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Pluriclosed flow on manifolds with globally generated bundles

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Abstract: We show global existence and convergence results for the pluriclosed flow on manifolds for which certain naturally associated tensor bundles are globally generated.

1 Introduction

Given (M^{2n}, J) a complex manifold, we say that a Hermitian metric g is pluriclosed if the associated Kähler form ω satisfies $\sqrt{-1}\partial\bar{\partial}\omega = 0$. For such metrics the author and Tian introduced [7] a parabolic flow generalizing the Kähler-Ricci flow (see §2.1 for definitions). Recently in [4] the author obtained global existence and convergence results for this flow and manifolds admitting special background metrics, for instance tori and manifolds with nonpositive bisectional curvature. In this short note we establish global existence and convergence results for this flow assuming conditions of a complex geometric nature as opposed to the differential geometric assumptions of metrics with certain curvature conditions. Thus these theorems are more natural from a complex geometry standpoint, and apply to a much wider class of manifolds. Moreover, our results have implications for the existence and moduli of generalized Kähler structures on these manifolds using the generalized Kähler-Ricci flow [9]. This note is a close companion to [4], and though we will review the most pertinent aspects, familiarty with that paper will help in reading this. Before stating our theorems we record several definitions.

Definition 1.1. Fix (M^{2n}, J) a complex manifold. Given g a Hermitian metric on M, by taking inverses and tensor products g defines a Hermitian metric on $(T_{1,0})^{\otimes p} \oplus (T^{\star}_{1,0})^{\otimes q}$. Then by restriction we obtain a natural metric on any subbundle $E \subset (T_{1,0})^{\otimes p} \oplus (T^{\star}_{1,0})^{\otimes q}$, which we will refer to as $F_E(g)$. We say that such a holomorphic subbundle E is

1. *covariant proper* if $E \subset (T_{1,0})^{\otimes p}$ for some $p \in \mathbb{N}$ and the natural map
$$F_E : \mathrm{Sym}^2(T^{\star}_{1,0}) \to \mathrm{Sym}^2(E^{\star})$$
is proper.
2. *covariant weakly proper* if $E \subset (T_{1,0})^{\otimes p}$ for some $p \in \mathbb{N}$, and if given a background metric h, the map
$$F_E : \mathrm{Sym}^2(T^{\star}_{1,0}) \cap \left\{ g \mid \frac{\det g}{\det h} \geq 1 \right\} \to \mathrm{Sym}^2(E^{\star})$$
is proper.
3. *contravariant proper* if $E \subset (T^{\star}_{1,0})^{\otimes p}$ for some $p \in \mathbb{N}$ and the natural map
$$F_E : \mathrm{Sym}^2(T_{1,0}) \to \mathrm{Sym}^2(E^{\star})$$
is proper.

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4. *globally generated* if it is generated by sections. That is, letting $H^0(M, E)$ denote the finite dimensional space of holomorphic sections of E, for all $p \in M$ the natural evaluation map

$$\mathrm{ev}_p : H^0(M, E) \to E_p$$

is surjective.

Remark 1.2. 1. An elementary interpretation of a bundle being proper is that an upper bound for the induced metric on the bundle implies an upper bound for the original metric. For weakly proper bundles the meaning is that an upper bound for the induced metric on the bundle combined with a lower bound for the determinant implies an upper bound for the metric.
2. The most basic examples of proper bundles are $T_{1,0}$, $T^*_{1,0}$. Other examples include $(T_{1,0})^{\otimes p}$, $(T^*_{1,0})^{\otimes p}$.
3. An example of a bundle which is weakly proper but not proper is $\Lambda^{n-1}(T_{1,0})$.
4. The question of when complex manifolds admit globally generated bundles has various relations to algebraic geometry. We direct the reader to [2] and the references therein for some further context.

Next we state our main theorems. They are stated in an overly general manner, but we supplement the discussion with concrete families of examples.

