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The Soliton-Ricci Flow with variable volume forms

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Abstract: We introduce a flow of Riemannian metrics and positive volume forms over compact oriented manifolds whose formal limit is a shrinking Ricci soliton. The case of a fixed volume form has been considered in our previous work. We still call this new flow, the Soliton-Ricci flow. It corresponds to a forward Ricci type flow up to a gauge transformation. This gauge is generated by the gradient of the density of the volumes. The new Soliton-Ricci flow exist for all times. It represents the gradient flow of Perelman's $\mathcal{W}$ functional with respect to a pseudo-Riemannian structure over the space of metrics and normalized positive volume forms. We obtain an expression of the Hessian of the $\mathcal{W}$ functional with respect to such structure. Our expression shows the elliptic nature of this operator in the orthogonal directions to the orbits obtained by the action of the group of diffeomorphism. In the case that initial data is Kähler, the Soliton-Ricci flow over a Fano manifold preserves the Kähler condition and the symplectic form. Over a Fano manifold, the space of tamed complex structures embeds naturally, via the Chern-Ricci map, into the space of metrics and normalized positive volume forms. Over such space the pseudo-Riemannian structure restricts to a Riemannian one. We perform a study of the sign of the restriction of the Hessian of the $\mathcal{W}$ functional over such space. This allows us to obtain a finite dimensional reduction of the stability problem for Kähler-Ricci solitons. This reduction represents the solution of this well known problem. A less precise and less geometric version of this result has been obtained recently by the author in [28].

Keywords: Bakry-Emery-Ricci tensor, Chern-Ricci form, Shrinking Ricci solitons, Kähler-Ricci solitons, Perelman's entropy functional, Variational stability of Perelman's entropy functional

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1 Introduction and statement of the main result

In this paper we provide a caracterisation of the second variation of Perelman's $\mathcal{W}$ more precise and more geometric that the one described recently in [28]. Indeed here we detect the kernel of the second variation. We represent also the second variation in terms of a Riemannian structure over the space of compatible complex structures. This structure is crucial for the computation of the kernel of the second variation. Indeed it allows a fundamental triple decomposition of the tangent cone of the space of compatible complex structures (embedded in the space of Riemannian metrics and positive normalised volume forms via the Chern-Ricci map).

This is the first of a series of papers whose purpose is the study the following problem.

Let (X, J_0) be a Fano manifold. We remind that the first Chern class $c_1(X, [J_0]) \in H^2_d(X, \mathbb{R})$ depends only on X and the coboundary class $[J_0]$ of the complex structure J.

Let also $\omega \in 2\pi c_1(X, [J_0])$ be an arbitrary J_0-invariant Kähler form over X. We want to find under which conditions on J_0 and ω there exists a smooth complex structure $J \in [J_0]$ and a smooth volume form $\Omega > 0$

over X such that

$$\begin{cases} \omega = \mathrm{Ric}_J(\Omega)\,, \\ \\ \overline{\partial}_{T_{X,J}}\left(\omega^{-1} d\log\frac{\omega^n}{\Omega}\right) = 0\,, \end{cases}$$

i.e. the Riemannian metric $g := -\omega J$, is a J-invariant Kähler-Ricci soliton.

This set up represents a particular case of the Hamilton-Tian conjecture with a stronger conclusion. Namely we avoid the singularities in the solution of the Kähler-Ricci soliton equation.

Proofs of the Hamilton-Tian conjecture have been posted on the arXiv server in (2013) by Tian-Zhang [38] in complex dimension 3 and quite recently by Chen-Wang [6] in arbitrary dimensions.

Our starting point of view is Perelman's twice contracted second Bianchi type identity introduced in [29].

We remind first what this is about. Let $\Omega > 0$ be a smooth volume form over an oriented compact and connected Riemannian manifold (X, g). We remind that the Ω-Bakry-Emery-Ricci tensor of g is defined by the formula

$$\mathrm{Ric}_g(\Omega) \quad := \quad \mathrm{Ric}(g) \quad + \quad \nabla_g d\log\frac{dV_g}{\Omega}.$$

A Riemannian metric g is called a Ω-Shrinking Ricci soliton if $g = \mathrm{Ric}_g(\Omega)$. We equip the set of smooth Riemannian metrics $\mathcal{M}$ with the scalar product

$$(u, v) \longmapsto \int_X \langle u, v\rangle_g\, \Omega, \tag{1.1}$$

for all $u, v \in \mathcal{H} := L^2(X, S^2_{\mathbb{R}} T^\star_X)$. Let $P^\star_g$ be the formal adjoint of some operator P with respect to a metric g. We observe that the operator

$$P^{\star_\Omega}_g \quad := \quad e^f P^\star_g\left(e^{-f}\bullet\right),$$

with $f := \log\frac{dV_g}{\Omega}$, is the formal adjoint of P with respect to the scalar product (1.1). We define also the Ω-Laplacian operator

$$\Delta^\Omega_g \quad := \quad \nabla^{\star_\Omega}_g\nabla_g = \Delta_g + \nabla_g f\neg\nabla_g.$$

It is also useful to introduce the Ω-divergence operator acting on vector fields as follows:

$$\mathrm{div}^\Omega\,\xi \quad := \quad \frac{d(\xi\neg\Omega)}{\Omega} = e^f\,\mathrm{div}_g\left(e^{-f}\xi\right) = \mathrm{div}_g\,\xi - g\left(\xi, \nabla_g f\right).$$

(We denote by $\neg$ the contraction operator). We infer in particular the identity $\mathrm{div}^\Omega\,\nabla_g u = -\Delta^\Omega_g u$, for all functions u. We observe also the integration by parts formula

$$-\int_X u\,\mathrm{div}^\Omega\,\xi\,\Omega \quad = \quad \int_X g(\nabla_g u, \xi)\,\Omega.$$

We define now the following fundamental objects

$$h \quad \equiv \quad h_{g,\Omega} := \mathrm{Ric}_g(\Omega) - g,$$

$$2H \quad \equiv \quad 2H_{g,\Omega} := -\Delta^\Omega_g f + \mathrm{Tr}_g\, h + 2f,$$

$$f \quad := \quad \log\frac{dV_g}{\Omega}.$$

An elementary computation made by Perelman [29] (see also [22]) shows that the maps h and H satisfy **Perelman's twice contracted second Bianchi type identity**

$$\nabla^{\star_\Omega}_g h^\star_{g,\Omega} + \nabla_g H_{g,\Omega} = 0, \tag{1.2}$$

where $h^{\star}_{g,\Omega} := g^{-1}h_{g,\Omega}$ is the endomorphism associated to $h_{g,\Omega}$. We remind now that for any symmetric 2-tensor u the tensor $\mathcal{R}_g \star u$, defined by the formula

$$(\mathcal{R}_g \star u)(\xi,\eta) \quad := \quad -\operatorname{Tr}_g\left[u\left(\mathcal{R}_g\left(\xi,\cdot\right)\eta,\cdot\right)\right],$$

is also symmetric (see section 3). For any smooth symmetric 2-tensor u we define the Ω-Lichnerowicz Laplacian $\Delta^{\Omega}_{L,g}$ as

$$\Delta^{\Omega}_{L,g}u \quad := \quad \Delta^{\Omega}_{g}u - 2\mathcal{R}_g \star u + u\operatorname{Ric}^{\star}_g(\Omega) + \operatorname{Ric}_g(\Omega)u^{\star}_g.$$

This operator is self-adjoint with respect to the scalar product (1.1) thanks to the identity

$$\langle \mathcal{R}_g \star u, v\rangle_g = \langle u, \mathcal{R}_g \star v\rangle_g, \tag{1.3}$$

for all symmetric 2-tensors u and v (see section 3). We define also the set of normalized volume forms $\mathcal{V}_1 := \left\{\Omega > 0 \mid \int_X \Omega = 1\right\}$. From now on we consider the maps h and H over $\mathcal{M}\times\mathcal{V}_1$. Notice that the tangent space of $\mathcal{M}\times\mathcal{V}_1$ is $T_{\mathcal{M}\times\mathcal{V}_1} = C^{\infty}(X, S^2T^{\star}_X)\oplus C^{\infty}(X,\Lambda^m T^{\star}_X)_0$, where $m = \dim_{\mathbb{R}} X$ and

$$C^{\infty}(X,\Lambda^m T^{\star}_X)_0 \quad := \quad \left\{ V \in C^{\infty}(X,\Lambda^m T^{\star}_X) \mid \int_X V = 0 \right\}.$$

We denote by $\operatorname{End}_g(T_X)$ the bundle of g-symmetric endomorphisms of T_X and by $C^{\infty}_{\Omega}(X,\mathbb{R})_0$ the space of smooth functions with zero integral with respect to Ω. We will systematically use the fact that for any $(g,\Omega)\in \mathcal{M}\times\mathcal{V}_1$ the tangent space $T_{\mathcal{M}\times\mathcal{V}_1,(g,\Omega)}$ identifies with $C^{\infty}(X,\operatorname{End}_g(T_X))\oplus C^{\infty}_{\Omega}(X,\mathbb{R})_0$ via the isomorphism

$$(v,V) \quad \longmapsto \quad \left(v^{\star}_g, V^{\star}_{\Omega}\right) := \left(g^{-1}v, V/\Omega\right).$$

With these notations holds the fundamental variation formulas

$$2D_{g,\Omega}h(v,V) = \Delta^{\Omega}_{L,g}v - L_{\nabla^{\star_{\Omega}}_g v^{\star}_g + \nabla_g V^{\star}_{\Omega}}g - 2v, \tag{1.4}$$

and

$$2D_{g,\Omega}H(v,V) = \Delta^{\Omega}_{g}V^{\star}_{\Omega} - \left(L_{\nabla^{\star_{\Omega}}_g v^{\star}_g + \nabla_g V^{\star}_{\Omega}}\Omega\right)^{\star}_{\Omega} - 2V^{\star}_{\Omega} - \left\langle v, h_{g,\Omega}\right\rangle_g, \tag{1.5}$$

where L_{ξ} denotes the Lie derivative in the direction ξ. (We will give a detailed proof in section 3). We infer that the variations of the non-linear operators h and H are strictly elliptic in restriction to the space

$$\mathbb{F}_{g,\Omega} \quad := \quad \left\{(v,V)\in T_{\mathcal{M}\times\mathcal{V}_1} \mid \nabla^{\star_{\Omega}}_g v^{\star}_g + \nabla_g V^{\star}_{\Omega} = 0\right\}.$$

This fact strongly suggests that the following flow represents a strictly parabolic system.

Definition 1. *The Soliton-Ricci flow is the smooth curve $(g_t,\Omega_t)_{t\geqslant 0}\subset \mathcal{M}\times\mathcal{V}_1$ solution of the evolution system*

$$\begin{cases} \dot{g}_t = -h_{g_t,\Omega_t}, \\ \dot{\Omega}_t = -\underline{H}_{g_t,\Omega_t}\Omega_t, \end{cases}$$

with

$$\underline{H}_{g,\Omega} \quad := \quad H_{g,\Omega} - \int_X H_{g,\Omega}\Omega.$$

Indeed this is the case. We show the strict parabolic statement in the proof of the following basic fact.

Lemma 1. *For every $(\check{g}_0, \Omega_0) \in \mathcal{M}\times\mathcal{V}_1$ there exists a unique smooth solution $(g_t, \Omega_t)_{t\geqslant 0} \subset \mathcal{M}\times\mathcal{V}_1$ of the Soliton-Ricci flow equation with initial data $(\check{g}_0/\lambda, \Omega_0)$, for some $\lambda > 0$. In the case (X, J_0) is a Fano variety and $\check{g}_0$ is J_0 invariant Kähler such that $\check{g}_0 J_0 \in 2\pi c_1(X)$, we can choose $\lambda = 1$. In this case the Soliton-Ricci-flow represents a smooth family of Kähler structures and normalized positive volumes $(J_t, g_t, \Omega_t)_{t\geqslant 0}$ uniquely determined by the evolution system*

$$\begin{cases} \dot{g}_t = -h_{g_t,\Omega_t}, \\ \dot{\Omega}_t = -\underline{H}_{g_t,\Omega_t}\Omega_t, \\ 2\dot{J}_t = \left[J_t, \dot{g}_t^{\star}\right]. \end{cases}$$

We call the latter the ***Soliton-Kähler-Ricci flow****.*

Let $\mathrm{Ric}_J(\Omega)$ be the Chern-Ricci form associated to the volume form Ω with respect to the complex structure J. We will show in section 3 that, if the initial data (J_0, g_0, Ω_0) satisfies

$$\omega := g_0 J_0 = \mathrm{Ric}_{J_0}(\Omega_0), \int_X \Omega_0 = 1,$$

then the Soliton-Kähler-Ricci flow equation is equivalent with the evolution system

$$\begin{cases} \omega = \mathrm{Ric}_{J_t}(\Omega_t)\ , \int_X \Omega_t = 1, \\ \dot{J}_t = \overline{\partial}_{T_{X,J_t}}\left(\omega^{-1} d\log\frac{\omega^n}{\Omega_t}\right). \end{cases} \tag{1.6}$$

Thus the Soliton-Kähler-Ricci flow preserves the initial symplectic structure ω.

