$\begin{aligned} -\,dJdu = &-A^2u_{xx}\alpha^2 \wedge \beta^2 - BAu_{xy}\alpha^2 \wedge \beta^2 - ACu_{xy}\beta^1 \wedge \beta^2 + ACu_{xy}\alpha^2 \wedge \alpha^1 \\ &+ CBu_{yy}\alpha^2 \wedge \alpha^1 + C^2u_{yy}\beta^1 \wedge \alpha^1 - (Au_x + Bu_y)\left(\lambda\,\alpha^1 \wedge \beta^1 + \mu\alpha^2 \wedge \beta^1\right) - d(Bu_y\beta^2)\,. \end{aligned}$$

Now,

$$(Au_x + Bu_y)\left(\lambda\,\alpha^1 \wedge \beta^1 + \mu\alpha^2 \wedge \beta^1\right) = \lambda(Au_x + Bu_y)\,\alpha^1 \wedge \beta^1 + d(\mu u\,\beta^1)$$

and we can write

$$\begin{aligned} -\,dJdu = &(-C^2u_{yy} - \lambda Au_x - \lambda Bu_y)\alpha^1 \wedge \beta^1 + (-A^2u_{xx} - BAu_{xy})\alpha^2 \wedge \beta^2 \\ &- ACu_{xy}\beta^1 \wedge \beta^2 - (ACu_{xy} + B^2u_{yy})\alpha^1 \wedge \alpha^2 - d(\mu u\beta^1 + Bu_y\beta^2) \end{aligned}$$

which implies the first part of the statement.

Moreover,

$$\begin{aligned} (\Omega + d\alpha)^2 &= \left((1 - A^2u_{xx})(1 - C^2u_{yy} - \lambda Au_x - \lambda Bu_y) - (ACu_{xy})^2\right)\,\Omega^2 \\ &= \frac{1}{l_1 l_2}\left((l_1 + u_{xx})(l_2 + u_{yy} + m_1u_x + m_2u_y) - (u_{xy})^2\right)\,\Omega^2 \end{aligned}$$

where

$$l_1 = -\frac{1}{A^2}\,, \quad l_2 = -\frac{1}{C^2}\,, \quad m_1 = -\lambda\frac{A}{C^2}\,, \quad m_2 = -\lambda\frac{B}{C^2}$$

and the claim follows. □

From propositions 2.3 and 2.4 it follows that if we see M as 2-torus over a 2-torus and we fix an invariant Lagrangian almost-Kähler structure (Ω, J) on M; then for every given F defined on the base of M and satisfying $\int_M \mathrm{e}^F\,\Omega^2 = \int_M \Omega^2$ the corresponding Calabi-Yau equation can be written in terms of an unknown function u on the base $\mathbb{T}^2_{xy}$ of M as

$$\frac{1}{l_1 l_2}\left((l_1 + u_{xx})(l_2 + u_{yy} + m_1u_x + m_2u_y) - (u_{xy})^2\right) = \mathrm{e}^F$$

where $l_1, l_2, m_1, m_2 \in \mathbb{R}$ and l_1 and l_2 are both positive or negative. This kind of equations are solvable in view of theorem 6.2 in [4].

3 The equation for non-invariant almost-complex structures

As pointed out in [13] it is interesting to extend the results described in the previous section to torus-invariant almost complex structures on the Kodaira-Thurston manifold M which are compatible to the "natural" symplectic form Ω_0 defined in (2.1). In this section we consider some basic cases. Let $h = h(x_1, y_1)$ be a function in $C^\infty(\mathbb{T}^2_{x_1y_1})$ and consider the family of Ω_0-compatible almost-complex structures J_h induced by the relations

$$J_h(e^1) = -\mathrm{e}^h f^1 \quad J_h(e^2) = -f^2 \,. \tag{3.1}$$

The following result is a generalization of proposition 2.1 to the family J_h.