Theorem 1.3. *Let (M^{2n}, J) be a compact complex manifold.*
1. *Suppose M admits a contravariant globally generated proper bundle. Given g a pluriclosed metric the solution to pluriclosed flow exists on $[0, \tau^*)$ (see Definition 2.1 for the definition of τ^*).*
2. *Suppose M admits a contravariant globally generated weakly proper bundle. If $c_1^{BC}(M, J) = 0$ and $[\partial\omega] = 0 \in H^{2,1}$ then the solution exists on $[0, \infty)$ and converges exponentially as $t \to \infty$ to a Calabi-Yau metric.*

Remark 1.4. 1. The bundle $T^*_{1,0}$ is proper. If it is globally generated then the anticanonical bundle is also globally generated and it follows that the formal existence time $\tau^* = \infty$.
2. Kähler manifolds with globally generated cotangent bundle are quite abundant. For instance, any product of Riemann surfaces of positive genus yields a manifold with globally generated $T^*_{1,0}$. Moreover, having a globally generated cotangent bundle is inherited by complex subvarieties, so in particular subvarieties of tori have globally generated cotangent bundles. This includes large families of manifolds of general type.
3. The hypothesis $[\partial\omega] = 0$ is satisfied automatically in some circumstances, such as of course if $h^{2,1} = 0$, or if (M^{2n}, J) satisfies the $\partial\overline{\partial}$-lemma.

Theorem 1.5. *Let (M^{2n}, J) be a complex manifold with a covariant globally generated weakly proper bundle and $c_1^{BC}(M, J) = 0$. Given g a pluriclosed metric with $[\partial\omega] = 0 \in H^{2,1}$ the solution to pluriclosed flow with initial condition g exists on $[0, \infty)$ and converges exponentially as $t \to \infty$ to a Calabi-Yau metric.*

Remark 1.6. 1. The two cohomological hypotheses $c_1^{BC} = 0$ and $[\partial\omega] = 0 \in H^{2,1}$ are natural to impose if one expects convergence to Calabi-Yau. The Hopf surface $S^3 \times S^1$ with standard complex structure has $c_1 = 0$, but $c_1^{BC} \neq 0$ and $[\partial\omega] \neq 0$ for any pluriclosed metric.
2. The bundle $T^{1,0}$ is proper, and this bundle being globally generated is equivalent to (M^{2n}, J) being complex homogeneous, as follows from elementary arguments (see [1]). Theorem 1.5 can be used to rule out the existence of pluriclosed metrics on certain backgrounds as well. For instance, compact quotients of $SL(2, \mathbb{C})$ are parallelizable and hence $c_1^{BC} = 0$. Moreover, they satisfy $h^{2,1} = 0$ ([1] Corollary 8.2.3). It follows that these manifolds admit no pluriclosed metric, since Theorem 1.5 then yields a Kähler metric, which quotients of $SL(2, \mathbb{C})$ cannot support since the only Kähler parallelizable manifolds are tori. This particular statement can be obtained directly by averaging a putative SKT metric and then performing direct calculations using the Lie algebra structure of $SL(2, \mathbb{C})$. Nonetheless we include this example to illustrate the nonexistence principle.

Corollary 1.7. *Let (M^{2n}, I) be a complex manifold with either a covariant or contravariant globally generated weakly proper bundle and $c_1^{BC} = 0$. Suppose (M^{2n}, I, J, g) is a generalized Kähler structure with $[\partial\omega_I] = 0 \in H_I^{2,1}$. Then (M^{2n}, I, J, g) is deformable through generalized Kähler structures to a structure $(M^{2n}, I, J_\infty, g_\infty)$ such that g_∞ is Calabi-Yau.*

The central new observation in proving the above results is that there are very clean evolution equations for the square norm of a holomorphic section of a vector bundle along pluriclosed flow. The hypotheses on the bundle allow one to turn these favorable evolution equations into upper or lower bounds for the metric. Combining these with prior results yields full regularity of the flow. We make use of the Perelman functionals for pluriclosed flow discovered in [8] to obtain the convergence statements. In §2 we provide relevant background information on the pluriclosed flow and generalized Kähler-Ricci flow. In §3 we derive evolution equations for holomorphic sections of tensor bundles over M. We combine our estimates in §4 and give the proofs of the theorems.

