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An extention of Nomizu's Theorem –A user's guide–

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Abstract: For a simply connected solvable Lie group G with a lattice Γ, the author constructed an explicit finite-dimensional differential graded algebra A^*_Γ which computes the complex valued de Rham cohomology $H^*(\Gamma\backslash G, \mathbb{C})$ of the solvmanifold $\Gamma\backslash G$. In this note, we give a quick introduction to the construction of such A^*_Γ including a simple proof of $H^*(A^*_\Gamma) \cong H^*(\Gamma\backslash G, \mathbb{C})$.

Keywords: Solvmanifold, Cohomology

1 Introduction

In [11], Nomizu proved that the de Rham cohomology of nilmanifolds can be computed by the left-invariant differential forms. Nomizu's Theorem is the most important tool for studying the geometry of nilmanifolds. Nomizu's Theorem was extended to solvmanifolds of completely solvable type by Hattori in [5]. However, there exist solvmanifolds whose de Rham cohomology can not computed by left-invariant differential forms.

This note is a self-contained quick reference for the following statement.

Theorem 1.1. *Let G be a simply connected solvable Lie group. We assume that G contains a lattice Γ. Then, we can construct an explicit finite-dimensional sub-DGA (differential graded algebra) A^*_Γ which computes the complex valued de Rham cohomology of the solvmanifold $\Gamma\backslash G$.*

In fact, on author's studies of extensions of Nomizu's theorem, this statement is only the tip of the iceberg. In order to gain a deeper understanding, we should study relations to rational homotopy theory, local system cohomology, linear algebraic groups, group cohomology, etc., (see [7], [8], [9], [10] for details). However, without such details, by using Theorem 1.1, we may produce many results on solvmanifolds, like use of Nomizu's theorem in [11].

2 Constructions of A^*_Γ

2.1 Cartan subalgebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. A subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is a *Cartan subalgebra* if:
– $\mathfrak{c}$ is nilpotent and
– the normalizer of $\mathfrak{c}$ in $\mathfrak{g}$ is equal to $\mathfrak{c}$.
It is known that any $\mathfrak{g}$ contains a Cartan subalgebra.

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Lemma 2.1. ([6, 15.4 Lemma A]) *Let $\phi : \mathfrak{g} \to \mathfrak{g}'$ be a surjective homomorphism between Lie algebras. If $\mathfrak{c} \subset \mathfrak{g}$ is a Cartan subalgebra in $\mathfrak{g}$, then the image $\phi(\mathfrak{c})$ is a Cartan subalgebra in $\mathfrak{g}'$.*

2.2 Semi-simple complements of solvable Lie algebras

Let $\mathfrak{g}$ be a solvable Lie algebra over $\mathbb{R}$. The *nilradical* $\mathfrak{n}$ of $\mathfrak{g}$ is the maximal nilpotent ideal in $\mathfrak{g}$. The nilradical $\mathfrak{n}$ is characterized by

$$\mathfrak{n} = \{X \in \mathfrak{g} : \mathrm{ad}_X \text{ is nilpotent}\}.$$

It is known that $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$.

Lemma 2.2. *If $\mathfrak{c} \subset \mathfrak{g}$ is a Cartan subalgebra in $\mathfrak{g}$, then*

$$\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$$

where the sum is not direct in general.

Proof. Consider the quotient $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$. By Lemma 2.1, $q(\mathfrak{c})$ is a Cartan subalgebra in $\mathfrak{g}/\mathfrak{n}$. On the other hand, by $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}/\mathfrak{n}$ is Abelian and so $q(\mathfrak{c}) = \mathfrak{g}/\mathfrak{n}$. Hence the lemma follows. □

Definition 2.3. A *semi-simple complement* of $\mathfrak{g}$ is a vector subspace (not necessarily Lie subalgebra) $\mathfrak{s} \subset \mathfrak{g}$ so that:
- $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ as a direct sum of vector spaces.
- For any $A, B \in \mathfrak{s}$, $(\mathrm{ad}_A)_s(B) = 0$ where $(\mathrm{ad}_A)_s$ is the semi-simple part for the Jordan decomposition.

Lemma 2.4. ([3, Section III.1]) For a semi-simple complement $\mathfrak{s}$, the map

$$\psi : \mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n} \ni A + X \mapsto (\mathrm{ad}_A)_s \in D(\mathfrak{g})$$

is a semi-simple representation where $D(\mathfrak{g})$ is the space of derivations on $\mathfrak{g}$.