Proposition 3.1. *Let $u\colon M \to \mathbb{R}$ be an S^1-invariant function and*

$$\alpha := -J_h du - ue^1 .$$

Then

$$d\alpha \text{ is of type } (1,1)$$

and

$$(\Omega_0 + d\alpha)^2 = \left(\det(I + \mathcal{A}_h(u)) - \mathrm{e}^{-h} u^2_{x_2y_1}\right) \Omega_0^2 \tag{3.2}$$

where I is the identity 2×2 matrix and

$$\mathcal{A}_h(u) = \begin{pmatrix} \mathrm{e}^h u_{x_1x_1} + \mathrm{e}^{-h} u_{y_1y_1} + u_{y_1} + \mathrm{e}^h h_{x_1} u_{x_1} - \mathrm{e}^{-h} h_{y_1} u_{y_1} & u_{x_1x_2} \\ \mathrm{e}^h u_{x_1x_2} & u_{x_2x_2} \end{pmatrix} \tag{3.3}$$

Proof. Let u be an S^1-invariant function. Then

$$-J_h du = \mathrm{e}^h u_{x_1} f^1 + u_{x_2} f^2 - \mathrm{e}^{-h} u_{y_1} e^1$$

and

$$\begin{aligned} -dJ_h du = &(\mathrm{e}^h u_{x_1})_{x_1} e^1 \wedge f^1 + \mathrm{e}^h u_{x_1x_2} e^2 \wedge f^1 + u_{x_1x_2} e^1 \wedge f^2 + u_{x_2x_2} e^2 \wedge f^2 \\ &+ u_{x_2y_1} f^1 \wedge f^2 + \mathrm{e}^{-h} u_{x_2y_1} e^1 \wedge e^2 + (\mathrm{e}^{-h} u_{y_1})_{y_1} e^1 \wedge f^1 - u_{x_2} e^1 \wedge e^2 \,, \end{aligned}$$

i.e.,

$$\begin{aligned} -dJ_h du = &\left(\mathrm{e}^h u_{x_1x_1} + \mathrm{e}^{-h} u_{y_1y_1} + \mathrm{e}^h h_{x_1} u_{x_1} - \mathrm{e}^{-h} h_{y_1} u_{y_1} + u_{y_1}\right) e^1 \wedge f^1 + u_{x_2x_2} e^2 \wedge f^2 \\ &+ \mathrm{e}^h u_{x_1x_2} e^2 \wedge f^1 + u_{x_1x_2} e^1 \wedge f^2 + u_{x_2y_1} f^1 \wedge f^2 + \left(\mathrm{e}^{-h} u_{x_2y_1} - u_{x_2}\right) e^1 \wedge e^2 \,. \end{aligned}$$

Therefore if $\alpha = -J_h du - ue^1$, then

$$\begin{aligned} d\alpha = &\left(\mathrm{e}^h u_{x_1x_1} + \mathrm{e}^{-h} u_{y_1y_1} + \mathrm{e}^h h_{x_1} u_{x_1} - \mathrm{e}^{-h} h_{y_1} u_{y_1} + u_{y_1}\right) e^1 \wedge f^1 + u_{x_2x_2} e^2 \wedge f^2 \\ &+ \mathrm{e}^h u_{x_1x_2} e^2 \wedge f^1 + u_{x_1x_2} e^1 \wedge f^2 + u_{x_2y_1} f^1 \wedge f^2 + \mathrm{e}^{-h} u_{x_2y_1} e^1 \wedge e^2 \,. \end{aligned}$$

which is of type $(1,1)$ and

$$(\Omega_0 + d\alpha)^2 = \det(I + \mathcal{A}_h(u)) - \mathrm{e}^{-h} u^2_{x_2y_1} \,,$$

as required. □

In view of proposition 3.1, the Calabi-Yau equation on (M, Ω_0, J_h), for an S^1-invariant function $F\colon M \to \mathbb{R}$ can be reduced to

$$\det(I + \mathcal{A}_h(u)) - \mathrm{e}^{-h}\, u^2_{x_2y_1} = \mathrm{e}^F \tag{3.4}$$

where $\mathcal{A}_h$ is given by (3.3) and $u\colon M \to \mathbb{R}$ is an unknown S^1-invariant function. Note that for $h = 0$, equation (3.4) reduces to equation (2.5) studied in [2]. We consider the following special cases:

If $h = h(x_1)$ and $F = F(x_1, x_2)$ we may assume u depending only on (x_1, x_2) and (3.4) reduces in the variables $x = x_1$, $y = x_2$ to

$$\det\begin{pmatrix} 1 + e^h u_{xx} + e^h h' u_x & u_{xy} \\ e^h u_{xy} & 1 + u_{yy} \end{pmatrix} = e^F$$

on the 2-dimensional torus $\mathbb{T}^2_{xy}$. Such an equation can be rewritten as

$$\det\begin{pmatrix} e^{-h} + u_{xx} + h' u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F-h} .$$

If $h = h(y_1)$ and $F = F(x_2, y_1)$, then we assume u depending only on (x_2, y_1) and (3.4) reduces in the variables $x = y_1$, $y = x_2$ to

$$\det\begin{pmatrix} 1 + e^{-h} u_{xx} + (1 - e^{-h} h') u_x & u_{xy} \\ e^{-h} u_{xy} & 1 + u_{yy} \end{pmatrix} = e^F$$

on $\mathbb{T}^2_{xy}$. Such an equation can be rewritten as

$$\det\begin{pmatrix} e^h + u_{xx} + (e^h - h') u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F+h} .$$

Both cases fit in the following class of equations on $\mathbb{T}^2_{xy}$

$$\det\begin{pmatrix} e^{-h} + u_{xx} + (ce^{-h} + h') u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F-h}$$

where $h = h(x)$ is a smooth 1-periodic functions on $\mathbb{R}$ and $c \in \mathbb{R}$. We will show the solvability of the last class of equations in the next section.

4 Solvability of the special cases

The aim of this section is to prove the following result

Theorem 4.1. *Let $h = h(x)$ be a smooth 1-periodic functions on $\mathbb{R}$, $c \in \mathbb{R}$ and let $F = F(x, y) \in C^\infty(\mathbb{T}^2)$ be such that*

$$\int_{\mathbb{T}^2} e^F \, dx \wedge dy = 1 .$$

Then equation

$$\det\begin{pmatrix} e^{-h} + u_{xx} + (ce^{-h} + h') u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F-h} \tag{4.1}$$

has a solution $u \in C^\infty(\mathbb{T}^2)$.

Before proving theorem 4.1 we consider the following preliminary lemma which is a slight generalization of lemma 6.3 in [4].

Lemma 4.2. *Let $h, v \in C^1(\mathbb{R})$ be 1-periodic functions satisfying*

$$e^h v' + (c + e^h h') v > -1 .$$

Assume there exists $s_0 \in [0, 1]$ such that $v(s_0) = 0$; then

$$\|v\|_{C^0} \leq C ,$$

where C is a constant depending only on c and h.

Proof. Let G be a primitive of $ce^{-h}+h'$ in $\mathbb{R}$. Since

$$v' + (ce^{-h}+h')v > -e^{-h}\,,$$

in terms of G we have

$$e^{G}(v'+G'v) > -e^{G-h}\,,$$

i.e.

$$\frac{d}{ds}(e^{G}v) > -e^{G-h}\,.$$

Since $v(s_0)=0$, we have

$$\int_{s_0}^{s}\frac{d}{ds}(e^{G}v)ds > -\int_{s_0}^{s} e^{G-h}d\tau\,,\quad \text{for every } s\geq 1\,,$$

which implies

$$v(s) > -e^{-G(s)}\int_{s_0}^{s} e^{G-h}d\tau\,,\quad \text{for every } s\in[1,2]\,.$$

On the other hand

$$\int_{s}^{s_0}\frac{d}{ds}(e^{G}v)ds > -\int_{s}^{s_0} e^{G-h}d\tau\,,\quad \text{for every } s\leq 0\,,$$

which implies

$$v(s) < e^{-G(s)}\int_{s}^{s_0} e^{G-h}d\tau\,,\quad \text{for every } s\in[-1,0]\,.$$