2 Background

In this section we give a very brief introduction to relevant aspects of pluriclosed flow and generalized Kähler-Ricci flow. The reader should refer to [4], [7], and [8] for more detail.

2.1 Pluriclosed flow

In this subsection we record some elementary properties of the pluriclosed flow. First we express the flow equation using differential operators appearing in Hodge theory. In particular, on a complex manifold (M^{2n}, J), a one-parameter family of Hermitian metrics g_t is a solution of pluriclosed flow if the corresponding Kähler forms ω_t satisfy

$$\frac{\partial}{\partial t}\omega = \partial\partial_g^\star\omega + \overline{\partial\partial}_g^\star\omega + \sqrt{-1}\partial\overline{\partial}\log\det g. \tag{2.1}$$

As shown in [7], this is a strictly parabolic equation with pluriclosed initial condition ω_0, and admits short-time solutions on compact manifolds.

It is also useful to express this flow using the Chern connection. Given (M^{2n}, J, g) a Hermitian manifold, the Chern connection is the unique connection ∇ on $T_{1,0}$ such that $\nabla g \equiv 0$, $\nabla J \equiv 0$ and the torsion of ∇ has vanishing $(1,1)$ piece. This torsion can be expressed in complex coordinates as

$$T_{ij\bar{k}} = g_{l\bar{k}}\left[\Gamma^l_{ij} - \Gamma^l_{ji}\right] = g_{j\bar{k},i} - g_{i\bar{k},j}.$$

The metric is Kähler if and only if $T \equiv 0$. Due to the fact that ∇, in general, has torsion, there are various "Ricci curvatures" which can be defined using this connection. First, one has

$$S_{i\bar{j}} = g^{\bar{l}k}\Omega_{k\bar{l}i\bar{j}},$$

where Ω is the Chern curvature. We will also use the representative of the first Chern class with respect to the Chern connection, which we will denote by

$$\rho_{i\bar{j}} = g^{\bar{l}k}\Omega_{i\bar{j}k\bar{l}}.$$

We also define a certain quadratic expression in torsion, namely

$$Q_{i\bar{j}} = g^{\bar{l}k}g^{\bar{n}m}T_{ik\bar{n}}T_{\bar{j}\bar{l}m}.$$

With these definitions made, we can express the pluriclosed flow equation ([7] Proposition 3.3) as

$$\frac{\partial}{\partial t}g = -S + Q. \tag{2.2}$$

2.2 Formal existence time

Important in understanding the existence time of solutions to (2.1) (equivalently 2.2) is a formal cohomological obstruction. Observe that a pluriclosed metric defines a positive class in Aeppli cohomology. Using (2.1), it is direct to see that this class evolves along the pluriclosed flow via

$$[\omega_t] = [\omega_0] - tc_1.$$

This allows us to define a formal maximal smooth existence time (cf. [8])

Definition 2.1. Given (M^{2n}, J) a compact complex manifold, and g_0 a pluriclosed metric, let

$$\tau^\star := \sup\{t > 0 \mid [\omega_0] - tc_1 \text{ admits pluriclosed metrics}\}.$$

For times $\tau < \tau^\star$ we can define a reduction of the pluriclosed flow to a flow on a $(1,0)$ form on $[0,\tau]$. First, fix a background Hermitian metric h. Since $\tau < \tau^\star$, there exists $\mu \in \Lambda^{1,0}$ such that

$$\hat{\omega}_\tau := \omega_0 - \tau\rho(h) + \overline{\partial}\mu + \partial\overline{\mu} > 0. \tag{2.3}$$

Now consider the smooth one-parameter family of Kähler forms

$$\hat{\omega}_t := \frac{t}{\tau}\hat{\omega}_\tau + \frac{(\tau - t)}{\tau}\omega_0.$$

Definition 2.2. Let (M^{2n}, g_t, J) be a smooth solution to pluriclosed flow on $[0,\tau]$. Given choices $\hat{g}_t, h, \mu$ as above, for a one parameter family $\alpha_t \in \Lambda^{1,0}$ let

$$\omega_\alpha := \hat{\omega}_t + \overline{\partial}\alpha_t + \partial\overline{\alpha_t}.$$