Proposition 2.5. *Any solvable Lie algebra $\mathfrak{g}$ contains a semi-simple complement.*

Proof. Take a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{g}$. Then, by Lemma 2.2, we can take a subspace $\mathfrak{s} \subset \mathfrak{c}$ so that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$. Since $\mathfrak{c}$ is nilpotent, the proposition follows.

□

2.3 Constructions of A^*_Γ

Let G be a simply connected solvable real Lie group with the Lie algebra $\mathfrak{g}$. We assume that G contains a lattice Γ. Take a semi-simple complement $\mathfrak{s}$ of $\mathfrak{g}$ and the semi-simple representation $\psi : \mathfrak{g} \to D(\mathfrak{g})$ as in Lemma 2.4. Take the extension $\Psi : G \to \mathrm{Aut}(\mathfrak{g})$. Since Ψ is semi-simple, we take a basis $X_1, \ldots, X_n$ of $\mathfrak{g} \otimes \mathbb{C}$ so that

$$\Psi = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$$

for complex characters $\alpha_1, \ldots, \alpha_n$ of G where $n = \dim \mathfrak{g}$. Take the dual basis $x_1, \ldots, x_n$ of $\mathfrak{g}^* \otimes \mathbb{C}$. Considering $x_1, \ldots, x_n$ as complex valued left-invariant differential forms on $\Gamma \backslash G$, we define the graded subspace $A^*_\Gamma \subset A^*_{\mathbb{C}}(\Gamma \backslash G)$ so that

$$A^p_\Gamma = \mathrm{span}\langle \alpha_I x_I | I \subset \{1, \ldots, n\} \text{ with } |I| = p,\ \ \alpha_I|_\Gamma = 1\rangle$$

where for a multi-index $I = \{i_1, \ldots, i_p\}$ we write $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$ and $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$ and for a character α of G whose restriction on Γ is trivial, α is considered as a function on $\Gamma \backslash G$.

Lemma 2.6. *$A^{\star}_{\Gamma}$ is a sub-DGA of the complex valued de Rham complex $A^{\star}_{\mathbb{C}}(\Gamma\backslash G)$ of $\Gamma\backslash G$.*

Proof. Take the weight space decomposition

$$\bigwedge \mathfrak{g}^{*} \otimes \mathbb{C} = \bigoplus_{\alpha} V_{\alpha}$$

for the dual representation of $\Psi : G \to \mathrm{Aut}(\mathfrak{g})$. We notice that $\bigwedge \mathfrak{s}^{*} \otimes \mathbb{C} \subset V_1$ where $\mathfrak{s}$ is a semi-simple complement. We have $V_{\alpha} \wedge V_{\beta} \subset V_{\alpha\beta}$ and $dV_{\alpha} \subset V_{\alpha}$ for any weight vector spaces V_{α}, V_{β}. In fact, we have

$$A^{\star}_{\Gamma} = \bigoplus_{\alpha|_{\Gamma}=1} \alpha^{-1} V_{\alpha}.$$

Hence, $A^{\star}_{\Gamma}$ is closed under the wedge product. Since $\alpha^{-1}d\alpha \in V_1$ for any character α of G, we have

$$d(\alpha^{-1}V_{\alpha}) \subset \alpha d\alpha^{-1} \wedge \alpha^{-1}V_{\alpha} + \alpha^{-1}dV_{\alpha} \subset \alpha^{-1}V_{\alpha}$$

and so

$$dA^{\star}_{\Gamma} \subset A^{\star}_{\Gamma}.$$

Hence the lemma follows. □

Theorem 2.7. *The inclusion $A^{\star}_{\Gamma} \subset A^{\star}_{\mathbb{C}}(\Gamma\backslash G)$ induces a cohomology isomorphism.*

We will prove this theorem in Section 5. The proof in this paper is simpler than the one which was given previously.

