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Hodge numbers and invariant complex structures of compact nilmanifolds

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Abstract: In this paper, we consider several invariant complex structures on a compact real nilmanifold, and we study relations between invariant complex structures and Hodge numbers.

Keywords: nilmanifold, Dolbeault cohomology group, complex structure

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1 Introduction

Let M be a compact Kählerian manifold, and $h^{p,q} = h^{p,q}(M)$ a Hodge number of M. Then, M satisfies $h^{p,q} = h^{q,p}$ for each p, q by the Hodge theory. In general, a compact complex manifold does not satisfy the relations. In the paper [10], we construct compact 4-dimensional complex manifolds M_1 and M_2 which satisfy that M_1 and M_2 are diffeomorphic, and $h^{p,q}(M_1) = h^{q,p}(M_2)$ for each p, q, namely, we consider two invariant complex structures on a compact real nilmanifold. In this paper, we consider several invariant complex structures on a compact real nilmanifold, and we study properties of Hodge numbers.

It is now well-known that if a compact nilmanifold $\Gamma\backslash N$ admits a Kähler structure, then N is an abelian group and $\Gamma\backslash N$ is a torus, where N is a simply connected real nilpotent Lie group, and Γ is a lattice in N ([1], [5]). If a compact complex parallelizable nilmanifold $\Gamma\backslash N$ admits a pseudo-Kähler structure, then N is an abelian group by a result of Dolbeault cohomology groups of compact complex parallelizable nilmanifolds ([4, Theorem 3.2], [6, Theorem 3], [9]). Thus, it is important to study properties of Hodge numbers of non-toral compact complex nilmanifolds. As an application of main theorems (Theorems 4.2 and 4.4), we have

Theorem 1.1. *Let $H_{\mathbb{R}}(n)$ be a $(2n+1)$-dimensional real Heisenberg group, and $\mathfrak{h}_{\mathbb{R}}(n)$ its Lie algebra. Let $\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}}$ be the complexification of $\mathfrak{h}_{\mathbb{R}}(n)$, and ${}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}})$ a real Lie algebra obtained from $\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}}$ by scalar restriction. Moreover, let ${}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}})$ be the simply connected nilpotent Lie group corresponding to ${}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}})$, and Γ a lattice in ${}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}})$. Then there exist invariant complex structures $\tilde{J}_0, \dots, \tilde{J}_n$ on $\Gamma\backslash{}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}})$ which satisfy*
(1) *If $k \neq h$, then $(\Gamma\backslash{}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_k)$ and $(\Gamma\backslash{}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_h)$ are not biholomorphic.*
(2) *$\sum_{p+q=r} h^{p,q}(\Gamma\backslash H(n;k)) = \sum_{p+q=r} h^{p,q}(\Gamma\backslash H(n;n-k))$ for each r, where $\Gamma\backslash H(n;h) = (\Gamma\backslash{}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_h)$ for each h.*

2 Preliminaries

Let H be a Lie group, and $\mathfrak{h}$ its Lie algebra. We denote by $H^*(\mathfrak{h}) = H^*(\mathfrak{h}, \mathbb{C})$ the cohomology of the complex $\wedge^*(\mathfrak{h}^*)^{\mathbb{C}}$ of left-invariant differential forms on the Lie group H. Note that if $\omega \in (\mathfrak{h}^{\mathbb{C}})^*$, and $X, Y \in \mathfrak{h}^{\mathbb{C}}$, then

$$d\omega(X, Y) = -\omega([X, Y]).$$

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Let $\mathfrak{g} = (\mathfrak{g}, J)$ be a Lie algebra with a complex structure, and $\mathfrak{g}_J^{\pm}$ the vector spaces of the $\pm\sqrt{-1}$ eigenvectors of the complex structure J, respectively. We denote by $H^{*,*}_{\bar\partial_J}(\mathfrak{g}^{\mathbb{C}})$ the cohomology ring of the differential bigraded algebra $\bigwedge^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$, associated to $\mathfrak{g}^{\mathbb{C}}$ with respect to the operator $\bar\partial_J$ in the canonical decomposition $d = \partial_J + \bar\partial_J$ on $\bigwedge^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$. We write $h^{p,q}(\mathfrak{g}_J) = \dim H^{p,q}_{\bar\partial_J}(\mathfrak{g}^{\mathbb{C}})$. Let $\omega_1, \dots, \omega_n$ be a basis of $(\mathfrak{g}_J^+)^*$. Since

$$(\bigwedge{}^{*,*}\mathfrak{g}^{\mathbb{C}}, \bar\partial_J) = (\bigwedge{}^{*,*}\langle \omega_1, \dots, \omega_n, \bar\omega_1, \dots, \bar\omega_n\rangle, \bar\partial_J),$$

it is important to investigate relations of Lie brackets and the operator $\bar\partial_J$. Note that also

$$(\bigwedge{}^{0,*}\mathfrak{g}^{\mathbb{C}}, \bar\partial_J) = (\bigwedge{}^{*}\langle \bar\omega_1, \dots, \bar\omega_n\rangle, \bar\partial_J).$$

For an arbitrary $X \in \mathfrak{g}^{\mathbb{C}}$, set

$$X_J^+ = (X - \sqrt{-1}JX)/2, \qquad X_J^- = (X + \sqrt{-1}JX)/2,$$

so that X_J^+ is holomorphic type and X_J^- is antiholomorphic type with respect to J. From now on, when there exist no possibilities of confusion, we omit the subscript J.

Lemma 2.1. *Let $\omega \in (\mathfrak{g}^+)^*$, $X^+, Y^+ \in \mathfrak{g}^+$, and $X^-, Y^- \in \mathfrak{g}^-$. Then,*
1. $(\bar\partial\omega)(X^+, Y^-) = -\omega([X^+, Y^-]^+)$,
2. $(\bar\partial\bar\omega)(X^-, Y^-) = -\bar\omega([X^-, Y^-])$.

Proof. Let $\eta \in (\mathfrak{g}^{\mathbb{C}})^*$, and $X, Y \in \mathfrak{g}^{\mathbb{C}}$. Since

$$(d\eta)(X, Y) = (\bar\partial\eta)(X, Y) + (\partial\eta)(X, Y) = -\eta([X, Y]),$$

we have

$$(\bar\partial\omega)(X^+, Y^-) = -\omega([X^+, Y^-]), (\bar\partial\bar\omega)(X^-, Y^-) = -\bar\omega([X^-, Y^-]).$$

□

Assume that

$$\bar\partial\omega_k = \sum_{i,j=1}^n c_{ij}^k \omega_i \wedge \bar\omega_j,\ \bar\partial\bar\omega_k = \sum_{i,j=1}^n d_{ij}^k \bar\omega_i \wedge \bar\omega_j,$$

for $k = 1, \dots, n$, where $c_{ij}^k, d_{ij}^k \in \mathbb{R}$. Let us consider a $2n$-dimensional Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}^*$ has a basis $\eta_1, \dots, \eta_{2n}$ which satisfies

$$d\eta_k = \sum_{i,j=1}^n c_{ij}^k \eta_i \wedge \eta_{n+j},\ d\eta_{n+k} = \sum_{i,j=1}^n d_{ij}^k \eta_{n+i} \wedge \eta_{n+j}\ (k = 1, \dots, n).$$