We say that a one-parameter family $\alpha_t \in \Lambda^{1,0}$ is a solution to $(\hat{g}_t, h, \mu)$-*reduced pluriclosed flow* if

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \overline{\partial}^\star_{g_\alpha}\omega_\alpha - \frac{\sqrt{-1}}{2}\partial \log\frac{\det g_\alpha}{\det h} - \frac{\mu}{\tau}, \\ \alpha_0 &= 0. \end{aligned} \tag{2.4}$$

Remark 2.3. This reduction generalizes the reduction of Kähler-Ricci flow to the complex parabolic Monge Ampere equation (with additional background terms). In the special case which we frequently consider where $c_1^{BC} = 0$ and $[\partial\omega_0] = 0$, it follows that $\tau^\star = \infty$, and moreover if one chooses the background metric h to satisfy $\rho(h) = 0$, then the reduction can be chosen so that $\hat{\omega}_t = \omega_0$, $\mu = 0$. This is relevant to obtaining certain a priori estimates below.

2.3 Generalized Kähler-Ricci flow

We will briefly summarize the generalized Kähler-Ricci flow here, referring the reader to [4, 9] for further detail. To begin we introduce generalized Kähler structures, referring the reader to [3] for further background. A generalized Kähler manifold is a quadruple (M^{2n}, I, J, g) consisting of a Riemannian metric with two compatible integrable complex structures I, J, such that the corresponding Kähler forms satisfy

$$d^c_I\omega_I = H = -d^c_J\omega_J, \qquad dH = 0.$$

In particular, the metric g is pluriclosed with respect to two different complex structures. This observation combined with the connection between pluriclosed flow and renormalization group flows from [8] leads one to the definition of generalized Kähler-Ricci flow:

$$\begin{aligned} &\frac{\partial}{\partial t}g = -2\,\mathrm{Rc} + \frac{1}{2}H^2 \qquad \frac{\partial}{\partial t}H = \Delta_d H, \\ &\frac{\partial}{\partial t}I = L_{\theta_I^\sharp}I, \qquad \frac{\partial}{\partial t}I = L_{\theta_J^\sharp}J. \end{aligned}$$

Here θ_I is the Lee form with respect to the Hermitian structure (g, I). Interestingly, both complex structures evolve by time-dependent, but distinct, diffeomorphisms. It is possible to gauge-modify this flow to freeze one complex structure, but not both. Choosing to freeze I, the resulting metric evolves by pluriclosed flow on (M, I), and so our results on pluriclosed flow will have immediate applications to this flow. We will refer to this as considering the flow "in the I-fixed gauge."

2.4 Evolution equations and technical results

We begin by recording two evolution equations relevant to what follows.

Lemma 2.4. *([4] Lemma 6.1) Let (M^{2n}, J, g_t) be a solution to pluriclosed flow, and let h denote another Hermitian metric on (M, J). Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \frac{\det g}{\det h} = |T|^2 - \mathrm{tr}_g \rho(h).$$

Lemma 2.5. *([4] Proposition 4.9,4.10) Let (M^{2n}, J, g_t) be a solution to pluriclosed flow. Fix background data $\hat{g}_t, h, \mu$ and a solution α_t to (2.4). Then*

$$\frac{\partial}{\partial t} |\partial\alpha|^2_{g_\alpha} = \Delta_{g_\alpha} |\partial\alpha|^2 - |\nabla\partial\alpha|^2 - \left|\overline{\nabla}\partial\alpha\right|^2 - 2\left\langle Q, \mathrm{tr}\, \partial\alpha \otimes \overline{\partial}\overline{\alpha}\right\rangle - 2\Re\left\langle \mathrm{tr}_{g_\alpha} \nabla^{g_\alpha} T_{\hat{g}} + \partial\mu, \overline{\partial}\overline{\alpha}\right\rangle.$$

Suppose furthermore that $\mu = 0$ and

$$\partial\hat{\omega}_t = \partial\hat{\omega}_0 = \overline{\partial}\eta.$$

Let $\phi = \partial\alpha - \eta$. Then

$$\left(\frac{\partial}{\partial t} - \Delta_{g_t}\right) |\phi|^2 = -|\nabla\phi|^2 - |T_{g_t}|^2 - 2\left\langle Q, \phi \otimes \overline{\phi}\right\rangle.$$

Next we record some background theorems on regularity and the existence and rigidity of limit points for pluriclosed flow relevant to what follows. Corollary 2.9 summarizes the situation and is the main technical tool.

