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An ℓ-th root of a test configuration of exponent ℓ

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Abstract: Let (X, L) be a polarized algebraic manifold. Then for every test configuration $\mu = (\mathcal{X}, \mathcal{L}, \psi)$ for (X, L) of exponent ℓ, we obtain an ℓ-th root (κ, D) of μ and $\mathbb{G}_m$-equivariant desingularizations $\iota : \hat{\mathcal{X}} \to \mathcal{X}$ and $\eta : \hat{\mathcal{X}} \to \mathcal{Y}$, both isomorphic on $\hat{\mathcal{X}} \setminus \hat{\mathcal{X}}_0$, such that

$$\iota^\star \mathcal{L} = \mathcal{O}_{\hat{\mathcal{X}}}(\ell D) \otimes \eta^\star \mathcal{Q}^{\otimes \ell},$$

where $\kappa = (\mathcal{Y}, \mathcal{Q}, \varphi)$ is a test configuration for (X, L) of exponent 1, and D is an effective $\mathbb{Q}$-divisor on $\hat{\mathcal{X}}$ such that ℓD is an integral divisor with support in the fiber $\mathcal{X}_0$. Then (κ, D) can be chosen in such a way that

$$\ell^{-n} \deg \iota_\star D + \deg \eta_\star D \leq C_1 \|\mu\|_\infty \quad \text{and} \quad \|\kappa\|_\infty \leq C_2 \|\mu\|_\infty,$$

where C_1 and C_2 are positive real constants independent of the choice of μ and ℓ. This plays an important role in our forthcoming papers on the existence of constant scalar curvature Kähler metrics (cf. [6]) and also on the compactified moduli space of test configurations (cf. [5],[7]).

1 Introduction

By a *polarized algebraic manifold* (X, L), we mean a pair of a smooth irreducible projective variety X, defined over $\mathbb{C}$, and a very ample line bundle L over X. Put $n := \dim_{\mathbb{C}} X$. For such a pair (X, L), replacing L by its power if necessary, we may assume without loss of generality that

$$H^i(X, L^{\otimes \ell}) = \{0\}, \qquad i \geq 1, \ \ell \geq 1,$$

and that the natural map $H^0(X, L)^{\otimes k} \to H^0(X, L^{\otimes k})$ is surjective. Fix a Hermitian metric h for L such that $\omega := c_1(L, \omega)$ is Kähler. Then for each positive integer ℓ, we define a Hermitian metric ρ_ℓ on $V_\ell := H^0(X, L^{\otimes \ell})$ by

$$\rho_\ell(\sigma, \sigma') := \int_X (\sigma, \sigma')_h \, \omega^n, \qquad \sigma, \sigma' \in V_\ell,$$

where $(\sigma, \sigma')_h$ denotes the pointwise Hermitian inner product of σ and σ' by the ℓ-multiple h^ℓ of h. Put $N_\ell :=$ $\dim V_\ell$. By choosing an orthonormal basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_\ell}\}$ for (V_ℓ, ρ_ℓ), we define

$$\omega_{\mathrm{FS},\ell} := \sqrt{-1}\partial\bar{\partial} \log \Sigma_{\alpha=1}^{N_\ell} |\sigma_\alpha|^2.$$

In this paper, we fix a polarized algebraic manifold (X, L) as above once for all, and we consider a *test configuration*

$$\mu = (\mathcal{X}, \mathcal{L}, \psi),$$

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of exponent ℓ, for (X, L). Let $\mathbb{G}_m = \{ t \in \mathbb{C}^\star \}$ be the 1-dimensional algebraic torus. Then we have an algebraic group homomorphism

$$\psi : \mathbb{G}_m \to \mathrm{GL}(V_\ell)$$

such that the maximal compact subgroup $S^1 \subset \mathbb{G}_m$ acts isometrically on (V_ℓ, ρ_ℓ), and that $(\mathcal{X}, \mathcal{L})$ is the *De Concini-Procesi family* for ψ. Namely, for the affine line $\mathbb{A}^1 = \{ z \in \mathbb{C} \}$, $\mathcal{X}$ is the closure in $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ of the graph

$$\Gamma_\psi := \left\{ (t, \psi(t)x) \, ; \, t \in \mathbb{C}^\star, x \in X \right\},$$

and $\mathcal{L}$ is the restriction to $\mathcal{X}$ of the pullback $\mathrm{pr}_2^\star \mathcal{O}_{\mathbb{P}^*(V_\ell)}(1)$ by the projection $\mathrm{pr}_2 : \mathbb{A}^1 \times \mathbb{P}^*(V_\ell) \to \mathbb{P}^*(V_\ell)$ to the second factor, where X is viewed as a subvariety of $\mathbb{P}^*(V_\ell)$ by the Kodaira embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}^*(V_\ell)$ associated to the complete linear system $|L^{\otimes \ell}|$ on X. Let

$$\pi : \mathcal{X} \to \mathbb{A}^1$$

be the restriction to $\mathcal{X}$ of the projection $\mathrm{pr}_1 : \mathbb{A}^1 \times \mathbb{P}^*(V_\ell) \to \mathbb{A}^1$ to the first factor. Then the fiber $\mathcal{X}_1 = \pi^{-1}(1)$ over $1 \in \mathbb{A}^1$ is naturally identified with X. Recall that μ is said to be *trivial*, if the algebraic group homomorphism

$$\psi^{\mathrm{PGL}} : \mathbb{G}_m \to \mathrm{PGL}(V_\ell)$$

induced by ψ is trivial. For the multiplicative real Lie group $\mathbb{R}_+$, we define a real Lie group homomorphism $\psi^{\mathrm{SL}} : \mathbb{R}_+ \to \mathrm{SL}(V_\ell)$ by

$$\psi^{\mathrm{SL}}(t) := \frac{\psi(t)}{\{\det \psi(t)\}^{1/N_\ell}}, \qquad t \in \mathbb{R}_+.$$

Let b_α, $\alpha = 1, 2, \dots, N_\ell$, be the weights of the $\mathbb{R}_+$-action on V_ℓ by ψ^{SL}. Then we define $0 \le \|\psi\|_\infty \in \mathbb{Q}$ by setting

$$\|\psi\|_\infty := \ell^{-1} \max\{ |b_\alpha| \, ; \, \alpha = 1, 2, \dots, N_\ell \},$$

and then put $\|\mu\|_\infty := \|\psi\|_\infty$. For an invertible sheaf $\hat{\mathcal{L}}$ over a smooth irreducible variety $\hat{\mathcal{X}}$, we call $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ a $\mathbb{G}_m$*-equivariant desingularization* of $(\mathcal{X}, \mathcal{L})$, if there exists a $\mathbb{G}_m$-equivariant proper birational morphism

$$\iota : \hat{\mathcal{X}} \to \mathcal{X},$$

isomorphic over $\mathcal{X} \setminus \mathcal{X}_0$, such that $\hat{\mathcal{L}} = \iota^\star \mathcal{L}$, where $\mathcal{X}_0$ denotes the scheme-theoretic fiber $\pi^{-1}(0)$ of π over the origin. Now, the main purpose of this paper is to show the following:

Main Theorem. *For a test configuration $\mu = (\mathcal{X}, \mathcal{L}, \psi)$ for (X, L), following the construction in Section 2, we obtain a $\mathbb{G}_m$-equivariant desingularization $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ and a test configuration $\kappa = (\mathcal{Y}, \mathcal{Q}, \varphi)$ for (X, L), of exponent 1, such that $\hat{\mathcal{L}}$ over $\hat{\mathcal{X}}$ is $\mathbb{G}_m$-equivariantly identified with*

$$\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}) \otimes \eta^\star \mathcal{Q}^{\otimes \ell},$$

where $\eta : \hat{\mathcal{X}} \to \mathcal{Y}$ is a $\mathbb{C}^\star$-equivariant proper birational morphism, isomorphic over $\mathcal{Y} \setminus \mathcal{Y}_0$, and $\hat{D}$ is an effective divisor on $\hat{\mathcal{X}}$ sitting in $\hat{\mathcal{X}}_0$ set-theoretically. Moreover, ι, η, κ and $D := \hat{D}/\ell$ can be chosen in such a way that

$$\ell^{-n} \deg \iota_\star D + \deg \eta_\star D \le C_1 \|\mu\|_\infty, \tag{a}$$

$$\|\kappa\|_\infty \le C_2 \|\mu\|_\infty, \tag{b}$$

where C_1 and C_2 are positive real constants independent of the choice of the test configuration μ and the exponent ℓ.

Here $\iota_\star D$ and $\eta_\star D$ are viewed as algebraic $\mathbb{Q}$-cycles on the projective spaces $\{0\} \times \mathbb{P}^*(V_\ell)$ and $\{0\} \times \mathbb{P}^*(V_1)$, respectively, for $V_1 := H^0(X, L)$. Note that the pair (κ, D) above is called an *ℓ-th root* of μ. This main theorem plays an important role in our forthcoming papers (cf. [5],[6],[7]).

2 Proof of Main Theorem

In this section, we shall prove Main Theorem except the inequalities (a) and (b), where these inequalities will be shown in later sections. Consider the relative Kodaira embedding

$$\mathcal{X} \hookrightarrow \mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$$

whose restriction $\mathcal{X}_z := \pi^{-1}(z) \hookrightarrow \{z\} \times \mathbb{P}^*(V_\ell)$ over each $z \in \mathbb{A}^1 \setminus \{0\}$ is the Kodaira embedding of $\mathcal{X}_z$ by the complete linear system $|\mathcal{L}_z|$. Take a member H in the complete linear system $|L|$ for the line bundle L on X. By identifying X with $\mathcal{X}_1$, we view H as a divisor on $\mathcal{X}_1$. Then on the trivial projective bundle $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ over the complex affine line $\mathbb{A}^1 = \{z \in \mathbb{C}\}$, we can choose a $\mathbb{G}_m$-invariant irreducible reduced divisor δ as a projective subbundle, of codimension 1, of the bundle $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ such that

$$\delta \cdot \mathcal{X}_1 \;=\; \ell H, \tag{2.1}$$

where ℓH is viewed as a member of the complete linear system $|\mathcal{L}_1| = |L^{\otimes \ell}|$ on $\mathcal{X}_1$ (= X). The restriction of the divisor δ to $\mathcal{X}$ is written in the form

$$\delta \cdot \mathcal{X} \;=\; \text{zero}(\zeta) \tag{2.2}$$

for a suitable choice of a quasi-invariant section ζ for $\mathcal{L}$ over $\mathcal{X}$ such that $\zeta_{|\mathcal{X}_1} \neq 0$. Recall that ζ is *quasi-invariant*, if there exists an integer β independent of t such that

$$\psi(t)\,\zeta \;=\; t^\beta \zeta, \qquad t \in \mathbb{G}_m,$$

i.e., $z^\beta \zeta$ is $\mathbb{G}_m$-invariant. We here explain how ζ is specified. By choosing an element $v \neq 0$ in V_ℓ associated to $\zeta_{|\mathcal{X}_1}$ via the identification

$$H^0(\mathcal{X}_1, \mathcal{L}_1) \;\cong\; V_\ell,$$

we obtain zero(v) = ℓH. Let $b_1 \geq b_2 \geq \cdots \geq b_{N_\ell}$ be the weights of the $\mathbb{G}_m$-action on V_ℓ by ψ. Then for a suitable basis $\{v_1, v_2, \dots, v_{N_\ell}\}$ for V_ℓ,

$$\psi(t)\, v_\alpha \;=\; t^{b_\alpha} v_\alpha, \qquad t \in \mathbb{G}_m,$$

where v is written as $\Sigma_{\alpha=1}^{N_\ell} a_\alpha v_\alpha$ for some $a_\alpha \in \mathbb{C}$. Let α_1 be the largest integer α in $\{1, 2, \dots, N_\ell\}$ such that $a_\alpha \neq 0$. Then ζ can be chosen in such a way that its restriction to each $\mathcal{X}_t$, $t \in \mathbb{A}^1 \setminus \{0\}$, is