Over a m-dimensional compact Riemannian manifold (X, g) we consider Perelman's $\mathcal{W}$-functional [29]

$$\begin{aligned} \mathcal{W}(g, f) &:= \int_X \left[|\nabla_g f|_g^2 + \mathrm{Scal}(g) + 2f - m\right] e^{-f} dV_g \\ &= \int_X \left[-\Delta_g f + \mathrm{Scal}(g) + 2f - m\right] e^{-f} dV_g. \end{aligned}$$

(We can use here the identity $\Delta_g e^{-f} = -(|\nabla_g f|_g^2 + \Delta_g f)e^{-f}$). If we use the identifications $f \longleftrightarrow \Omega := e^{-f} dV_g$ and $\mathcal{W}(g, f) \equiv \mathcal{W}(g, \Omega)$, then

$$\mathcal{W}(g, \Omega) = \int_X \left[\mathrm{Tr}_g\, h_{g,\Omega} + 2\log\frac{dV_g}{\Omega}\right]\Omega = 2\int_X H_{g,\Omega}\Omega.$$

With these notations Perelman's first variation formula for the functional $\mathcal{W} : \mathcal{M}\times\mathcal{V}_1 \longrightarrow \mathbb{R}$ in [29] writes as

$$D_{g,\Omega}\mathcal{W}(v, V) = -\int_X \left[\left\langle v, h_{g,\Omega}\right\rangle_g - 2V_{\Omega}^{\star}\underline{H}_{g,\Omega}\right]\Omega.$$

We consider the pseudo-Riemannian structure over the space $\mathcal{M}\times\mathcal{V}_1$ given by the formula $(g, \Omega) \in \mathcal{M}\times\mathcal{V}_1 \longmapsto G_{g,\Omega}$, with

$$G_{g,\Omega}(u, U; v, V) = \int_X \left[\langle u, v\rangle_g - 2\, U_{\Omega}^{\star} V_{\Omega}^{\star}\right]\Omega,$$

for all $(u, U), (v, V) \in T_{\mathcal{M}\times\mathcal{V}_1}$. We infer the identity

$$\nabla_G \mathcal{W}(g, \Omega) = -\left(h_{g,\Omega}, \underline{H}_{g,\Omega}\,\Omega\right).$$

This shows that the Soliton-Ricci flow is the gradient flow of the $\mathcal{W}$ functional with respect to the pseudo-Riemmanian structure G. Perelman's twice contracted second Bianchi identity (1.2) implies the equality

$$\left\{(g,\Omega)\in\mathcal{M}\times\mathcal{V}_1 \mid D_{g,\Omega}\mathcal{W}=0\right\}=\left\{(g,\Omega)\in\mathcal{M}\times\mathcal{V}_1 \mid h_{g,\Omega}=0\right\},$$

i.e the critical points of $\mathcal{W}$ are precisely the shrinking Ricci solitons. We provide at this point a geometric interpretation of the space $\mathbb{F}_{g,\Omega}$. Let

$$[g,\Omega]=\mathrm{Diff}_0(X)\cdot(g,\Omega),$$

be the orbit of the point (g,Ω) under the action of the identity component $\mathrm{Diff}_0(X)$ of the group of smooth diffeomorphisms of X. Then $\mathbb{F}_{g,\Omega}$ represents the orthogonal space, with respect to G, to the tangent space $T_{[g,\Omega],(g,\Omega)}$ at the point $(g,\Omega)\in\mathcal{M}\times\mathcal{V}_1$ of the orbit $[g,\Omega]$. In formal terms holds the equality

$$T^{\perp_G}_{[g,\Omega],(g,\Omega)}=\mathbb{F}_{g,\Omega}. \tag{1.7}$$

We define the anomaly space of the pseudo-Riemannian structure G at an arbitrary point (g,Ω) as the vector space

$$\mathbb{A}^{\Omega}_{g}:=\mathbb{F}_{g,\Omega}\cap T_{[g,\Omega],g,\Omega}.$$

In the case (g,Ω) is a shrinking Ricci-Soliton then the map

$$\begin{aligned}\mathrm{Ker}(\Delta^{\Omega}_{g}-2\mathbb{I}) &\longrightarrow \mathbb{A}^{\Omega}_{g}\\ u &\longmapsto 2\left(\nabla_g du,-u\Omega\right),\end{aligned}$$

is an isomorphism (see section 8). In the case (J,g,Ω) is a Kähler-Ricci soliton then $\mathbb{A}^{\Omega}_{g}$ is canonically isomorphic with the space of Killing vector fields of g. This is a consequence of a non trivial result (see corollary 5).

We denote by $\nabla^2_G\mathcal{W}(g,\Omega)$ the Hessian endomorphism of the $\mathcal{W}$ functional with respect to the pseudo-Riemannian structure G at the point $(g,\Omega)\in\mathcal{M}\times\mathcal{V}_1$. We show in lemma 7 that its restriction to the space $\mathbb{F}_{g,\Omega}$ is a strictly elliptic operator for any point (g,Ω). A simple consequence of Perelman's twice contracted second Bianchi type identity (1.2) is that the map

$$\nabla^2_G\mathcal{W}(g,\Omega):\mathbb{F}_{g,\Omega}\longrightarrow\mathbb{F}_{g,\Omega}, \tag{1.8}$$

is well defined in the case (g,Ω) is a shrinking Ricci-Soliton (see section 10). In this case holds also the inclusion

$$\mathbb{A}^{\Omega}_{g}\subseteq\mathbb{F}_{g,\Omega}\cap\mathrm{Ker}\,\nabla^2_G\mathcal{W}(g,\Omega).$$

(See lemma 8). In general (see section 10) for any point (g,Ω) holds the fundamental and deep property

$$\nabla^2_G\mathcal{W}(g,\Omega)(h_{g,\Omega},\underline{H}_{g,\Omega}\Omega)\in\mathbb{F}_{g,\Omega}. \tag{1.9}$$

This is quite crucial for the stability of the Soliton-Kähler-Ricci flow (see [27]). The following basic fact is a meaningful geometric reformulation of the monotony statement for Perelman's $\mathcal{W}$ functional discovered by the author in 2006 [21] and published in 2008.

Lemma 2. *Let (X,J) be a Fano manifold, let g be a J-invariant Kähler metric with symplectic form $\omega:=gJ\in 2\pi c_1(X,[J])$ and let $\Omega>0$ be the unique smooth volume form with $\int_X\Omega=1$ such that $\omega=\mathrm{Ric}_J(\Omega)$. Then Perelman's $\mathcal{W}$ functional is monotone increasing along the Soliton-Kähler-Ricci flow with initial data $(J_0,g_0,\Omega_0)=(J,g,\Omega)$. The monotony is strict unless (J,g) is a Kähler-Ricci soliton.*

From now on we will refer to the Soliton-Kähler-Ricci flow only if the initial data are as in the previous lemma. Let $\mathcal{J}_{\text{int}}$ be the space of smooth integrable complex structures over X. We consider the space of ω-compatible complex structures

$$\mathcal{J}_\omega \quad := \quad \{J \in \mathcal{J}_{\text{int}} \mid -\omega J \in \mathcal{M}\}\,.$$

Over a Fano manifold, the Chern-Ricci map provides a natural embedding of $\mathcal{J}_\omega$ inside $\mathcal{M} \times \mathcal{V}_1$. The image $\mathcal{S}_\omega \subset \mathcal{M} \times \mathcal{V}_1$ of this embedding is

$$\mathcal{S}_\omega \quad := \quad \left\{(g, \Omega) \in \mathcal{M}_\omega \times \mathcal{V}_1 \mid \omega = \text{Ric}_J(\Omega), J = g^{-1}\omega\right\},$$

with $\mathcal{M}_\omega := -\omega \cdot \mathcal{J}_\omega \subset \mathcal{M}$. The fact that the Soliton-Kähler-Ricci flow preserves the symplectic form ω strongly suggests the study of the restriction of Perelman's $\mathcal{W}$ functional over $\mathcal{S}_\omega$.

The space $\mathcal{J}_\omega$ may be singular in general. This implies that also the space $\mathcal{S}_\omega$ may be singular. We denote by $\text{TC}_{\mathcal{S}_\omega,(g,\Omega)}$ the tangent cone of $\mathcal{S}_\omega$ at an arbitrary point $(g, \Omega) \in \mathcal{S}_\omega$. This is by definition the union of all tangent vectors of $\mathcal{S}_\omega$ at the point (g, Ω). We notice that (see for example [23]) the tangent cone $\text{TC}_{\mathcal{M}_\omega,g}$ of $\mathcal{M}_\omega$ at an arbitrary point $g \in \mathcal{M}_\omega$ satisfies the inclusion

$$\text{TC}_{\mathcal{M}_\omega,g} \subseteq \mathbb{D}^J_{g,[0]}, \tag{1.10}$$

with

$$\mathbb{D}^J_{g,[0]} := \left\{ v \in C^\infty\left(X, S^2_{\mathbb{R}} T^\star_X\right) \;\mid\; v = -J^\star v J \;,\; \overline{\partial}_{T_{X,J}} v^\star_g = 0 \right\}.$$

The first variation of the Chern-Ricci form (see lemma 17) shows that for any $(g, \Omega) \in \mathcal{S}_\omega$ hold the inclusion

$$\text{TC}_{\mathcal{S}_\omega,(g,\Omega)} \subseteq \mathbb{T}^J_{g,\Omega}, \tag{1.11}$$

with

$$\mathbb{T}^J_{g,\Omega} := \left\{(v, V) \in \mathbb{D}^J_{g,[0]} \times T\mathcal{V}_1 \mid L_{\nabla^{\star_\Omega}_g v^\star_g + \nabla_g V^\star_\Omega}\, \omega = 0\right\}.$$

We consider also its sub-space

$$\mathbb{F}^J_{g,\Omega}[0] := \left\{(v, V) \in \mathbb{F}_{g,\Omega} \mid v \in \mathbb{D}^J_{g,[0]}\right\}.$$

In the case (X, J, g) is a compact Kähler-Ricci soliton then the map

$$\nabla^2_G \mathcal{W}(g, \Omega) : \mathbb{F}^J_{g,\Omega}[0] \longrightarrow \mathbb{F}^J_{g,\Omega}[0], \tag{1.12}$$

is well defined. Furthermore for any $(g, \Omega) \in \mathcal{S}_\omega$ the fundamental property (1.9) implies

$$\nabla^2_G \mathcal{W}(g, \Omega)(h_{g,\Omega}, \underline{H}_{g,\Omega}\Omega) \in \mathbb{F}^J_{g,\Omega}[0]. \tag{1.13}$$

This is precisely the key statement needed for the study of the stability of the Soliton-Kähler-Ricci flow in [27]. For any point $(g, \Omega) \in \mathcal{S}_\omega$ we denote by

$$[g, \Omega]_\omega := \text{Symp}^0(X, \omega) \cdot (g, \Omega) \subset \mathcal{S}_\omega,$$

the orbit of (g, Ω) under the action of the identity component $\text{Symp}^0(X, \omega)$ of the group of smooth symplectomorphisms of X. With these notations hold the property

$$T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} \cap \mathbb{T}^J_{g,\Omega} = \mathbb{F}^J_{g,\Omega}[0]\,. \tag{1.14}$$

This combined with (1.11) implies directly the geometric identity

$$T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} \cap \text{TC}_{\mathcal{S}_\omega,(g,\Omega)} = \mathbb{F}^J_{g,\Omega}[0] \cap \text{TC}_{\mathcal{S}_\omega,(g,\Omega)}, \tag{1.15}$$

for any $(g, \Omega) \in \mathcal{S}_\omega$. An other remarkable fact is that for any $(g, \Omega) \in \mathcal{S}_\omega$ the restriction of the symmetric form $G_{g,\Omega}$ over the vector space $\mathbb{T}^J_{g,\Omega}$, with $J := g^{-1}\omega$, is positive definite. This implies the G-orthogonal decomposition (see corollary 6 and the sub-section 18.2)

$$\mathbb{T}^J_{g,\Omega} \quad = \quad T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G \mathbb{F}^J_{g,\Omega}[0]. \tag{1.16}$$

The vector space of Ω-harmonic $T_{X,J}$-valued $(0,1)$-forms $\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$ embeds naturally inside $\mathbb{F}^J_{g,\Omega}[0]$ via the map $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \longmapsto (gA, 0)$. By abuse of notations we still denote by $\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \subset \mathbb{F}^J_{g,\Omega}[0]$ the image of this embedding. There exists an infinite dimensional vector space $\mathbb{E}^J_{g,\Omega}[0] \subseteq \mathbb{F}^J_{g,\Omega}[0]$, (see the sub-section 18.2 for its definition) such that the G-orthogonal decomposition holds true

$$\mathbb{F}^J_{g,\Omega}[0] = \mathbb{E}^J_{g,\Omega}[0] \oplus_G \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right).$$

We can explain now a more precise property of the tangent cone $\mathrm{TC}_{\mathcal{S}_\omega,(g,\Omega)}$. For this purpose we consider the Kuranishi space $\mathcal{K}_{J,g} \subset \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$, $0 \in \mathcal{K}_{J,g}$ of X. (See theorem 3 in the sub-section 21.4.4 of appendix B for its definition and properties.) In the sub-section 21.4.5 we define also the Kuranishi space of ω-polarized complex deformations $\mathcal{K}^\omega_J \subseteq \mathcal{K}_{J,g}$ of the Fano manifold (X, J, ω). (See the definition 2). Then holds the inclusions

$$T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G \mathbb{E}^J_{g,\Omega}[0] \oplus_G \mathrm{TC}_{\mathcal{K}^\omega_J,0}$$

$$\subseteq \quad \mathrm{TC}_{\mathcal{S}_\omega,(g,\Omega)} \tag{1.17}$$

$$\subseteq \quad T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G \mathbb{E}^J_{g,\Omega}[0] \oplus_G \mathrm{TC}_{\mathcal{K}_{J,g},0}. \tag{1.18}$$

Let $F := f - \int_X f\,\Omega$. We define the non-negative cone of Ω-harmonic variations

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_{\geqslant 0} \quad := \quad \left\{ A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \mid \int_X |A|^2_g\, F\,\Omega \geqslant 0 \right\},$$

and the sub-cone

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_0 \quad := \quad \left\{ A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \mid \int_X |A|^2_g\, F\,\Omega = 0 \right\}.$$

In the Kähler-Einstein case holds the obvious identities

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_{\geqslant 0} = \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_0 = \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right).$$

In the Dancer-Wang Kähler-Ricci soliton case $\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_0 \neq \{0\}$, thanks to a result in Hall-Murphy [16]. Let $H_{T_{X,J}}$ be the L^2_Ω-projector over the space $\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$. We define also the non-negative cone

$$\mathbb{T}^{J,\geqslant 0}_{g,\Omega} \quad := \quad \left\{ (v, V) \in \mathbb{T}^J_{g,\Omega} \mid H_{T_{X,J}} v^\star_g \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_{\geqslant 0} \right\},$$

and in a similar way $\mathbb{T}^{J,0}_{g,\Omega}$. An interesting non-negative cone from the geometric point of view is also

$$\mathrm{TC}^{\geqslant 0}_{\mathcal{S}_\omega,(g,\Omega)} \quad := \quad \mathrm{TC}_{\mathcal{S}_\omega,(g,\Omega)} \cap \mathbb{T}^{J,\geqslant 0}_{g,\Omega}.$$

Let now KRS_ω be the set of all Kähler-Ricci solitons inside $\mathcal{S}_\omega$. We observe that Perelman's twice contracted second Bianchi type identity implies

$$\mathrm{KRS}_\omega \quad = \quad \left\{ (g, \Omega) \in \mathcal{S}_\omega \mid \underline{H}_{g,\Omega} = 0 \right\}.$$

Notice also that for any $(g, \Omega) \in \mathrm{KRS}_\omega$ holds the inclusions $[g, \Omega]_\omega \subseteq \mathrm{KRS}_\omega$ and

$$T_{[g,\Omega]_\omega,(g,\Omega)} \subseteq \mathrm{TC}_{\mathrm{KRS}_\omega,(g,\Omega)} \subseteq \mathrm{Ker}\, D_{g,\Omega}\underline{H} \cap \mathrm{TC}_{\mathcal{S}_\omega,(g,\Omega)}\,.$$

The following statement provides a finite dimensional reduction of the stability problem for Kähler-Ricci solitons. This reduction represents a different and more sharp formulation of the solution of this well known problem obtained recently in [28].