Theorem 2.6. *([4] Theorem 1.8) Let (M^{2n}, J) be a compact complex manifold. Suppose g_t is a solution to the pluriclosed flow on $[0, \tau)$, with α_t a solution to the $(\hat{g}_t, h, \mu)$-reduced flow. Assume there is a constant λ such that for all $t \in [0, \tau)$,*

$$\lambda g_0 \leq g_t.$$

There exists a constant $\Lambda = \Lambda(n, g_0, \hat{g}, h, \mu, \lambda)$ such that for all $t \in [0, \tau)$,

$$g_t \leq \Lambda(1 + t)g_0, \qquad |\partial\alpha|^2 \leq \Lambda.$$

Theorem 2.7. *([4] Theorem 1.7) Let (M^{2n}, J) be a compact complex manifold. Suppose g_t is a solution to the pluriclosed flow on $[0, \tau)$, $\tau \leq 1$, with α_t a solution to the $(\hat{g}_t, h, \mu)$-reduced flow as in (2.4). Suppose there exist constants λ, Λ such that*

$$\lambda g_0 \leq g_t \leq \Lambda g_0, \qquad |\partial\alpha|^2 \leq \Lambda. \tag{2.5}$$

Given $k \in \mathbb{N}$ there exists a constant $C = C(n, k, g_0, \hat{g}, h, \mu, \lambda, \Lambda)$ such that

$$\sup_{M\times\{t\}} t \sum_{j=0}^{k} \left|\nabla^j_g Y(g, h)\right|^{\frac{2}{1+j}} \leq C,$$

where $Y(g, h) = \nabla^g - \nabla^h$ is the difference of the Chern connections associated to g and h.

Lemma 2.8. *([4] Lemma 6.3) Let (M^{2n}, J, h) be a compact Hermitian manifold with $\rho(h) \leq 0$. Suppose g is a pluriclosed metric which is a steady gradient soliton. Then g is a Calabi-Yau metric.*

Corollary 2.9. *Let (M^{2n}, J, h) be a compact Hermitian manifold with $\rho(h) \leq 0$. Suppose g_t is a solution to pluriclosed flow on $[0, \infty)$ satisfying*

$$C^{-1}h \leq g \leq Ch, \qquad \left|\nabla_h^k Y(g,h)\right|^2 \leq C,$$

where $Y(g,h) = \nabla^g - \nabla^h$ is the difference of the Chern connections associated to g and h. Then g_t converges exponentially to a Calabi-Yau metric.

Proof. This argument is implicit in the proof of ([4] Theorem 1.1), though not stated explicitly and so we repeat it for convenience. With the assumed uniform estimates, any sequence of times $t_j \to \infty$ admits a smooth subsequential limiting metric on the same complex manifold. Moreover, the assumed uniform estimates imply that the Perelman-type $\mathcal{F}$ functional for the pluriclosed flow ([8] Theorem 1.1) has a uniform upper bound for all times. It follows from a standard argument that any subsequential limit as described above is a pluriclosed steady soliton, and hence by Lemma 2.8 Calabi-Yau. It now follows from the linear/dynamic stability result of ([6] Theorem 1.2) that the whole flow converges exponentially to g_∞, as required. □