3 Examples

Example 1. Let $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^2$ so that $\phi(t) = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}$. Then G has a lattices $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2$. We have

$$\mathfrak{g} \otimes \mathbb{C} = \mathrm{span}\left\langle \frac{\partial}{\partial t}, e^{2\pi\sqrt{-1}t}\frac{\partial}{\partial z}, e^{-2\pi\sqrt{-1}t}\frac{\partial}{\partial \bar{z}} \right\rangle$$

where $z = x + \sqrt{-1}y \in \mathbb{R}^2$. We take

$$\mathfrak{s} = \mathrm{span}\left\langle \frac{\partial}{\partial t} \right\rangle$$

as a semi-simple complement. Then the map Ψ is given by

$$\Psi = \mathrm{diag}(1, e^{2\pi\sqrt{-1}t}, e^{-2\pi\sqrt{-1}t}).$$

For the dual basis $dt, e^{-2\pi\sqrt{-1}t}dz, e^{2\pi\sqrt{-1}t}d\bar{z}$, we have

$$A^{\star}_{\Gamma} = \bigwedge \mathrm{span}\langle dt, dz, d\bar{z}\rangle$$

since the characters $e^{-2\pi\sqrt{-1}t}, e^{2\pi\sqrt{-1}t}$ are restricted to the trivial character 1 on the lattice Γ.

In fact, we have an isomorphism $\Gamma \cong \mathbb{Z}^3$ and hence $\Gamma\backslash G$ is a 3-torus. By standard computations on tori, we can also see that the DGA $\bigwedge \mathrm{span}\langle dt, dz, d\bar{z}\rangle$ computes the complex valued de Rham cohomology.

Example 2. Let $G = \mathbb{C} \ltimes_{\phi} \mathbb{C}^2$ such that

$$\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^{x+\sqrt{-1}y} & 0 \\ 0 & e^{-x-\sqrt{-1}y} \end{pmatrix}.$$

Then we have a lattice $\Gamma = (a\mathbb{Z} + 2\pi\sqrt{-1}) \ltimes \Gamma''$ for some $a \in \mathbb{R}^*$ and some lattice Γ'' in $\mathbb{C}^2$. For example, $a = \log(2 + \sqrt{3})$. Considering $\mathfrak{g}$ as a real 6-dimensional Lie algebra $\mathbb{C} \ltimes \mathbb{C}^2 = \mathbb{R}^2 \ltimes \mathbb{R}^4$, we have

$$\mathfrak{g} \otimes \mathbb{C} = \mathrm{span}\left\langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, e^{z_1}\frac{\partial}{\partial z_2}, e^{\bar{z}_1}\frac{\partial}{\partial \bar{z}_2}, e^{-z_1}\frac{\partial}{\partial z_3}, e^{-\bar{z}_1}\frac{\partial}{\partial \bar{z}_3} \right\rangle.$$

We take

$$\mathfrak{s} = \mathrm{span}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

as a semi-simple complement where $z_1 = x + \sqrt{-1}y$. Then the map Ψ is given by

$$\Psi = \mathrm{diag}(1, 1, e^{z_1}, e^{\bar{z}_1}, e^{-z_1}, e^{-\bar{z}_1}).$$

For the dual basis $dz_1, d\bar{z}_1, e^{-z_1}dz_2, e^{-\bar{z}_1}d\bar{z}_2, e^{z_1}dz_3, e^{\bar{z}_1}d\bar{z}_3$, we can write A^*_Γ. (Left as an exercise for the reader.) We remark that the character $e^{z_1-\bar{z}_1} = e^{2\pi\sqrt{-1}y}$ is restricted to 1 on the lattice Γ.

4 Some useful properties of A^*_Γ

We use the same setting as in Section 2.3.

4.1 Completely solvable case

Proposition 4.1. *If G is completely solvable (i.e. for any $g \in G$ all eigenvalues of Ad_g are real), then*

$$A^*_\Gamma \subset \bigwedge \mathfrak{g}^* \otimes \mathbb{C}.$$

Proof. In this case, the characters $\alpha_1, \dots, \alpha_n$ are real valued. Hence, for any $I \subset \{1, \dots, n\}$, if the restriction $(\alpha_I)|_\Gamma$ is trivial, then α_I is also trivial. Hence we have

$$A^*_\Gamma = \mathrm{span}\langle x_I | I \subset \{1, \dots, n\},\ \ \alpha_I = 1\rangle \subset \bigwedge \mathfrak{g}^* \otimes \mathbb{C}.$$

□

This implies that if G is completely solvable, then every de Rham cohomology class of $\Gamma\backslash G$ is represented by a left-invariant differential form. This statement is Hattori's theorem [5].