Let F be a homomorphism

$$F : \bigoplus_r (\bigoplus_{p+q=r} \bigwedge{}^{p,q}(\mathfrak{g}^{\mathbb{C}})^*) \longrightarrow \bigoplus_r \bigwedge{}^r(\mathfrak{h}^*)^{\mathbb{C}}$$

induced by a linear isomorphism $(\mathfrak{g}^{\mathbb{C}})^* \longrightarrow (\mathfrak{h}^*)^{\mathbb{C}}$ defined by $\omega_k \mapsto \eta_k$, $\bar\omega_k \mapsto \eta_{n+k}$ $(k = 1, \dots, n)$. Then, F is an isomorphism of differential graded algebras from $(\bigoplus_r(\bigoplus_{p+q=r} \bigwedge^{p,q}(\mathfrak{g}^{\mathbb{C}})^*), \bar\partial_J)$ to $(\bigoplus_r \bigwedge^r(\mathfrak{h}^*)^{\mathbb{C}}, d)$ by Lemma 2.1. We use this fact in the proof of Theorem 4.4.

Let N be a simply connected real nilpotent Lie group whose Lie algebra $\mathfrak{n}$ has a rational Lie subalgebra $\mathfrak{n}_{\mathbb{Q}}$ such that $\mathfrak{n} \cong \mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$, and Γ a lattice in N. A complex structure J on $\mathfrak{n}$ is called *rational* if $J(\mathfrak{n}_{\mathbb{Q}}) \subset \mathfrak{n}_{\mathbb{Q}}$ ([2]).

Theorem 2.2 ([2]). *Let N be a simply connected nilpotent Lie group with a rational complex structure J. Then,*

$$H^{p,q}_{\bar\partial}(\Gamma\backslash N) \cong H^{p,q}_{\bar\partial}(\mathfrak{n}^{\mathbb{C}})$$

for each p, q.

3 Complex structures on nilpotent Lie groups

In this section, we consider invariant complex structures on nilpotent Lie groups.

We consider the following Lie algebra $\mathfrak{g}$ over $\mathbb{R}$:

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b},$$

where $\mathfrak{a}$ is Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Take bases of the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$:

$$\mathfrak{a} = \operatorname{span}_{\mathbb{R}}\{U_1^1, \dots, U_p^1\},$$
$$\mathfrak{b} = \operatorname{span}_{\mathbb{R}}\{V_1^1, \dots, V_q^1\}.$$

Consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Since $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ has the following basis:

$$\{U_1^1, \dots, U_p^1, V_1^1, \dots, V_q^1, U_1^2, \dots, U_p^2, V_1^2, \dots, V_q^2\},$$

where $U_i^2 = \sqrt{-1}U_i^1$, $V_j^2 = \sqrt{-1}V_j^1$.

Let J be the complex structure on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ defined by

$$JU_i^1 = U_i^2 \ (JU_i^2 = -U_i^1), \ JV_j^1 = V_j^2 \ (JV_j^2 = -V_j^1)$$

for each i, j. Note that $({}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}), J)$ is a complex Lie algebra.

We define other complex structure $\tilde{J}$ on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by

$$\tilde{J}U_i^1 = -U_i^2 \ (\tilde{J}U_i^2 = U_i^1), \ \tilde{J}V_j^1 = V_j^2 \ (\tilde{J}V_j^2 = -V_j^1)$$

for each i, j.

Let ${}_{\mathbb{R}}(G^{\mathbb{C}})$ be the simply connected real Lie group corresponding to ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Then, we have the following proposition:

Proposition 3.1 ([10]). *$\tilde{J}$ is integrable on ${}_{\mathbb{R}}(G^{\mathbb{C}})$. If J is a rational complex structure, then $\tilde{J}$ is also a rational complex structure.*

Example 3.2. Let $H_{\mathbb{R}}(n)$ be a $(2n+1)$-dimensional real Heisenberg group and $\mathfrak{h}_{\mathbb{R}}(n)$ its Lie algebra. Then $\mathfrak{h}_{\mathbb{R}}(n)$ has a basis $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ satisfying $[X_i, Y_i] = Z$ $(i = 1, \dots, n)$ with other brackets vanishing. Consider the following Lie subalgebras of $\mathfrak{h}_{\mathbb{R}}(n)$:

$$\mathfrak{a}_k = \operatorname{span}\{X_1, \dots, X_k\}$$
$$\mathfrak{b}_k = \operatorname{span}\{X_{k+1}, \dots, X_n, Y_1, \dots, Y_n, Z\}$$

for each $0 \le k \le n$. Then, $\mathfrak{b}_k$ is an ideal of $\mathfrak{h}_{\mathbb{R}}(n)$. Moreover, $\mathfrak{a}_k$ and $\mathfrak{b}_k$ satisfies $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$. Hence, we have a rational complex structure $\tilde{J}_k$ corresponding to the decomposition $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$. For example, in the case of $n = 1$ we have

$$({}_{\mathbb{R}}(H_{\mathbb{R}}(1)^{\mathbb{C}}), \tilde{J}_1) = \left\{ \left. \begin{pmatrix} 1 & \bar{x} & 0 & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & \bar{x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{C} \right\} \cong \left\{ \left. \begin{pmatrix} 1 & \bar{x} & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{C} \right\}$$
$$\cong \left\{ \left. \begin{pmatrix} 1 & \bar{x} & 0 & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{C} \right\}.$$

See also section 8. We denote by $\Gamma\backslash H(n;k)$ is a compact complex manifold $(\Gamma\backslash {}_{\mathbb{R}}(H(n)^{\mathbb{C}}), \tilde{J}_k)$, where Γ is a lattice in ${}_{\mathbb{R}}(H(n)^{\mathbb{C}})$. Then, by a result of Console-Fino (Theorem 2.2), we have

$$h^{p,q}(\Gamma\backslash H(n;k)) = h^{p,q}(({}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_k)^{\mathbb{C}})$$

for each p, q.

4 Invariant complex structures and Hodge numbers of compact nilmanifolds

In this section, we assume that $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{b}$, J, and $\tilde{J}$ are as in section 3. Let $J' = J$ or $\tilde{J}$. Then, we define

$$\mathfrak{a}^{\pm}_{J'} = \mathfrak{g}^{\pm}_{J'} \cap \mathfrak{a}^{\mathbb{C}}, \quad \mathfrak{b}^{\pm}_{J'} = \mathfrak{g}^{\pm}_{J'} \cap \mathfrak{b}^{\mathbb{C}}.$$

Note that $\mathfrak{a}^{\pm}_{J'}$, $\mathfrak{b}^{\pm}_{J'}$ are the vector spaces of the $\pm\sqrt{-1}$ eigenvectors of complex structure J', respectively.