3 Evolution of holomorphic sections

Lemma 3.1. *([5] Lemma 4.7) Let (M^{2n}, J, g_t) be a solution to pluriclosed flow, and suppose $\beta_t, \mu_t \in (T^\star_{1,0})^{\otimes p}$ are one-parameter families satisfying*

$$\frac{\partial}{\partial t}\beta = \Delta_{g_t}\beta + \mu. \tag{3.1}$$

Then

$$\frac{\partial}{\partial t}|\beta|^2 = \Delta|\beta|^2 - |\nabla\beta|^2 - \left|\overline{\nabla}\beta\right|^2 - p\left\langle Q, \mathrm{tr}_g\left(\beta\otimes\overline{\beta}\right)\right\rangle + 2\Re\langle\beta,\mu\rangle.$$

Remark 3.2. Lemma 4.7 of [5] is stated only for $\beta \in \Lambda^{p,0}$ but the proof easily applies to this more general case.

Corollary 3.3. *Let (M^{2n}, J, g_t) be a solution to pluriclosed flow, and suppose $\beta \in (T^\star_{1,0})^{\otimes p}$ is holomorphic. Then*

$$\frac{\partial}{\partial t}|\beta|^2 = \Delta|\beta|^2 - |\nabla\beta|^2 - p\left\langle Q, \mathrm{tr}_g\left(\beta\otimes\overline{\beta}\right)\right\rangle. \tag{3.2}$$

Proof. If β is holomorphic then $\beta_t = \beta$ is a solution of (3.3) with $\mu = 0$, and so from Lemma 3.1 we conclude

$$\begin{aligned}\frac{\partial}{\partial t}|\beta|^2 &= \Delta|\beta|^2 - |\nabla\beta|^2 - \left|\overline{\nabla}\beta\right|^2 - p\left\langle Q, \mathrm{tr}_g\left(\beta\otimes\overline{\beta}\right)\right\rangle \\ &= \Delta|\beta|^2 - |\nabla\beta|^2 - p\left\langle Q, \mathrm{tr}_g\left(\beta\otimes\overline{\beta}\right)\right\rangle,\end{aligned}$$

as required. □

Lemma 3.4. *Let (M^{2n}, J, g_t) be a solution to pluriclosed flow, and suppose $A_t, B_t \in T^{p,0}$ are one-parameter families satisfying*

$$\frac{\partial}{\partial t}A = \Delta_{g_t}A + B. \tag{3.3}$$

Then

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - |\nabla A|^2 - \left|\overline{\nabla}A\right|^2 + p\left\langle Q, \mathrm{tr}_g\left(A\otimes\overline{A}\right)\right\rangle + 2\Re\langle A, B\rangle.$$

Proof. By direct computation we have

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}\left(g_{i_1\bar{j}_1}\cdots g_{i_p\bar{j}_p}A^{i_1\ldots i_p}\overline{A}^{\bar{j}_1\ldots\bar{j}_p}\right)\\ &= p(-S+Q)_{i_1\bar{j}_1}g_{i_2\bar{j}_2}\cdots g_{i_p\bar{j}_p}A^{i_1\ldots i_p}\overline{A}^{\bar{j}_1\ldots\bar{j}_p}+\left\langle\Delta A,\overline{A}\right\rangle+\left\langle A,\overline{\Delta A}\right\rangle+2\Re\left\langle A,B\right\rangle\\ &= p\left\langle -S+Q,\mathrm{tr}_g\left(A\otimes\overline{A}\right)\right\rangle+\left\langle\Delta A,\overline{A}\right\rangle+\left\langle A,\overline{\Delta A}\right\rangle+2\Re\left\langle A,B\right\rangle.\end{aligned}$$

Next we observe the commutation formula

$$\begin{aligned}\overline{\Delta A}^{\bar{j}_1\ldots\bar{j}_p} &= g^{\bar{k}l}\nabla_{\bar{k}}\nabla_l\overline{A}^{\bar{j}_1\ldots\bar{j}_p}\\ &= g^{\bar{k}l}\nabla_l\nabla_{\bar{k}}\overline{A}_{\bar{j}_1\ldots\bar{j}_p}+\sum_{r=1}^{p}g^{\bar{k}l}\Omega^{\bar{j}_r}_{\bar{k}l\overline{m}}\overline{A}^{\bar{j}_1\ldots\bar{j}_{r-1}\overline{m}\bar{j}_{r+1}\ldots\bar{j}_p}\\ &= \Delta\overline{A}_{\bar{j}_1\ldots\bar{j}_p}+\sum_{r=1}^{p}S^{\bar{j}_r}_{\overline{m}}\overline{A}^{\bar{j}_1\ldots\bar{j}_{r-1}\overline{m}\bar{j}_{r+1}\ldots\bar{j}_p}.\end{aligned}$$