$$\left(\Sigma_{\alpha=1}^{\alpha_1} t^{b_\alpha - b_{\alpha_1}} a_\alpha v_\alpha\right)_{|\mathcal{X}_t}$$

where $\mathcal{X}_t$ is viewed as a smooth subvariety of $\mathbb{P}^*(V_\ell) \cong \{t\} \times \mathbb{P}^*(V_\ell)$. Let α_0 be the smallest α in $\{1, 2, \dots, N_\ell\}$ such that b_α coincides with b_{α_1}. It then follows that the restriction of ζ to $\mathcal{X}_0$ is

$$\left(\Sigma_{\alpha=\alpha_0}^{\alpha_1} a_\alpha v_\alpha\right)_{|\mathcal{X}_0}$$

where $\mathcal{X}_0$ is viewed as a subscheme of $\mathbb{P}^*(V_\ell) \cong \{0\} \times \mathbb{P}^*(V_\ell)$. Now for the complex variety $\mathcal{X}$, we choose its proper $\mathbb{G}_m$-equivariant desinguralization

$$\iota : \hat{\mathcal{X}} \to \mathcal{X},$$

isomorphic over $\mathcal{X} \setminus \mathcal{X}_0$, such that the support of the fiber $\hat{\mathcal{X}}_0$ over the origin is simple normal crossing as a divisor on $\hat{\mathcal{X}}$. Put $\hat{\pi} := \pi \circ \iota$. By a suitable choice of ι, we may assume that $\hat{\pi}$ is a projective morphism. We then consider the $\mathbb{G}_m$-invariant irreducible reduced divisor $\mathcal{H}$ on $\hat{\mathcal{X}}$ obtained as the closure in $\hat{\mathcal{X}}$ of the preimage of

$$\bigcup_{t \in \mathbb{C}^*} \{t\} \times \psi(t) H$$

under the mapping ι, where H on X is viewd as a subset of $\mathbb{P}^*(V_\ell)$ via the Kodaira embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}^*(V_\ell)$ in the introduction. Then

$$\iota^*(\delta \cdot \mathcal{X}) \;=\; \hat{D} + \ell \mathcal{H}, \tag{2.3}$$

where $\hat{D}$ is an effective divisor on $\hat{\mathcal{X}}$ with support in $\hat{\mathcal{X}}_0$. Here by viewing $\mathcal{X}_0$ as a subscheme of $\mathbb{P}^*(V_\ell) \cong \{0\} \times \mathbb{P}^*(V_\ell)$, we can characterize the restriction of $\hat{D} + \ell\mathcal{H}$ to $\hat{\mathcal{X}}_0$ as the zeroes of $\iota^*(\Sigma_{\alpha=\alpha_0}^{\alpha_1} a_\alpha v_\alpha)_{|\hat{\mathcal{X}}_0}$. Since the divisor $\mathcal{H}$ on $\hat{\mathcal{X}}$ is $\mathbb{G}_m$-invariant, the $\mathbb{G}_m$-action on $\hat{\mathcal{X}}$ lifts to a $\mathbb{G}_m$-linearization of

$$\hat{\mathcal{Q}} := \mathcal{O}_{\hat{\mathcal{X}}}(\mathcal{H}). \tag{2.4}$$

In this paper, we use locally free sheaves and vector bundles interchangeably. By (2.3) and (2.4), the pullback $\hat{\mathcal{L}} := \iota^*\mathcal{L}$ of $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\delta \cdot \mathcal{X})$ is written as

$$\hat{\mathcal{L}} \;=\; \mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}) \otimes \hat{\mathcal{Q}}^{\otimes \ell}. \tag{2.5}$$

In particular, the $\mathbb{G}_m$-linearizations of $\hat{\mathcal{L}}$ and $\hat{\mathcal{Q}}$ induce a $\mathbb{G}_m$-linearization of $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D})$ making the identification (2.5) $\mathbb{G}_m$-equivariant. For the direct image sheaf $F := \hat{\pi}_*\hat{\mathcal{Q}}$ over $\mathbb{A}^1$, let F_z be the fiber of F over each $z \in \mathbb{A}^1$. Then we have a $\mathbb{G}_m$-equivariant rational map

$$\eta : \hat{\mathcal{X}} \to \mathbb{P}^*(F)$$

whose restriction over each $z \in \mathbb{A}^1 \setminus \{0\}$ is the Kodaira embedding $\eta_z : \hat{\mathcal{X}}_z \hookrightarrow \mathbb{P}^*(F_z)$ associated to the complete linear system $|\hat{\mathcal{Q}}_z|$ on $\hat{\mathcal{X}}_z$. Put

$$\mathcal{Y}_z := \eta_z(\hat{\mathcal{X}}_z).$$

Then by this η, we can naturally identify the open subset $\mathcal{X} \setminus \mathcal{X}_0$ of $\hat{\mathcal{X}}$ with the $\mathbb{G}_m$-invariant subset

$$\mathcal{Y}^\star := \bigcup_{0 \neq z \in \mathbb{A}^1} \mathcal{Y}_z$$

of $\mathbb{P}^*(F)$. Let $\mathcal{Y}$ be the $\mathbb{G}_m$-invariant subvariety of $\mathbb{P}^*(F)$ obtained as the closure of $\mathcal{Y}^\star$ in $\mathbb{P}^*(F)$, i.e., $\mathcal{Y}$ is the meromorphic image of $\hat{\mathcal{X}}$ under the rational map η. Then the restriction

$$\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{A}^1$$

to $\mathcal{Y}$ of the natural projection of $\mathbb{P}^*(F)$ onto $\mathbb{A}^1$ is a $\mathbb{G}_m$-equivariant projective morphism with a relatively very ample invertible sheaf

$$\mathcal{Q} \;:=\; \mathcal{O}_{\mathbb{P}^*(F)}(1)_{|\mathcal{Y}} \tag{2.6}$$

on the fiber space $\mathcal{Y}$ over $\mathbb{A}^1$. Note that $\hat{\pi} = \pi_{\mathcal{Y}} \circ \eta$. The $\mathbb{G}_m$-action on $\hat{\mathcal{Q}}$ naturally induces a $\mathbb{G}_m$-action on F. It then induces a $\mathbb{G}_m$-action on $\mathcal{Q}$ covering the $\mathbb{G}_m$-action on $\mathcal{Y}$. By the affirmative solution of $\mathbb{G}_m$-equivariant Serre's conjecture, we have a $\mathbb{G}_m$-equivariant trivialization

$$F \;\cong\; \mathbb{A}^1 \times F_0, \tag{2.7}$$

where this isomorphism can be chosen in such a way that the Hermitian metric ρ_1 (= $\rho_{\ell\,|\ell=1}$) as in the introduction for the vector space

$$F_1 \;=\; V_1 \;=\; H^0(X, L)$$

corresponds to a Hermitian metric on F_0 which is preserved by the action of the compact subgroup $S^1 \subset \mathbb{G}_m$ (see [2]). By this trivialization, F_0 is identified with F_1 (= V_1), so that the $\mathbb{G}_m$-action on F_0 induces a representation

$$\varphi : \mathbb{G}_m \to \mathrm{GL}(V_1).$$

Hence $(\mathcal{Y}, \mathcal{Q}, \varphi)$ is a test configuration for (X, L) of exponent 1. By (2.4), the base point set B for the linear subsystem F_0 of $H^0(\hat{\mathcal{X}}_0, \hat{\mathcal{Q}}_0)$ contains no components of dimension n. Replacing $\hat{\mathcal{X}}$ by its suitable birational model obtained from $\hat{\mathcal{X}}$ by a sequence of $\mathbb{G}_m$-equivariant proper birational morphisms with exceptional sets sitting over B, we may assume from the beginning that B is empty (see for instance [3], p.114). Hence the rational map $\eta : \hat{\mathcal{X}} \to \mathcal{Y} \subset \mathbb{P}^*(F)$ is holomorphic satisfying

$$\hat{\mathcal{Q}} \;=\; \eta^*\mathcal{Q}, \tag{2.8}$$

as required. In view of (2.8), we now conclude from (2.5) that Main Theorem is true except (a) and (b).

Remark 2.9. If the $\mathbb{G}_m$-action on $\mathbb{P}^*(V_\ell)$ by ψ preserves the subset $X_\ell := \Phi_\ell(X)$, then $\mathcal{X} = \mathbb{A}^1 \times X_\ell$, and hence we can choose $(\hat{\mathcal{X}}, \hat{\mathcal{L}}, \hat{D}) = (\mathcal{X}, \mathcal{L}, 0)$.

Remark 2.10. For a test configuration μ in Main Theorem, κ should be taken in such a way that $\|\kappa\|_\infty$ becomes as large as possible. In some sense, our κ constructed as above satisfies such a requirement.

3 Proof of the inequality (a)

In this section, we compactify $\mathbb{A}^1 = \{z \in \mathbb{C}\}$ to the complex projective line $\mathbb{P}^1$ by viewing $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Since $\mathcal{X}$ sits in $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$, we consider the subvariety $\bar{\mathcal{X}}$ in $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ obtained as the closure of $\mathcal{X}$ in $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$. Then $\mathcal{L}$ over $\mathcal{X}$ extends to a holomorphic line bundle

$$\bar{\mathcal{L}} := \mathrm{pr}_2^* \, \mathcal{O}_{\mathbb{P}^*(V_\ell)}(1)_{|\bar{\mathcal{X}}}, \tag{3.1}$$

over $\bar{\mathcal{X}}$, where $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^*(V_\ell) \to \mathbb{P}^*(V_\ell)$ is the projection to the second factor. By setting $d := \ell^n c_1(L)^n[X]$, we consider the tensor space

$$W_\ell := \{\mathrm{Sym}^d(V_\ell)\}^{\otimes n+1}, \tag{3.2}$$

where $\mathrm{Sym}^d(V_\ell)$ is the d-th symmetric tensor product of V_ℓ. We now put $\mathcal{N} := \mathcal{O}_{\mathbb{P}^*(W_\ell)}(1)$. For the projection $p_2 : \mathbb{P}^1 \times \mathbb{P}^*(W_\ell) \to \mathbb{P}^*(W_\ell)$ to the second factor, we consider the composite

$$\bar{Z} : \mathbb{P}^1 \to \mathbb{P}^*(W_\ell)$$

of the projection p_2 with the Chow section (see [9], 1.3) for $\bar{\mathcal{X}}$ over $\mathbb{P}^1$. Note that each $\bar{Z}(z)$, $z \in \mathbb{P}^1$, is the Chow point for $\bar{\mathcal{X}}_z$, where we view $\bar{\mathcal{X}}_z$ as a cycle on the projective space $\{z\} \times \mathbb{P}^*(V_\ell)\, (\cong \mathbb{P}^*(V_\ell))$. For $z = 1$, an element $Z(1)$ of $W_\ell^* \setminus \{0\}$ which lies above $\bar{Z}(1) \in \mathbb{P}^*(W_\ell)$ is called the *Chow form* for $\bar{\mathcal{X}}_1 = \mathcal{X}_1 = \Phi_\ell(X)$. Now by Theorem 1.4 in [9],

$$\bar{\mathcal{L}}^{\langle n+1 \rangle} \;\cong\; \bar{Z}^* \mathcal{N}, \tag{3.3}$$

where $\bar{\mathcal{L}}^{\langle n+1 \rangle} := \langle \bar{\mathcal{L}}, \dots, \bar{\mathcal{L}} \rangle (\bar{\mathcal{X}}/\mathbb{P}^1(\mathbb{C}))$ is the Deligne pairing of $(n+1)$-pieces of $\bar{\mathcal{L}}$. For the $\mathbb{G}_m$-action by ψ on V_ℓ, let b' and b'' be the maximal weight and the minimal weight, respectively. Then by (3.2), the difference between the maximal weight and the minimal weight for the $\mathbb{G}_m$-action on W_ℓ is

$$d(n+1)(b' - b''), \tag{3.4}$$

If $\bar{Z}$ is a trivial map, then by Remark 2.9, we may assume $\hat{D} = 0$, so that (a) holds in this case. Hence we may assume without loss of generality that $\bar{Z}$ is a non-trivial map. Then $\bar{Z}(\mathbb{P}^1)$ is a rational curve obtained as the closure in $\mathbb{P}^*(W_\ell)$ of the $\mathbb{G}_m$-orbit

$$\mathbb{G}_m \cdot \bar{Z}(1) \;:=\; \{\, [\psi(t) \cdot Z(1)]\,;\; t \in \mathbb{C}^* \,\},$$

where for each $w \in W_\ell^* \setminus \{0\}$, its natural image in $\mathbb{P}^*(W_\ell)$ is written as $[w]$. Hence by taking a general hyperplane Σ in $\mathbb{P}^*(W_\ell)$, we see from (3.4) that $\gamma := \deg \bar{Z}^*(\Sigma)$ satisfies

$$0 < \gamma \le d(n+1)(b' - b''), \tag{3.5}$$

where γ is characterized also by the equality $\bar{Z}^*(\mathcal{N}) \;=\; \mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(\gamma)$. Now for the algebraic cycle $\mathcal{W} := (\mathrm{pr}_2)_* \bar{\mathcal{X}}$ on $\mathbb{P}^*(V_\ell)$, by (3.1), (3.3) and (3.5) combined with the projection formula applied to $\mathrm{pr}_{2|\bar{\mathcal{X}}}$, we obtain

$$\gamma \;=\; c_1(\bar{\mathcal{L}}^{\langle n+1 \rangle})[\mathbb{P}^1(\mathbb{C})] \;=\; \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}})^{n+1} \;=\; \deg \mathcal{W}. \tag{3.6}$$