Lemma 4.1.

$$\begin{aligned} &[\mathfrak{a}^+_{\tilde{J}}, \mathfrak{a}^+_{\tilde{J}}] \subset \mathfrak{a}^+_{\tilde{J}}, \quad [\mathfrak{b}^+_{\tilde{J}}, \mathfrak{b}^+_{\tilde{J}}] \subset \mathfrak{b}^+_{\tilde{J}}, \quad [\mathfrak{a}^+_{\tilde{J}}, \mathfrak{b}^+_{\tilde{J}}] = 0, \\ &[\mathfrak{a}^+_{\tilde{J}}, \mathfrak{a}^-_{\tilde{J}}] = 0, \quad [\mathfrak{b}^+_{\tilde{J}}, \mathfrak{b}^-_{\tilde{J}}] = 0, \quad [\mathfrak{a}^+_{\tilde{J}}, \mathfrak{b}^-_{\tilde{J}}] \subset \mathfrak{b}^-_{\tilde{J}}, \quad [\mathfrak{b}^+_{\tilde{J}}, \mathfrak{a}^-_{\tilde{J}}] \subset \mathfrak{b}^+_{\tilde{J}}, \\ &[\mathfrak{a}^-_{\tilde{J}}, \mathfrak{a}^-_{\tilde{J}}] \subset \mathfrak{a}^-_{\tilde{J}}, \quad [\mathfrak{b}^-_{\tilde{J}}, \mathfrak{b}^-_{\tilde{J}}] \subset \mathfrak{b}^-_{\tilde{J}}, \quad [\mathfrak{a}^-_{\tilde{J}}, \mathfrak{b}^-_{\tilde{J}}] = 0. \end{aligned}$$

Proof. Since $\mathfrak{a}^{\pm}_{\tilde{J}} = \mathfrak{a}^{\mp}_{J}$, $\mathfrak{b}^{\pm}_{\tilde{J}} = \mathfrak{b}^{\pm}_{J}$, and $[\mathfrak{g}^+_J, \mathfrak{g}^-_J] = 0$,

$$[\mathfrak{a}^+_{\tilde{J}}, \mathfrak{b}^+_{\tilde{J}}] = [\mathfrak{a}^-_J, \mathfrak{b}^+_J] = 0.$$

The other cases are similar, and hence we omit proof of the other cases. □

Theorem 4.2. *For each q,*

$$h^{0,q}(\mathfrak{g}_{\tilde{J}}) = \dim H^q(\mathfrak{a} \times \mathfrak{b}).$$

Proof. By Lemma 4.1,

$$\bigwedge{}^{0,*}(\mathfrak{g}^{\mathbb{C}}_{\tilde{J}})^* = \bigwedge{}^*(\mathfrak{a}^-_{\tilde{J}} + \mathfrak{b}^-_{\tilde{J}})^* = \bigwedge{}^*(\mathfrak{a}^-_{\tilde{J}} \times \mathfrak{b}^-_{\tilde{J}})^* = \bigwedge{}^*((\mathfrak{a}^-_{\tilde{J}})^* \times (\mathfrak{b}^-_{\tilde{J}})^*).$$

Since $\mathfrak{a}^-_{\tilde{J}}$ and $\mathfrak{a}$ are isomorphic, and $\mathfrak{b}^-_{\tilde{J}}$ and $\mathfrak{b}$ are isomorphic by natural homomorphisms

$$f : \mathfrak{a}^-_{\tilde{J}} \longrightarrow \mathfrak{a}^{\mathbb{C}}\,; X^-_{\tilde{J}} \mapsto X,$$
$$g : \mathfrak{b}^-_{\tilde{J}} \longrightarrow \mathfrak{b}^{\mathbb{C}}\,; Y^-_{\tilde{J}} \mapsto Y,$$

we can consider isomorphisms

$$(f^{-1})^* : (\mathfrak{a}^-_{\tilde{J}})^* \longrightarrow (\mathfrak{a}^*)^{\mathbb{C}}, \quad (g^{-1})^* : (\mathfrak{b}^-_{\tilde{J}})^* \longrightarrow (\mathfrak{b}^*)^{\mathbb{C}}.$$

Let

$$F : \bigwedge{}^{0,*}(\mathfrak{g}^{\mathbb{C}}_{\tilde{J}})^* = \bigwedge{}^*((\mathfrak{a}^-_{\tilde{J}})^* \times (\mathfrak{b}^-_{\tilde{J}})^*) \longrightarrow \bigwedge{}^*((\mathfrak{a}^*)^{\mathbb{C}} \times (\mathfrak{b}^*)^{\mathbb{C}})$$

be a homomorphism induced by $(f^{-1})^*$ and $(g^{-1})^*$. Then, by Lemma 2.1, $(\bigwedge^{0,*}(\mathfrak{g}^{\mathbb{C}}_{\tilde{J}})^*, \bar{\partial})$ and $(\bigwedge^*((\mathfrak{a}^*)^{\mathbb{C}} \times (\mathfrak{b}^*)^{\mathbb{C}}), d)$ are isomorphic as differential graded algebras by F. □

Corollary 4.3. *If $\mathfrak{a}$ and $\mathfrak{b}$ are rational nilpotent Lie algebras, then*

$$h^{0,q}(\mathfrak{g}_{\tilde{J}}) = h^{0,\dim \mathfrak{g}-q}(\mathfrak{g}_{\tilde{J}})$$

for each q.

Proof. Let A, B be simply connected nilpotent Lie groups corresponding to $\mathfrak{a}$, $\mathfrak{b}$, respectively. Let Γ_A and Γ_B be lattices in A, B, respectively. Since, by Nomizu's theorem,

$$\dim H^q(\mathfrak{a} \times \mathfrak{b}) = \dim H^q(\Gamma_A\backslash A \times \Gamma_B\backslash B)$$

and Poincaré's duality, we have

$$h^{0,q}(\mathfrak{g}_{\tilde{J}}) = \dim H^q(\mathfrak{a} \times \mathfrak{b}) = \dim H^{\dim \mathfrak{g}-q}(\mathfrak{a} \times \mathfrak{b}) = h^{0,\dim \mathfrak{g}-q}(\mathfrak{g}_{\tilde{J}})$$

for each q. □

Assume that

$$[U_i^1, U_j^1] = \sum_{k=1}^{p} C_{ij}^k U_k^1,\ [U_i^1, V_s^1] = \sum_{t=1}^{q} D_{is}^t V_t^1,\ [V_s^1, V_t^1] = \sum_{k=1}^{q} E_{st}^k V_k^1$$

for each i, j, s and t. Let $\mathfrak{g}_s$ be a Lie algebra defined by

$$\mathfrak{g}_s = \mathrm{span}\{U_1, \dots, U_p, V_1, \dots, V_q\}$$

which satisfies

$$[U_i, U_j] = \sum_{k=1}^{p} C_{ij}^k U_k, \quad [U_i, V_s] = \sum_{t=1}^{q} D_{is}^t V_t \ (i, j = 1, \dots, p, s = 1, \dots, q)$$

with other brackets vanishing.