4.2 Harmonic forms

Since the representation $\Psi : G \to \mathrm{Aut}(\mathfrak{g})$ is real valued, we can choose $X_1, \dots, X_n$ so that for some k, l with $k + 2l = n$, $X_1, \dots X_k$ are real and $X_{k+i} = \overline{X_{k+l+i}}$ for each $1 \le i \le l$. Consider the left-invariant Riemannian metric g on $\Gamma\backslash G$ so that

$$g = \sum_{i=1}^{k} x_i \cdot x_i + \sum_{i=1}^{l} x_{k+i} \cdot x_{k+l+i}.$$

Proposition 4.2. *The space of the harmonic forms on $\Gamma\backslash G$ associated with the left-invariant Riemannian metric g is contained in the DGA A^*_Γ.*

Proof. For any complex character α of G, if the restriction of α on Γ is trivial, then α is unitary since $\Gamma\backslash G$ is compact. Thus, for the anti-linear Hodge star operator $\bar{\star}$, we have

$$\bar{\star}(\alpha_I x_I) = \bar{\alpha}_I x_{\hat{I}} = \alpha_I^{-1} x_{\hat{I}}$$

for $I \subset \{1, \dots, n\}$ with $(\alpha_I)|_\Gamma = 1$ where $\hat{I} = \{1, \dots, n\} - I$. It is known that any Lie group admitting a lattice is unimodular and so $\alpha_1 \cdots \alpha_n = 1$. Thus, $\alpha_I^{-1} = \alpha_{\hat{I}}$ for any $I \subset \{1, \dots, n\}$. Thus $\bar{\star}$ preserves A^*_Γ. By this, the proposition easily follows from Theorem 2.7. □

4.3 Nilpotence

For the semi-simple representation $\psi : \mathfrak{g} \to D(\mathfrak{g})$ as in Lemma 2.4, we define the new bracket $[,]_{\mathfrak{u}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ so that

$$[X, Y]_{\mathfrak{u}} = [X, Y] - \psi(X)Y + \psi(Y)X$$

for $X, Y \in \mathfrak{g}$. Then this is a Lie bracket and we denote by $\mathfrak{u}$ the new Lie algebra. It is known that $\mathfrak{u}$ is nilpotent and the Lie algebra structure on $\mathfrak{u}$ is independent of the choice of a semi-simple complement (see [3, Section III.2, III.3]). We call $\mathfrak{u}$ the *nilshadow* of $\mathfrak{g}$.

Proposition 4.3. *A^*_Γ is a sub-DGA of $\bigwedge \mathfrak{u}^* \otimes \mathbb{C}$.*

Proof. For the basis $X_1, \dots, X_n$, the Lie bracket $[,]_{\mathfrak{u}}$ on the complexification is given by

$$[X_i, X_j]_{\mathfrak{u}} = [X_i, X_j] - \alpha_j^{-1} X_i(\alpha_j)X_j + \alpha_i^{-1} X_j(\alpha_i)X_i.$$

Consider the subspace

$$\bar{\mathfrak{g}} = \mathrm{span}\langle \alpha_1^{-1}X_1, \dots, \alpha_n^{-1}X_n\rangle$$

of Lie algebra $\chi(G) \otimes \mathbb{C}$ of all complex valued vector fields on G. Then, we can say that $\bar{\mathfrak{g}}$ is a Lie subalgebra of $\chi(G) \otimes \mathbb{C}$. By the weight space decomposition $\mathfrak{g} \otimes \mathbb{C} = \bigoplus_\alpha \mathfrak{g}_\alpha$ for the representation $\Psi : G \to \mathrm{Aut}(\mathfrak{g})$, we can check this claim. In fact, we can easily check $[\alpha^{-1}\mathfrak{g}_\alpha, \beta^{-1}\mathfrak{g}_\beta] \subset \alpha^{-1}\beta^{-1}\mathfrak{g}_{\alpha\beta}$ and hence $\bar{\mathfrak{g}} = \bigoplus_\alpha \alpha^{-1}\mathfrak{g}_\alpha$ is closed under the Lie bracket.