For the divisor δ on $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ as in (2.1), let $\bar{\delta}$ be the irreducible reduced effective divisor on $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ obtained as the closure of δ in $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$. Note that $\bar{\delta}$ is a projective subbundle, of codimension 1, of the trivial projective bundle $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ over $\mathbb{P}^1$. Let Ψ be a hyperplane in $\mathbb{P}^*(V_\ell)$. We further choose a curve C on $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ written in the form

$$C = \mathbb{P}^1 \times \{p\},$$

where p is a general point in $\mathbb{P}^*(V_\ell)$. Let z be a general point in $\mathbb{P}^1$. Then $\bar{\delta}$ as a cycle on $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ is homologous to

$$\mathbb{P}^1 \times \Psi + a\,\{z\} \times \mathbb{P}^*(V_\ell), \tag{3.7}$$

where a is the intersection number $\bar{\delta} \cdot C$ on $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$. Since ζ in (2.2) is quasi-invariant, we see that

$$0 \le a \le b' - b''. \tag{3.8}$$

In view of (2.4), (2.6) and (2.8), we have a quasi-invariant section $\sigma \neq 0$ for $\mathcal{Q}$ over $\mathcal{Y}$ such that the divisor $\mathcal{H}$ on $\hat{\mathcal{X}}$ in (2.3) is written as

$$\mathcal{H} = \eta^* \operatorname{zero}(\sigma), \tag{3.9}$$

where $\eta : \hat{\mathcal{X}} \to \mathcal{Y}$ is the holomorphic map as in the previous section. By (2.7), we have $\mathbb{G}_m$-equivariant identifications

$$F = \mathbb{A}^1 \times F_0 \quad \text{and} \quad \mathbb{P}^*(F) = \mathbb{A}^1 \times \mathbb{P}^*(F_0).$$

For an arbitrary element $f_0 \neq 0$ in F_0, we extend it to a section f for $F = \hat{\pi}_*\hat{\mathcal{Q}}$ by sending each $z \in \mathbb{A}^1$ to

$$f(z) := (z, f_0) \in \mathbb{A}^1 \times F_0.$$

Then by (2.8), we obtain a nonzero section $\tau = \tau(f_0)$ for $\mathcal{Q}$ over $\mathcal{Y}$ such that the section f for $F = \hat{\pi}_*\hat{\mathcal{Q}}$ comes from the section $\eta^*\tau$ for $\hat{\mathcal{Q}}$. Next by taking a $\mathbb{G}_m$-equivariant desingularization

$$\iota' : \mathcal{X}' \to \bar{\mathcal{X}} \tag{3.10}$$

whose restriction over $\mathbb{A}^1$ coincides with $\iota : \hat{\mathcal{X}} \to \mathcal{X}$, we may further assume that $\eta : \hat{\mathcal{X}} \to \mathcal{Y} \subset \mathbb{P}^*(F) = \mathbb{A}^1 \times \mathbb{P}^*(F_0)$ extends to a $\mathbb{G}_m$-equivariant holomorphic map

$$\eta' : \mathcal{X}' \to \mathbb{P}^*(F'),$$

where $F' := \mathbb{P}^1 \times F_0$ and $\mathbb{P}^*(F') := \mathbb{P}^1 \times \mathbb{P}^*(F_0)$. Let $\mathcal{Y}'$ be the closure of $\mathcal{Y}$ in $\mathbb{P}^*(F')$. Note that $\mathcal{Y}'$ is the image of $\mathcal{X}'$ under the holomorphic map η'. Now for $\tau = \tau(f_0)$, we consider the following divisor on $\mathcal{X}'$:

$$\mathcal{H}'(f_0) := (\eta')^* \operatorname{zero}(\tau'), \tag{3.11}$$

where $\tau' = \tau'(f_0)$ denotes the section for $\mathcal{Q}' := \mathcal{O}_{\mathbb{P}^*(F')}(1)_{|\mathcal{Y}'}$ which coincides with $\tau = \tau(f_0)$ when restricted to $\mathcal{Y}$. Let $\mathcal{X}'_\infty$ be the fiber of $\mathcal{X}'$ over ∞. Since the vector space F_0 is generated by quasi-invariant elements, there exists a quasi-invariant element $f_0^\#$ in F_0 nonvanishing at some point in the image $\eta'(\mathcal{X}'_\infty)$. Consider the case when $f_0 = f_0^\#$. Then the corresponding f and τ, denoted by $f^\#$ and $\tau^\#$, respectively, are quasi-invariant. For a suitable choice of H, replacing σ in (3.9) by $\tau^\#$, we may assume that

$$\mathcal{H} = \eta^* \operatorname{zero}(\tau^\#), \tag{3.12}$$

and on the space $\mathcal{X}'$ over $\mathbb{P}^1$, by (3.11) applied to $\tau' = \tau'(f_0^\#)$, we have the divisor $\mathcal{H}'(f_0^\#) := (\eta')^* \operatorname{zero}(\tau'(f_0^\#))$.

On the other hand, for a general element f_0 in F_0, we easily see that the divisor $\mathcal{H}'(f_0)$ is *horizontal* in the sense that no components in the divisor $\mathcal{H}'(f_0)$ sit in the fibers of $\mathcal{X}'$ over $\mathbb{P}^1$. Hence we have a sequence of elements

$$f_0^{(j)}, \qquad j = 1, 2, \ldots,$$

in F_0 converging to $f_0^{\#}$, as $j \to \infty$, such that the divisors $\mathcal{H}'(f_0^{(j)})$ are all horizontal. Then for each j, the corresponding f and τ will be denoted by $f^{(j)}$ and $\tau^{(j)}$, respectively. For each $z \in \mathbb{A}^1 \setminus \{0\}$, the inclusion $\mathcal{X}'_z \hookrightarrow \mathbb{P}^*(V_\ell)$ allows us to obtain hyperplanes $\delta_z^{(j)}$, $\delta_z^{\#}$ in $\mathbb{P}^*(V_\ell)$ such that

$$\delta_z^{(j)} \cdot \mathcal{X}'_z \;=\; \ell\,\mathcal{H}'(f_0^{(j)})_{|\mathcal{X}'_z} \quad \text{and} \quad \delta_z^{\#} \cdot \mathcal{X}'_z \;=\; \ell\,\mathcal{H}'(f_0^{\#})_{|\mathcal{X}'_z},$$

where on the right-hand side of both equalities above, the intersection is taken in the projective space $\mathbb{P}^*(V_\ell)$. For the projective subbundles

$$\delta^{(j)} := \bigcup_{z \in \mathbb{C}^*} \{z\} \times \delta_z^{(j)} \quad \text{and} \quad \delta^{\#} := \bigcup_{z \in \mathbb{C}^*} \{z\} \times \delta_z^{\#}$$

of the trivial bundle $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{P}^*(V_\ell)$, let $\bar{\delta}^{(j)}$ and $\bar{\delta}^{\#}$ be the irreducible reduced divisors on $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ obtained as the closures of $\delta^{(j)}$ and $\delta^{\#}$, respectively, in $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$. Then $\bar{\delta}^{(j)}$ and $\bar{\delta}^{\#}$ are viewed as projective subbundles, of codimension 1, of the trivial projective bundle $\mathbb{P}^1 \times \mathbb{P}^*(V_\ell)$ over $\mathbb{P}^1$. Since the divisors $\mathcal{H}'(f_0^{(j)})$ are horizontal, we obtain

$$\ell\,\mathcal{H}'(f_0^{(j)}) \;\le\; (\iota')^*(\bar{\delta}^{(j)} \cdot \bar{\mathcal{X}}), \qquad j = 1, 2, \dots, \tag{3.13}$$

where the inequality means that the right-hand side minus the left-hand side is an effective divisor. Let $j \to \infty$ in (3.13). Then, since $f_0^{(j)} \to f_0^{\#}$, we have

$$\ell\,\mathcal{H}'(f_0^{\#}) \;\le\; (\iota')^*(\bar{\delta}^{\#} \cdot \bar{\mathcal{X}}). \tag{3.14}$$

From now on, the divisor $\mathcal{H}'(f_0^{\#})$ on $\mathcal{X}'$ will be written simply as $\mathcal{H}'$ by abuse of terminology. Then by (2.3) and (3.14), there exists an effective divisor D' on $\mathcal{X}'$ sitting over $\infty \in \mathbb{P}^1$ such that

$$(\iota')^*(\bar{\delta}^{\#} \cdot \bar{\mathcal{X}}) \;=\; \hat{D} + \ell\mathcal{H}' + D', \tag{3.15}$$

where $\hat{D}$ is an effective divisor as in (2.3) on $\hat{\mathcal{X}} \subset \mathcal{X}'$ sitting over the origin. Now by (3.7) and (3.8) applied to $\bar{\delta} = \bar{\delta}^{\#}$, we see that $\delta' := (\iota')^*(\bar{\delta}^{\#} \cdot \bar{\mathcal{X}})$ is a nef divisor on $\mathcal{X}'$. Note also that $\mathcal{H}'$ is nef. Let i be an integer such that $1 \le i \le n$. Then by (3.15),

$$\begin{cases} \delta'^{\,n-i+1} \cdot \ell^i\,\mathcal{H}'^{\,i} \\ = \delta'^{\,n-i} \cdot \ell^{i+1}\,\mathcal{H}'^{\,i+1} + \ell^i\,\delta'^{\,n-i} \cdot \mathcal{H}'^{\,i} \cdot (\hat{D} + D') \\ \ge \delta'^{\,n-i} \cdot \ell^{i+1}\,\mathcal{H}'^{\,i+1} + \ell^i\,\delta'^{\,n-i} \cdot \mathcal{H}'^{\,i} \cdot \hat{D} \\ \ge \delta'^{\,n-i} \cdot \ell^{i+1}\,\mathcal{H}'^{\,i+1}. \end{cases} \tag{3.16}$$

In particular, by applying this to $i = n$, we obtain

$$\delta' \cdot \ell^n \mathcal{H}'^n \ge \ell^{n+1}\,\mathcal{H}'^{n+1} + (\ell\,\mathcal{H}')^n \cdot \hat{D} \;=\; \ell^{n+1}\,\mathcal{H}'^{n+1} + \ell^n \deg \eta_\star \hat{D}. \tag{3.17}$$

Hence by (3.7), (3.15), (3.16) and (3.17), we see that

$$\begin{cases} \delta'^{\,n+1} = \delta'^{\,n} \cdot (\hat{D} + \ell\mathcal{H}' + D') \\ = \deg \iota_\star \hat{D} + \delta'^{\,n} \cdot \ell\mathcal{H}' + \deg \iota'_\star D' \\ \ge \deg \iota_\star \hat{D} + \delta'^{\,n} \cdot \ell\mathcal{H}' \ge \deg \iota_\star \hat{D} + \delta' \cdot \ell^n \mathcal{H}'^n \\ \ge \deg \iota_\star \hat{D} + \ell^{n+1}\,\mathcal{H}'^{n+1} + \ell^n \deg \eta_\star \hat{D} \\ \ge \deg \iota_\star \hat{D} + \ell^n \deg \eta_\star \hat{D}. \end{cases} \tag{3.18}$$

Now by (3.7), δ'^{n+1} is written as

$$(n+1)\,a\,\Psi^n[\mathcal{X}_z] + \mathrm{pr}_2^\star\,\Psi^{n+1} \cdot \bar{\mathcal{X}} \;=\; d\,(n+1)\,a + \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}})^{n+1}$$

and hence by (3.5) and (3.6) together with (3.8), we obtain

$$\delta'^{n+1} \le 2d(n+1)(b'-b''). \tag{3.19}$$

Then by $\hat{D} = \ell D$ and $b' - b'' \le 2\ell\|\mu\|_\infty$, we see from (3.18) and (3.19) that

$$\ell \deg \iota_\star D + \ell^{n+1} \deg \eta_\star D \le 2d(n+1)(b'-b'') \le 4(n+1)d\,\ell\,\|\mu\|_\infty,$$

and hence by setting $C_1 := 4(n+1)\,c_1(L)^n[X]$, we obtain the inequality $\ell^{-n} \deg \iota_\star D + \deg \eta_\star D \le C_1\,\|\mu\|_\infty$, as required.