Then we have

Theorem 4.4. *For each r,*

$$\sum_{p+q=r} h^{p,q}(\mathfrak{g}_{\tilde{J}}) = \dim H^r(\mathfrak{g}_s \times \mathfrak{b} \times \mathbb{R}^{\dim \mathfrak{a}}).$$

Proof. By Lemma 4.1, we have

$$d(\mathfrak{a}_{\tilde{J}}^+)^* \subset \bigwedge^2 (\mathfrak{a}_{\tilde{J}}^+)^*, \quad d(\mathfrak{b}_{\tilde{J}}^+)^* \subset (\mathfrak{a}_{\tilde{J}}^-)^* \wedge (\mathfrak{b}_{\tilde{J}}^+)^* + \bigwedge^2 (\mathfrak{b}_{\tilde{J}}^+)^*$$
$$d(\mathfrak{a}_{\tilde{J}}^-)^* \subset \bigwedge^2 (\mathfrak{a}_{\tilde{J}}^-)^*, \quad d(\mathfrak{b}_{\tilde{J}}^-)^* \subset \bigwedge^2 (\mathfrak{b}_{\tilde{J}}^-)^* + (\mathfrak{a}_{\tilde{J}}^+)^* \wedge (\mathfrak{b}_{\tilde{J}}^-)^*.$$

Thus, we have

$$\bar\partial(\mathfrak{a}_{\tilde{J}}^-)^* \subset \bigwedge^2 (\mathfrak{a}_{\tilde{J}}^-)^*, \quad \bar\partial(\mathfrak{b}_{\tilde{J}}^+)^* \subset (\mathfrak{a}_{\tilde{J}}^-)^* \wedge (\mathfrak{b}_{\tilde{J}}^+)^*,$$
$$\bar\partial(\mathfrak{b}_{\tilde{J}}^-)^* \subset \bigwedge^2 (\mathfrak{b}_{\tilde{J}}^-)^*,$$
$$\bar\partial(\mathfrak{a}_{\tilde{J}}^+)^* = \{0\}.$$

Hence, we obtain our claim by Lemma 2.1 and the argument after the proof of Lemma 2.1. □

5 Invariant complex structures and Hodge numbers of compact nilmanifolds of a Heisenberg group

In this section, we consider Hodge numbers of a compact complex nilmanifold $\Gamma\backslash H(n; k) = (\Gamma\backslash_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_k)$, where $H_{\mathbb{R}}(n)$ is a $(2n + 1)$-dimensional real Heisenberg group. We consider the following Lie subalgebras of $\mathfrak{h}_{\mathbb{R}}(n)$:

$$\mathfrak{a}_k = \mathrm{span}\{X_1, \dots, X_k\}$$
$$\mathfrak{b}_k = \mathrm{span}\{X_{k+1}, \dots, X_n, Y_1, \dots, Y_n, Z\},$$

which are considered in Example 3.2. We write $\mathfrak{h}(n; k) = (_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_k)$, where $\tilde{J}_k$ is the complex structure corresponding to a decomposition $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$.

Proposition 5.1. *For each q,*

$$h^{0,q}(\mathfrak{h}(n; k)) = \dim H^q(\mathfrak{h}_{\mathbb{R}}(n - k) \times \mathbb{R}^{2k}).$$

Proof. Since $\mathfrak{a}_k \cong \mathbb{R}^k$, and $\mathfrak{b}_k \cong \mathfrak{h}_{\mathbb{R}}(n - k) \times \mathbb{R}^k$ as Lie algebras, we have

$$h^{0,q}(\mathfrak{h}(n; k)) = \dim H^q(\mathfrak{a}_k \times \mathfrak{b}_k) = \dim H^q(\mathbb{R}^k \times \mathfrak{h}_{\mathbb{R}}(n - k) \times \mathbb{R}^k)$$

by Theorem 4.2. □

Proposition 5.2. *For each r,*

$$\sum_{p+q=r} h^{p,q}(\mathfrak{h}(n;k)) = \dim H^r(\mathfrak{h}_{\mathbb{R}}(k) \times \mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^{2n}).$$

Proof. Since $\mathfrak{b} \cong \mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^k$, and $\mathfrak{g}_s = \mathfrak{h}_{\mathbb{R}}(k) \times \mathbb{R}^{2(n-k)}$, we have

$$\begin{aligned}\sum_{p+q=r} h^{p,q}(\mathfrak{h}(n;k)) &= \dim H^r(\mathfrak{g}_s \times \mathfrak{b} \times \mathbb{R}^{\dim \mathfrak{a}}) \\ &= \dim H^r((\mathfrak{h}_{\mathbb{R}}(k) \times \mathbb{R}^{2(n-k)}) \times (\mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^k) \times \mathbb{R}^k)\end{aligned}$$

by Theorem 4.4. □

Corollary 5.3. *For each r,*

$$\sum_{p+q=r} h^{p,q}(\mathfrak{h}(n;k)) = \sum_{p+q=r} h^{p,q}(\mathfrak{h}(n;n-k)).$$

Next, we consider $h^{p,0}(\mathfrak{h}(n;k))$. Since

$$H(n;k) \cong \left\{ \begin{pmatrix} 1 & \bar{x}_1 & \dots & \bar{x}_k & x_{k+1} & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 & y_1 \\ \vdots & 0 & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 & y_n \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \middle| \; x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{C} \right\},$$

we have that

$$\lambda_i = dx_i, \mu_i = dy_i, \nu = dz - \sum_{s=1}^{k} \bar{x}_s dy_s - \sum_{t=k+1}^{n} x_t dy_t \; (i = 1, \dots, n)$$

is a basis of left-invariant $(1,0)$-forms on $H(n;k)$. Put $\omega_i = \lambda_i$, $\omega_{n+i} = \mu_i$ for $i = 1, \dots, n$, and $\omega_{2n+1} = \nu$. Then, we have

$$\begin{cases} \bar{\partial}\omega_1 = \dots = \bar{\partial}\omega_{2n} = 0, \\ \bar{\partial}\omega_{2n+1} = -\bar{\omega}_1 \wedge \omega_{n+1} - \dots - \bar{\omega}_k \wedge \omega_{n+k}, \\ \bar{\partial}\bar{\omega}_1 = \dots = \bar{\partial}\bar{\omega}_{2n} = 0, \\ \bar{\partial}\bar{\omega}_{2n+1} = -\bar{\omega}_{k+1} \wedge \bar{\omega}_{n+k+1} - \dots - \bar{\omega}_n \wedge \bar{\omega}_{2n}. \end{cases}$$

Proposition 5.4. *For each p,*

$$h^{p,0}(\mathfrak{h}(n;k)) = \dim Z^{p,0}_{\bar{\partial}}(\mathfrak{h}(n;k)^{\mathbb{C}}) = \binom{2n}{p} + \binom{2n-k}{p-k-1}$$

for $p \leq 2n$.