Theorem 1. (***Main result. The stability of Kähler-Ricci solitons***)

Let (X, J, g) be a compact Kähler-Ricci soliton and let $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\omega := gJ = \mathrm{Ric}_J(\Omega)$. Then for all $(v, V) \in \mathbb{T}^{J,\geqslant 0}_{g,\Omega}$ the Hessian form of Perelman's $\mathcal{W}$ functional with respect to the pseudo-Riemannian structure G at the point (g, Ω), in the direction (v, V) satisfies the inequality

$$\nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) \quad \leqslant \quad 0\,, \tag{1.19}$$

with equality if and only if

$$(v, V) \quad \in \quad \mathrm{Ker}\, D_{g,\Omega}\underline{H} \cap \mathbb{T}^{J,0}_{g,\Omega} \tag{1.20}$$

$$= \quad T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_0 \tag{1.21}$$

$$\supseteq \quad \mathrm{TC}_{\mathrm{KRS}_\omega,(g,\Omega)}\,. \tag{1.22}$$

In more explicit/classic terms the previous statement shows that for any smooth curve $(g_t, \Omega_t)_{t\in\mathbb{R}} \subset \mathcal{M} \times \mathcal{V}_1$ (not necessarily in $\mathcal{S}_\omega$!) with $(g_0, \Omega_0) = (g, \Omega)$ a Kähler-Ricci soliton and with $(\dot{g}_0, \dot{\Omega}_0) = (v, V) \in \mathbb{T}^{J,\geqslant 0}_{g,\Omega}$ holds the inequality

$$\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) \quad \leqslant \quad 0\,,$$

with equality if and only if $(v, V) \in \mathrm{Ker}\, D_{g,\Omega}\underline{H} \cap \mathbb{T}^{J,0}_{g,\Omega}$. (This is precisely the statement obtained in [28]). The identity (1.21) and the inclusion (1.22) are part of the statement in the main theorem 1. We deduce in particular that the second variation of $\mathcal{W}$ is istrictly negative in the directions $\mathbb{E}^J_{g,\Omega}[0]$.

In section 17 we obtain also a quite sharp second variation formula for Perelman's $\mathcal{W}$ functional with respect to more general variations $(v, V) \in \mathbb{F}_{g,\Omega}$ over a Kähler-Ricci soliton point (g, Ω). These variations arise from variations of Kähler structures preserving the first Chern class of X.

This formula provide a precise control of the sign of the second variation of Perelman's $\mathcal{W}$ functional over a Kähler-Ricci soliton point. This can be of independent interest for experts. (In particular we will see below some general consequences for the classical stability of Kähler-Einstein metrics.) For our geometric applications the most striking particular case is the one corresponding to the main theorem 1.

The highly geometric nature of the Soliton-Kähler-Ricci flow combined with the main theorem 1, suggest to the author the following version of the Hamilton-Tian conjecture (compare with the statements made in [38] and [6]).

Conjecture 1. *Let (X, J_0) be a Fano manifold and let $\omega \in 2\pi c_1(X, [J_0])$ be an arbitrary J_0-invariant Kähler form. Then there exists an analytic subset $\Sigma \subset X$ of real codimension greater or equal to 4, which is empty for generic choices of J_0 inside $[J_0]$ and ω inside $2\pi c_1(X, [J_0])$, there exists a smooth complex structure $J \in [J_0]$ outside Σ and a smooth volume form $\Omega > 0$ outside Σ such that;*

$$\begin{cases} \omega \quad = \quad \mathrm{Ric}_J(\Omega)\,, \\ \\ \overline{\partial}_{T_{X,J}}\left(\omega^{-1} d\log\frac{\omega^n}{\Omega}\right) \quad = \quad 0\,, \end{cases}$$

outside Σ, i.e. the Riemannian metric $g := -\omega J$, is a smooth J-invariant Kähler-Ricci soliton outside Σ. The triple (J, g, Ω) is obtained as the limit in the smooth topology of $X \setminus \Sigma$, as $t \to +\infty$, of the Soliton-Kähler-Ricci flow with initial data (J_0, g_0, Ω_0) where $g_0 := -\omega J_0$ and $\omega = \mathrm{Ric}_{J_0}(\Omega_0)$, with $\int_X \Omega_0 = 1$.

We explain now a very particular consequence of our study of the stability problem. This consequence provides a result on the stability in the classical sense of Kähler-Einstein manifolds. We introduce first a few basic notations.

Let (X, J) be a compact Kähler manifold and let $c_1 \equiv c_1(X, [J]) \in H^2_d(X, \mathbb{R})$. We denote by $\mathcal{KS}$ the space of Kähler structures over X and we set

$$\mathcal{KS}_{2\pi c_1} \quad := \quad \left\{(J, g) \in \mathcal{KS} \mid gJ \in 2\pi c_1\right\}.$$

We define also the set $\mathbb{KV}^J_g(2\pi c_1)$ of symmetric variations of Kähler structures preserving the first Chern class of X. The latter is defined as the set of elements $v \in C^\infty\left(X, S^2_{\mathbb{R}} T^\star_X\right)$ such that there exists a smooth curve $(J_t, g_t)_t \subset \mathcal{KS}_{2\pi c_1}$ with $(J_0, g_0) = (J, g)$, $\dot{g}_0 = v$ and $\dot{J}_0 = (\dot{J}_0)^T_g$. In section 14 we show the inclusion

$$\mathbb{KV}^J_g(2\pi c_1) \subseteq \mathbb{D}^J_{g,0}\,, \tag{1.23}$$

with

$$\mathbb{D}^J_{g,0} := \left\{v \in C^\infty\left(X, S^2_{\mathbb{R}} T^\star_X\right) \mid \partial^g_{T_{X,J}}(v'_J)^\star_g = 0, \overline{\partial}_{T_{X,J}}(v''_J)^\star_g = 0, \left\{v'_J J\right\}_d = 0\right\},$$

where v'_J and v''_J denote respectively the J-invariant and J-anti-invariant parts of v and $\{\alpha\}_d$ denotes the De Rham cohomology class of any d-closed form α. We introduce also the classical stability operator (see [2])

$$\mathcal{L}_g \quad := \quad \Delta_g - 2\mathcal{R}_g\star\,,$$

acting on smooth symmetric 2-tensors. With these notations we can state the following stability (in the classical sense) result.

Theorem 2. *Let (X, J, g) be a Fano Kähler-Einstein manifold. Then for any $v \in \operatorname{Ker}\nabla^\star_g \cap \mathbb{D}^J_{g,0}$, holds the inequality*

$$\int_X \left\langle \mathcal{L}_g v, v\right\rangle_g dV_g \quad \geqslant \quad 0\,,$$

with equality if and only if $v^\star_g \in \mathcal{H}^{0,1}_g\left(T_{X,J}\right)$.

(See sub-section 17.1 for the proof). A similar result in the case of negative or vanishing first Chern class has been proved in the remarkable paper [9] (see also [8]). The statement about the equality case holds also under more general assumptions (see lemma 29 in the appendix B). We wish to point out to the readers that the sections 4 and 5 are not needed for the proof of the main result. However section 4 is crucial for the general gradient flow picture explained in this paper. In the next section we enlighten the results obtained by other authors in the long standing problem of the stability of Kähler-Ricci solitons and on the Hamilton-Tian conjecture.

2 Other works on the subject

A question of central importance in complex differential geometry is the Hamilton-Tian conjecture.

Solutions of this conjecture have been posted on the arXiv server in (2013) by Tian-Zhang [38] in complex dimension 3 and quite recently by Chen-Wang [6] in general.

Since we have learned about this conjecture in 2004 we immediately asked ourself which one is the precise notion of gauge needed for the convergence. (The Kähler-Ricci flow $\left(J_0, \hat{g}_t\right)_{t\geqslant 0}$ needs to be modified since its **formal** limit $\left(J_0, \hat{g}_\infty\right)$ as $t \to +\infty$ is a is a Kähler-Einstein metric, but Fano manifolds do not always admit such ones!)

It turns out that the Soliton-Kähler-Ricci flow introduced in this paper corresponds to a modification of the Kähler-Ricci flow via the gauge provided by the gradient of the Ricci potentials.

To the very best of our knowledge the Soliton-Kähler-Ricci flow with variable volume forms introduced in this paper does not appear nowhere in the literature.

In our previous works [24] and [25], we introduced also the notion of Soliton-Kähler-Ricci flow with fixed volume form. This leads to a complete different approach which conducts naturally to the study of the existence of ancient solutions of the Kähler-Ricci flow and their modified (according to [24] and [25]) convergence as $t \to -\infty$. This approach requires some particular geometric conditions (which imply some strong regularity) on the initial data. The key point in [24] and [25] is that these conditions represent a conservative law along the Soliton-Kähler-Ricci flow with fixed volume form. These conditions imply good convexity properties for the convergence of this flow.

We review now the modifications of the Kähler-Ricci flow made by other authors. We can find two frequent approaches in the literature. One is based on the gauge transformation generated by a holomorphic vector field with imaginary part generating an S^1-action on the manifold (see [34] and [31] for a very elegant construction). A Kähler-Ricci-soliton vector field provides such example.

The second approach, which has been used quite intensively in the last years is based on the gauge modification constructed via the minimizers of Perelman's $\mathcal{W}$ functional (see [36] and [33]). As far as known the minimizers are unique only in a small neighborhood of the Kähler-Ricci soliton. Therefore the "modified Kähler-Ricci flow" in [36] and [33] exists only in such small neighborhood.

For historical reasons it is important to remind that Hamilton [17] pointed out first that to any flow of Kähler structures with fixed complex structure corresponds an other flow of Kähler structures which preserves the symplectic form (see also Donaldson [10] for the same remark). He suggested this approach for the study of the Kähler-Ricci flow. As far as we know he did not pursuit on this idea.

As explained in the introduction our definition of the Soliton-Ricci flow with variable volume forms was inspired to us from Perelman's twice contracted second Bianchi type identity and from the strict ellipticity of the first variation of the maps h and H in the directions $\mathbb{F}$.

It was surprising for us to discover that the corresponding Soliton-Kähler-Ricci flow with variable volume forms (from now on we will refer only to this flow) preserves the symplectic structure.

We realized quickly the power of this fact. It allows indeed the application of Futaki's weighted complex Bochner identity and the uniform lower bound on the first eigenvalue of the complex weighted Laplacian [13]. The main feature of the Soliton-Kähler-Ricci flow in this paper is the jumping of the complex structure at the limit when $t \to +\infty$. This phenomenon is necessary for the existence of Kähler-Ricci solitons in general. We learned for the first time about this key phenomenon in the Pioneer work of [30]. In this fundamental work the authors introduce a condition on stability (is the condition (B) in [30]) witch is the key phenomenon occurring in the convergence of the Kähler-Ricci flow. We refer also to [32] for further developments.

We remind now that by definition, the stability of a critical point of a functional corresponds to determine a sign of its second variation in determinate directions.

The stability of critical metrics for natural geometric functionals was naturally born with differential geometry (see [2]). The main classic example is the Einstein metric. In the case of this metric the corresponding functional is the integral of the scalar curvature.

In 2003 Grigory Perelman astonished the mathematical community with his spectacular proof of the Poincaré conjecture. In this celebrated paper [29] he introduced various entropy functionals for Ricci-solitons. Shrinking Ricci-solitons correspond to critical points of his $\mathcal{W}$ functional or his entropy functional ν.

Since then, the second variation of Perelman's functionals $\mathcal{W}$ and ν has been studied quite intensively. It started in 2004 with the works of Cao-Hamilton-Imlanen [3], [4] and Tian-Zhu [35] independently. It continued with [5] and [15], [16].

We wish to point out that the results in this paper and in [23] are of completely different nature with respect to the previous works. The reason is that in our work we compute the second variation of Perelman's $\mathcal{W}$ functional with respect to the pseudo-Riemannian structure G. (The work [23] is a particular case.)

An important fact about Kähler-Ricci solitons is that once they exist, one can obtain the Einstein condition by proving the vanishing of the Futaki invariant [12]. From our point of view they provide a natural and necessary generalization in order to control the Einstein condition.

The stability of Kähler-Ricci solitons is important in order to understand the convergence of the Kähler-Ricci flow. The first work on the subject is due to Tian-Zhu, see [35].

In 2009 Sun-Wang [33] posted on the arXiv server a stability result for the Kähler-Ricci flow basing on the Lojasiewicz inequality (see [7]). In this paper the authors use the modified flow in [36]. The same method was used in Ache [1], where a uniform bound assumption on the curvature is made. We report finally a quite recent work on the same subject by Kröncke [19]. This author combines the technical details in [33], [1] and [7] in the Riemannian set up.

The statements made in this section are based on the very best of our knowledge and understanding of the subject. We sincerely apologize to other authors in case of inaccuracies or omissions in the claims of this section.

3 Proof of the first variation formulas for the maps h and H

3.1 The first variation of the Bakry-Emery-Ricci tensor

We remind (see [23]) that the first variation of the Bakry-Emery-Ricci tensor with fixed volume form $\Omega > 0$ is given by the formula

$$2\frac{d}{dt}\operatorname{Ric}_{g_t}(\Omega) = -\nabla^{\star_\Omega}_{g_t}\mathcal{D}_{g_t}\dot{g}_t, \tag{3.1}$$

where $\mathcal{D}_g := \hat{\nabla}_g - 2\nabla_g$, with $\hat{\nabla}_g$ being the symmetrization of ∇_g acting on symmetric 2-tensors. Explicitly

$$\hat{\nabla}_g\alpha(\xi_0,\ldots,\xi_p) \quad := \quad \sum_{j=0}^{p}\nabla\alpha(\xi_j,\xi_0,\ldots,\hat{\xi}_j,\ldots,\xi_p),$$

for all p-tensors α. Fixing an arbitrary time τ and time deriving at $t=\tau$ the decomposition

$$\operatorname{Ric}_{g_t}(\Omega_t) \quad = \quad \operatorname{Ric}_{g_t}(\Omega_\tau) - \nabla_{g_t} d\log\frac{\Omega_t}{\Omega_\tau},$$

we deduce, thanks to (3.1), the general variation formula

$$2\frac{d}{dt}\operatorname{Ric}_{g_t}(\Omega_t) = -\nabla^{\star_{\Omega_t}}_{g_t}\mathcal{D}_{g_t}\dot{g}_t - 2\nabla_{g_t} d\frac{\dot{\Omega}_t}{\Omega_t}. \tag{3.2}$$

This formula implies directly Perelman's general first variation formula for the $\mathcal{W}$ functional (see appendix A). We define the Hodge Laplacian (resp. the Ω-Hodge Laplacian) operators acting on q-forms as

$$\Delta_{T_{X,g}} \quad := \quad \nabla_{T_{X,g}}\nabla^\star_g + \nabla^\star_g\nabla_{T_{X,g}},$$

$$\Delta^\Omega_{T_{X,g}} \quad := \quad \nabla_{T_{X,g}}\nabla^{\star_\Omega}_g + \nabla^{\star_\Omega}_g\nabla_{T_{X,g}}.$$

We remind also the following Weitzenböck type formula proved in [24]

Lemma 3. *Let (X,g) be a orientable Riemannian manifold, let $\Omega > 0$ be a smooth volume form and let $A \in C^\infty(X,\operatorname{End}(T_X))$. Then*

$$\Delta^\Omega_{T_{X,g}}A \quad = \quad \Delta^\Omega_g A - \mathcal{R}_g \star A + A\operatorname{Ric}^\star_g(\Omega),$$

Where $(\mathcal{R}_g\star A)\,\xi := \operatorname{Tr}_g[(\xi\neg\mathcal{R}_g)A]$ for all $\xi \in T_X$.