4 Proof of the inequality (b)

For the irreducible components $\hat{D}_\alpha$, $\alpha = 1, 2, \dots, r$, of $\mathrm{Supp}(\hat{\mathcal{X}}_0)$, we can write the divisor $\hat{D}$ on $\hat{\mathcal{X}}$ in the form

$$\hat{D} = \sum_{\alpha=1}^{r} \hat{e}_\alpha \hat{D}_\alpha,$$

where $\hat{e}_\alpha$ is the multiplicity ≥ 0 of $\hat{D}_\alpha$ in $\hat{D}$. Then we may assume that, for some integer r_0 with $1 \le r_0 \le r$,

$$\eta_\star \hat{D}_\alpha = 0 \quad \text{if and only if} \quad r_0 < \alpha \le r, \tag{4.1}$$

where we view $\eta_\star \hat{D}_\alpha$ as an n-dimensional algebraic cycle on $\mathbb{P}^\star(F_0) \cong \mathbb{P}^\star(V_1)$. For the normalization $\nu_{\mathcal{Y}} : \hat{\mathcal{Y}} \to \mathcal{Y}$ of $\mathcal{Y}$, it follows from the Stein factorization that $\eta : \hat{\mathcal{X}} \to \mathcal{Y}$ factor through $\hat{\mathcal{Y}}$, i.e.,

$$\eta = \nu_{\mathcal{Y}} \circ \hat{\eta}$$

for some $\mathbb{G}_m$-equivariant birational morphism $\hat{\eta} : \hat{\mathcal{X}} \to \hat{\mathcal{Y}}$ with connected fibers. Then by Zariski's Main Theorem, r_0 in (4.1) is expressible as

$$r_0 = n_0,$$

where n_0 is the number of the irreducible components in $\mathrm{Supp}(\hat{\mathcal{Y}}_0)$. Hence, we see from (4.1) that

$$\deg \eta_\star \hat{D} = \sum_{\alpha=1}^{n_0} \hat{e}_\alpha \deg \eta_\star \hat{D}_\alpha \ge \sum_{\alpha=1}^{n_0} \hat{e}_\alpha,$$

where $\deg \eta_\star \hat{D} = \ell \deg \eta_\star D \le C_1\,\ell\,\|\mu\|_\infty$ by (a) in Main Theorem. Then the nonnegative rational numbers $e_\alpha := \hat{e}_\alpha/\ell$ satisfy

$$0 \le \sum_{\alpha=1}^{n_0} e_\alpha \le C_1\,\|\mu\|_\infty. \tag{4.2}$$

For $\alpha = 1, 2, \dots, n_0$, we consider the irreducible reduced effective divisor $D_\alpha := \hat{\eta}(\hat{D}_\alpha) \ne 0$ on $\hat{\mathcal{Y}}$. The divisor $\hat{\eta}_\star D$ on $\hat{\mathcal{Y}}$ is witten in the form

$$\hat{\eta}_\star D = \sum_{\alpha=1}^{n_0} e_\alpha D_\alpha.$$

Let m_α be the multiplicity of D_α in the scheme-theoretic fiber $\hat{\mathcal{Y}}_0$. Then for all $\alpha = 1, 2, \dots, n_0$, we obtain

$$m_\alpha \le m_1 \deg \eta_\star D_1 + \dots + m_{n_0} \deg \eta_\star D_{n_0} = c_1(L)^n[X]. \tag{4.3}$$

Let $b_1 \ge b_2 \ge \dots \ge b_{N_\ell}$ be the weights of the $\mathbb{G}_m$-action on V_ℓ by ψ. To the test configuration $\mu = (\mathcal{X}, \mathcal{L}, \psi)$, we assign a new test configuration

$$\tilde{\mu} := (\mathcal{X}, \mathcal{L}, \tilde{\psi})$$

obtained from μ by replacing ψ by the algebraic group homomorphism $\tilde{\psi}: \mathbb{G}_m \to \mathrm{GL}(V_\ell)$ defined by

$$\tilde{\psi}(t) = t^{-b_1}\psi(t) \qquad t \in \mathbb{C}^*.$$

Let $\tilde{b}_1 \geq \tilde{b}_2 \geq \cdots \geq \tilde{b}_{N_\ell}$ be the weights of the $\mathbb{G}_m$-action on V_ℓ by $\tilde{\psi}$, so that we have an orthonormal basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_\ell}\}$ for (V_ℓ, ρ_ℓ) satisfying

$$\tilde{\psi}(t)\cdot\sigma_i = t^{\tilde{b}_i}\sigma_i \tag{4.4}$$

for all $i \in \{1, 2, \dots, N_\ell\}$ and $t \in \mathbb{G}_m$. Then $\tilde{b}_i = b_i - b_1$, and hence

$$0 = \tilde{b}_1 \geq \tilde{b}_2 \geq \cdots \geq \tilde{b}_{N_\ell}. \tag{4.5}$$

Let $\nu_{\mathcal{X}}: \underline{\mathcal{X}} \to \mathcal{X}$ be the normalization of $\mathcal{X}$. Then for $\underline{\mathcal{L}} := \nu_{\mathcal{X}}^*\mathcal{L}$, we have the $\mathbb{G}_m$-action on $(\underline{\mathcal{X}}, \underline{\mathcal{L}})$ induced by that on $(\mathcal{X}, \mathcal{L})$ via $\tilde{\psi}$. Let $\underline{\pi}: \underline{\mathcal{X}} \to \mathbb{A}^1$ be the natural projection. For the direct image sheaves

$$E := \pi_*\mathcal{L} \quad \text{and} \quad \underline{E} := \underline{\pi}_*\underline{\mathcal{L}}$$

over $\mathbb{A}^1$, the algebraic torus $\mathbb{G}_m$ acts on E and $\underline{E}$ via $\tilde{\psi}$ preserving the fibers E_0 and $\underline{E}_0$ over the origin $0 \in \mathbb{A}^1$. Since $\mathcal{L}_0 := \mathcal{L}_{|\mathcal{X}_0}$ is generated by E_0 over all points of $\mathcal{X}_0$, so is $\underline{\mathcal{L}}_0 := \underline{\mathcal{L}}_{|\underline{\mathcal{X}}_0}$ by $\underline{E}_0$ over all points of $\underline{\mathcal{X}}_0$. In view of $V_\ell = H^0(\mathbb{P}^*(V_\ell), \mathcal{O}_{\mathbb{P}^*(V_\ell)}(1))$ and $\mathcal{L} = \mathrm{pr}_2^*\,\mathcal{O}_{\mathbb{P}^*(V_\ell)}(1)_{|\mathcal{X}}$, consider the pullback

$$\iota_0^*: V_\ell \to H^0(\mathcal{X}_0, \mathcal{L}_0)$$

by the inclusion $\iota_0: \mathcal{X}_0 \hookrightarrow \{0\}\times\mathbb{P}^*(V_\ell)$, where $\{0\}\times\mathbb{P}^*(V_\ell)$ is identified with $\mathbb{P}^*(V_\ell)$. We here observe that

$$\mathcal{L}_0 \text{ is generated by } \iota_0^* V_\ell \text{ over all points of } \mathcal{X}_0. \tag{4.6}$$

Lemma 4.7. *Every weight $\underline{b}$ of the $\mathbb{G}_m$-action on $\underline{E}_0$ by $\tilde{\psi}$ is nonpositive.*

Proof: For a weight $\underline{b}$ as above, we have a nonzero element $\underline{e}_0$ in $\underline{E}_0$ such that $\tilde{\psi}(t)\cdot\underline{e}_0 = t^{\underline{b}}\,\underline{e}_0$ for all $t \in \mathbb{G}_m$. For the $\mathbb{G}_m$-action on $\underline{E}$ via $\tilde{\psi}$, we have a $\mathbb{G}_m$-equivariant identification

$$\underline{E} \cong \mathbb{A}^1 \times \underline{E}_0, \tag{4.7}$$

taking the Hermitian metric ρ_ℓ on $\underline{E}_1$ $(= V_\ell)$ to a Hermitian metric on $\underline{E}_0$ preserved by the maximal compact group S^1 in $\mathbb{G}_m$. Let τ denote $\mathbb{A}^1 \times \{\underline{e}_0\}$ viewed as a section of $\underline{E}$ over $\mathbb{A}^1$. Then $\tau \in H^0(\mathbb{A}^1, \underline{E})$ satisfies

$$\tilde{\psi}(t)\cdot\tau = t^{\underline{b}}\tau, \qquad t \in \mathbb{G}_m, \tag{4.8}$$

and also $\tau(0) = \underline{e}_0 \neq 0$ in $\underline{E}_0$. For the embedding $\mathcal{X} \hookrightarrow \mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$, the restriction to $\mathcal{X}$ of the projection $\mathrm{pr}_2: \mathbb{A}^1 \times \mathbb{P}^*(V_\ell) \to \mathbb{P}^*(V_\ell)$ to the second factor will be denoted by p_2. Then

$$\mathcal{L} = p_2^*\mathcal{O}_{\mathbb{P}^*(V_\ell)}(1). \tag{4.9}$$

From a Hermitian metric h for L over X, we obtain a Hermitian metric ρ_ℓ for V_ℓ as in the introduction. Hence by (4.9), ρ_ℓ induces pointwise Hermitian norms for $\mathcal{L}$, $\underline{\mathcal{L}}$ and their powers, denoted both by $|\ \ |_h$ by abuse of terminology. We now view τ above as an element in $H^0(\underline{\mathcal{X}}, \underline{\mathcal{L}})$ by the isomorphism $H^0(\mathbb{A}^1, \underline{E}) \cong H^0(\underline{\mathcal{X}}, \underline{\mathcal{L}})$. We then have a rational number ε satisfying $0 \leq \varepsilon < 1$ such that

$$|\tau|_h^2 = |z|^{2\varepsilon}\xi,$$

where ξ is a real-valued nonnegative C^∞ function on $\underline{\mathcal{X}}$ such that $\xi(x) > 0$ for some point x in $\underline{\mathcal{X}}_0$. Let m_0 be the smallest positive integer such that $m_0\,\varepsilon$ is an integer. Then $\tilde{\tau} := \tau^{\otimes m_0}/z^{\varepsilon\, m_0}$ is a section in $H^0(\underline{\mathcal{X}}, \underline{\mathcal{L}}^{\otimes m_0})$ such that $\tilde{\tau}(x) \neq 0$. In view of (4.8), by the $\mathbb{G}_m$-actions via $\tilde{\psi}$, we obtain

$$\tilde{\psi}(t)\cdot\tilde{\tau}(x) = t^{m_0(\underline{b}+\varepsilon)}\tilde{\tau}(\tilde{\psi}(t)\cdot x), \qquad t \in \mathbb{G}_m, \tag{4.10}$$

so that $\tilde{\tau}$ is non-vanishing along the $\mathbb{G}_m$-orbit through x. Consider the point

$$x' := \lim_{t\to 0} \tilde{\psi}(t)\cdot x \in \underline{\mathcal{X}}_0 \tag{4.11}$$

fixed by the $\mathbb{G}_m$-action. For the restriction $(\nu_{\mathcal{X}})_0 : \underline{\mathcal{X}}_0 \to \mathcal{X}_0$ of $\nu_{\mathcal{X}} : \underline{\mathcal{X}} \to \mathcal{X}$ to $\underline{\mathcal{X}}_0$, we consider the pullback

$$(\nu_{\mathcal{X}})_0^{\star} : H^0(\mathcal{X}_0, \mathcal{L}_0) \to H^0(\underline{\mathcal{X}}_0, \underline{\mathcal{L}}_0).$$