Proof. Any element of

$$\bigwedge\nolimits^p \langle \omega_1, \dots, \omega_n, \omega_{n+1}, \dots, \omega_{2n} \rangle$$

is $\bar{\partial}$-closed but not $\bar{\partial}$-exact. Moreover,

$$\omega_{2n+1} \wedge \omega_{n+1} \wedge \dots \wedge \omega_{n+k} \wedge \omega_{j_1} \wedge \dots \wedge \omega_{j_{p-k-1}}$$

is also $\bar{\partial}$-closed but not $\bar{\partial}$-exact, where $1 \leq j_1 < \cdots < j_{p-k-1} \leq 2n$. Thus,

$$h^{p,0}(\mathfrak{h}(n;k)) \geq \binom{2n}{p} + \binom{2n-k}{p-k-1}.$$

Let

$$\alpha = \sum_{i_1<\cdots<i_{p-1}\leq 2n} a_{i_1\cdots i_{p-1}} \omega_{2n+1} \wedge \omega_{i_1} \wedge \ldots \wedge \omega_{i_{p-1}}$$

be a $\bar{\partial}$-closed $(p,0)$-form. Since

$$\bar{\partial}\alpha = -\sum_{i_1<\cdots<i_{p-1}} \sum_{s=1}^{k} a_{i_1\cdots i_{p-1}} \bar{\omega}_s \wedge \omega_{n+s} \wedge \omega_{i_1} \wedge \ldots \wedge \omega_{i_{p-1}},$$

we see that

$$\sum_{i_1<\cdots<i_{p-1}} \sum_{s=1}^{k} a_{i_1\cdots i_{p-1}} \bar{\omega}_s \wedge \omega_{n+s} \wedge \omega_{i_1} \wedge \ldots \wedge \omega_{i_{p-1}} = 0$$

for any $1 \leq s \leq k$. Thus, if $n+s \notin \{i_1, \ldots, i_{p-1}\}$, then $a_{i_1\cdots i_{p-1}} = 0$. Hence, if $\{n+1, \ldots, n+k\} \not\subset \{i_1, \ldots, i_{p-1}\}$, then $a_{i_1\cdots i_{p-1}} = 0$. On the other hand, if $\{n+1, \ldots, n+k\} \subset \{i_1, \ldots, i_{p-1}\}$, then $\omega_{2n+1} \wedge \omega_{i_1} \wedge \ldots \wedge \omega_{i_{p-1}}$ is $\bar{\partial}$-closed. Thus, we obtain our claim. □

Corollary 5.5. *For each p,*

$$h^{p,0}(\mathfrak{h}(n;1)) = h^{2n+1-p,0}(\mathfrak{h}(n;1)).$$

Corollary 5.6. *If $k_1 \neq k_2$, then $\Gamma\backslash H(n;k_1)$ and $\Gamma\backslash H(n;k_2)$ are not biholomorphic, where Γ is a lattice in the underlying real Lie group of $H(n;k_1)$ and $H(n;k_2)$.*

Proof. We may assume that $k_1 > k_2$. By a result of Console-Fino (Theorem 2.2),

$$H^{p,q}_{\bar{\partial}}(\Gamma\backslash H(n;k_i)) \cong H^{p,q}_{\bar{\partial}}(\mathfrak{h}(n;k_i)^{\mathbb{C}})$$

for each i. Since

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

we have

$$\binom{2n-k_2}{p-k_2-1} = \binom{2n-k_1}{p-k_1-1} + \sum_{s=1}^{k_1-k_2} \binom{2n-(k_2+s)}{p-(k_2+s)}.$$

Thus, if $k_2 < p-1$, then $h^{p,0}(\mathfrak{h}(n;k_1)) \neq h^{p,0}(\mathfrak{h}(n;k_2))$. □

Recall that the minimal model for the de Rham complex $(\Omega^*(\Gamma\backslash N), d)$ of a nilmanifold $\Gamma\backslash N$ is $(\bigwedge^* \mathfrak{n}^*, d)$ by Nomizu's theorem (see [5, 7]).

The dual space $\mathfrak{h}_{\mathbb{R}}(k)^*$ of $\mathfrak{h}_{\mathbb{R}}(k)$ has a basis $\gamma_1, \ldots, \gamma_{2k}, \gamma_{2k+1}$ which satisfies the relations

$$d\gamma_1 = \cdots = d\gamma_{2k} = 0,\ d\gamma_{2k+1} = -\sum_{s=1}^{k} \gamma_s \wedge \gamma_{k+s}.$$

Lemma 5.7. *For $k \geq 2$, and $p \leq k$,*

$$\dim H^p(\mathfrak{h}_{\mathbb{R}}(k)) = \binom{2k}{p} - \binom{2k}{p-2}.$$

Proof. Since any element of
$$\bigwedge^{p}\langle \gamma_1, \dots, \gamma_{2k}\rangle$$
is d-closed, we see
$$\dim Z^p(\mathfrak{h}_{\mathbb{R}}(k)) \geq \binom{2k}{p}.$$

Let α be a q-form such that
$$\alpha = \sum_{i_1<\dots<i_{q-1}} a_{i_1\cdots i_{q-1}} \gamma_{2k+1}\wedge\gamma_{i_1}\wedge\dots\wedge\gamma_{i_{q-1}} = \gamma_{2k+1}\wedge\Big(\sum_{i_1<\dots<i_{q-1}} a_{i_1\cdots i_{q-1}}\gamma_{i_1}\wedge\dots\wedge\gamma_{i_{q-1}}\Big).$$
Put $\beta = \sum_{i_1<\dots<i_{q-1}} a_{i_1\cdots i_{q-1}}\gamma_{i_1}\wedge\dots\wedge\gamma_{i_{q-1}}$. Then,
$$d\alpha = \omega\wedge\beta,$$
where $\omega = -\sum_{s=1}^{k}\gamma_s\wedge\gamma_{k+s}$. Let us consider
$$\Big(\bigwedge^{*}\langle\gamma_1,\dots,\gamma_{2k}\rangle, d\Big) \quad \text{and} \quad \omega = -\sum_{s=1}^{k}\gamma_s\wedge\gamma_{k+s}.$$
Then we can identify the pair as the minimal model for de Rham complex of $2k$-dimensional torus T^{2k} and an invariant symplectic form on T^{2k}. Thus, we can use an $\mathfrak{sl}(2)$-representation (see [11, Corollaries 2.5, 2.7, and 2.8]). Hence, $L_\omega : \bigwedge^{q-1} \longrightarrow \bigwedge^{q+1}; \beta \mapsto \omega\wedge\beta$ is injective for $q \leq k$. We have that if $d\alpha = \omega\wedge\beta = 0$, then $\alpha = 0$ for $q \leq k$. Therefore,
$$\dim Z^p(\mathfrak{h}_{\mathbb{R}}(k)) = \binom{2k}{p}, \ \dim B^p(\mathfrak{h}_{\mathbb{R}}(k)) = \binom{2k}{p-2}.$$
□