In analogy to the Ω-Hodge Laplacian we can define the Laplace type operator

$$\hat{\Delta}^\Omega_g := \nabla^{\star_\Omega}_g\hat{\nabla}_g - \hat{\nabla}_g\nabla^{\star_\Omega}_g : C^\infty(X,S^pT^\star_X) \longrightarrow C^\infty(X,S^2T^\star_X).$$

Using this notation we observe that for any $u \in C^\infty(X, S^2T_X^\star)$ hold the identities

$$\begin{aligned} -\nabla_g^{\star_\Omega}\mathcal{D}_g u &= \left(2\Delta_g^\Omega - \hat{\Delta}_g^\Omega\right)u - \hat{\nabla}_g\nabla_g^{\star_\Omega}u \\ &= \left(2\Delta_g^\Omega - \hat{\Delta}_g^\Omega\right)u - L_{\nabla_g^{\star_\Omega}u_g^\star}g, \end{aligned}$$

The last one follows from the equalities $\nabla_g^{\star_\Omega}u = g\nabla_g^{\star_\Omega}u_g^\star$ and $\hat{\nabla}_g(g\xi) = L_\xi g$, $\xi \in C^\infty(X, T_X)$. We observe now that for any symmetric 2-tensor u the tensor $\mathcal{R}_g \star u$ is also symmetric. In fact let $(e_k)_k$ be a $g(x)$-orthonormal base of $T_{X,x}$. Then

$$-(\mathcal{R}_g \star u)(\xi,\eta) = R_g(\xi, e_k, u_g^\star e_k, \eta) = R_g(\eta, u_g^\star e_k, e_k, \xi).$$

Furthermore if we choose the $g(x)$-orthonormal base $(e_k)_k$ such that u is diagonal with respect to this one, then

$$R_g(\eta, u_g^\star e_k, e_k, \xi) = R_g(\eta, e_k, u_g^\star e_k, \xi) = -(\mathcal{R}_g \star u)(\eta,\xi).$$

We observe also that the previous computation shows the identity

$$\begin{aligned} (\mathcal{R}_g \star u)(\xi,\eta) &= R_g(\xi, e_k, \eta, u_g^\star e_k) \\ &= g(\mathcal{R}_g(\xi, e_k)u_g^\star e_k, \eta) \\ &= g\left(\left(\mathcal{R}_g \star u_g^\star\right)\xi, \eta\right), \end{aligned}$$

i.e

$$(\mathcal{R}_g \star u)_g^\star = \mathcal{R}_g \star u_g^\star. \tag{3.3}$$

We deduce in particular the equality

$$\mathcal{R}_g \star u_g^\star = \left(\mathcal{R}_g \star u_g^\star\right)_g^T. \tag{3.4}$$

We remind that the Ω-Lichnerowicz Laplacian $\Delta_{L,g}^\Omega$ is self-adjoint with respect to the scalar product (1.1) thanks to the identity (1.3) that we show now.

We pick a $g(x)$-orthonormal base $(e_k)_k \subset T_{X,x}$ such that v is diagonal with respect to this one at the point x. Using (3.3) we infer

$$\begin{aligned} \langle \mathcal{R}_g \star u, v\rangle_g &= \mathrm{Tr}_{\mathbb{R}}\left[\left(\mathcal{R}_g \star u_g^\star\right)v_g^\star\right] \\ &= R_g(v_g^\star e_l, e_k, e_l, u_g^\star e_k) \\ &= R_g(e_l, e_k, v_g^\star e_l, u_g^\star e_k) \\ &= R_g(e_k, e_l, u_g^\star e_k, v_g^\star e_l) \\ &= \langle \mathcal{R}_g \star v, u\rangle_g, \end{aligned}$$

since these identities are independent of the choice of the $g(x)$-orthonormal base $(e_k)_k \subset T_{X,x}$.

Lemma 4. *For any $g \in \mathcal{M}$ and $u \in C^\infty(X, S^2T_X^\star)$ holds the Weitzenböck type formula*

$$-\nabla_g^{\star_\Omega}\mathcal{D}_g u = \Delta_{L,g}^\Omega u - L_{\nabla_g^{\star_\Omega}u_g^\star}g.$$

Proof. The required formula follows from the identity

$$\Delta^{\Omega}_{L,g}u = \left(2\Delta^{\Omega}_{g} - \hat{\Delta}^{\Omega}_{g}\right)u. \tag{3.5}$$

In order to show this identity we expand $\hat{\Delta}^{\Omega}_{g}u = \nabla^{\star_\Omega}_{g}\hat{\nabla}_g u - \hat{\nabla}_g\nabla^{\star_\Omega}_{g}u$. We observe first

$$\nabla^{\star_\Omega}_{g}\hat{\nabla}_g u(\xi,\eta) \quad = \quad \nabla^{\star}_{g}\hat{\nabla}_g u(\xi,\eta) + \hat{\nabla}_g u(\nabla_g f,\xi,\eta).$$

We fix an arbitrary point $x_0 \in X$ and we choose the vector fields ξ and η such that $0 = \nabla_g\xi(x_0) = \nabla_g\eta(x_0)$. Let $(e_k)_k$ be a g-orthonormal local frame such that $\nabla_g e_k(x_0) = 0$. Then at the point x_0 hold the identities

$$\begin{aligned}
\nabla^{\star}_{g}\hat{\nabla}_g u(\xi,\eta) &= -\nabla_{g,e_k}\hat{\nabla}_g u(e_k,\xi,\eta)\\
&= -\nabla_{g,e_k}\left[\hat{\nabla}_g u(e_k,\xi,\eta)\right]\\
&= -\nabla_{g,e_k}\left[\nabla_g u(e_k,\xi,\eta) + \nabla_g u(\xi,e_k,\eta) + \nabla_g u(\eta,e_k,\xi)\right]\\
&= -\nabla_{g,e_k}\nabla_{g,e_k}u(\xi,\eta) - \nabla_{g,e_k}\nabla_{g,\xi}u(e_k,\eta) - \nabla_{g,e_k}\nabla_{g,\eta}u(e_k,\xi),
\end{aligned}$$

and

$$\hat{\nabla}_g u(\nabla_g f,\xi,\eta) \quad = \quad \nabla_g u(\nabla_g f,\xi,\eta) + \nabla_g u(\xi,\nabla_g f,\eta) + \nabla_g u(\eta,\nabla_g f,\xi).$$

Moreover

$$\hat{\nabla}_g\nabla^{\star_\Omega}_{g}u(\xi,\eta) \quad = \quad \hat{\nabla}_g\nabla^{\star}_{g}u(\xi,\eta) + \hat{\nabla}_g\left(\nabla_g f\neg u\right)(\xi,\eta),$$

and at the point x_0 hold the identities

$$\begin{aligned}
\hat{\nabla}_g\nabla^{\star}_{g}u(\xi,\eta) &= \nabla_{g,\xi}\nabla^{\star}_{g}u\cdot\eta + \nabla_{g,\eta}\nabla^{\star}_{g}u\cdot\xi\\
&= \nabla_{g,\xi}\left[\nabla^{\star}_{g}u\cdot\eta\right] + \nabla_{g,\eta}\left[\nabla^{\star}_{g}u\cdot\xi\right]\\
&= -\nabla_{g,\xi}\left[\nabla_{g,e_k}u(e_k,\eta)\right] - \nabla_{g,\eta}\left[\nabla_{g,e_k}u(e_k,\xi)\right]\\
&= -\nabla_{g,\xi}\nabla_{g,e_k}u(e_k,\eta) - \nabla_{g,\eta}\nabla_{g,e_k}u(e_k,\xi),
\end{aligned}$$

and

$$\begin{aligned}
\hat{\nabla}_g\left(\nabla_g f\neg u\right)(\xi,\eta) &= \nabla_{g,\xi}\left(\nabla_g f\neg u\right)\cdot\eta + \nabla_{g,\eta}\left(\nabla_g f\neg u\right)\cdot\xi\\
&= \nabla_{g,\xi}\left[u(\nabla_g f,\eta)\right] + \nabla_{g,\eta}\left[u(\nabla_g f,\xi)\right]\\
&= \nabla_g u(\xi,\nabla_g f,\eta) + \left(u\nabla^2_g f\right)(\xi,\eta) + \nabla_g u(\eta,\nabla_g f,\xi) + \left(\nabla_g df u^{\star}_{g}\right)(\xi,\eta).
\end{aligned}$$

Let now $A \in C^{\infty}(X,\mathrm{End}(T_X))$. We denote by $A\neg u$ the 2-tensor defined by the formula

$$(A\neg u)(\xi,\eta) \quad := \quad u(A\xi,\eta) + u(\xi,A\eta).$$

We observe that if μ,ζ are two germs of vector fields near x_0 such that $[\mu,\zeta](x_0) = 0$ then holds the identity at the point x_0

$$\nabla_{g,\mu}\nabla_{g,\zeta}u - \nabla_{g,\zeta}\nabla_{g,\mu}u \quad = \quad -\mathcal{R}_g(\mu,\zeta)\neg u.$$

Using this identity we infer the equalities at the point x_0

$$\begin{aligned}\left(\nabla_{g,\xi}\nabla_{g,e_k}u-\nabla_{g,e_k}\nabla_{g,\xi}u\right)(e_k,\eta) &= -u\left(\mathcal{R}_g(\xi,e_k)e_k,\eta\right)-u\left(e_k,\mathcal{R}_g(\xi,e_k)\eta\right)\\ &= -\left(u\,\mathrm{Ric}^{\star}(g)\right)(\xi,\eta)+(\mathcal{R}_g\star u)(\xi,\eta),\\ \left(\nabla_{g,\eta}\nabla_{g,e_k}u-\nabla_{g,e_k}\nabla_{g,\eta}u\right)(e_k,\xi) &= -\left(\mathrm{Ric}(g)u_g^{\star}\right)(\xi,\eta)+(\mathcal{R}_g\star u)(\xi,\eta),\end{aligned}$$

by obvious symmetries. Combining the identities obtained so far and simplifying we obtain the identity

$$\hat{\Delta}_g^{\Omega}u = \Delta_g^{\Omega}u+2\mathcal{R}_g\star u-u\,\mathrm{Ric}_g^{\star}(\Omega)-\mathrm{Ric}_g(\Omega)u_g^{\star},$$

which in its turn implies the required identity (3.5). □

The Weitzenböck type identity in lemma 4 combined with the variation formula (3.2) implies directly the variation formula (1.4).

3.2 Proof of the first variation formula for Perelman's H-function

We show now the variation formula (1.5). For this purpose let $0<(g_t,\Omega_t)_t\subset\mathcal{M}\times\mathcal{V}_1$ be a smooth family and set as usual $f_t:=\log\frac{dV_{g_t}}{\Omega_t}$. We start time deriving the identity

$$-\Delta_{g_t}^{\Omega_t}f_t = \mathrm{div}^{\Omega_t}\nabla_{g_t}f_t.$$

We compute first the variation of the Ω-divergence operator. Set $u_t:=\dot{\Omega}_t^{\star}$ and time derive the definition identity

$$d\left(\xi\neg\Omega_t\right) = (\mathrm{div}^{\Omega_t}\xi)\Omega_t.$$

We infer

$$d\left(\xi\neg u_t\Omega_t\right) = \left(\frac{d}{dt}\,\mathrm{div}^{\Omega_t}\xi\right)\Omega_t+u_t(\mathrm{div}^{\Omega_t}\xi)\Omega_t.$$

Moreover expanding the left hand side we obtain

$$d\left(\xi\neg u_t\Omega_t\right) = (\xi.u_t)\Omega_t+u_t d\left(\xi\neg\Omega_t\right),$$

which implies the formula

$$\left(\frac{d}{dt}\,\mathrm{div}^{\Omega_t}\right)\xi = g\left(\nabla_g\dot{\Omega}_t^{\star},\xi\right).$$

We observe also the variation formulas

$$\frac{d}{dt}\left(\nabla_{g_t}f_t\right)=\nabla_{g_t}\dot{f}_t-\dot{g}_t^{\star}\nabla_{g_t}f_t, \tag{3.6}$$

and

$$\dot{f}_t=\frac{1}{2}\,\mathrm{Tr}_{g_t}\dot{g}_t-\dot{\Omega}_t^{\star}. \tag{3.7}$$

Combining all these formulas we obtain

$$\begin{aligned}-\frac{d}{dt}\Delta_{g_t}^{\Omega_t}f_t &= \left(\frac{d}{dt}\,\mathrm{div}^{\Omega_t}\right)\nabla_{g_t}f_t+\mathrm{div}^{\Omega_t}\frac{d}{dt}\left(\nabla_{g_t}f_t\right)\\ &= g_t\left(\nabla_{g_t}\dot{\Omega}_t^{\star},\nabla_{g_t}f_t\right)+\Delta_{g_t}^{\Omega_t}\left(\dot{\Omega}_t^{\star}-\frac{1}{2}\,\mathrm{Tr}_{g_t}\dot{g}_t\right)-\mathrm{div}^{\Omega_t}\left(\dot{g}_t^{\star}\nabla_{g_t}f_t\right).\end{aligned}$$

We expand last term using the identity

$$\operatorname{div}^{\Omega}\xi \;=\; \operatorname{Tr}_{\mathbb{R}}\left(\nabla_g\xi\right) - g\left(\xi, \nabla_g f\right).$$

We obtain with respect to a $g_t(x)$-orthonormal basis $(e_k)_k \subset T_{X,x}$ at an arbitrary space-time point (x, t)

$$\begin{aligned}
\operatorname{div}^{\Omega_t}\left(\dot g_t^{\star}\nabla_{g_t} f_t\right) &= g_t(\nabla_{g,e_k}\left(\dot g_t^{\star}\nabla_{g_t} f_t\right), e_k) - g_t\left(\dot g_t^{\star}\nabla_{g_t} f_t, \nabla_{g_t} f_t\right)\\
&= g_t\left(\nabla_{g,e_k}\dot g_t^{\star}\cdot\nabla_{g_t} f_t + \dot g_t^{\star}\nabla_{g,e_k}\nabla_{g_t} f_t, e_k\right) - g_t\left(\dot g_t^{\star}\nabla_{g_t} f_t, \nabla_{g_t} f_t\right)\\
&= g_t\left(\nabla_{g_t} f_t, \nabla_{g,e_k}\dot g_t^{\star}\cdot e_k\right) + g_t\left(\nabla^2_{g,e_k} f_t, \dot g_t^{\star} e_k\right) - g_t\left(\nabla_{g_t} f_t, \dot g_t^{\star}\nabla_{g_t} f_t\right)\\
&= -g_t\left(\nabla^{\star_{\Omega_t}}_{g_t}\dot g_t^{\star}, \nabla_{g_t} f_t\right) + \langle\nabla_{g_t} df_t, \dot g_t\rangle_{g_t}.
\end{aligned}$$