In view of (4.6), by setting $\underline{\sigma}_i := (\nu_{\mathcal{X}})_0^{\star}(\iota_0^{\star}\sigma_i)$, we obtain $\underline{\sigma}_i(x') \neq 0$ for some $i \in \{1, 2, \dots, N_\ell\}$. Fix such an i until the end of this proof. Then by the $\mathbb{G}_m$-equivariance of $\iota_0^{\star}$ and $(\nu_{\mathcal{X}})_0^{\star}$, it follows from (4.4) that

$$\tilde{\psi}(t)\cdot \underline{\sigma}_i(x)^{\otimes k} = t^{k\tilde{b}_i}\underline{\sigma}_i(\tilde{\psi}(t)\cdot x)^{\otimes k}, \qquad t \in \mathbb{G}_m, \tag{4.12}$$

where k is an arbitrary positive integer. Now by $\underline{\sigma}_i(x') \neq 0$, we see that $\underline{\sigma}_i$ is non-vanishing at every points in a neighborhood of x'. In view of (4.12) applied to $k = 1$, it follows from (4.11) that $\underline{\sigma}_i$ is non-vanishing along the $\mathbb{G}_m$-orbit through x. In the fiber of the line bundle $\underline{\mathcal{L}}^{\otimes m_0}$ over x, we have

$$\underline{\sigma}_i(x)^{\otimes m_0} \neq 0 \neq \tilde{\tau}(x).$$

Replacing σ_i by its suitable constant multiple if necessary, we may assume without loss of generality that $\underline{\sigma}_i(x)^{\otimes m_0} = \tilde{\tau}(x)$. Then

$$\tilde{\psi}(t)\cdot \underline{\sigma}_i(x)^{\otimes m_0} = \tilde{\psi}(t)\cdot \tilde{\tau}(x), \qquad t \in \mathbb{G}_m. \tag{4.13}$$

Hence by (4.10) together with (4.12) applied to $k = m_0$, we can rewrite the equality (4.13) in the form

$$\tilde{\tau}(\tilde{\psi}(t)\cdot x) = t^{m_0(\tilde{b}_i - \underline{b} - \varepsilon)}\underline{\sigma}_i(\tilde{\psi}(t)\cdot x)^{\otimes m_0}, \qquad t \in \mathbb{G}_m.$$

Then by letting $t \to 0$, we obtain the convergences $\tilde{\tau}(\tilde{\psi}(t)\cdot x) \to \tilde{\tau}(x')$ and $\underline{\sigma}_i(\tilde{\psi}(t)\cdot x)^{\otimes m_0} \to \underline{\sigma}_i(x')^{\otimes m_0} \neq 0$. Hence we obtain $\tilde{b}_i - \underline{b} - \varepsilon \geq 0$, so that by (4.5) and $\varepsilon \geq 0$, we now conclude that $\underline{b} \leq \tilde{b}_i - \varepsilon \leq 0$, as required.

For each weight $\tilde{b}$ of the $\mathbb{G}_m$-action on V_ℓ by $\tilde{\psi}$, we define a subspace $S_0 = S_0(\tilde{b})$, depending on $\tilde{b}$, of V_ℓ by

$$S_0 := \{\, \sigma \in V_\ell \,;\, \tilde{\psi}(t)\cdot\sigma = t^{\tilde{b}}\sigma \,\}.$$

Endow S_0 with the Hermitian metric induced by ρ_ℓ on V_ℓ. Now, we inductively define a strictly decreasing sequence of $\mathbb{G}_m$-invariant linear subspaces

$$S_0 \supset S_1 \supset \cdots \supset S_k \supset S_{k+1} \supset \cdots$$

as follows. In view of the identification $V_\ell = H^0(\mathbb{P}^*(V_\ell), \mathcal{O}_{\mathbb{P}^*(V_\ell)}(1))$, we consider the $\mathbb{G}_m$-equivariant linear map

$$p_2^{\star} : V_\ell \hookrightarrow H^0(\mathcal{X}, \mathcal{L}) = H^0(\mathbb{A}^1, E).$$

Let $k \geq 0$ be an integer. For each $0 \neq \sigma \in S_k$, let $\gamma(\sigma)$ denote the largest integer $\gamma \geq 0$ such that $p_2^{\star}\sigma$ is divisible by z^γ in the space $H^0(\mathbb{A}^1, E)$. Put

$$a_k := \max_{0\neq\sigma\in S_k} \gamma(\sigma).$$

For the linear subspace $S_{k+1}^{\perp} := \{\, 0 \neq \sigma \in S_k \,;\, \gamma(\sigma) = a_k \,\} \cup \{0\}$ of S_k, we define S_{k+1} as the orthogonal complement of $S_{k+1}^{\perp}$ in S_k. Since $S_k \neq S_{k+1}$, and since S_0 is finite dimensional, the above decreasing sequence stops at some finite k, so that for some positive integer k_0

$$S_{k_0-1} \neq \{0\} = S_{k_0}.$$

Hence $S_0 = S_0(\tilde{b})$ is expressible as $\oplus_{k=1}^{k_0} S_k^{\perp}$. Let Σ_k be an orthonormal basis for $S_k^{\perp}$. Then $\Sigma(\tilde{b}) := \cup_{k=1}^{k_0} \Sigma_k$ is an orthonormal basis for $S_0(\tilde{b})$. Let

$$\Sigma := \cup_{\tilde{b}}\, \Sigma(\tilde{b}),$$

where $\tilde{b}$ runs through the set of all weights of the $\mathbb{G}_m$-action on V_ℓ by $\tilde{\psi}$. We can choose Σ as the orthonormal basis $\{\sigma_1, \sigma_2, \dots, \sigma_{N_\ell}\}$ for V_ℓ associated to the weights $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{N_\ell}$ in (4.4). Then we have integers $\gamma_i \geq 0$ satisfying

$$p_2^\star(\sigma_i) \;=\; z^{\gamma_i}\sigma_i', \qquad i = 1, 2, \dots, N_\ell, \tag{4.14}$$

where $\sigma_i' \in H^0(\mathbb{A}^1, E)$ are such that $\{\sigma_1'(0), \sigma_2'(0), \dots, \sigma_{N_\ell}'(0)\}$ is a basis for E_0. Since $p_2^\star$ is $\mathbb{G}_m$-equivariant, by (4.4) and (4.14), we obtain

$$\tilde{\psi}(t)\cdot\sigma_i'(0) \;=\; t^{\tilde{b}_i+\gamma_i}\sigma_i'(0),$$

i.e., the associated weight β_i of the $\mathbb{G}_m$-action on E_0 by $\tilde{\psi}$ is $\tilde{b}_i + \gamma_i$, so that by $\gamma_i \geq 0$, we have the inequalities

$$\beta_i \;\geq\; \tilde{b}_i, \qquad i = 1, 2, \dots, N_\ell. \tag{4.15}$$

We next consider a $\mathbb{G}_m$-equivariant identification $E \cong \mathbb{A}^1 \times E_0$ similar to the identification $\underline{E} \cong \mathbb{A}^1 \times \underline{E}_0$ in the beginning of the proof of Lemma 4.7. Then to each $e_0 \in E_0$, we assign the subset $\mathbb{A}^1 \times \{e_0\}$ of E viewed as a section in $H^0(\mathbb{A}^1, E)$. Hence E_0 is regarded as a $\mathbb{G}_m$-invariant subspace of $H^0(\mathbb{A}^1, E)$. By setting $p := \nu^\star_{\mathcal{X}|E_0}$, we have the restriction to E_0,

$$p:\; E_0 \;\hookrightarrow\; H^0(\mathbb{A}^1, \underline{E}),$$

of the $\mathbb{G}_m$-equivariant pullback $\nu_{\mathcal{X}}^\star : H^0(\mathbb{A}^1, E) \hookrightarrow H^0(\mathbb{A}^1, \underline{E})$. Then we consider the weights $\beta_1, \beta_2, \dots, \beta_{N_\ell}$ of the $\mathbb{G}_m$-action on E_0 by $\tilde{\psi}$. Note that we have an orthonormal basis $\{\tau_1, \tau_2, \dots, \tau_{N_\ell}\}$ for E_0 such that

$$\tilde{\psi}(t)\cdot\tau_i \;=\; t^{\beta_i}\tau_i, \tag{4.16}$$

for all $i \in \{1, 2, \dots, N_\ell\}$ and $t \in \mathbb{G}_m$. In (4.14), we obtain $\sigma_i' \in H^0(\mathbb{A}^1, E)$, $i = 1, 2, \dots, N_\ell$, for the $\mathbb{G}_m$-equivariant linear map $p_2^\star : V_\ell \hookrightarrow H^0(\mathbb{A}^1, E)$. Similarly, applying the same argument to the $\mathbb{G}_m$-equivariant linear map $p : E_0 \hookrightarrow H^0(\mathbb{A}^1, \underline{E})$, we see for a suitable choice of $\{\tau_1, \tau_2, \dots, \tau_{N_\ell}\}$ that there exist integers $\epsilon_i \geq 0$ satisfying

$$p(\tau_i) \;=\; z^{\epsilon_i}\underline{\tau}_i, \qquad i = 1, 2, \dots, N_\ell, \tag{4.17}$$

where $\underline{\tau}_i \in H^0(\mathbb{A}^1, \underline{E})$ are such that $\{\underline{\tau}_1(0), \underline{\tau}_2(0), \dots, \underline{\tau}_{N_\ell}(0)\}$ is a basis for $\underline{E}_0$. Since p is $\mathbb{G}_m$-equivariant, by (4.16) and (4.17), we obtain

$$\tilde{\psi}(t)\cdot\underline{\tau}_i(0) \;=\; t^{\beta_i+\epsilon_i}\underline{\tau}_i(0),$$

i.e., the associated weight $\underline{b}_i$ of the $\mathbb{G}_m$-action on $\underline{E}_0$ by $\tilde{\psi}$ is $\beta_i + \epsilon_i$, so that by $\epsilon_i \geq 0$, we have the inequalities

$$\underline{b}_i \;\geq\; \beta_i, \qquad i = 1, 2, \dots, N_\ell. \tag{4.18}$$

By (4.15) and (4.18) together with Lemma 4.7, the weights $\underline{b}_1, \underline{b}_2, \dots, \underline{b}_{N_\ell}$ of the $\mathbb{G}_m$-action on $\underline{E}_0$ satisfy

$$0 \;\geq\; \underline{b}_i \;\geq\; \tilde{b}_i, \qquad i = 1, 2, \dots, N_\ell. \tag{4.19}$$