By Propositions 5.1, 5.2, and Lemma 5.7, we can compute $\sum_{p+q=r} h^{p,q}(\mathfrak{h}(n;k))$ and $h^{0,q}(\mathfrak{h}(n;k))$. For example, we have
$$\sum_{p+q=1} h^{p,q}(\mathfrak{h}(n;k)) = \begin{cases} 4n+1 & k=0,n \\ 4n & k\neq 0,n \end{cases}$$
$$\sum_{p+q=2} h^{p,q}(\mathfrak{h}(2;k)) = \begin{cases} 35 & k=0,2 \\ 30 & k=1 \end{cases}$$
and
$$\sum_{p+q=2} h^{p,q}(\mathfrak{h}(n;k)) = \begin{cases} 8n^2+2n-1 & k=0,n \\ 8n^2-2n & k=1,n-1 \\ 8n^2-2n-2 & k\neq 0,1,n-1,n \end{cases}$$
$$h^{0,2}(\mathfrak{h}(n;k)) = \begin{cases} 2n^2-n+1 & k=n-1 \\ 2n^2+n & k=n \\ 2n^2-n-1 & k\neq n,n-1 \end{cases}$$
for $n \geq 3$. Moreover, we have
$$h^{2,0}(\mathfrak{h}(n;k)) = \begin{cases} 2n^2+n & k=0 \\ 2n^2-n+1 & k=1 \\ 2n^2-n & k\neq 0,1 \end{cases}$$

$$h^{1,1}(\mathfrak{h}(n;k)) = \begin{cases} 4n^2+2n & k=0 \\ 4n^2 & k=1 \\ 4n^2+2n-1 & k=n \\ 4n^2-1 & k\neq 0,1,n. \end{cases}$$

Remark 5.8. (i) By a straightforward computation, we see that $h^{p,q}(\mathfrak{h}(2;1))$ satisfy the following interesting relations:

$$h^{p,q}(\mathfrak{h}(2;1)) = h^{p,0}(\mathfrak{h}(2;1)) \cdot h^{0,q}(\mathfrak{h}(2;1)),$$
$$h^{p,0}(\mathfrak{h}(2;1)) = h^{0,p}(\mathfrak{h}(2;1)),$$
$$h^{p,q}(\mathfrak{h}(2;1)) = h^{q,p}(\mathfrak{h}(2;1))$$

for each p,q, where $h^{1,0} = h^{4,0} = 4$ and $h^{2,0} = h^{3,0} = 7$. Moreover, we see $\sum_{p+q=r} h^{p,q}(\mathfrak{h}(2;1)) < \sum_{p+q=r} h^{p,q}(\mathfrak{h}(2;0))$ for any $1 \le r \le 4$.

(ii) We can directly check Propositions 5.1 and 5.2. Indeed, since

$$\begin{cases} \bar\partial\omega_1 = \cdots = \bar\partial\omega_{2n} = 0, \\ \bar\partial\omega_{2n+1} = -\bar\omega_1\wedge\omega_{n+1} - \cdots - \bar\omega_k\wedge\omega_{n+k}, \\ \bar\partial\bar\omega_1 = \cdots = \bar\partial\bar\omega_{2n} = 0, \\ \bar\partial\bar\omega_{2n+1} = -\omega_{k+1}\wedge\bar\omega_{n+k+1} - \cdots - \bar\omega_n\wedge\bar\omega_{2n}. \end{cases}$$

for $\mathfrak{h}(n;k)^*$, we have that a differential graded algebra

$$(\bigwedge^{*}\langle\bar\omega_{k+1},\dots,\bar\omega_n,\bar\omega_{n+k+1},\dots,\bar\omega_{2n},\bar\omega_{2n+1}\rangle,\bar\partial)$$

and a differential graded algebra $(\bigwedge^*(\mathfrak{h}_{\mathbb{R}}(n-k)^*)^{\mathbb{C}}, d)$ are isomorphic as differential graded algebras. Moreover,

$$(\bigwedge^{*}\langle\bar\omega_1,\dots\bar\omega_k,\bar\omega_{n+1},\dots,\bar\omega_{n+k}\rangle,\bar\partial)$$

and $(\bigwedge^*((\mathbb{R}^{2k})^{\mathbb{C}})^*, d)$ are isomorphic as differential graded algebras. Since

$$\bigwedge^{0,*}(\mathfrak{h}(n;k)^{\mathbb{C}})^* = \bigwedge^{*}\langle\bar\omega_{k+1},\dots,\bar\omega_n,\bar\omega_{n+k+1},\dots,\bar\omega_{2n},\bar\omega_{2n+1},\bar\omega_1,\dots\bar\omega_k,\bar\omega_{n+1},\dots,\bar\omega_{n+k}\rangle,$$

we have $h^{0,q}(\mathfrak{h}(n;k)) = \dim H^q(\mathfrak{h}_{\mathbb{R}}(n-k)\times\mathbb{R}^{2k})$.

6 Invariant complex structures and Hodge numbers of compact nilmanifolds of a generalized Heisenberg group

In this section, we consider the case that $\mathfrak{g}$ in Section 4 is the Lie algebra of a real generalized Heisenberg group. Let $H_{\mathbb{R}}(1,n)$ be a $(2n+1)$-dimensional real generalized Heisenberg group and $\mathfrak{h}_{\mathbb{R}}(1,n)$ its Lie algebra (see [3]). Then, $\mathfrak{h}_{\mathbb{R}}(1,n)$ has a basis $X_1,\dots,X_n,Y,Z_1,\dots,Z_n$ satisfying $[X_i,Y]=Z_i$ $(i=1,\dots,n)$ with other brackets vanishing. Let us consider the following Lie subalgebras of $\mathfrak{h}_{\mathbb{R}}(1,n)$:

$$\mathfrak{a}_k = \operatorname{span}\{X_1,\dots,X_k\},$$
$$\mathfrak{b}_k = \operatorname{span}\{X_{k+1},\dots,X_n,Y,Z_1,\dots,Z_n\}.$$

We write $\mathfrak{h}(1,n;k) = ({}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}(1,n)^{\mathbb{C}}), \tilde{J}_k)$, where $\tilde{J}_k$ is the complex structure corresponding to a decomposition $\mathfrak{h}_{\mathbb{R}}(1,n) = \mathfrak{a}_k + \mathfrak{b}_k$. We write $H(1,n;k) = ({}_{\mathbb{R}}(H_{\mathbb{R}}(1,n)^{\mathbb{C}}), \tilde{J}_k)$. Then,

$$H(1,n;k) \cong \left\{ \left. \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 & \bar{x}_1 & z_1 \\ 0 & 1 & 0 & \dots & \dots & 0 & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots & \bar{x}_k & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & x_{k+1} & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & x_n & z_n \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 & y \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \right| x_1, \dots, x_n, y, z_1, \dots, z_n \in \mathbb{C} \right\}.$$

Similarly as in the case of Heisenberg group, we have the following propositions. Thus, we omit these proofs.