It follows that

$$\begin{aligned}\left\langle A,\overline{\Delta A}\right\rangle &= g_{i_1\bar{j}_1}\cdots g_{i_p\bar{j}_p}A^{i_1\ldots i_p}\overline{\Delta A}^{\bar{j}_1\ldots\bar{j}_p}\\ &= g_{i_1\bar{j}_1}\cdots g_{i_p\bar{j}_p}A^{i_1\ldots i_p}\left[\Delta\overline{A}^{\bar{j}_1\ldots\bar{j}_p}+\sum_{r=1}^{p}S^{\bar{j}_r}_{\overline{m}}A^{\bar{j}_1\ldots\bar{j}_{r-1}\overline{m}\bar{j}_{r+1}\ldots\bar{j}_p}\right]\\ &= \left\langle A,\Delta\overline{A}\right\rangle+p\left\langle S,\mathrm{tr}_g A\otimes\overline{A}\right\rangle.\end{aligned}$$

Lastly observe the identity

$$\Delta|A|^2=\left\langle\Delta A,\overline{A}\right\rangle+\left\langle A,\Delta\overline{A}\right\rangle+|\nabla A|^2+\left|\overline{\nabla}A\right|^2.$$

Combining the above calculations yields the lemma. □

Corollary 3.5. *Let (M^{2n},J,g_t) be a solution to pluriclosed flow, and suppose $A\in T^{p,0}$ is holomorphic. Then*

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2-|\nabla A|^2+p\left\langle Q,\mathrm{tr}_g\left(A\otimes\overline{A}\right)\right\rangle,\\ \frac{\partial}{\partial t}\log|A|^2 &\le \Delta\log|A|^2+p|T|^2.\end{aligned}\tag{3.4}$$

Proof. If A is holomorphic then $A_t=A$ is a solution of (3.3) with $B=0$, and so from Lemma 3.1 we conclude

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2-|\nabla A|^2-\left|\overline{\nabla}A\right|^2+p\left\langle Q,\mathrm{tr}_g\left(A\otimes\overline{A}\right)\right\rangle\\ &= \Delta|A|^2-|\nabla A|^2+p\left\langle Q,\mathrm{tr}_g\left(A\otimes\overline{A}\right)\right\rangle,\end{aligned}$$

as required. A further elementary calculation yields that

$$\frac{\partial}{\partial t}\log|A|^2=\Delta\log|A|^2+\frac{\left|\nabla|A|^2\right|^2}{|A|^2}-|\nabla A|^2+p\frac{\left\langle Q,\mathrm{tr}_g(A\otimes\overline{A})\right\rangle}{|A|^2}.$$

Since A is holomorphic it follows from Kato's inequality that

$$\left|\nabla|A|^2\right|^2\le|\nabla A|^2|A|^2.$$

Also, by the Cauchy-Schwarz inequality it follows that

$$\left\langle Q,\mathrm{tr}_g(A\otimes\overline{A})\right\rangle\le|Q||A|^2\le|T|^2|A|^2.$$

The corollary follows. □

4 Proofs of Theorems

Proof of Theorem 1.3. Let (M^{2n}, J) be a compact complex manifold and let E denote a contravariant globally generated proper bundle. We claim that, given a background Hermitian metric h, one has

$$g_t \geq C^{-1}h, \tag{4.1}$$

for some uniform constant C and any smooth existence time t. First we observe that since M is compact, the space of holomorphic sections of E is finite dimensional, and we choose a basis $\{\sigma_i\}$. Declaring $\sigma_i(x,t) = \sigma_i(x)$ one has that

$$\frac{\partial}{\partial t}\sigma_i = \Delta\sigma_i.$$

It follows directly from the maximum principle applied to the result of Corollary 3.3 that

$$\sup_{M\times\{t\}} |\sigma_i|^2_{g_t} \leq \sup_{M\times\{0\}} |\sigma_i|^2_{g_0} \leq C.$$