The claim follows since v is 1-periodic. □

Now we can prove theorem 4.1

Proof of Theorem 4.1. Fix $0<\alpha<1$ and let $C^{2,\alpha}_0(\mathbb{T}^2)$ be the space of $C^{2,\alpha}$-functions u on $\mathbb{T}^2$ satisfying

$$\int_{\mathbb{T}^2} u\, dx\wedge dy = 0\,.$$

Then we consider the operator $T\colon C^{2,\alpha}_0(\mathbb{T}^2)\times[0,1]\to C^{0,\alpha}_0(\mathbb{T}^2)$ defined by

$$T(u,t)=\det\begin{pmatrix} e^{-h}+u_{xx}+(c\,e^{-h}+h')u_x & u_{xy}\\ u_{xy} & 1+u_{yy}\end{pmatrix} - e^{-h}\left(te^{F}+1-t\right)$$

in order that $u\in C^{2,\alpha}_0(\mathbb{T}^2)$ solves (4.1) if and only if $T(u,1)=0$. Then we define the set

$$S:=\{t\in[0,1]\ :\ \text{there exists } u\in C^{2,\alpha}_0(\mathbb{T}^2) \text{ such that } T(u,t)=0\}\,.$$

Note that S is not empty since $u\equiv 0$ satisfies $T(u,0)=0$. We will show that $1\in S$ by proving that S is open and closed in $[0,1]$. In this way we get that (4.1) has a solution u in $C^{2,\alpha}(\mathbb{T}^2)$ and theorem 3 in [9] implies that u is in fact C^∞. Note that if $(u,t)\in C^2_0(\mathbb{T}^2)\times[0,1]$ is such that $T(u,t)=0$, then the matrix

$$\mathcal{A}_h:=\det\begin{pmatrix} e^{-h}+u_{xx}+(ce^{-h}+h')u_x & u_{xy}\\ u_{xy} & 1+u_{yy}\end{pmatrix}$$

is positive-defined. Indeed, since $\int_{\mathbb{T}^2} e^{F}dx\wedge dy=0$, then $\mathcal{A}_h(u)$ is non-singular and at a minimum point of u all the eigenvalues of $\mathcal{A}_h$ are positive.

Now we prove that S is closed. First of all we observe that if $u \in C^2_0(\mathbb{T}^2)$ satisfies $T(u,t) = 0$ for some $t \in [0,1]$, then

$$e^h u_{xx} + (c + e^h h')u_x > -1\,, \tag{4.2}$$
$$1 + u_{yy} > -1. \tag{4.3}$$

Indeed, since

$$(1 + e^h u_{xx} + (c + e^h h')u_x)(1 + u_{yy}) > 0$$

the two terms have the same sign, and they are both positive at a point (x_0, y_0) where u reaches its minimum value. Lemma 4.2 then implies

$$\|u_x\|_{C^0} \leq C \quad \text{and} \quad \|u_y\|_{C^0} \leq C \tag{4.4}$$

where C is a constant depending on c, h and k. Now we focus on the C^0 estimate on u. Let (x_0, y_0) be a point in $[0,1] \times [0,1]$ where u vanishes, then

$$u(x,y) = (x - x_0)\int_0^1 u_x((1-t)x + tx_0, (1-t)y + ty_0)\,dt + (y - y_0)\int_0^1 u_y((1-t)x + tx_0, (1-t)y + ty_0))\,dt,$$

and by using (4.4) we get

$$|u(x,y)| \leq C(x - x_0) + C(y - y_0)$$

which readily implies

$$\|u\|_{C^0} \leq C\,.$$

Hence u satisfies a C^1 a priori bound. Furthermore, if $t \in [0,1]$ is fixed, equation