Proposition 6.1. *For each q,*

$$h^{0,q}(\mathfrak{h}(1,n;k)) = \dim H^q(\mathfrak{h}_{\mathbb{R}}(1,n-k) \times \mathbb{R}^{2k}).$$

Proposition 6.2. *For each r,*

$$\sum_{p+q=r} h^{p,q}(\mathfrak{h}(1,n;k)) = \dim H^r(\mathfrak{h}_{\mathbb{R}}(1,k) \times \mathfrak{h}_{\mathbb{R}}(1,n-k) \times \mathbb{R}^{2n}).$$

Corollary 6.3. *For each r,*

$$\sum_{p+q=r} h^{p,q}(\mathfrak{h}(1,n;k)) = \sum_{p+q=r} h^{p,q}(\mathfrak{h}(1,n;n-k)).$$

Proposition 6.4.

$$h^{p,0}(\mathfrak{h}(1,n;k)) = \binom{2n}{p-1} + \binom{2n-k}{p}$$

for each $1 \le p \le 2n+1$.

Corollary 6.5. *For each p,*

$$h^{p,0}(\mathfrak{h}(1,n;1)) = h^{2n+1-p,0}(\mathfrak{h}(1,n;1)).$$

Corollary 6.6. *If $k_1 \neq k_2$, then $\Gamma\backslash H(1,n;k_1)$ and $\Gamma\backslash H(1,n;k_2)$ are not biholomorphic, where Γ is a lattice in the underlying real Lie group of $H(1,n;k_1)$ and $H(1,n;k_2)$.*

Proof. Since $h^{1,0}(\mathfrak{h}(1,n;k)) = 2n-k+1$, we have our claim. □

Remark 6.7. (i) We can easily check that

$$h^{1,1}(\mathfrak{h}(1,n;k)) = 2n^2 + kn - \frac{1}{2}k^2 + 3n - \frac{1}{2}k + 1.$$

(ii) We can consider other complex structures on ${}_{\mathbb{R}}(H_{\mathbb{R}}(1,n)^{\mathbb{C}})$. For example, consider the following Lie subalgebras :

$$\mathfrak{a} = \mathrm{span}\{X_1, \dots, X_k, Y, Z_1, \dots, Z_k\},$$
$$\mathfrak{b} = \mathrm{span}\{X_{k+1}, \dots, X_n, Z_{k+1}, \dots, Z_n\},$$

where $0 \le k \le n$. Then, $\mathfrak{b}$ is an ideal such that $\mathfrak{h}_{\mathbb{R}}(1,n) = \mathfrak{a} + \mathfrak{b}$.

7 Example of high dimensional complex nilmanifolds which have duality

In this section, we construct examples of pairs of high dimensional complex nilmanifolds M_1, M_2 which have duality of Hodge's numbers, i.e., M_1 and M_2 are complex manifolds which satisfy that M_1 and M_2 are diffeomorphic, and $h^{p,q}(M_1) = h^{q,p}(M_2)$ for each p, q.

Proposition 7.1. *Let M_1, M_2, M_1' and M_2' be compact complex manifolds which satisfy $h^{p,q}(M_1) = h^{q,p}(M_2)$ and $h^{p,q}(M_1') = h^{q,p}(M_2')$ for each p, q such that $p + q \leq k$. Let $L_i = M_i \times M_i'$ $(i = 1, 2)$. Then, $h^{p,q}(L_1) = h^{q,p}(L_2)$ for each p, q such that $p + q \leq k$.*

Proof. By the assumption,

$$\begin{aligned} h^{p,q}(L_1) &= \sum_{\substack{a+c=p\\ b+d=q}} h^{a,b}(M_1)h^{c,d}(M_1') \\ &= \sum_{\substack{b+d=q\\ a+c=p}} h^{b,a}(M_2)h^{d,c}(M_2') = h^{q,p}(L_2) \end{aligned}$$

for each p, q. □

As an application of Proposition 7.1, we have examples of pairs of high dimensional complex nilmanifolds which have duality of Hodge's numbers.

Example 7.2. Let us consider the following nilpotent Lie groups defined by

$$N_1 = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; z_i \in \mathbb{C} \right\}, \quad N_2 = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; z_i \in \mathbb{C} \right\}.$$

Then, by a straightforward computation, we see $h^{p,q}(\Gamma\backslash N_1) = h^{q,p}(\Gamma\backslash N_2)$ for each p, q. Hence, we have

$$h^{p,q}(\overset{n}{\times}\, \Gamma\backslash N_1) = h^{q,p}(\overset{n}{\times}\, \Gamma\backslash N_2)$$

for each p, q, where Γ is a lattice in the underlying real Lie group $N_1 \cong N_2$, and $\overset{n}{\times}\, \Gamma\backslash N_1 = \Gamma\backslash N_1 \times \ldots \times \Gamma\backslash N_1$.

Moreover, we have

$$h^{p,q}(L\backslash(\overset{n}{\times} N_1)) = h^{q,p}(L\backslash(\overset{n}{\times} N_2))$$

for each p, q, where L is lattice in the underlying real Lie group $\overset{n}{\times} N_1 \cong \overset{n}{\times} N_2$. Because $h^{p,q}(L\backslash(\overset{n}{\times} N_i)) = h^{p,q}(\overset{n}{\times}\, \Gamma\backslash N_i) = h^{p,q}(\overset{n}{\times} \mathfrak{n}_i^{\mathbb{C}})$ $(i = 1, 2)$ by a result of Console-Fino (Theorem 2.2, see also [9]).

Remark 7.3. Recall that $\mathbb{C} \cong \{(x, y) \mid x, y \in \mathbb{R}\}$ has a lattice $\mathbb{Z} + \tau\mathbb{Z} \cong \{(m + na, nb) \mid m, n \in \mathbb{Z}\}$, where $\tau = a + \sqrt{-1}b$, and $a, b \in \mathbb{R}$ such that $b > 0$. Let

$$N = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x_i \in \mathbb{R} \right\}, A = \left\{ \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix} \middle| \; x_4 \in \mathbb{R} \right\}.$$

Then,

$$N \times A \cong \left\{ \begin{pmatrix} 1 & x_1 & 0 & x_3 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x_i \in \mathbb{R} \right\}$$

has a lattice

$$\Gamma \cong \left\{ \begin{pmatrix} 1 & m_1 & 0 & m + na \\ 0 & 1 & 0 & m_2 \\ 0 & 0 & 1 & nb \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; m_i, m, n \in \mathbb{Z} \right\},$$

which is not direct product of lattices in N and A. Similarly, we can construct a lattice in $\overset{n}{\times}N$ which is not the direct product of lattices in N.

8 Fibration and modification of a complex structure

In this section, we see the complex structure $\tilde{J}$ from a viewpoint of fibrations.