By computing

$$[\alpha_i^{-1}X_i, \alpha_j^{-1}X_j] = \alpha_i^{-1}\alpha_j^{-1}\left([X_i, X_j] - \alpha_j^{-1}X_i(\alpha_j)X_j + \alpha_i^{-1}X_j(\alpha_i)X_i\right),$$

the correspondence $\bar{\mathfrak{g}} \ni \alpha_i^{-1}X_i \mapsto X_i \in \mathfrak{u} \otimes \mathbb{C}$ gives an isomorphism $\bar{\mathfrak{g}} \cong \mathfrak{u} \otimes \mathbb{C}$ by the above formula on the bracket $[,]_{\mathfrak{u}}$. Define the subspace

$$A^* = \mathrm{span}\langle \alpha_I x_I | I \subset \{1, \dots, n\}\rangle$$

of the complex valued de Rham complex $A^*(G)$ of the Lie group G. Then A^* is identified with the DGA $\bigwedge \bar{\mathfrak{g}}^*$ and so it is isomorphic to $\bigwedge \mathfrak{u}^* \otimes \mathbb{C}$. This implies the proposition. □

4.4 Averaging

A Lie group containing a lattice is unimodular (see [12]). Since G is unimodular and $\Gamma\backslash G$ is compact, there exists a bi-invariant Haar measure dV so that $\int_{\Gamma\backslash G} dV = 1$.

Proposition 4.4. *Define $\pi : A^*_{\mathbb{C}}(\Gamma\backslash G) \to A^*_\Gamma$ so that for $A^*_{\mathbb{C}}(\Gamma\backslash G) \ni \omega = \sum_I f_I x_I$,*

$$\pi(\omega) = \sum_{\alpha_I|_\Gamma = 1} \left(\int_{\Gamma\backslash G} \alpha_I^{-1} f_I dV \right) \alpha_I x_I.$$

Then, π is a cochain complex homomorphism and for the natural inclusion $\iota : A^*_\Gamma \to A^*_{\mathbb{C}}(\Gamma\backslash G)$, we have $\pi \circ \iota = \mathrm{id}$.

Proof. For a complex character α of G with $\alpha|_\Gamma = 1$, we have the sub-complex

$$\alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C}$$

of $A^*_{\mathbb{C}}(\Gamma\backslash G)$. Define the map $\tau_\alpha : A^*_{\mathbb{C}}(\Gamma\backslash G) \to \alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C}$ so that for $A^*_{\mathbb{C}}(\Gamma\backslash G) \ni \omega = \sum_I f_I x_I$,

$$\tau_\alpha(\omega) = \sum_I \left(\int_{\Gamma\backslash G} \alpha^{-1} f_I dV \right) \alpha x_I.$$

Then, for the inclusion $\iota : \alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C} \to A^*_{\mathbb{C}}(\Gamma\backslash G)$, we have $\tau_\alpha \circ \iota = \mathrm{id}$. It is known that for any function f on $\Gamma\backslash G$ and any left-invariant vector field X, we have $\int_{\Gamma\backslash G} X(f)dV = 0$. (see [1]). By this formula, we can easily check that τ_α is a cochain complex homomorphism.

Let Ξ be the set of characters α_I for all multi-indices $I \subset \{1, \dots, n\}$ satisfying $\alpha_I|_\Gamma = 1$. Then A^*_Γ is a sub-complex of $\bigoplus_{\alpha\in\Xi} \alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C}$. Moreover, we have

$$\bigoplus_{\alpha\in\Xi} \alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C} = \mathrm{span}\langle \alpha x_I | I \subset \{1, \dots, n\},\ \alpha \in \Xi \text{ where } \alpha \neq \alpha_I \rangle \oplus A^*_\Gamma$$

and this is a direct sum of cochain complexes. Thus the projection $p : \bigoplus_{\alpha\in\Xi} \alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C} \to A^*_\Gamma$ is a cochain complex homomorphism. For the map $\tau = \bigoplus \tau_\alpha : A^*_{\mathbb{C}}(\Gamma\backslash G) \to \bigoplus_{\alpha\in\Xi} \alpha \cdot \bigwedge \mathfrak{g}^* \otimes \mathbb{C}$, we have $\pi = p \circ \tau$ and so π is a cochain complex homomorphism so that $\pi \circ \iota = \mathrm{id}$.

□

5 Proof of Theorem 2.7

Let N be the connected normal subgroup of G which corresponds to the nilradical $\mathfrak{n}$. Then, it is known that the group $\Gamma_N = \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is a closed subgroup in G and $\Gamma N\backslash G$ is a torus (see [12]). By this, the solvmanifold $\Gamma\backslash G$ is a fiber bundle over the torus $\Gamma N\backslash G$ with the nilmanifold fiber $\Gamma_N\backslash N$. We call this fiber bundle structure the *Mostow fibration*. By this fiber bundle structure, we have the spectral sequence $E_r^{p,q}$ so that

$$E_2^{p,q} \cong H^p(\Gamma N\backslash G, H^q(\Gamma_N\backslash N, \mathbb{C}))$$

and it converges to $H^{p+q}(\Gamma\backslash G, \mathbb{C})$.