$$T(u,t) = 0$$

belongs to the class of equations studied in [7] and theorem 2 in [7] implies that if $u \in C^{2,\alpha}_0(\mathbb{T}^2)$ solves $T(u,t) = 0$ for some t and satisfies a priori C^1 bound, then it also satisfies a $C^{2,\alpha}$ bound. This implies that S is closed in $[0,1]$. Indeed, let t_n be a sequence in S converging to $\bar{t}$ in $[0,1]$. To each t_n corresponds a function $u_n \in C^{2,\alpha}_0(\mathbb{T}^2)$ such that $T(u_n, t_n) = 0$. The $C^{2,\alpha}$ a priori bound on solutions to $T(u,t) = 0$ implies that the sequence u_n is bounded in $C^{2,\alpha}_0(\mathbb{T}^2)$ and so it admits a subsequence, which we still denote by u_n, which converges in $C^2_0(\mathbb{T}^2)$ to a function $\bar{u} \in C^2_0(\mathbb{T}^2)$. Since T is continuos, $T(\bar{u}, \bar{t}) = 0$ and so, in view of [7], $\bar{u}$ in $C^{2,\alpha}(\mathbb{T}^2)$. Hence $\bar{t} \in S$ and S is closed.

Next we show that S is open . Let $t_0 \in S$. Then there exists $u \in C^{2,\alpha}_0(\mathbb{T}^2)$ such that $T(u, t_0) = 0$. Let $L\colon C^{2,\alpha}_0(\mathbb{T}^2) \to C^{0,\alpha}_0(\mathbb{T}^2)$ be defined as

$$L(w) := T_{*|(u,t_0)}(w, 0)\,.$$

A direct computation yields that

$$L(w) = (w_{xx} + (ce^{-h} + h')w_x)(1 + u_{yy}) + (e^{-h} + u_{xx} + (c\,e^{-h} + h')u_x)(w_{yy}) - 2u_{xy}w_{xy} \tag{4.5}$$

and so L is uniformly elliptic. L is injective by maximum principle and it is surjective in view of elliptic theory (see e.g. [5]). Therefore the implicit function theorem implies that $\bar{t}$ has a open neighborhood contained in S, and so S is open, as required. □

5 A generalization to higher dimensions

In this section we consider a generalization of the Kodaira-Thurston manifold in dimension greater than 4. Assume $n \geq 3$. Let G_n be the Lie group $(\mathbb{R}^{2n}, *_n)$, where

$$(x_1,\dots,x_n,y_1,\dots,y_n) *_n (x'_1,\dots,x'_n,y'_1,\dots,y'_n) = \\ (x_1 + x'_1,\dots,x_n + x'_n, y_1 + y'_1, y_2 + y'_2 - x_2x'_1,\dots,y_{n-1} + y'_{n-1} - x_nx'_1)$$

and let $M_n = \Gamma_n \backslash G_n$, where Γ_n is $\mathbb{Z}^{2n}$ with the multiplication induced by $\star_n$. Then M_n is a 2-step nilmanifold and the projection $\pi\colon \mathbb{R}^{2n} \to \mathbb{R}^{n+1}$ onto the first $(n+1)$-coordinates induces to M_n a structure of principal $(n-1)$–torus bundle on an $(n+1)$–torus $\mathbb{T}^{n+1}$. M_n is parallelizable and

$$e^i = dx_i\,, \quad i = 1\dots, n, \quad f^j = dy_j - x_1 dx_j\,, \quad j = 1\dots, n$$

defines a global coframe which satisfies

$$de^k = 0\,,\ k = 1,\dots, n\,, \quad df^1 = 0\,, \quad df^k = e^k \wedge e^1\,, \quad k = 2,\dots, n\,.$$

We then consider on M_n the symplectic form

$$\Omega_n = \sum_{k=1}^{n} \alpha^k \wedge \beta^k$$

and the Ω_n-compatible almost-complex structure J_n induced by Ω_n and the natural metric

$$g_n = \sum_{k=1}^{n} \alpha^k \otimes \alpha^k + \beta^k \otimes \beta^k\,.$$

In terms of the basis $\mathcal{B} = \{e^1, \dots, e^n, f^1, \dots, f^n\}$, J_n is defined by

$$J_n e^k = -f^k\,, \quad J_n f^k = e^k\,.$$

Let u be a T^{n+1}-invariant function on M_n; then

$$du = \sum_{s=1}^{n} u_{x_s}\, e^s + u_{y_1} f^1\,, \quad -J_n du = \sum_{s=1}^{n} u_{x_s} f^s - u_{y_1} e^1$$