Since the σ_i form a finite spanning set at each point p, it follows that the induced metric on E is bounded above. Since E is proper this implies that the metric on T^*, i.e. g^{-1}, is bounded above. Thus the claim of (4.1) follows. The statement of existence on $[0, \tau^*)$ follows directly from Theorem 2.6 and Theorem 2.7.

Now we establish the statement of convergence. Let h denote a background Hermitian metric for which $\rho(h) = 0$. Combining Lemmas 2.4 and 2.5 we obtain that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left[\log\frac{\det g}{\det h} + |\partial\alpha|^2\right] \leq 0.$$

It follows from the maximum principle that

$$g_t \leq Ch, \qquad |\partial\alpha|^2 \leq C.$$

It now follows directly from Theorem 2.7 and Corollary 2.9 that the flow exists smoothly for all time and converges to a Calabi-Yau metric, as claimed. □

Proof of Theorem 1.5. Let (M^{2n}, J) be a compact complex manifold with $c_1^{BC} = 0$, and let E denote a covariant globally generated weakly proper bundle. We claim that, given a background Hermitian metric h, one has

$$g_t \leq Ch, \tag{4.2}$$

for some uniform constant C and any smooth existence time t. As above, since M is compact, the space of holomorphic sections of E is finite dimensional, and we choose a basis $\{\sigma_i\}$. Declaring $\sigma_i(x,t) = \sigma_i(x)$ one has that

$$\frac{\partial}{\partial t}\sigma_i = \Delta\sigma_i.$$

Since $c_1^{BC} = 0$ and $[\partial\omega_0] = 0$ we can choose η and ϕ as in Lemma 2.5 so that

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\phi|^2 \leq -|T|^2.$$

Now define

$$\Phi = \log|\sigma_i|^2 + p\,|\phi|^2$$

It follows from Lemma 2.5 and Corollary 3.5 that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi \leq 0.$$

Note that we can still apply the maximum principle to Φ at maximum points even though it approaches $-\infty$ at the vanishing locus of σ_i. It follows that

$$\sup_{M\times\{t\}} |\sigma_i|^2_{g_t} \leq \sup_{M\times\{0\}} |\sigma_i|^2_{g_0} \leq C.$$

Since the σ_i form a finite spanning set at each point p, it follows that the induced metric on E is bounded above. Since $c_1^{BC} = 0$ we choose a Hermitian metric h such that $\rho(h) = 0$. It follows from Lemma 2.4 that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \frac{\det g}{\det h} = |T|^2 \geq 0.$$

It follows from the maximum principle that

$$\inf_{M\times\{t\}} \frac{\det g_t}{\det h} \geq \inf_{M\times\{0\}} \frac{\det g_0}{\det h} \geq C^{-1}.$$

As we have established an upper bound for the induced metric on E and a lower bound on the volume form, since the bundle E is weakly proper it now follows that the metric g_t is bounded above. But again since the volume form is bounded below it follows that g_t is bounded below as well. It follows directly from Theorem 2.7 and 2.9 that the flow exists smoothly for all time and converges to a Calabi-Yau metric. □

Proof of Corollary 1.7. Let (M^{2n}, I, J_t, g_t) be the solution to generalized Kähler-Ricci flow in the I-fixed gauge, as explained in §2.3. This means that (M, I, g_t) is a solution to pluriclosed flow. In either case of the Corollary, using Theorem 1.3 or 1.5 we obtain the long time existence and exponential convergence of the flow to a Calabi-Yau manifold. In particular, the torsion is decaying exponentially to zero, and so the vector field defining the diffeomorphisms ϕ_t such that $J_t = \phi_t^* J$ are converging exponentially fast to a limiting diffeomorphism ϕ_∞, and hence J_t is converging to a limit J_∞. The corollary follows. □

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