Theorem 8.1 ([8], Theorem 1.13 in Chapter I). *Let G be a second countable locally compact group and Γ a lattice in G. Let H be a closed subgroup. Then if H ∩ Γ is a lattice in H, ΓH is closed in G; equivalently the natural injection*

$$H \cap \Gamma \backslash H \longrightarrow \Gamma \backslash G$$

is proper. If H is normal in G or if Γ is uniform then ΓH is closed in G if and only if H ∩ Γ is a lattice in H.

Let us consider the following nilpotent Lie group defined by

$$G_1 = \left\{ \left. \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_i \in \mathbb{C} \right\} \times \left\{ \left. \begin{pmatrix} 1 & z_4 \\ 0 & 1 \end{pmatrix} \right| z_4 \in \mathbb{C} \right\}.$$

Then,

$$\Gamma = \left\{ \left. \begin{pmatrix} 1 & \mu_1 & \mu_3 \\ 0 & 1 & \mu_2 \\ 0 & 0 & 1 \end{pmatrix} \right| \mu_i \in \mathbb{Z}[\sqrt{-1}] \right\} \times \left\{ \left. \begin{pmatrix} 1 & \mu_4 \\ 0 & 1 \end{pmatrix} \right| \mu_4 \in \mathbb{Z}[\sqrt{-1}] \right\}$$

is a lattice in G_1. Moreover,

$$Z_1 = \frac{\partial}{\partial z_1}, Z_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, Z_3 = \frac{\partial}{\partial z_3}, Z_4 = \frac{\partial}{\partial z_4}$$

is a basis of left-invariant holomorphic vector fields on G_1, and

$$\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 - z_1 dz_2, \omega_4 = dz_4$$

is its dual basis. Then a left-invariant holomorphic 2-form

$$\begin{aligned} \Omega &= \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 \\ &= dz_1 \wedge dz_3 + dz_2 \wedge dz_4 - z_1 dz_1 \wedge dz_2 \end{aligned}$$

on G_1 yields an invariant holomorphic symplectic structure on $\Gamma \backslash G_1$.

Put

$$H = \left\{ \left. \begin{pmatrix} 1 & 0 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_2, z_3 \in \mathbb{C} \right\} \times \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then H is a closed normal subgroup of G_1, and $H \cap \Gamma$ is lattice in H. Thus, $H\Gamma = \Gamma H$ is closed in G_1 by Theorem 8.1. Consider the following fibration:

$$\Gamma \backslash H\Gamma \longrightarrow \Gamma \backslash G_1 \longrightarrow H\Gamma \backslash G_1.$$

Since

$$H \backslash G_1 \cong \left\{ \left. \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| z_1 \in \mathbb{C} \right\} \times \left\{ \left. \begin{pmatrix} 1 & z_4 \\ 0 & 1 \end{pmatrix} \right| z_4 \in \mathbb{C} \right\},$$

$$H\backslash\Gamma H \cong \left\{ \left.\begin{pmatrix} 1 & \mu_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| \mu_1 \in \mathbb{Z}[\sqrt{-1}] \right\} \times \left\{ \left.\begin{pmatrix} 1 & \mu_4 \\ 0 & 1 \end{pmatrix} \right| \mu_4 \in \mathbb{Z}[\sqrt{-1}] \right\},$$

we can use (z_1, z_4) as a local coordinate system on a neighborhood of each point of $H\Gamma\backslash G_1 \cong (H\backslash H\Gamma)\backslash(H\backslash G_1)$.

More carefully, let

$$N = \left\{ \left.\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_i \in \mathbb{C} \right\}, K = \left\{ \left.\begin{pmatrix} 1 & 0 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_2, z_3 \in \mathbb{C} \right\}.$$

Consider the projection $\pi : N \longrightarrow K\backslash N$;

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, it is obvious that

$$K\backslash N \cong \left\{ \left.\begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| z_1 \in \mathbb{C} \right\}.$$

The group K transitively acts $\pi^{-1}(a_1)$ on the left. Then

$$\begin{aligned} T_{(a_1,a_2,a_3)}\pi^{-1}(a_1) &= \operatorname{span}\left\{(\partial/\partial z_2)_{(a_1,a_2,a_3)} + a_1(\partial/\partial z_3)_{(a_1,a_2,a_3)}, (\partial/\partial z_3)_{(a_1,a_2,a_3)}\right\} \\ &= \operatorname{span}\left\{(\partial/\partial z_2)_{(a_1,a_2,a_3)}, (\partial/\partial z_3)_{(a_1,a_2,a_3)}\right\} \end{aligned}$$

with respect to the natural coordinate system (z_1, z_2, z_3) of N. Thus, we have that each fiber of $\varpi : \Gamma\backslash G_1 \longrightarrow H\Gamma\backslash G_1$ is a holomorphic Lagrangian submanifold of $(\Gamma\backslash G_1, \Omega)$.

Hence, to consider the following modification G_2 of G_1:

$$G_2 = \left\{ \left.\begin{pmatrix} 1 & z_1 & \bar{z}_3 \\ 0 & 1 & \bar{z}_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_i \in \mathbb{C} \right\} \times \left\{ \left.\begin{pmatrix} 1 & z_4 \\ 0 & 1 \end{pmatrix} \right| z_4 \in \mathbb{C} \right\};$$

$$(w_1, w_2, w_3, w_4) \star (z_1, z_2, z_3, z_4) = (z_1 + w_1, z_2 + w_2, z_3 + \bar{w}_1 z_2 + w_3, z_4 + w_4)$$

geometrically corresponds to take the conjugate of a local coordinate system (z_2, z_3) on each fiber, which is a holomorphic Lagrangian submanifold of $(\Gamma\backslash G_1, \Omega)$.

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References

[1] C. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), 513–518.

[2] S. Console and A. Fino: *Dolbeault cohomology of compact nilmanifolds* , Transform. Groups **6** (2001), 111–124.

[3] L.A. Cordero, M, Fernández and A. Gray, *Symplectic manifolds with no Kahler structure*, Topology **25** (1986), 375–380.

[4] L.A. Cordero, M, Fernández, and L. Ugarte, *Lefschetz complex conditions for complex manifolds*, Ann. Global Anal. Geom. **22** (2002), 355–373.

[5] K. Hasegawa: *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), 65–71.

[6] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, J. Differential Geom. **10** (1975), 85–112.

[7] K. Nomizu : *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. **59** (1954), 531–538.

[8] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68, Springer-Verlag, New York-Heidelberg, 1972. ix+227 pp.

[9] Y. Sakane: *On compact complex parallelisable solvmanifolds*, Osaka J. Math. **13** (1976), 187–212.

[10] T. Yamada: *Duality of Hodge numbers of compact complex nilmanifolds*, Complex Manifolds **2** (2015), 168–177.

[11] D. Yan: *Hodge structure on symplectic manifolds*, Adv. in Math. **120** (1996), 143–154.