and so

$$\begin{aligned} -dJ_n du &= \sum_{r,s=1}^{n} u_{x_r x_s} e^r \wedge f^s - \sum_{k=1}^{n} u_{x_k y_1} e^k \wedge e^1 + u_{x_k y_1} f^1 \wedge f^k + u_{y_1 y_1} e^1 \wedge f^1 + \sum_{k=2}^{n} u_{x_r} e^r \wedge e^1 \\ &= \sum_{r,s=1}^{n} u_{x_r x_s} e^r \wedge f^s - \sum_{k=1}^{n} u_{x_k y_1} e^k \wedge e^1 + u_{x_k y_1} f^1 \wedge f^k + (u_{y_1 y_1} - u_{y_1}) e^1 \wedge f^1 + d(u e^1)\,, \end{aligned}$$

and so if

$$\alpha = -J_n du - u e^1\,,$$

then $d\alpha$ is of type $(1,1)$ with respect to J_n. Furthermore,

$$\begin{aligned} (\Omega_n + d\alpha)^n = &\left(\sum_{r,s=1}^{n} (\delta_{rs} + u_{x_r x_s}) e^r \wedge f^s + (1 + u_{y_1 y_1} - u_{y_1}) e^1 \wedge f^1 \right)^n \\ &- n! \sum_{k,m=2}^{n} \left(\prod_{r,s=2,(r,s)\neq(k,m)}^{n} u_{x_r x_s} u_{x_k y_1} u_{x_m y_1} \right) e^1 \wedge f^1 \wedge \dots \wedge e^n \wedge f^n \end{aligned}$$

and if F is a given T^{n+1}-invariant map, the Calabi-Yau equation $(\Omega + d\alpha)^n = \mathrm{e}^F\, \Omega^n$ reads in terms of u as

$$\det(I + \mathcal{A}(u)) - \sum_{k,m=2}^{n} \left(\prod_{r,s=2,(r,s)\neq(k,m)}^{n} u_{x_r x_s} u_{x_k y_1} u_{x_m y_1} \right) = \mathrm{e}^F\,, \tag{5.1}$$

where $\mathcal{A}(u) = (A_{ij})$ is the $n \times n$ matrix

$$A_{11} = u_{x_1 x_1} + u_{y_1 y_1} - u_{y_1}\,, \quad A_{ij} = u_{x_i x_j}\,, \quad \text{if } (i,j) \neq (1,1)\,.$$

Example 5.1. For $n = 3$, equation (5.1) reads as

$$\det(I + A(u)) - u_{x_3x_3}u^2_{x_2y_1} - u_{x_2x_2}u^2_{x_3y_1} - 2u_{x_2x_3}u_{x_2y_1}u_{x_3y_1} = \mathrm{e}^F\,,$$

this kind of equations has been considered in [6].

In analogy to the case $n = 2$, we can obtain special cases by regarding M_n as a principal T^n-bundle over a $\mathbb{T}^n$ and assuming F to be T^n-invariant. It is not restrictive considering only the following two cases:

$$F = F(x_1, \dots, x_n)\,, \qquad \text{or } F = F(x_2, \dots, x_n, y_1).$$

– In the first case $F = F(x_1, \dots, x_n)$, equation (5.1) reduces to the Monge-Ampère equation

$$\det(I + \mathcal{H}(u)) = \mathrm{e}^F$$

on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, where $\mathcal{H}(u)$ is the Hessian metric of u. In this case the equation has a solution in view of [8].

– In the second case, $F = F(x_2, \dots, x_n, y_1)$, in the variables

$$z_1 = y_1\,, z_2 = x_2, \dots, z_n = x_n$$

equation (5.1) takes the following expression

$$\det(I + \mathcal{B}(u)) = \mathrm{e}^F$$

where $\mathcal{B}(u) = (B_{ij})$ is given by

$$B_{11} = u_{z_1z_1} + u_{z_1}\,, \quad B_{ij} = u_{z_iz_j}\,, \text{ if } i, j \neq 1\,.$$

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