For a semi-simple complement $\mathfrak{s}$ of $\mathfrak{g}$, we define the filtration $F^p(\bigwedge \mathfrak{g}^* \otimes \mathbb{C})$ of $\bigwedge \mathfrak{g}^* \otimes \mathbb{C}$ so that

$$F^p(\bigwedge^r \mathfrak{g}) = \bigoplus_{i\geq p} \bigwedge^i \mathfrak{s} \otimes \bigwedge^{r-i} \mathfrak{n}.$$

Then, for $A^*_{\mathbb{C}}(\Gamma\backslash G) = \mathcal{C}^\infty(\Gamma\backslash G) \otimes \bigwedge \mathfrak{g}^* \otimes \mathbb{C}$, the filtration $F^p(A^*_{\mathbb{C}}(\Gamma\backslash G)) = \mathcal{C}^\infty(\Gamma\backslash G) \otimes F^p(\bigwedge \mathfrak{g}^*) \otimes \mathbb{C}$ induces the spectral sequence $E_r^{p,q}$. We consider the filtration $F^p(A^*_\Gamma)$ of the sub-DGA A^*_Γ of $A^*_{\mathbb{C}}(\Gamma\backslash G)$ by the restriction of $F^p(A^*_{\mathbb{C}}(\Gamma\backslash G))$. Then, we obtain the spectral sequence $\overline{E}_r^{p,q}$ which is induced by the filtration $F^p(A^*_\Gamma)$. By the standard fact on the spectral sequence, we can reduce Theorem 2.7 to proving the following statement.

Theorem 5.1. *The inclusion $A^*_\Gamma \subset A^*_{\mathbb{C}}(\Gamma\backslash G)$ induces an isomorphism*

$$\overline{E}_2^{p,q} \cong E_2^{p,q}.$$

By the map $\pi : A^*_{\mathbb{C}}(\Gamma\backslash G) \to A^*_\Gamma$ as in Proposition 4.4, we can easily show that the induced map $\overline{E}_2^{p,q} \to E_2^{p,q}$ is injective. Hence, it is sufficient to show that there exists an isomorphism (not necessarily canonical)

$$H^p(\Gamma N\backslash G, H^q(\Gamma_N\backslash N, \mathbb{C})) \cong \overline{E}_2^{p,q}.$$

We consider the sub-representation $\Psi_{\mathfrak{n}} : G \to \mathrm{Aut}(\mathfrak{n})$ of the representation $\Psi : G \to \mathrm{Aut}(\mathfrak{g})$. Take the weight space decomposition $\bigwedge^* \mathfrak{n} \otimes \mathbb{C} = \bigoplus W_\alpha$ for the dual of this representation. Then we have $W_\alpha \wedge W_\beta \subset W_{\alpha\beta}$ and $dW_\alpha \subset W_\alpha$ for any weight vector spaces W_α, W_β.

Lemma 5.2. *There exists an isomorphism*

$$H^p(\Gamma N\backslash G, H^q(\Gamma_N\backslash N, \mathbb{C})) \cong \bigoplus_{\alpha|_\Gamma=1} H^p(\mathfrak{g}/\mathfrak{n}, H^q(W_\alpha) \otimes E_{\alpha^{-1}})$$

where we regard the cohomology $H^q(W_\alpha)$ as $\mathfrak{g}/\mathfrak{n}$-module induced by the adjoint representation and we denote by $E_{\alpha^{-1}}$ the complex 1-dimensional module induced by a character α^{-1}.

Proof. By Nomizu's Theorem in [11], we have an isomorphism

$$H^\star(\Gamma_N\backslash N,\mathbb{C})\cong H^\star(\mathfrak{n}\otimes\mathbb{C}).$$

Hence, $H^p(\Gamma N\backslash G, H^q(\Gamma_N\backslash N,\mathbb{C}))$ is the cohomology of the torus $\Gamma N\backslash G$ with values in the local system associated with the action of $N\backslash G$ on $H^q(\mathfrak{n}\otimes\mathbb{C})$ via the adjoint representation. We can check that each $H^q(W_\alpha)$ is the generalized α-eigenspace of the action of $N\backslash G$ on $H^q(\mathfrak{n}\otimes\mathbb{C})$. By this, the lemma follows from the following proposition.