Let $\tilde{\varphi} : \mathbb{G}_m \to \mathrm{GL}(V_1)$ be the $\mathbb{G}_m$-action on F_0 ($= V_1$) induced by $\tilde{\psi}$, where we identify V_1 with F_0 as in Section 2. Let Γ ($\subset \mathbb{Z}$) be the set of all weights of the $\mathbb{G}_m$-action on F_0 ($= V_1$) by $\tilde{\varphi}$. If Γ consists of a single element, then $\|\kappa\|_\infty = \|\varphi\|_\infty = \|\tilde{\varphi}\|_\infty = 0$, so that $\|\kappa\|_\infty \leq C_2\|\mu\|_\infty$, i.e., (b) holds. Hence we may assume that Γ has more than one element. Then

$$\|\kappa\|_\infty \;=\; \|\tilde{\varphi}\|_\infty \;\leq\; \gamma_{\max} - \gamma_{\min}, \tag{4.20}$$

where $\gamma_{\max}$ (resp. $\gamma_{\min}$) is the maximal (resp. minimal) element in Γ. Note that $\gamma_{\max} - \gamma_{\min} \geq 1$. Let $\gamma \in \Gamma$. Then we have some $0 \neq \zeta_0 \in F_0$ such that

$$\tilde{\psi}(t)\cdot\zeta_0 \;=\; t^{\gamma}\zeta_0, \qquad t \in \mathbb{G}_m.$$

For each γ as above, we fix such a ζ_0 once for all. In view of the isomorphism (2.7), $\mathbb{A}^1 \times \{\zeta_0\}$ defines a section $\zeta \in H^0(\mathbb{A}^1, F)$ such that

$$\tilde{\psi}(t)\cdot\zeta \;=\; t^{\gamma}\zeta, \qquad t \in \mathbb{G}_m, \tag{4.21}$$

and that $\zeta(0) = \zeta_0$. For a Hermitian metric ρ on $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^*(F)}(1)_{|\mathcal{Y}}$, we consider its pullback $\hat{\rho} := \nu_{\mathcal{Y}}^*\rho$ on $\mathcal{Q}_{\hat{\mathcal{Y}}} := \nu_{\mathcal{Y}}^*\mathcal{Q}$. From now on until the end of this section, by the identification

$$H^0(\mathbb{A}^1, F) \;=\; H^0(\hat{\mathcal{X}}, \hat{\mathcal{Q}}) \;=\; H^0(\hat{\mathcal{Y}}, \mathcal{Q}_{\hat{\mathcal{Y}}}),$$

we view ζ as a section in $H^0(\hat{\mathcal{Y}}, \mathcal{Q}_{\hat{\mathcal{Y}}})$. For $\alpha = 1, 2, \dots, n_0$, we see from (4.3) that $1 - m_\alpha^{-1} \le 1 - \delta$, where $\delta := \{c_1(L)^n[X]\}^{-1} > 0$. Hence

$$|\zeta|^2_{\hat{\rho}} \;=\; |z|^{2\epsilon}\xi_1, \tag{4.22}$$

where ϵ is a rational number satisfying $0 \le \epsilon \le 1-\delta$, and ξ_1 is a real-valued nonnegative continuous function on $\hat{\mathcal{Y}}$ such that

$$\xi_1(y) > 0 \tag{4.23}$$

for general points y on some irreducible component of $\hat{\mathcal{Y}}_0$. By the Stein factorization, the birational morphism $\iota : \hat{\mathcal{X}} \to \mathcal{X}$ factors through the normalization $\underline{\mathcal{X}}$ of $\mathcal{X}$, so that we naturally have a birational morphism

$$\underline{\iota} : \hat{\mathcal{X}} \to \underline{\mathcal{X}}$$

with connected fibers. For each $\alpha = 1, 2, \dots, r$, let $\hat{\tau}_\alpha$ be the natural section for the line bundle $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}_\alpha)$ over $\hat{\mathcal{X}}$ such that the zeroes of $\hat{\tau}_\alpha$ on $\hat{\mathcal{X}}$ is the divisor $\hat{D}_\alpha$. Then by (2.5) and (2.8), we have an integer $a(\gamma) \ge 0$ and a section $\underline{\theta} \neq 0$ in $H^0(\mathbb{A}^1, \underline{E})$ satisfying $0 \neq \underline{\theta}(0) \in \underline{E}_0$ such that

$$(\hat{\eta}^*\zeta)^{\otimes \ell}\, \Pi_{\alpha=1}^r\, \hat{\tau}_\alpha^{\hat{e}_\alpha} \;=\; z^{a(\gamma)}\, \underline{\iota}^*\underline{\theta}, \tag{4.24}$$

where we view $\underline{\theta}$ as a section in $H^0(\underline{\mathcal{X}}, \underline{\mathcal{L}})$ by the identification $H^0(\mathbb{A}^1, \underline{E}) = H^0(\underline{\mathcal{X}}, \underline{\mathcal{L}})$. Let $\hat{\mathcal{Y}}^{\mathrm{reg}}$ be the set of all smooth points in $\hat{\mathcal{Y}}$. We may assume

$$\hat{\tau}_\alpha \;=\; \hat{\eta}^*\tau_\alpha \;\text{ for some } \tau_\alpha \in H^0(\hat{\mathcal{Y}}^{\mathrm{reg}}, \mathcal{O}_{\hat{\mathcal{Y}}^{\mathrm{reg}}}(D_\alpha)),\;\; \alpha = 1, 2, \dots, n_0,$$

with simple zeroes of τ_α along D_α on $\hat{\mathcal{Y}}^{\mathrm{reg}}$. Hence, outside the preimage $\hat{\eta}^{-1}(Z)$ by $\hat{\eta}$ of some algebraic subset Z of codimension ≥ 2 in $\hat{\mathcal{Y}}$, we can write (4.24) in the form

$$\hat{\eta}^*\left(\zeta^{\otimes \ell}\, \Pi_{\alpha=1}^{n_0}\, \tau_\alpha^{\hat{e}_\alpha}\right) \cdot \Psi \;=\; z^{a(\gamma)}\underline{\iota}^*\underline{\theta}, \tag{4.25}$$

where the term $\Psi := \Pi_{n_0<\alpha\le r}\, \hat{\tau}_\alpha^{\hat{e}_\alpha}$ is non-vanishing outside $\hat{\eta}^{-1}(Z)$. Recall that m_α is the multiplicity of D_α in the scheme-theoretic fiber $\hat{\mathcal{Y}}_0$ viewed as an algebraic cycle. By (4.25) together with (4.22) and (4.23), we obtain

$$a(\gamma) \le \epsilon\,\ell + \max\{\hat{e}_\alpha/m_\alpha;\, \alpha = 1, 2, \dots, n_0\} \le (1-\delta)\ell + \Sigma_{\alpha=1}^{n_0}\hat{e}_\alpha. \tag{4.26}$$

Since $\mathbb{G}_m$ acts on $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}_\alpha)$, $\alpha = 1, 2, \dots, r$, we have an integer λ_α independent of t and γ such that

$$\tilde{\psi}(t) \cdot \hat{\tau}_\alpha \;=\; t^{\lambda_\alpha}\,\hat{\tau}_\alpha, \qquad t \in \mathbb{G}_m. \tag{4.27}$$

We now set $\underline{b}(\gamma) := a(\gamma) + \ell\gamma + \Sigma_{\alpha=1}^r \lambda_\alpha\hat{e}_\alpha$. Then by (4.24) together with (4.21) and (4.27), we see that

$$\tilde{\psi}(t) \cdot \underline{\theta}(0) \;=\; t^{\underline{b}(\gamma)}\,\underline{\theta}(0).$$

From the definition of $\underline{b}(\gamma)$ applied to $\gamma = \gamma_{\max}$ and $\gamma = \gamma_{\min}$, we obtain

$$\underline{b}(\gamma_{\max}) - \underline{b}(\gamma_{\min}) \;=\; a(\gamma_{\max}) - a(\gamma_{\min}) + \ell(\gamma_{\max} - \gamma_{\min}). \tag{4.28}$$

Since we have $a(\gamma) \ge 0$ for all $\gamma \in \Gamma$, in view of (4.2), it follows from (4.26) and (4.28) that

$$\begin{aligned}\ell(\gamma_{\max} - \gamma_{\min}) \;&\le\; \underline{b}(\gamma_{\max}) - \underline{b}(\gamma_{\min}) + a(\gamma_{\min}) \\ &\le\; \underline{b}(\gamma_{\max}) - \underline{b}(\gamma_{\min}) + (1-\delta)\ell + C_1\,\ell\,\|\mu\|_\infty.\end{aligned}$$

By $\gamma_{\max} - \gamma_{\min} \geq 1$, we have $1 - \delta \leq (1-\delta)(\gamma_{\max} - \gamma_{\min})$. Hence

$$\ell\,(\gamma_{\max} - \gamma_{\min})\delta \;\leq\; \underline{b}(\gamma_{\max}) \;-\; \underline{b}(\gamma_{\min}) \;+\; C_1\,\ell\,\|\mu\|_\infty. \tag{4.29}$$

Since we have the inequality $\gamma_{\max} - \gamma_{\min} \geq \|\tilde{\varphi}\|_\infty = \|\kappa\|_\infty$, and since (4.5) and (4.19) imply that

$$\underline{b}(\gamma_{\max}) \;-\; \underline{b}(\gamma_{\min}) \;\leq\; |\tilde{b}_{N_\ell}| \;\leq\; 2\ell\|\tilde{\psi}\|_\infty \;=\; 2\ell\|\psi\|_\infty \;=\; 2\ell\|\mu\|_\infty,$$

it follows from (4.29) that $\ell\,\|\kappa\|_\infty\delta \;\leq\; (2 + C_1)\,\ell\,\|\mu\|_\infty$. Then by setting

$$C_2 := \delta^{-1}(2 + C_1) \;=\; c_1(L)^n[X]\;\{\,2 \;+\; 4(n+1)c_1(L)^n[X]\,\}\,,$$

we now conclude that $\|\kappa\|_\infty \;\leq\; C_2\|\mu\|_\infty$, as required.

Appendix

In this appendix, we shall give a uniform upper bound for some seminorm of the $\mathbb{Q}$-divisor $D = \hat{D}/\ell$ on $\hat{\mathcal{X}}$ in Main Theorem. We write the scheme-theoretic fiber $\hat{\mathcal{X}}_0$ over the origin as a divisor

$$\hat{\mathcal{X}}_0 \;=\; \sum_{\alpha=1}^{r} m_\alpha \hat{D}_\alpha$$

on $\hat{\mathcal{X}}$, where m_α is the multiplicity of $\hat{D}_\alpha$ in $\hat{\mathcal{X}}_0$. As in Section 2, $\operatorname{Supp}(\hat{\mathcal{X}}_0)$ is simple normal crossing. Put $D_\alpha := \hat{\eta}(\hat{D}_\alpha)$ for $\alpha \leq n_0$ as in Section 4. Then by Zariski's Main Theorem,

$$\hat{\mathcal{Y}}_0 \;=\; \sum_{\alpha=1}^{n_0} m_\alpha D_\alpha.$$

In view of the expression $\hat{D} := \Sigma_{\alpha=1}^r\, \hat{e}_\alpha D_\alpha$ at the beginning of Section 4, since $\hat{e}_\alpha$ is nonnegative, we can define nonnegative rational numbers

$$\hat{q}_\alpha := \hat{e}_\alpha/m_\alpha \qquad \text{and} \qquad q_\alpha := \hat{q}_\alpha/\ell,$$

where $\alpha = 1, 2, \dots, r$. By setting $\bar{q} := \max\{q_\alpha\,;\ \alpha = 1, 2, \dots, r\}$, we consider the nonnegative rational numbers

$$\bar{\Delta}_\alpha \;:=\; \bar{q} - q_\alpha, \qquad \alpha = 1, 2, \dots, r.$$

Then the seminorm $\|D\|_\infty \;:=\; \max\{\,\bar{\Delta}_\alpha\,;\ \alpha \;=\; 1, 2, \dots, n_0\,\}$ for D will be shown to be uniformly bounded as follows. In addition, we can show that the maximum $\bar{q}$ is attained by q_α for some α satisfying $1 \leq \alpha \leq n_0$.

Theorem A. (A.1) $\bar{q} = q_\alpha$ *for some* α *satisfying* $1 \leq \alpha \leq n_0$.