We infer the variation formula

$$-\frac{d}{dt}\Delta^{\Omega_t}_{g_t} f_t \;=\; \Delta^{\Omega_t}_{g_t}\left(\dot\Omega_t^{\star} - \frac{1}{2}\operatorname{Tr}_{g_t}\dot g_t\right) + g_t\left(\nabla^{\star_{\Omega_t}}_{g_t}\dot g_t^{\star} + \nabla_{g_t}\dot\Omega_t^{\star}, \nabla_{g_t} f_t\right) - \langle\dot g_t, \nabla_{g_t} df_t\rangle_{g_t}. \tag{3.8}$$

We observe next the identity

$$\begin{aligned}
2\frac{d}{dt}h_t^{\star} &= 2\dot h_t^{\star} - 2\dot g_t^{\star} h_t^{\star}\\
&= \Delta^{\Omega_t}_{g_t}\dot g_t^{\star} - 2(\mathcal{R}_{g_t}\star\dot g_t)_t^{\star} + \dot g_t^{\star}\operatorname{Ric}^{\star}_{g_t}(\Omega_t) + \operatorname{Ric}^{\star}_{g_t}(\Omega_t)\dot g_t^{\star} - \left(L_{\nabla^{\star_{\Omega_t}}_{g_t}\dot g_t^{\star} + \nabla_{g_t}\dot\Omega_t^{\star}} g_t\right)^{\star}_t - 2\dot g_t^{\star} - 2\dot g_t^{\star} h_t^{\star},
\end{aligned}$$

thanks to the variation formula (1.4). We deduce the formula

$$\frac{d}{dt}\operatorname{Tr}_{g_t} h_t = \frac{1}{2}\Delta^{\Omega_t}_{g_t}\operatorname{Tr}_{g_t}\dot g_t - \left\langle\dot g_t, \operatorname{Ric}(g_t)\right\rangle_{g_t} - \operatorname{div}_{g_t}\left(\nabla^{\star_{\Omega_t}}_{g_t}\dot g_t^{\star} + \nabla_{g_t}\dot\Omega_t^{\star}\right), \tag{3.9}$$

thanks to the identities

$$\operatorname{Tr}_g\left(\mathcal{R}_g\star\nu\right) \;=\; \left\langle\nu, \operatorname{Ric}(g)\right\rangle_g, \tag{3.10}$$

$$\operatorname{Tr}_g(L_\xi g) \;=\; 2\operatorname{Tr}_{\mathbb{R}}(\nabla_g\xi) \;=\; 2\operatorname{div}_g\xi. \tag{3.11}$$

In order to show the identity (3.10) we expand with respect to a $g(x)$-orthonormal basis $(e_k)_k \subset T_{X,x}$ the term

$$\begin{aligned}
\operatorname{Tr}_g\left(\mathcal{R}_g\star\nu\right) &= \left(\mathcal{R}_g\star\nu\right)(e_k, e_k)\\
&= -\nu(\mathcal{R}_g(e_k, e_l)e_k, e_l)\\
&= g\left(\nu_g^{\star} e_l, \operatorname{Ric}^{\star}(g)e_l\right)\\
&= \left\langle\nu, \operatorname{Ric}(g)\right\rangle_g.
\end{aligned}$$

The first equality in (3.11) follows from the elementary identity

$$(L_\xi g)^{\star}_g \;=\; \nabla_g\xi + (\nabla_g\xi)^T_g,$$

where A^T_g denotes the transpose of an endomorphism A of T_X with respect to g.

In conclusion combining the variation formulas (3.8), (3.9) and (3.7) we infer the variation identity

$$2D_{g,\Omega}H(\nu, V) = \Delta^{\Omega}_g V^{\star}_{\Omega} - \operatorname{div}^{\Omega}\left(\nabla^{\star_{\Omega}}_g\nu_g^{\star} + \nabla_g V^{\star}_{\Omega}\right) - 2V^{\star}_{\Omega} - \left\langle\nu, h_{g,\Omega}\right\rangle_g,$$

and thus the required variation formula (1.5).

4 The Soliton-Kähler-Ricci Flow with variable volume forms

4.1 Existence of the Soliton-Kähler-Ricci flow

We prove in this sub-section lemma 1.

Proof. From now on we will set for notation simplicity $h_t \equiv h_{g_t,\Omega_t}$, $H_t \equiv H_{g_t,\Omega_t}$ and $\underline{H}_t \equiv \underline{H}_{g_t,\Omega_t}$. We observe that for any smooth curve $(g_t, \Omega_t)_{t\geqslant 0} \subset \mathcal{M} \times \mathcal{V}_1$ the identity

$$f_t = \log \frac{dV_{g_t}}{\Omega_t},$$

is equivalent to the evolution equation

$$2\dot{f}_t = \mathrm{Tr}_{g_t} \dot{g}_t - 2\dot{\Omega}_t^{\star}. \tag{4.1}$$

with initial data $f_0 := \log \frac{dV_{g_0}}{\Omega_0}$. Along the Soliton-Ricci flow, the equation (4.1) rewrites as

$$\begin{aligned} 2\dot{f}_t &= -\mathrm{Tr}_{g_t} h_t + 2\underline{H}_t \\ &= -\Delta_{g_t}^{\Omega_t} f_t + 2f_t - 2\int_X H_t \Omega_t. \end{aligned}$$

We infer that the Soliton-Ricci flow equation is equivalent to the evolution system

$$\begin{cases} \dot{g}_t = g_t - \mathrm{Ric}(g_t) \quad - \nabla_{g_t} df_t \ , \\ 2\,\dot{f}_t \quad = -\Delta_{g_t} f_t - |\nabla_{g_t} f_t|_{g_t}^2 + 2f_t - \mathcal{W}(g_t, f_t), \end{cases} \tag{4.2}$$

with $f_0 := \log \frac{dV_{g_0}}{\Omega_0}$. We consider now the flow of diffeomorphisms $(\varphi_t)_{t\geqslant 0}$ solution of the equation

$$2\dot{\varphi}_t = (\nabla_{g_t} f_t) \circ \varphi_t,$$

with $\varphi_0 = \mathrm{Id}_X$ and we define $(\hat{g}_t, \hat{f}_t) := \varphi_t^{\star}(g_t, f_t)$. We observe the evolution formulas

$$\begin{aligned} \frac{d}{dt}\hat{g}_t &= \varphi_t^{\star}\left(\dot{g}_t + \frac{1}{2}L_{\nabla_{g_t} f_t} g_t\right) \\ &= \varphi_t^{\star}\left[g_t - \mathrm{Ric}(g_t)\right] \\ &= \hat{g}_t - \mathrm{Ric}(\hat{g}_t), \end{aligned}$$

and

$$\begin{aligned} 2\frac{d}{dt}\hat{f}_t &= 2\dot{f}_t \circ \varphi_t + 2d_{\varphi_t} f_t \cdot \dot{\varphi}_t \\ &= 2\dot{f}_t \circ \varphi_t + d_{\varphi_t} f_t \cdot [(\nabla_{g_t} f_t) \circ \varphi_t] \\ &= \left(2\dot{f}_t + df_t \cdot \nabla_{g_t} f_t\right) \circ \varphi_t \\ &= \left(2\dot{f}_t + |\nabla_{g_t} f_t|_{g_t}^2\right) \circ \varphi_t, \end{aligned}$$

We deduce thanks to the diffeomorphism invariance of the $\mathcal{W}$ functional that the evolution system (4.2) is equivalent to

$$\begin{cases} \frac{d}{dt}\hat{g}_t = \hat{g}_t - \mathrm{Ric}(\hat{g}_t), \\ 2\,\frac{d}{dt}\hat{f}_t = -\Delta_{\hat{g}_t}\hat{f}_t + 2\hat{f}_t - \mathcal{W}(\hat{g}_t, \hat{f}_t), \end{cases} \tag{4.3}$$

with initial data $(\hat{g}_0, \hat{f}_0) := (g_0, f_0)$. Notice indeed that we can obtain (4.2) from (4.3) by performing the inverse gauge transformation $(g_t, f_t) := \psi_t^\star(\hat{g}_t, \hat{f}_t)$ with $\psi_t = \varphi_t^{-1}$ being characterized by the evolution equation

$$2\dot{\psi}_t = -\left(\nabla_{\hat{g}_t}\hat{f}_t\right) \circ \psi_t,$$

$\psi_0 = \mathrm{Id}_X$. In order to show all time existence and uniqueness of the solutions of the evolution system (4.3) we consider a solution of the Ricci flow $(\check{g}_t)_{t\in[0,T)}$,

$$\frac{d}{dt}\check{g}_t = -2\,\mathrm{Ric}(\check{g}_t),$$

with initial data $\check{g}_0$ and $0 < T < +\infty$. Then $(\hat{g}_t)_{t\geqslant 0}$ defined by

$$\hat{g}_t := \frac{e^t}{2T}\check{g}_{T(1-e^{-t})},$$

satisfies the evolution equation relative to the metrics in (4.3). Then we set $\lambda := 2T$. In the case (X, J_0) is a Fano variety and $\check{g}_0 J_0 \in 2\pi c_1(X)$ we can choose $\lambda = 1$ since the the evolution equation of $\hat{g}_t$ in (4.3) represents a solution of the Kähler-Ricci flow equation.

The existence and uniqueness of the solutions of the evolution equation for $\hat{f}_t$ in (4.3) follows directly from standard parabolic theory with respect to Hölder spaces. Notice indeed that the presence of the integral term $\mathcal{W}(\hat{g}_t, \hat{f}_t)$ (we consider the expression involving the $H^1(X)$ norm of f) does not produce any issue in this theory.

In the Fano set up we define the complex structures $J_t := \psi_t^\star J_0$. Then the family $(J_t, g_t)_{t\geqslant 0}$ represents a flow of Kähler structures since $(J_0, \hat{g}_t)_{t\geqslant 0}$ is also a flow of Kähler structures. The identity $\varphi_t^\star J_t \equiv J_0$ is equivalent to the equality

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\varphi_t^\star J_t\right) \\ &= \varphi_t^\star\left(\dot{J}_t + \frac{1}{2}\,L_{\nabla_{g_t} f_t} J_t\right) \\ &= \varphi_t^\star\left(\dot{J}_t + J_t\,\overline{\partial}_{T_{X,J_t}}\nabla_{g_t} f_t\right), \end{aligned}$$

i.e to the equation

$$\dot{J}_t = -J_t\,\overline{\partial}_{T_{X,J_t}}\nabla_{g_t} f_t.$$

This combined with the J_t-linearity of the first two terms in the right hand side of the complex decomposition

$$\mathrm{Ric}^\star_{g_t}(\Omega_t) = \mathrm{Ric}^\star(g_t) + \partial^{g_t}_{T_{X,J_t}}\nabla_{g_t} f_t + \overline{\partial}_{T_{X,J_t}}\nabla_{g_t} f_t,$$

implies the required characterization $2\dot{J}_t = \left[J_t, \dot{g}_t^\star\right]$ of the evolution of the complex structures J_t.

□

4.2 Monotony of Perelman's $\mathcal{W}$-functional along the Soliton-Kähler-Ricci flow

We observe first the following elementary fact.

Lemma 5. *Let (X, J) be a Fano manifold and let g be a J-invariant Kähler metric with symplectic form $\omega := gJ \in 2\pi c_1(X, [J])$. Then (J, g) is a Kähler-Ricci soliton if and only if there exists a smooth volume form $\Omega > 0$ with $\int_X \Omega = 1$ such that*

$$(S)\begin{cases} \omega = \mathrm{Ric}_J(\Omega), \\ \\ \Delta_g^\Omega f - 2f + 2\int_X f\Omega = 0, f := \log\frac{\omega^n}{n!\Omega}. \end{cases}$$

Proof. We assume first that (J, g) is a Kähler-Ricci soliton. Then Perelman's twice contracted Bianchi type identity (1.2) implies $\underline{H}_{g,\Omega} = 0$. The latter is equivalent to the second equation of the system (S) thanks to the identity $\mathrm{Tr}_g\, h_{g,\Omega} = 0$. We show now that the solution of the system (S) implies that (J, g) is a Kähler-Ricci soliton. Indeed multiplying by $\nabla_g f$ both sides of the identity (1.2) and integrating by parts we obtain the general formula

$$\int_X \left\langle h^{\star}_{g,\Omega}, \nabla^2_g f \right\rangle_g \Omega = -\int_X \underline{H}_{g,\Omega} \Delta^{\Omega}_g f \Omega. \tag{4.4}$$

In our case this rewrites as

$$2\int_X \left|\overline{\partial}_{T_{X,J}} \nabla_g f\right|^2_g \Omega = \int_X (\Delta^{\Omega}_g - 2\mathbb{I}) f \Delta^{\Omega}_g f \Omega, \tag{4.5}$$

thanks to the condition $\omega = \mathrm{Ric}_J(\Omega)$ and the complex decomposition of the Bakry-Emery-Ricci tensor. We infer the required conclusion. □

We provide now a proof of the monotony statement in lemma 2.