Proposition 5.3. *Let A be a simply connected Abelian Lie group with the Lie algebra $\mathfrak{a}$ and Λ be a lattice in A. Consider the torus $T=\Lambda\backslash A$. Let L be a complex local system over T. Then:*
- *If L contains no trivial local subsystem, then $H^\star(T,L)=0$ where we call a local system trivial if its monodromy representation is trivial.*
- *If L comes from the restriction of a complex unipotent representation $\rho:A\to GL(V)$, then we have an isomorphism $H^\star(\mathfrak{a},V)\cong H^\star(T,L)$.*

We can easily show this proposition by induction. We omit details.

□

Finally, the theorem follows from the following lemma.

Lemma 5.4. *There exists an isomorphism*

$$\overline{E}_2^{p,q}\cong\bigoplus_{\alpha|_\Gamma=1}H^p(\mathfrak{g}/\mathfrak{n},H^q(W_\alpha)\otimes E_{\alpha^{-1}}).$$

Proof. We can write $A_\Gamma^\star=\bigwedge\mathfrak{s}^\star\otimes(\bigoplus_{\alpha|_\Gamma=1}\alpha^{-1}W_\alpha)$. Thus, we have

$$F^p(A_\Gamma^\star)=\bigoplus_{i\geq p}\bigwedge^i\mathfrak{s}^\star\otimes(\bigoplus_{\alpha|_\Gamma=1}\alpha^{-1}W_\alpha).$$

The lemma follows from the straightforward computation of the second term of the spectral sequence. □

6 Dolbeault cohomology of complex parallelizable solvmanifolds

We give an extension of Sakane's theorem in [13].

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Then the arguments in Section 2.2 are extended for complex coefficients without any problem. We take a semi-simple complement $\mathfrak{s}$ and define the representation $\psi:\mathfrak{g}\to D(\mathfrak{g})$ as in Lemma 2.4. Let G be a simply connected solvable complex Lie group with the complex Lie algebra $\mathfrak{g}$. We have the holomorphic representation $\Psi:G\to\mathrm{Aut}(\mathfrak{g})$. Since Ψ is diagonalizable, we take a basis $X_1,\dots,X_n$ of $\mathfrak{g}$ so that

$$\Psi=\mathrm{diag}(\alpha_1,\dots,\alpha_n)$$

for holomorphic characters $\alpha_1,\dots,\alpha_n$ of G. Take the dual basis $x_1,\dots,x_n$ of $\mathfrak{g}^\star$.

We assume that G has a lattice Γ. Then we can consider $\mathfrak{g}^\star$ as the space of holomorphic 1-forms on the complex parallelizable solvmanifold $\Gamma\backslash G$. Let B_Γ^q be the subspace of $(0,q)$-forms $A^{0,q}(\Gamma\backslash G)$ on $\Gamma\backslash G$ defined as

$$B_\Gamma^q=\mathrm{span}\left\langle\frac{\bar\alpha_I}{\alpha_I}\bar x_I\,\middle|\,I\subset\{1,\dots,n\}\text{ with }|I|=q,\ \left(\frac{\bar\alpha_I}{\alpha_I}\right)_{|\Gamma}=1\right\rangle$$

where $\bar x_I$ is the complex conjugation of $x_I\in\bigwedge^q\mathfrak{g}^\star$. Then, we can easily show that $B_\Gamma^\star$ is a sub-DGA of $A^{0,\star}(\Gamma\backslash G)$ with the Dolbeault operator $\bar\partial$. We have:

Theorem 6.1. *The inclusion $B^{\star}_{\Gamma} \subset A^{0,\star}(\Gamma\backslash G)$ induces a cohomology isomorphism*

$$H^{\star}(B^{\star}_{\Gamma}) \cong H^{0,\star}(\Gamma\backslash G).$$

Hence we have an isomorphism

$$\bigwedge^{p} \mathfrak{g}^{\star} \otimes H^{q}(B^{\star}_{\Gamma}) \cong H^{p,q}(\Gamma\backslash G).$$

We can prove this theorem as similar to Theorem 2.7. We have the spectral sequence of the Dolbeault cohomology associated with the holomorphic Mostow fibration (see [4]). We must remark that the second term of such spectral sequence is the Dolbeault cohomology of a complex torus with values in holomorphic flat bundles. Details are left to the reader. See [2] for more general case.

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