(A.2) *There exists a positive real constant* C_3 *independent of the choice of the test configuration* μ *and the exponent* ℓ *such that* $\|D\|_\infty \leq C_3$.

Remark A.3. By definition, we easily see from (A.1) that $\|D\|_\infty = 0$ if and only if $\hat{\eta}_*D$ is a rational multiple of $\hat{\mathcal{Y}}_0$ as a divisor on $\hat{\mathcal{Y}}$.

Remark A.4. By setting $\underline{q} := \min\{q_\alpha\,;\ \alpha = 1, 2, \dots, n_0\}$, we consider the rational numbers $\underline{\Delta}_\alpha := q_\alpha - \underline{q}$, $\alpha = 1, 2, \dots, r$. Then by (A.1), we can write

$$\|D\|_\infty := \max\{\,\underline{\Delta}_\alpha\,;\ \alpha = 1, 2, \dots, n_0\,\} \;=\; \bar{q} - \underline{q}\,.$$

Proof of (A.1): By setting $\bar{q}' := \max\{q_\alpha\,;\ \alpha = 1, 2, \dots, n_0\}$, we have $\bar{q}' \leq \bar{q}$. Hence it suffices to show that $\bar{q}' = \bar{q}$. For contradiction, we assume the contrary, i.e., assume $\bar{q}' < \bar{q}$. Then

$$\bar{q} \;=\; q_{\alpha_0} \tag{A.5}$$

for some α_0 with $n_0 < \alpha_0 \leq r$. In view of the fact that $\bar{q}'$ is a rational number, we take the smallest positive integer j such that $j\ell\bar{q}'$ is an integer. For a sufficiently small $\varepsilon \ll 1$, by choosing an open disc $U_\varepsilon := \{|z| < \varepsilon\}$ in $\mathbb{A}^1$, we consider the preimage $\hat{\mathcal{X}}_\varepsilon := \hat{\pi}^{-1}(U_\varepsilon)$. We now take a general point x in $\hat{D}_{\alpha_0}$. Since the restriction of $\hat{\mathcal{L}}$ to $\hat{\mathcal{X}}_0$ is generated by the sections in $(\hat{\pi}_\star\hat{\mathcal{L}})_0$ $(\subseteq H^0(\hat{\mathcal{X}}_0, \hat{\mathcal{L}}_0))$, and since

$$\hat{\mathcal{L}} = \mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}) \otimes \hat{\mathcal{Q}}^{\otimes \ell},$$

we obtain a holomorphic section $\hat{\sigma}$ for $\hat{\mathcal{L}}$ over $\hat{\mathcal{X}}_\varepsilon$ with $\hat{\sigma}(x) \neq 0$ which can be viewed as a meromorphic section (denoted also by $\hat{\sigma}$ by abuse of terminology) for $\hat{\mathcal{Q}}^{\otimes \ell}$ over $\hat{\mathcal{X}}_\varepsilon$, holomorphic outside $\mathrm{Supp}(\hat{D})$, with a pole of order $\hat{e}_{\alpha_0}$ along $\hat{D}_{\alpha_0}$ and possibly with poles of order $\leq \hat{e}_\alpha$ along $\hat{D}_\alpha$ for all $\alpha \neq \alpha_0$. Let $\hat{\tau}$ be the meromorphic section for $\hat{\mathcal{Q}}^{\otimes j\ell}$ over $\hat{\mathcal{X}}_\varepsilon$ defined by

$$\hat{\tau} := z^{j\ell\bar{q}'}\hat{\sigma}^{\otimes j}.$$

Let γ_α be the order of the possible pole of $\hat{\tau}$ along $\hat{D}_\alpha$. For $\alpha = \alpha_0$, in view of (A.5) together with the definition of q_α, we obtain

$$\gamma_{\alpha_0} = -j\ell\bar{q}' m_{\alpha_0} + j\hat{e}_{\alpha_0} = j\ell m_{\alpha_0}(-\bar{q}' + q_{\alpha_0}) = j\ell m_{\alpha_0}(-\bar{q}' + \bar{q}) > 0,$$

so that $\hat{\tau}$ actually has a pole along $\hat{D}_{\alpha_0}$. If $\alpha \in \{1, 2, \dots, n_0\}$, then

$$\gamma_\alpha = -j\ell\bar{q}' m_\alpha + j\hat{e}_\alpha = j\ell m_\alpha(-\bar{q}' + q_\alpha) \leq 0,$$

and in this case $\hat{\tau}$ is holomorphic along $\hat{D}_\alpha$. Note that $\hat{\eta}_\star\hat{D}_\alpha$ vanishes as a cycle on $\hat{\mathcal{Y}}$ for $\alpha > n_0$. Recall that $\mathcal{Q}_{\hat{\mathcal{Y}}} := \nu_{\mathcal{Y}}^\star\mathcal{Q}$. Put $\pi_{\hat{\mathcal{Y}}} := \pi_{\mathcal{Y}} \circ \nu_{\mathcal{Y}}$. Since $\hat{\mathcal{Q}} = \hat{\eta}^\star\mathcal{Q}_{\hat{\mathcal{Y}}}$, it follows from the Hartogs extension theorem that there exists a holomorphic section τ for $\mathcal{Q}_{\hat{\mathcal{Y}}}^{\otimes \ell j}$ over $\hat{\mathcal{Y}}_\varepsilon := \pi_{\hat{\mathcal{Y}}}^{-1}(U_\varepsilon)$ such that

$$\hat{\tau} = \hat{\eta}^\star\tau.$$

Hence the section $\hat{\tau}$ for $\hat{\mathcal{Q}}^{\otimes \ell j}$ over $\hat{\mathcal{X}}_\varepsilon$ is holomorphic. On the other hand, $\hat{\tau}$ has a pole along $\hat{D}_{\alpha_0}$. This is a contradiction, as required.

Proof of (A.2): Let $\mu_j = (\mathcal{X}_j, \mathcal{L}_j, \psi_j)$, $j = 1, 2, \dots$, be a sequence of test configurations for (X, L). Let ℓ_j be the exponent of μ_j. Then for each j, we take an ℓ_j-th root (κ_j, D_j) of μ_j. Hence, we have test configurations

$$(\mathcal{Y}_j, \mathcal{Q}_j, \varphi_j), \qquad j = 1, 2, \dots,$$

for (X, L), of exponent 1, and $\mathbb{G}_m$-equivariant desingularizations $(\hat{\mathcal{X}}_j, \hat{\mathcal{L}}_j)$ of $(\mathcal{X}_j, \mathcal{L}_j)$ such that, by setting $\hat{D}_j = \ell_j D_j$, we have

$$\hat{\mathcal{L}}_j = \mathcal{O}_{\hat{\mathcal{X}}_j}(\hat{D}_j) \otimes \hat{\mathcal{Q}}_j^{\otimes \ell_j}, \tag{A.6}$$

where $\eta_j : \hat{\mathcal{X}}_j \to \mathcal{Y}_j$ and $\iota_j : \hat{\mathcal{X}}_j \to \mathcal{X}_j$ are $\mathbb{G}_m$-equivariant proper birational morphisms with $\hat{\mathcal{Q}}_j = \eta_j^\star\mathcal{Q}_j$ and $\hat{\mathcal{L}}_j = \iota_j^\star\mathcal{L}_j$. For contradiction, assume that

$$\|D_j\|_\infty \to +\infty \qquad \text{as } j \to \infty. \tag{A.7}$$

Let $(\mathcal{Q}_j)_0$ be the restriction of the line bundle $\mathcal{Q}_j$ to the central fiber $(\mathcal{Y}_j)_0$. Take the normalization $\nu_{\mathcal{Y}_j} : \hat{\mathcal{Y}}_j \to \mathcal{Y}_j$ of $\mathcal{Y}_j$. Let $(D_j)_\alpha$, $1 \leq \alpha \leq n_0(j)$, be the irreducible components of $(\hat{\mathcal{Y}}_j)_0$, and let $(\hat{D}_j)_\alpha$, $1 \leq \alpha \leq r(j)$, be the irreducible components of $(\hat{\mathcal{X}}_j)_0$. Then we can write

$$\begin{cases} (\hat{\mathcal{X}}_j)_0 = \Sigma_{\alpha=1}^{r(j)} m_\alpha(j)\,(\hat{D}_j)_\alpha, \\ (\hat{\mathcal{Y}}_j)_0 = \Sigma_{\alpha=1}^{n_0(j)} m_\alpha(j)\,(D_j)_\alpha, \end{cases}$$

where $m_\alpha = m_\alpha(j)$ is the multiplicity of $(\hat{D}_j)_\alpha$ in $(\hat{\mathcal{X}}_j)_0$. Here each $(D_j)_\alpha$ with $1 \leq \alpha \leq n_0(j)$ is the image of $(\hat{D}_j)_\alpha$ under the $\mathbb{G}_m$-equivariant birational morphism $\hat{\eta}_j : \hat{\mathcal{X}}_j \to \hat{\mathcal{Y}}_j$ induced by η_j. Let $\hat{e}_\alpha(j)$ be the multiplicity of $(\hat{D}_j)_\alpha$ in $\hat{D}_j$. For $1 \leq \alpha \leq r(j)$, we put

$$\hat{q}_\alpha(j) := \hat{e}_\alpha(j)/m_\alpha(j) \quad \text{and} \quad q_\alpha(j) := \hat{q}_\alpha(j)/\ell_j.$$

In view of (A.1) above and Remark A.4, we set $\underline{q}(j) := \min\{q_\alpha(j)\,;\,\alpha = 1, 2, \dots, n_0(j)\}$ and $\underline{\Delta}_\alpha(j) := q_\alpha(j) - \underline{q}(j)$, $\alpha \in \{1, 2, \dots, r(j)\}$. Now for $\alpha \neq \beta$ in $\{1, 2, \dots, n_0(j)\}$, consider the complex $(n-1)$-dimensional cycle

$$(D_j)_{\alpha\beta} \;:=\; (D_j)_\alpha \cdot (D_j)_\beta$$

on $(D_j)_\beta$ with multiplicities defined by the ideal sheaf $\mathcal{I}_{\alpha|(D_j)_\beta}$ on $(D_j)_\beta$, where $\mathcal{I}_\alpha$ denotes the ideal sheaf of $(D_j)_\alpha$ in $\hat{\mathcal{Y}}_j$. Then for the $\mathbb{G}_m$-equivariant embedding $\mathcal{Y}_j \hookrightarrow \mathbb{P}^*(F) = \mathbb{A}^1 \times \mathbb{P}^*(V_1)$, we put

$$\mathcal{H}^{(j)} \;:=\; \mathcal{Y}_j \cdot (\mathbb{A}^1 \times H), \tag{A.8}$$

where H is a general hyperplane in $\mathbb{P}^*(V_1)$. Put $\hat{\mathcal{H}}^{(j)} := \nu_{\mathcal{Y}_j}^* \mathcal{H}^{(j)}$ for the normalization $\nu_{\mathcal{Y}_j} : \hat{\mathcal{Y}}_j \to \mathcal{Y}_j$. Since we can view each

$$y_j \;=\; ((\mathcal{Y}_j)_0, (\mathcal{Q}_j)_0)$$

as an element of the Hilbert scheme for the projective subschemes of $\mathbb{P}^*(V_1)$ with the Hilbert polynomial

$$P(k) \;=\; \dim H^0(X, L^{\otimes k}), \qquad k \gg 1.$$

Replacing the sequence $\{\mu_j\}$ by its suitable subsequence if necessary, we may assume from the projectivity of the Hilbert scheme that $n_0(j)$ and $m_\alpha(j)$ with $1 \le \alpha \le n_0(j)$ are independent of the choice of j, and that

$$(D_j)_{\alpha\beta} \cdot (\hat{\mathcal{H}}^{(j)})^{n-1} \;\le\; C_4, \tag{A.9}$$

where C_4 is a positive real constant independent of α, β and j. Hence $n_0(j)$ and $m_\alpha(j)$ as above can be written simply as n_0 and m_α. Then by (A.7), replacing $\{\mu_j\}$ by its subsequence if necessary, we may assume that

$$\underline{\Delta}_{\alpha_0}(j) \to +\infty, \quad \text{as } j \to \infty,$$

for some $\alpha_0 \in \{1, 2, \dots, n_0\}$ independent of the choice of j. Similarly by (A.1) above, replacing $\{\mu_j\}$ by its subsequence if necessary, we may assume that there exist nonempty complementary subsets A, B of $\{1, 2, \dots, n_0\}$ with $A \cup B = \{1, 2, \dots, n_0\}$ satisfying the following:

$$\begin{cases} \text{If } \alpha \in A, \text{ then } \underline{\Delta}_\alpha(j) \to +\infty, \quad \text{as } j \to \infty. \\ \text{If } \beta \in B, \text{ then } \underline{\Delta}_\beta(j),\ j = 1, 2, \dots, \text{ are bounded.} \end{cases}$$