Proof. **STEP I.** Let $(J, \hat{g}_t)_{t\geqslant 0}$ be a solution of the Kähler-Ricci flow and observe that this equation rewrites in the equivalent form

$$\begin{cases} \frac{d}{dt}\hat{\omega}_t = i\,\partial_J\overline{\partial}_J \log \frac{\hat{\omega}^n_t}{\hat{\Omega}_t}, \\ \\ \hat{\omega}_t = \mathrm{Ric}_J(\hat{\Omega}_t),\ \int_X \hat{\Omega}_t = 1, \end{cases} \tag{4.6}$$

with $\hat{\omega}_t := \hat{g}_t J$, and $\hat{\omega}_0 := \omega$. We define the function

$$\hat{f}_t \quad := \quad \log \frac{\hat{\omega}^n_t}{\hat{\Omega}_t n!},$$

and we observe the analogue of (4.1)

$$2\frac{d}{dt}\hat{f}_t \quad = \quad \mathrm{Tr}_{\hat{\omega}_t} \frac{d}{dt}\hat{\omega}_t - 2\left(\frac{d}{dt}\hat{\Omega}_t\right)^{\star}_t.$$

This combined with the first equation in (4.6) implies

$$2\frac{d}{dt}\hat{f}_t = -\Delta_{\hat{g}_t}\hat{f}_t - 2\left(\frac{d}{dt}\hat{\Omega}_t\right)^{\star}_t. \tag{4.7}$$

On the other hand time differentiating the identity $\hat{\omega}_t = \mathrm{Ric}_J(\hat{\Omega}_t)$ in (4.6) we obtain

$$2\frac{d}{dt}\hat{\omega}_t \quad = \quad 2\frac{d}{dt}\mathrm{Ric}_J(\hat{\Omega}_t) = -2i\,\partial_J\overline{\partial}_J\left(\frac{d}{dt}\hat{\Omega}_t\right)^{\star}_t,$$

which combined with (4.7) implies

$$2i\,\partial_J\overline{\partial}_J\hat{f}_t \quad = \quad i\,\partial_J\overline{\partial}_J\left(2\frac{d}{dt}\hat{f}_t + \Delta_{\hat{g}_t}\hat{f}_t\right),$$

i.e.

$$2\frac{d}{dt}\hat{f}_t \quad = \quad -\Delta_{\hat{g}_t}\hat{f}_t + 2\hat{f}_t + C_t,$$

for some time dependent constant C_t which can be determined time deriving the integral condition $\int_X \hat{\Omega}_t = 1$. Indeed using (4.7) we obtain

$$\begin{aligned} 0 \quad &= \quad -2\int_X \left(\frac{d}{dt}\hat{\Omega}_t\right)^{\star}_t \hat{\Omega}_t \\ &= \quad \int_X \left[2\frac{d}{dt}\hat{f}_t + \Delta_{\hat{g}_t}\hat{f}_t\right]\hat{\Omega}_t \\ &= \quad C_t + 2\int_X \hat{f}_t\hat{\Omega}_t. \end{aligned}$$

We infer the evolution formula

$$2\frac{d}{dt}\hat{f}_t = -\Delta_{\hat{g}_t}\hat{f}_t + 2\hat{f}_t - 2\int_X \hat{f}_t e^{-\hat{f}_t} dV_{\hat{g}_t}, \tag{4.8}$$

with initial data

$$\hat{f}_0 \quad := \quad \log\frac{\omega^n}{\Omega n!}.$$

We observe now that the identity $\hat{\omega}_t = \mathrm{Ric}_J(\hat{\Omega}_t)$ in (4.6) implies

$$\hat{g}_t = -\mathrm{Ric}_J(\hat{\Omega}_t)J = \mathrm{Ric}_{\hat{g}_t}(\hat{\Omega}_t) - \hat{g}_t\overline{\partial}_{T_{X,J}}\nabla_{\hat{g}_t}\hat{f}_t, \tag{4.9}$$

and thus $\mathrm{Tr}_{\hat{g}_t} h_{\hat{g}_t,\hat{\Omega}_t} = 0$. We deduce the equality

$$\mathcal{W}(\hat{g}_t, \hat{f}_t) = 2\int_X \hat{f}_t e^{-\hat{f}_t} dV_{\hat{g}_t}. \tag{4.10}$$

We infer by Cauchy's uniqueness that the evolution equation (4.8) is equivalent with the second evolution equation in (4.3). We obtain, as in the proof of lemma 1, a Soliton-Kähler-Ricci flow $(J_t, \omega_t, \Omega_t)_{t\geqslant 0}$ with initial data $(J_0, \omega_0, \Omega_0) = (J, \omega, \Omega)$. We observe that thanks to (4.9) and (4.10) the Soliton-Ricci flow evolution system (4.2) writes in our case as

$$\begin{cases} \dot{g}_t = -g_t\overline{\partial}_{T_{X,J_t}}\nabla_{g_t}f_t, \\ \\ 2\dot{f}_t = -\Delta^{\Omega_t}_{g_t}f_t + 2f_t - 2\int_X f_t e^{-f_t} dV_{g_t}. \end{cases} \tag{4.11}$$

Time deriving the identity $\omega_t = g_t J_t$ and using the evolution formula for the complex structure $2\dot{J}_t = \left[J_t, \dot{g}^\star_t\right]$ in the Soliton-Kähler-Ricci flow equation we infer

$$\begin{aligned} \dot{\omega}_t &= \dot{g}_t J_t + g_t\dot{J}_t \\ &= \frac{1}{2}g_t\left(\dot{g}^\star_t J_t + J_t\dot{g}^\star_t\right) \\ &= \frac{1}{2}\omega_t\left(\dot{g}^\star_t - J_t\dot{g}^\star_t J_t\right) \\ &= \omega_t(\dot{g}^\star_t)^{1,0}_J \\ &= 0, \end{aligned}$$

thanks to the first equation in (4.11). We deduce $\omega_t \equiv \omega$ and thus the identity in time

$$\omega = \mathrm{Ric}_{J_t}(\Omega_t). \tag{4.12}$$

STEP IIa. We provide now a first proof of the monotony statement for the Soliton-Kähler-Ricci flow. The equality (4.10) rewrites as

$$\mathcal{W}(g_t, \Omega_t) = 2\int_X f_t e^{-f_t}\frac{\omega^n}{n!} \equiv 2\int_X f_t\Omega_t, \tag{4.13}$$

thanks to the invariance by diffeomorphisms of $\mathcal{W}$. Let

$$F_t \quad := \quad f_t - \int_X f_t\Omega_t,$$

and observe that the second evolution equation in (4.11) rewrites as

$$2\dot{f}_t = -\Delta^{\Omega_t}_{g_t}F_t + 2F_t. \tag{4.14}$$

Time deriving the expression (4.13) and using the evolution equation (4.14) we infer

$$\begin{aligned}\frac{d}{dt}\mathcal{W}(g_t,\Omega_t) &= 2\int_X \left(\dot f_t - f_t\dot f_t\right)\Omega_t\\ &= -2\int_X f_t\dot f_t\,\Omega_t\\ &= \int_X \left(\Delta_{g_t}^{\Omega_t}F_t - 2F_t\right)F_t\,\Omega_t \geqslant 0,\end{aligned}$$

thanks to the estimate $\lambda_1(\Delta_g^\Omega) \geqslant 2$ for the first eigenvalue $\lambda_1(\Delta_g^\Omega)$ of the weighted Laplacian Δ_g^Ω in the case $gJ = \mathrm{Ric}_J(\Omega)$. (See the estimate (13.15) in the section 13.) Indeed by the variational characterization of the first eigenvalue holds the estimate

$$2 \leqslant \lambda_1(\Delta_g^\Omega) = \inf\left\{\int_X \Delta_g^\Omega u\,u\,\Omega \mid u\in C^\infty_\Omega(X,\mathbb{R})_0 : \int_X u^2\Omega = 1\right\}, \tag{4.15}$$

which implies

$$0 \leqslant \int_X \left(\Delta_g^\Omega F - 2F\right)F\Omega, \tag{4.16}$$

with

$$F := f - \int_X f\Omega, \quad f := \log\frac{dV_g}{\Omega}.$$

We assume now equality in (4.16). We assume also $F \neq 0$ otherwise g will be a J-invariant Kähler-Einstein metric. Equality in (4.16) implies $2 = \lambda_1(\Delta_g^\Omega)$ and

$$u_0 := F\left[\int_X F^2\Omega\right]^{-1/2},$$

attains the infinitum in (4.15). Thus we can apply the method of Lagrange multipliers to the functionals

$$\begin{aligned}\Phi(u) &:= \int_X \Delta_g^\Omega u u\Omega,\\ \Psi(u) &:= \int_X u^2\Omega,\end{aligned}$$

over the space $C^\infty_\Omega(X,\mathbb{R})_0$. We have the equalities

$$2 = \min_{\Psi=1}\Phi = \Phi(u_0),$$

which imply $D_{u_0}\Phi = \mu D_{u_0}\Psi$, i.e. $\Delta_g^\Omega u_0 = \mu u_0$, with $\mu = 2$. The latter is equivalent to the equation $\Delta_g^\Omega F = 2F$. Then the required conclusion follows from lemma 5.

STEP IIb. We give here a different proof of the monotony statement. We remind first that Perelman's first variation formula for the $\mathcal{W}$ functional [29] writes as

$$D_{g,\Omega}\mathcal{W}(v,V) = -\int_X \left[\left\langle v, h_{g,\Omega}\right\rangle_g - 2V^\star_\Omega \underline{H}_{g,\Omega}\right]\Omega.$$

Thus along the Soliton-Ricci flow holds the identity

$$\frac{d}{dt}\mathcal{W}(g_t,\Omega_t) = \int_X \left[\left|h_{g_t,\Omega_t}\right|^2_{g_t} - 2\underline{H}^2_{g_t,\Omega_t}\right]\Omega_t.$$

Then the conclusion follows from the identity (4.12) combined with the elementary lemma below. □

Lemma 6. *Let (X, J) be a Fano manifold, let g be a J-invariant Kähler metric with symplectic form $\omega := gJ \in 2\pi c_1(X, [J])$ and let $\Omega > 0$ be a smooth volume form with $\int_X \Omega = 1$ such that $\omega = \mathrm{Ric}_J(\Omega)$. Then*

$$\int_X \left|h_{g,\Omega}\right|_g^2 \Omega \geqslant 2 \int_X \underline{H}_{g,\Omega}^2 \Omega, \tag{4.17}$$

with equality if and only if (J, g) is a Kähler-Ricci soliton.

Proof. The condition $\omega = \mathrm{Ric}_J(\Omega)$ and the complex decomposition of the Bakry-Emery-Ricci tensor in [25] imply

$$h_{g,\Omega} = g\overline{\partial}_{T_{X,J}} \nabla_g f,$$

and thus $\mathrm{Tr}_g\, h_{g,\Omega} = 0$. We deduce

$$2\underline{H}_{g,\Omega} = -(\Delta_g^\Omega - 2\mathbb{I})F. \tag{4.18}$$

Then

$$\begin{aligned} \int_X \left[\left|h_{g,\Omega}\right|_g^2 - 2\underline{H}_{g,\Omega}^2\right] \Omega &= \int_X [\left|\overline{\partial}_{T_{X,J}} \nabla_g f\right|_g^2 - \frac{1}{2}\left|(\Delta_g^\Omega - 2\mathbb{I})F\right|^2]\Omega \\ &= \int_X (\Delta_g^\Omega - 2\mathbb{I})F \cdot F\Omega, \end{aligned}$$

thanks to the integral identity (4.5). The conclusion follows from the variational argument at the end of step IIa. □

Remark 1. *We observe that the elementary identities*

$$\nabla_g f = J\omega^{-1} df = 2\omega^{-1} d_J^c f,$$

with $2d_J^c f := -df \cdot J$, allow to rewrite the Soliton-Ricci flow evolution system (4.11) as

$$\begin{cases} \dot{J}_t = \overline{\partial}_{T_{X,J_t}} \left(\omega^{-1} df_t\right), \\ 2\dot{f}_t = \mathrm{Tr}_\omega \left(dd_{J_t}^c f_t - df_t \wedge d_{J_t}^c f_t\right) + 2f_t - 2\int_X f_t e^{-f_t} \frac{\omega^n}{n!}. \end{cases} \tag{4.19}$$

We notice also that the Soliton-Kähler-Ricci flow evolution system with initial data $(J_0, g_0, \Omega_0) = (J, g, \Omega)$ such that $\omega := gJ = \mathrm{Ric}_J(\Omega)$ is equivalent to the system (1.6). Indeed the argument in step I of the proof of lemma 2 shows that our Soliton-Kähler-Ricci flow is equivalent to the Kähler-Ricci flow equation (4.6) via the gauge transformation given by the diffeomorphisms φ_t. But (1.6) is also equivalent to (4.6) via the same gauge transformation. Notice in fact the identities

$$\frac{d}{dt}\hat{\omega}_t = \frac{1}{2}\varphi_t^\star \left(L_{\nabla_{g_t} f_t}\omega\right) = \varphi_t^\star \left(i\,\partial_{J_t}\overline{\partial}_{J_t} f_t\right) = i\,\partial_J \overline{\partial}_J \hat{f}_t.$$

The corresponding identities for the transformation of the complex structure have been considered at the end of the proof of lemma 1. We infer the equivalence of our Soliton-Kähler-Ricci flow with (1.6).

Remark 2. *Let $(g_t, \Omega_t)_{t\geqslant 0}$ be the Soliton-Ricci flow and set for notation simplicity $\mathcal{W}_t := \mathcal{W}(g_t, \Omega_t)$, $h_t := h_{g_t,\Omega_t}$, $\underline{H}_t := \underline{H}_{g_t,\Omega_t}$. Perelman's twice contracted differential Bianchi identity (1.2) implies*

$$\nabla_{g_t}^{\star_{\Omega_t}} \dot{g}_t^\star + \nabla_{g_t} \dot{\Omega}_t^\star = 0. \tag{4.20}$$

Then the fundamental variation formula (1.5) implies the evolution equation along the Soliton-Ricci flow

$$2\frac{d}{dt}\underline{H}_t = -(\Delta_{g_t}^{\Omega_t} - 2\mathbb{I})\underline{H}_t + \left|h_t\right|_{g_t}^2 - \dot{\mathcal{W}}_t. \tag{4.21}$$

This combined with the monotony statement in lemma 2 or in [21] implies the inequality

$$2\frac{d}{dt}\underline{H}_t \leqslant -(\Delta_{g_t}^{\Omega_t} - 2\mathbb{I})\underline{H}_t + |h_t|_{g_t}^2\,, \tag{4.22}$$

along the Soliton-Kähler-Ricci flow.

The following section is not needed for the proof of the main result.

5 The second variation of the $\mathcal{W}$ functional along the Soliton-Kähler-Ricci flow

Let $(J_t, g_t, \Omega_t)_{t\geqslant 0}$ be the Soliton-Kähler-Ricci flow. In the proof of step I of lemma 2 we obtained the identity

$$\dot{\mathcal{W}}_t = -2\int_X f_t\dot{f}_t e^{-f_t}\frac{\omega^n}{n!}.$$

Time deriving this we obtain

$$\ddot{\mathcal{W}}_t = -2\int_X \dot{f}_t^2\Omega_t - 2\int_X f_t\left(\ddot{f}_t - \dot{f}_t^2\right)\Omega_t.$$

Time deriving the identity

$$0 = \int_X \dot{f}_t\Omega_t \equiv \int_X \dot{f}_t e^{-f_t}\frac{\omega^n}{n!},$$

we deduce

$$0 = \int_X \left(\ddot{f}_t - \dot{f}_t^2\right)\Omega_t,$$

and thus the evolution formula

$$\ddot{\mathcal{W}}_t = -2\int_X \dot{f}_t^2\Omega_t - 2\int_X F_t\left(\ddot{f}_t - \dot{f}_t^2\right)\Omega_t. \tag{5.1}$$

We observe now that the second evolution equation in the system (4.11) rewrites as

$$2\dot{f}_t = -\Delta_{g_t}^{\Omega_t} f_t + 2f_t - \mathcal{W}_t,$$

thanks to (4.13). Time deriving this we infer

$$-2\ddot{f}_t = \frac{d}{dt}\Delta_{g_t}^{\Omega_t} f_t - 2\dot{f}_t + \dot{\mathcal{W}}_t. \tag{5.2}$$

Plugging the identity (4.20) in the variation formula (3.8) and using the first equation in the system (4.11) we obtain

$$\begin{aligned}\frac{d}{dt}\Delta_{g_t}^{\Omega_t} f_t &= \Delta_{g_t}^{\Omega_t}\underline{H}_t - \left|\overline{\partial}_{T_{X,J_t}}\nabla_{g_t} f_t\right|_{g_t}^2 \\ &= \Delta_{g_t}^{\Omega_t}\underline{H}_t - |h_t|_{g_t}^2\,,\end{aligned}$$

with $2\underline{H}_t = -(\Delta_{g_t}^{\Omega_t} - 2\mathbb{I})F_t$. Thus $\dot{f}_t = \underline{H}_t$ thanks to (4.14). Using (5.2) we infer

$$-2\ddot{f}_t = (\Delta_{g_t}^{\Omega_t} - 2\mathbb{I})\underline{H}_t - |h_t|_{g_t}^2 + \dot{\mathcal{W}}_t.$$

(This last follows also from the general evolution formula (4.21).) Integrating by parts we obtain the identity

$$-2\int_X F_t\ddot{f}_t\Omega_t \;=\; -\int_X \left[2\underline{H}_t^2 + F_t\,|h_t|^2_{g_t}\right]\Omega_t,$$

(since $\int_X F_t\Omega_t = 0$). Plunging this identity in the evolution formula (5.1) we deduce the simple second variation formula

$$\begin{aligned}\ddot{\mathcal{W}}_t &= -\int_X \left[4\,\underline{H}_t^2 + \left(|h_t|^2_{g_t} - 2\,\underline{H}_t^2\right)F_t\right]\Omega_t \\ &= -\int_X \left[\left|\overline{\partial}_{T_X,J_t}\nabla_{g_t}f_t\right|^2_{g_t}F_t + \frac{1}{2}\left|(\Delta_{g_t}^{\Omega_t} - 2\mathbb{I})\right|^2(2-F_t)\right]\Omega_t.\end{aligned}$$

6 The Levi-Civita connection of the pseudo-Riemannian structure *G*

In this section we compute the Levi-Civita connection of the pseudo-Riemannian structure G. This is needed for the computation of the second variation of the $\mathcal{W}$ functional with respect to such structure. We set for notations simplicity $\mathcal{T} := T_{\mathcal{M}\times\mathcal{V}_1}$ and we compute the first variation of G at an arbitrary point (g,Ω),

$$D_{g,\Omega}G : \mathcal{T}\times\mathcal{T} \longrightarrow \mathcal{T}^\star.$$

In a direction $(\theta,\Theta)\in\mathcal{T}$ this is given by the identity

$$D_{g,\Omega}G(\theta,\Theta;u,V)(v,V) \;=\; \frac{d}{dt}_{|_{t=0}} G_{g_t,\Omega_t}(u,U;v,V),$$

where $(g_t,\Omega_t)_{t\in(-\varepsilon,\varepsilon)}\subset\mathcal{M}\times\mathcal{V}_1$ is a smooth curve with $(g_0,\Omega_0)=(g,\Omega)$ and $(\dot{g}_0,\dot{\Omega}_0)=(\theta,\Theta)$. For notation simplicity let denote $u_t^\star := g_t^{-1}u$ and $U_t^\star := U/\Omega_t$. Then holds the equality

$$\begin{aligned}D_{g,\Omega}G(\theta,\Theta;u,V)(v,V) &= \frac{d}{dt}_{|_{t=0}}\left[\int_X \mathrm{Tr}_{\mathbb{R}}(u_t^\star\,v_t^\star)\Omega_t - 2\int_X U_t^\star V\right] \\ &= \int_X\left[\frac{d}{dt}_{|_{t=0}}\mathrm{Tr}_{\mathbb{R}}(u_t^\star\,v_t^\star)\right]\Omega_t + \int_X \mathrm{Tr}_{\mathbb{R}}(u_t^\star\,v_t^\star)\Theta - 2\int_X \frac{d}{dt}_{|_{t=0}} U_t^\star V.\end{aligned}$$

Using the identity $\frac{d}{dt}\,u_t^\star = -\dot{g}_t^\star\,u_t^\star$, which follows from the formula

$$\frac{d}{dt}\,g_t^{-1} = -g_t^{-1}\,\dot{g}_t\,g_t^{-1},$$

we obtain

$$\begin{aligned}\frac{d}{dt}\,\mathrm{Tr}_{\mathbb{R}}(u_t^\star\,v_t^\star) &= \mathrm{Tr}_{\mathbb{R}}\left(\frac{d}{dt}u_t^\star\,v_t^\star \;+\; u_t^\star\,\frac{d}{dt}\,v_t^\star\right) \\ &= -\,\mathrm{Tr}_{\mathbb{R}}(\dot{g}_t^\star\,u_t^\star\,v_t^\star \;+\; u_t^\star\,\dot{g}_t^\star\,v_t^\star) \\ &= -\;2\,\mathrm{Tr}_{\mathbb{R}}(\dot{g}_t^\star\,u_t^\star\,v_t^\star)\,,\end{aligned}$$

since $\dot{g}_t$ is also symmetric. Indeed we observe the elementary identities

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{R}}\left[(u_t^\star\ \dot{g}_t^\star)v_t^\star\right] &= \mathrm{Tr}_{\mathbb{R}}\left[v_t^\star(u_t^\star\ \dot{g}_t^\star)\right] \\
&= \mathrm{Tr}_{\mathbb{R}}\left[v_t^\star(u_t^\star\ \dot{g}_t^\star)\right]_t^T \\
&= \mathrm{Tr}_{\mathbb{R}}\left[(u_t^\star\ \dot{g}_t^\star)_t^T v_t^\star\right] \\
&= \mathrm{Tr}_{\mathbb{R}}(\dot{g}_t^\star\ u_t^\star\ v_t^\star),
\end{aligned}$$

where A_t^T denotes the transpose of A with respect to g_t. Time deriving the identity $U = U_t^\star \Omega_t$ we infer

$$0 = \frac{dU_t^\star}{dt}\Omega_t + U_t^\star \dot{\Omega}_t,$$

and thus

$$\frac{dU_t^\star}{dt} = -U_t^\star \dot{\Omega}_t^\star.$$

Summing up we infer the expression of the variation of G at the point (g, Ω) in the direction (θ, Θ)

$$D_{g,\Omega}G(\theta, \Theta; u, U)(v, V) = \int_X \{\mathrm{Tr}_{\mathbb{R}}[(\Theta_\Omega^\star - 2\theta_g^\star)u_g^\star\ v_g^\star] + 2\Theta_\Omega^\star U_\Omega^\star V_\Omega^\star\}\ \Omega.$$

We can compute now the Levi-Civita connection $\nabla_G = D + \Gamma_G$ of the pseudo-Riemannian structure G. At a point (g, Ω) the symmetric bilinear form

$$\Gamma_G(g, \Omega) : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T},$$

is identified by the expression

$$\begin{aligned}
2G_{g,\Omega}\left(\Gamma_G(g, \Omega)(u, U; v, V); \theta, \Theta\right) &= [D_{g,\Omega}G(u, U; v, V) + D_{g,\Omega}G(v, V; u, U)]\,(\theta, \Theta) \\
&- D_{g,\Omega}G(\theta, \Theta; u, U)(v, V).
\end{aligned}$$

Expanding and arranging the terms of the right hand side we obtain

$$\begin{aligned}
2G_{g,\Omega}\left(\Gamma_G(g, \Omega)(u, U; v, V); \theta, \Theta\right) &= \int_X \left\{\mathrm{Tr}_{\mathbb{R}}\left[(U_\Omega^\star - 2u_g^\star)v_g^\star\ \theta_g^\star\right] + 2U_\Omega^\star V_\Omega^\star \Theta_\Omega^\star\right\}\ \Omega \\
&+ \int_X \left\{\mathrm{Tr}_{\mathbb{R}}\left[(V_\Omega^\star - 2v_g^\star)u_g^\star\ \theta_g^\star\right] + 2V_\Omega^\star U_\Omega^\star \Theta_\Omega^\star\right\}\ \Omega \\
&- \int_X \left\{\mathrm{Tr}_{\mathbb{R}}\left[(\Theta_\Omega^\star - 2\theta_g^\star)u_g^\star\ v_g^\star\right] + 2\Theta_\Omega^\star U_\Omega^\star V_\Omega^\star\right\}\ \Omega \\
&= \int_X \left\{\mathrm{Tr}_{\mathbb{R}}\left[(U_\Omega^\star - 2u_g^\star)v_g^\star\ \theta_g^\star + V_\Omega^\star u_g^\star\ \theta_g^\star - \Theta_\Omega^\star u_g^\star\ v_g^\star\right] + 2U_\Omega^\star V_\Omega^\star \Theta_\Omega^\star\right\}\Omega \\
&= \int_X \mathrm{Tr}_{\mathbb{R}}\left[(u_g^\star(V_\Omega^\star - v_g^\star) + v_g^\star(U_\Omega^\star - u_g^\star))\ \theta_g^\star\right]\ \Omega \\
&- \int_X \left[\mathrm{Tr}_{\mathbb{R}}(u_g^\star\ v_g^\star) - 2U_\Omega^\star V_\Omega^\star\right]\ \Theta_\Omega^\star\ \Omega
\end{aligned}$$

$$= \int_X \left\langle u(V_\Omega^\star - v_g^\star) + v(U_\Omega^\star - u_g^\star), \theta \right\rangle_g \Omega$$

$$- 2\int_X \left[\frac{1}{2}\langle u, v\rangle_g - U_\Omega^\star V_\Omega^\star - \frac{1}{2}G_{g,\Omega}(u, U; v, V)\right]\Theta_\Omega^\star\,\Omega,$$

since $\int_X \Theta = 0$. We infer the expression

$$(\psi, \Psi) \equiv \Gamma_G(g, \Omega)(u, U; v, V),$$

$$\psi = \frac{1}{2}\left[u(V_\Omega^\star - v_g^\star) + v(U_\Omega^\star - u_g^\star)\right],$$

$$\Psi = \frac{1}{4}\left[\langle u, v\rangle_g - 2U_\Omega^\star V_\Omega^\star - G_{g,\Omega}(u, U; v, V)\right]\Omega.$$

This concludes the computation of the Levi-Civita connection ∇_G.

7 The second variation of the $\mathcal{W}$ functional with respect to the pseudo-Riemannian structure G

We justify first the geometric interpretation of $\mathbb{F}_{g,\Omega}$ provided by the identity (1.7). We observe indeed that $(v, V) \in T^{\perp_G}_{[g,\Omega],(g,\Omega)}$ if and only if

$$G_{g,\Omega}(L_\xi g, L_\xi \Omega; v, V) = 0,$$

for all $\xi \in C^\infty(X, T_X)$, i.e

$$0 = \int_X \left[\left\langle L_\xi g, v\right\rangle_g - 2(L_\xi\Omega)_\Omega^\star V_\Omega^\star\right]\Omega$$

$$= 2\int_X \left[\left\langle \nabla_g \xi, v_g^\star\right\rangle_g - (\mathrm{div}^\Omega\,\xi)V_\Omega^\star\right]\Omega$$

$$= 2\int_X \left\langle \xi, \nabla_g^{\star_\Omega} v_g^\star + \nabla_g V_\Omega^\star\right\rangle_g \Omega,$$

which shows the required conclusion. We introduce now the operator

$$\mathcal{L}_g^\Omega : C^\infty(X, \mathrm{End}(T_X)) \longrightarrow C^\infty(X, \mathrm{End}(T_X)),$$

defined by the formula

$$\mathcal{L}_g^\Omega A := \Delta_g^\Omega A - 2\mathcal{R}_g \star A.$$

By abuse of notations we define also

$$\mathcal{L}_g^\Omega : C^\infty(X, S^2T_X^\star) \longrightarrow C^\infty(X, S^2T_X^\star),$$

defined by the same formula

$$\mathcal{L}_g^\Omega v := \Delta_g^\Omega v - 2\mathcal{R}_g \star v.$$

We observe that (3.3) implies the identity $(\mathcal{L}_g^\Omega v)_g^\star = \mathcal{L}_g^\Omega v_g^\star$. We show now the second variation formula for the $\mathcal{W}$ functional.

Lemma 7. *The Hessian endomorphism $\nabla^2_G \mathcal{W}(g,\Omega)$ of the $\mathcal{W}$ functional with respect to the pseudo-Riemannian structure G at the point $(g,\Omega) \in \mathcal{M} \times \mathcal{V}_1$ in the directions $(v,V) \in \mathbb{F}_{g,\Omega}$ is given by the expressions*

$$\begin{aligned}
(u,U) &\equiv \nabla^2_G \mathcal{W}(g,\Omega)(v,V),\\
u &:= -\frac{1}{2}\left(\mathcal{L}^{\Omega}_g + \underline{H}_{g,\Omega}\right) v - \frac{1}{2} V^{\star}_{\Omega} h_{g,\Omega},\\
U^{\star}_{\Omega} &:= -\frac{1}{2}\left(\Delta^{\Omega}_g + \underline{H}_{g,\Omega} - 2\mathbb{I}\right) V^{\star}_{\Omega} + \frac{1}{4}\left\langle h_{g,\Omega}, v\right\rangle_g + \frac{1}{4} D_{g,\Omega}\mathcal{W}(v,V).
\end{aligned}$$

In particular if $h_{g,\Omega} = 0$ then

$$\begin{aligned}
u &= -\frac{1}{2}\mathcal{L}^{\Omega}_g v,\\
U^{\star}_{\Omega} &= -\frac{1}{2}(\Delta^{\Omega}_g - 2\mathbb{I}) V^{\star}_{\Omega}.
\end{aligned}$$

Proof. We consider a smooth curve $(g_t,\Omega_t)_{t\in\mathbb{R}} \subset \mathcal{M}\times\mathcal{V}_1$ with $(g_0,\Omega_0) = (g,\Omega)$ and with arbitrary speed $(\dot g_0, \dot\Omega_0) = (v,V)$. We observe that the G-covariant derivative of its speed is given by the expressions

$$\begin{aligned}
(\theta_t,\Theta_t) &\equiv \nabla_G(\dot g_t,\dot\Omega_t)(\dot g_t,\dot\Omega_t) = (\ddot g_t,\ddot\Omega_t) + \Gamma(g_t,\Omega_t)(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t),\\
\theta_t &:= \ddot g_t + \dot g_t\left(\dot\Omega^{\star}_t - \dot g^{\star}_t\right),\\
\Theta_t &:= \ddot\Omega_t + \frac{1}{4}\left[|\dot g_t|^2_t - 2(\dot\Omega^{\star}_t)^2 - G_{g_t,\Omega_t}(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t)\right]\Omega_t.
\end{aligned}$$