Since $\cup_{\alpha=1}^{n_0} (D_j)_\alpha$ set-theoretically coincides with the connected fiber $(\hat{\mathcal{Y}}_j)_0$, some $\alpha(j) \in A$ and some $\beta(j) \in B$ are neighboring in the sense that

$$(D_j)_{\alpha(j)} \cap (D_j)_{\beta(j)} \;\neq\; \emptyset.$$

Replacing $\{\mu_j\}$ by its subsequence if necessary, we may assume that both $\alpha(j)$ and $\beta(j)$ are independent of the choice of j. Hence, such $\alpha(j)$ and $\beta(j)$ are written as $\alpha^\#$ and $\beta^\#$, respectively. Let $\mathcal{X}'_j$ be the smooth compactification of $\hat{\mathcal{X}}_j$ as in Section 3. Then by (A.6),

$$c_1(\hat{\mathcal{L}}_j) \;=\; \ell_j\, \eta_j^* c_1(\mathcal{Q}_j) + \sum_{\alpha=1}^{r(i)} \hat{e}_\alpha(j)\, [(\hat{D}_j)_\alpha], \tag{A.10}$$

where $[(\hat{D}_j)_\alpha] \in H^2(\hat{\mathcal{X}}_j, \mathbb{Q})$ is the restriction to $\hat{\mathcal{X}}_j$ of the Poincaré dual $\in H^2(\mathcal{X}'_j, \mathbb{Q})$ of the algebraic cycle $(\hat{D}_j)_\alpha$ on $\mathcal{X}'_j$. On the other hand,

$$\sum_{\alpha=1}^{r(j)} m_\alpha\, [(\hat{D}_j)_\alpha] \;=\; 0. \tag{A.11}$$

From now on, we replace $\{\mu_j\}$ by its suitable subsequence if necessary. In view of (A.1), by renumbering $(\hat{D}_j)_1, (\hat{D}_j)_2, \dots, (\hat{D}_j)_{n_0}$ if necessary, we may assume that $\underline{q}(j) = q_1(j)$ for all i. Hence $1 \in B$. Multiply (A.11) by $q_1(j)$. Then by subtracting it from $1/\ell_j$ times (A.10), we obtain

$$c_1(\hat{\mathcal{L}}_j)/\ell_j \;=\; \hat{\eta}_j^* c_1(\mathcal{Q}_{\hat{\mathcal{Y}}_j}) + \sum_{\alpha=1}^{r(j)} m_\alpha\, \underline{\Delta}_\alpha(j)\, [(\hat{D}_j)_\alpha],$$

since $\eta_j^\star c_1(\mathcal{Q}_j) = \hat{\eta}_j^\star c_1(\mathcal{Q}_{\hat{\mathcal{Y}}_j})$ by $\hat{\eta}_j^\star \mathcal{Q}_{\hat{\mathcal{Y}}_j} = \hat{\eta}_j^\star \nu_{\mathcal{Y}_j}^\star \mathcal{Q}_j = \eta_j^\star \mathcal{Q}_j$. By (A.8), the divisor $\mathcal{H}^{(j)}$ on $\mathcal{Y}_j$ is viewed as a hyperplane section obtained as the pullback to $\mathcal{Y}_j$ of a hyperplane in $\mathbb{P}^*(V_1)$, while $c_1(\mathcal{Q}_j)$ on $\mathcal{Y}_j$ is the pullback to $\mathcal{Y}_j$ of the first Chern class of the hyperplane bundle on $\mathbb{P}^*(V_1)$. To general hyperplanes H_k, $1 \le k \le n-1$, in $\mathbb{P}^*(V_1)$, we associate the hyperplane sections $\mathcal{H}_k^{(j)}$ on $\mathcal{Y}_j$ as in (A.8). By setting

$$\hat{\mathcal{H}}_k^{(j)} := \nu_{\mathcal{Y}_j}^\star \mathcal{H}_k^{(j)}, \qquad 1 \le k \le n-1.$$

we consider the restriction to $(\hat{\mathcal{Y}}_j)_0$ of the intersection $\hat{\mathcal{H}}_1^{(j)} \cdot \hat{\mathcal{H}}_2^{(j)} \cdots \hat{\mathcal{H}}_{n-1}^{(j)}$ written in the form

$$\hat{\mathcal{H}}_1^{(j)} \cdot \hat{\mathcal{H}}_2^{(j)} \cdots \hat{\mathcal{H}}_{n-1}^{(j)}{}_{|(\hat{\mathcal{Y}}_j)_0} = \sum_{\alpha=1}^{n_0} \gamma_\alpha,$$

where $\gamma_\alpha := m_\alpha \hat{\mathcal{H}}_1^{(j)} \cdot \hat{\mathcal{H}}_2^{(j)} \cdots \hat{\mathcal{H}}_{n-1}^{(j)} \cdot (D_j)_\alpha$ is a nontrivial effective algebraic cycle of complex dimension 1 on $(D_j)_\alpha$. Then for $0 \neq t \in \mathbb{A}^1 \setminus \{0\}$,

$$\begin{aligned}
c_1(L)^n[X] &= \langle\, c_1(\hat{\mathcal{L}}_j)/\ell_j,\ \hat{\eta}_j^\star(\hat{\mathcal{H}}_1^{(j)} \cdot \hat{\mathcal{H}}_2^{(j)} \cdots \hat{\mathcal{H}}_{n-1}^{(j)}{}_{|(\hat{\mathcal{Y}}_j)_t})\,\rangle \\
&= \langle\, c_1(\hat{\mathcal{L}}_j)/\ell_j,\ \hat{\eta}_j^\star(\hat{\mathcal{H}}_1^{(j)} \cdot \hat{\mathcal{H}}_2^{(j)} \cdots \hat{\mathcal{H}}_{n-1}^{(j)}{}_{|(\hat{\mathcal{Y}}_j)_0})\,\rangle \\
&= \langle\, c_1(\hat{\mathcal{L}}_j)/\ell_j,\ \Sigma_{\alpha=1}^{n_0} \hat{\eta}_j^\star \gamma_\alpha \,\rangle \ \ge\ \langle\, c_1(\hat{\mathcal{L}}_j)/\ell_j,\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle \\
&= \langle\, c_1(\mathcal{Q}_{\hat{\mathcal{Y}}_j}),\ \gamma_{\beta^\#}\,\rangle + \Sigma_{\alpha=1}^{r(j)} m_\alpha \underline{\Delta}_\alpha(j) \langle\, [(\hat{D}_j)_\alpha],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle \\
&\ge \Sigma_{\alpha=1}^{r(j)} m_\alpha \underline{\Delta}_\alpha(j) \langle\, [(\hat{D}_j)_\alpha],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle \\
&= J_{\alpha^\#}(j) + J_{\beta^\#}(j) + \Sigma_{\alpha^\# \neq \alpha \neq \beta^\#} J_\alpha(j),
\end{aligned}$$

where we put $J_\alpha(j) := m_\alpha \underline{\Delta}_\alpha(j) \langle [(\hat{D}_j)_\alpha], \hat{\eta}_j^\star \gamma_{\beta^\#}\rangle$ for all $\alpha \in \{1, 2, \dots, r(j)\}$, and the pairing $\langle [(\hat{D}_j)_\alpha],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\rangle$ is taken on $(\hat{D})_{\beta^\#}$ with $[(\hat{D}_j)_\alpha]$ viewed as its restriction to $(\hat{D})_{\beta^\#}$. Moreover, the summation $\Sigma_{\alpha^\# \neq \alpha \neq \beta^\#}$ is taken over all α in $\{1, 2, \dots, r(j)\}$ such that $\alpha^\# \neq \alpha \neq \beta^\#$. Since, by (4.1), we have

$$(\hat{\eta}_j)_\star (\hat{D}_j)_\alpha = \begin{cases} (D_j)_\alpha & \text{if } 1 \le \alpha \le n_0\,; \\ 0 & \text{if } \alpha > n_0, \end{cases}$$

we obtain the following from the projection formula 5.6.16 in [8], p.254, applied to the holomorphic mapping $\hat{\eta}_j : (\hat{D}_j)_{\beta^\#} \to (D_j)_{\beta^\#}$:

$$\langle\, [(\hat{D}_j)_\alpha],\ \hat{\eta}_i^\star \gamma_{\beta^\#}\,\rangle = \begin{cases} \langle\, [(D_j)_\alpha],\ \gamma_{\beta^\#}\,\rangle & \text{if } 1 \le \alpha \le n_0\,; \\ 0 & \text{if } \alpha > n_0. \end{cases} \tag{A.12}$$

In particular $\Sigma_{\alpha^\# \neq \alpha \neq \beta^\#} J_\alpha(j) \ge 0$. Hence $c_1(L)^n[X] \ge J_{\alpha^\#}(j) + J_{\beta^\#}(j)$. Since $\underline{\Delta}_{\beta^\#}(j)$, $j = 1, 2, \dots$, is a bounded sequence, in view of (A.9), (A.11) and (A.12), it now follows that

$$\begin{cases} |J_{\beta^\#}(j)| &= |\, m_{\beta^\#} \underline{\Delta}_{\beta^\#}(j) \langle\, [(\hat{D}_j)_{\beta^\#}],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle\,| \\ &= \underline{\Delta}_{\beta^\#}(j)\, |\, \langle\, \Sigma_{1 \le \alpha \le r(j),\, \alpha \neq \beta^\#}\, m_\alpha [(\hat{D}_j)_\alpha],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle\,| \\ &= \underline{\Delta}_{\beta^\#}(j)\, |\, \Sigma_{n_0 \ge \alpha \neq \beta^\#}\, m_\alpha \langle\, [(D_j)_\alpha],\ \gamma_{\beta^\#}\,\rangle\,| \\ &= \underline{\Delta}_{\beta^\#}(j)\, \Sigma_{n_0 \ge \alpha \neq \beta^\#}\, m_\alpha (D_j)_{\alpha\beta^\#} \cdot (\hat{\mathcal{H}}^{(j)})^{n-1} \ \le\ C_5, \end{cases} \tag{A.13}$$

where C_5 is a positive constant independent of the choice of j. On the other hand, since $\alpha^\#$ and $\beta^\#$ are neighboring, we see from (A.12) that

$$\langle\, [(\hat{D}_j)_{\alpha^\#}],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle = \deg\, (\nu_{\mathcal{Y}_j})_\star (D_j)_{\alpha^\# \beta^\#} \tag{A.14}$$

is a positive integer, where $(\mathcal{Y}_j)_0$ is viewed as a subvariety in $\mathbb{P}^*(V_1)$. Since $\underline{\Delta}_{\alpha^\#}(j) \to +\infty$ as $j \to \infty$, and since $m_{\alpha^\#} \ge 1$, it follows from (A.14) that

$$J_{\alpha^\#}(j) = m_{\alpha^\#} \underline{\Delta}_{\alpha^\#}(j) \langle\, [(\hat{D}_j)_{\alpha^\#}],\ \hat{\eta}_j^\star \gamma_{\beta^\#}\,\rangle \to +\infty,$$

in contradiction to $J_{\alpha^\#}(j) + J_{\beta^\#}(j) \le c_1(L)^n[X]$ and (A.13), as required.

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