We infer

$$\begin{aligned}
\theta^{\star}_t &= \frac{d}{dt}\dot g^{\star}_t + \dot\Omega^{\star}_t \dot g^{\star}_t,\\
\Theta^{\star}_t &= \frac{d}{dt}\dot\Omega^{\star}_t + \frac{1}{2}(\dot\Omega^{\star}_t)^2 + \frac{1}{4}|\dot g_t|^2_{g_t} - \frac{1}{4}G_{g_t,\Omega_t}(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t).
\end{aligned}$$

Using this expressions and Perelman's first variation formula we expand the Hessian form

$$\begin{aligned}
\nabla_G D\mathcal{W}(g_t,\Omega_t)(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t) &= \frac{d^2}{dt^2}\mathcal{W}(g_t,\Omega_t) - D_{g_t,\Omega_t}\mathcal{W}(\theta_t,\Theta_t)\\
&= -\frac{d}{dt}\int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(\dot g^{\star}_t h^{\star}_t\right) - 2\dot\Omega^{\star}_t H_t\right]\Omega_t\\
&+ \int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(\theta^{\star}_t h^{\star}_t\right) - 2\Theta^{\star}_t H_t\right]\Omega_t\\
&= -\int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(\frac{d}{dt}\dot g^{\star}_t h^{\star}_t + \dot g^{\star}_t \frac{d}{dt} h^{\star}_t\right) - 2\frac{d}{dt}\dot\Omega^{\star}_t H_t - 2\dot\Omega^{\star}_t \dot H_t\right]\Omega_t\\
&- \int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(\dot g^{\star}_t h^{\star}_t\right) - 2\dot\Omega^{\star}_t H_t\right]\dot\Omega_t\\
&+ \int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(\theta^{\star}_t h^{\star}_t\right) - 2\Theta^{\star}_t H_t\right]\Omega_t\\
&= -\int_X \left\{\mathrm{Tr}_{\mathbb{R}}\left[\dot g^{\star}_t\left(\dot h^{\star}_t - \dot g^{\star}_t h^{\star}_t\right)\right] - 2\dot\Omega^{\star}_t \dot H_t\right\}\Omega_t
\end{aligned}$$

$$- \quad \frac{1}{2}\int_X \left[|\dot{g}_t|^2_{g_t} - 2(\dot{\Omega}^\star_t)^2 - G_{g_t,\Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t)\right] H_t \Omega_t.$$

Using the variation formulas (1.4) and (1.5) and evaluating the previous identity at time $t = 0$ we obtain the expression

$$\begin{aligned}
\nabla_G D\mathcal{W}(g,\Omega)(v,V;v,V) &= -\frac{1}{2}\int_X \left[\left\langle \mathcal{L}^\Omega_g v - L_{\nabla^{\star_\Omega}_g v^\star_g + \nabla_g V^\star_\Omega} g, v\right\rangle_g\right]\Omega \\
&- \frac{1}{2}\int_X \{-2V^\star_\Omega[(\Delta^\Omega_g - 2\mathbb{I})V^\star_\Omega - \mathrm{div}^\Omega\left(\nabla^{\star_\Omega}_g v^\star_g + \nabla_g V^\star_\Omega\right) - \left\langle v, h_{g,\Omega}\right\rangle_g]\}\Omega \\
&- \frac{1}{2}\int_X [|v|^2_g - 2(V^\star_\Omega)^2]\underline{H}_{g,\Omega}\Omega,
\end{aligned}$$

since $\int_X \underline{H}_{g,\Omega}\Omega = 0$. Arranging symmetrically the integrand terms via the identity

$$\begin{aligned}
\nabla_G D\mathcal{W}(g,\Omega)(v,V;v,V) &= G_{g,\Omega}(u,U;v,V), \\
(u,U) &\equiv \nabla^2_G \mathcal{W}(g,\Omega)(v,V),
\end{aligned}$$

we infer the general expressions

$$u = -\frac{1}{2}\left(\mathcal{L}^\Omega_g + \underline{H}_{g,\Omega}\right)v + \frac{1}{2}L_{\nabla^{\star_\Omega}_g v^\star_g + \nabla_g V^\star_\Omega} g - \frac{1}{2}V^\star_\Omega h_{g,\Omega},$$

$$U^\star_\Omega = -\frac{1}{2}\left(\Delta^\Omega_g + \underline{H}_{g,\Omega} - 2\mathbb{I}\right)V^\star_\Omega + \frac{1}{2}\left(L_{\nabla^{\star_\Omega}_g v^\star_g + \nabla_g V^\star_\Omega}\Omega\right)^\star_\Omega + \frac{1}{4}\left\langle h_{g,\Omega}, v\right\rangle_g + \frac{1}{4}D_{g,\Omega}\mathcal{W}(v,V).$$

Then the required expression of the Hessian of $\mathcal{W}$ follows from the assumption $(v,V) \in \mathbb{F}_{g,\Omega}$. If $h_{g,\Omega} = 0$ then the required conclusion follows from Perelman's twice contracted second Bianchi identity (1.2) which implies $\underline{H}_{g,\Omega} = 0$. □

8 The anomaly space of the pseudo-Riemannian structure *G*

Let $\mathrm{Isom}^0_{g,\Omega}$ be the identity component of the group

$$\mathrm{Isom}_{g,\Omega} := \left\{\varphi \in \mathrm{Diff}(X) \mid \varphi^\star g = g, \varphi^\star \Omega = \Omega\right\},$$

and let

$$\mathrm{Kill}_{g,\Omega} := \mathrm{Lie}\left(\mathrm{Isom}^0_{g,\Omega}\right) \equiv \left\{\xi \in C^\infty(X, T_X) \mid L_\xi g = 0, L_\xi \Omega = 0\right\}.$$

We define the anomaly space of the pseudo-Riemannian structure G at an arbitrary point (g,Ω) as the vector space

$$\mathbb{A}^\Omega_g := \mathbb{F}_{g,\Omega} \cap T_{[g,\Omega],g,\Omega}.$$

We will study some properties of this space. It is clear by definition that this space is generated by the vector fields $\xi \in C^\infty(X, T_X)$ such that

$$0 = \left[\nabla^{\star_\Omega}_g \hat{\nabla}_g + d\,\mathrm{div}^\Omega_g\right](g\xi) = \hat{\Delta}^\Omega_g(g\xi).$$

More precisely there exists the exact sequence of finite dimensional vector spaces

$$0 \longrightarrow \mathrm{Kill}_{g,\Omega} \longrightarrow \quad \mathrm{Ker}\,\hat{\Delta}_g^{\Omega} \quad \longrightarrow \mathbb{A}_g^{\Omega} \longrightarrow 0$$

$$\xi \mapsto \quad g\xi = \alpha \quad \mapsto \left(\hat{\nabla}_g \alpha, (\mathrm{div}_g^{\Omega}\,\alpha)\Omega\right).$$

We observe that if $\alpha = du \in \mathrm{Ker}\,\hat{\Delta}_g^{\Omega}$ then the function u satisfies the equation

$$2\Delta_g^{\Omega}\nabla_g u - \nabla_g \Delta_g^{\Omega} u \quad = \quad 0,$$

which is equivalent to the equation

$$\left[\Delta_g^{\Omega} - \mathrm{Ric}_g^{\star}(\Omega)\right]\nabla_g u = 0, \tag{8.1}$$

thanks to the general identity

$$\nabla_g \Delta_g^{\Omega} u = \Delta_g^{\Omega}\nabla_g u + \mathrm{Ric}_g^{\star}(\Omega)\nabla_g u. \tag{8.2}$$

We set

$$\begin{aligned}\mathbb{V}_{g,\Omega} \quad &:= \quad \left\{\alpha \in \mathrm{Ker}\,\hat{\Delta}_g^{\Omega} \mid \alpha = du\right\} \\ &\cong \quad \left\{u \in C_{\Omega}^{\infty}(X,\mathbb{R})_0 \mid \left[\Delta_g^{\Omega} - \mathrm{Ric}_g^{\star}(\Omega)\right]\nabla_g u = 0\right\}.\end{aligned}$$

We observe that in the soliton case $h_{g,\Omega} = 0$ we have

$$\mathbb{V}_{g,\Omega} \cong \mathrm{Ker}(\Delta_g^{\Omega} - 2\mathbb{I}) \subset C_{\Omega}^{\infty}(X,\mathbb{R})_0, \tag{8.3}$$

thanks to the identity (8.2). By duality we can consider $\mathrm{Kill}_{g,\Omega} \subset \mathrm{Ker}\,\hat{\Delta}_g^{\Omega}$ and we observe the inclusion

$$\mathbb{V}_{g,\Omega} \subseteq \mathrm{Kill}_{g,\Omega}^{\perp_{g,\Omega}}, \tag{8.4}$$

where the symbol $\perp_{g,\Omega}$ indicates the orthogonal space inside $\mathrm{Ker}\,\hat{\Delta}_g^{\Omega}$ with respect to the scalar product (1.1) at the level of 1-forms. The previous inclusion holds true for any (g,Ω) since

$$\int_X \langle du, \beta\rangle_g\,\Omega \quad = \quad -\int_X \left\langle u, \mathrm{div}_g^{\Omega}\,\beta\right\rangle_g \Omega = 0,$$

for any $\beta \in \mathrm{Kill}_{g,\Omega}$. We infer that in the soliton case the previous exact sequence can be reduced to the sequence

$$0 \longrightarrow \mathrm{Ker}(\Delta_g^{\Omega} - 2\mathbb{I}) \quad \longrightarrow \quad \mathbb{A}_g^{\Omega}$$

$$u \quad \mapsto \quad 2\left(\nabla_g du, -u\Omega\right).$$

In order to show that the previous map is also surjective we need to show a few differential identities. We show first the Weitzenböck type formula

$$\hat{\Delta}_g^{\Omega}\alpha = \Delta_g^{\Omega}\alpha - \alpha\,\mathrm{Ric}_g^{\star}(\Omega). \tag{8.5}$$

(This implies in particular the identification of $\mathbb{V}_{g,\Omega}$ in terms of functions). We decompose the expression

$$\hat{\Delta}_g^{\Omega}\alpha = \left[\nabla_g^{\star_{\Omega}}\hat{\nabla}_g - d\nabla_g^{\star_{\Omega}}\right]\alpha. \tag{8.6}$$

We decompose first the term

$$\nabla_g^{\star_{\Omega}}\hat{\nabla}_g\alpha\cdot\xi \quad = \quad \nabla_g^{\star}\hat{\nabla}_g\alpha\cdot\xi + \hat{\nabla}_g\alpha(\nabla_g f,\xi).$$

We fix an arbitrary point p and we choose the vector fields ξ and η such that $0 = \nabla_g\xi(p) = \nabla_g\eta(p)$. Let $(e_k)_k$ be a g-orthonormal local frame such that $\nabla_g e_k(p) = 0$. Then at the point p hold the identities

$$\begin{aligned}\nabla_g^{\star}\hat{\nabla}_g\alpha\cdot\xi &= -\nabla_{g,e_k}\hat{\nabla}_g\alpha(e_k,\xi)\\ &= -\nabla_{g,e_k}\left[\hat{\nabla}_g\alpha(e_k,\xi)\right]\\ &= -\nabla_{g,e_k}\left[\nabla_{g,e_k}\alpha\cdot\xi+\nabla_{g,\xi}\alpha\cdot e_k\right]\\ &= -\nabla_{g,e_k}\nabla_{g,e_k}\alpha\cdot\xi-\nabla_{g,e_k}\nabla_{g,\xi}\alpha\cdot e_k.\end{aligned}$$

We infer the expression

$$\nabla_g^{\star_\Omega}\hat{\nabla}_g\alpha\cdot\xi = \Delta_g^{\Omega}\alpha-\nabla_{g,e_k}\nabla_{g,\xi}\alpha\cdot e_k+\nabla_g\alpha(\xi,\nabla_g f).$$

Moreover

$$d\nabla_g^{\star_\Omega}\alpha(\xi) = -\nabla_{g,\xi}\nabla_{g,e_k}\alpha\cdot e_k+\nabla_{g,\xi}\alpha\cdot\nabla_g f+\alpha\cdot\nabla^2_{g,\xi}f.$$

Summing up we deduce

$$\begin{aligned}\hat{\Delta}_g^{\Omega}\alpha\cdot\xi &= \Delta_g^{\Omega}\alpha\cdot\xi+\left(\nabla_{g,\xi}\nabla_{g,e_k}\alpha-\nabla_{g,e_k}\nabla_{g,\xi}\alpha\right)\cdot e_k-\alpha\cdot\nabla^2_{g,\xi}f\\ &= \Delta_g^{\Omega}\alpha\cdot\xi-\alpha\cdot\mathcal{R}_g(\xi,e_k)e_k-\alpha\cdot\nabla^2_{g,\xi}f,\end{aligned}$$

thanks to the dual identity

$$\nabla_{g,\xi}\nabla_{g,\eta}\alpha-\nabla_{g,\eta}\nabla_{g,\xi}\alpha=\nabla_{g,[\xi,\eta]}\alpha-\alpha\cdot\mathcal{R}_g(\xi,\eta),\tag{8.7}$$

and to the fact that $[\xi,e_k](p)=0$. We infer the required formula (8.5). We deduce that in the soliton case $h_{g,\Omega}=0$ holds the equality

$$\operatorname{Ker}\hat{\Delta}_g^{\Omega}=\operatorname{Ker}(\Delta_g^{\Omega}-\mathbb{I})\subset C^{\infty}(X,T_X^{\star}).\tag{8.8}$$

We define now the Ω-Hodge Laplacian acting on scalar valued differential forms as the operator

$$\Delta_{d,g}^{\Omega} := d\nabla_g^{\star_\Omega}+\nabla_g^{\star_\Omega}d.$$

At the level of scalar valued 1-forms we observe the identities

$$\begin{aligned}\left(\Delta_{d,g}^{\Omega}+\hat{\Delta}_g^{\Omega}\right)\alpha &= \nabla_g^{\star_\Omega}\left(d+\hat{\nabla}_g\right)\alpha\\ &= 2\nabla_g^{\star_\Omega}\nabla_g\alpha\\ &= 2\Delta_g^{\Omega}\alpha.\end{aligned}$$

We infer thanks to the identity (8.5) that for any scalar valued 1-form α holds the Weitzenböck type formula

$$\Delta_g^{\Omega}\alpha=\Delta_{d,g}^{\Omega}\alpha-\alpha\operatorname{Ric}_g^{\star}(\Omega).\tag{8.9}$$

Applying the $\nabla_g^{\star_\Omega}$-operator to both sides of this identity and using the fact that $(\nabla_g^{\star_\Omega})^2=0$ at the level of scalar valued differential forms we obtain

$$\nabla_g^{\star_\Omega}\Delta_g^{\Omega}\alpha = \Delta_g^{\Omega}\nabla_g^{\star_\Omega}\alpha-\nabla_g^{\star_\Omega}\left[\alpha\operatorname{Ric}_g^{\star}(\Omega)\right].$$

In the soliton case $h_{g,\Omega} = 0$ this implies the formula

$$\nabla_g^{\star_\Omega} \Delta_g^\Omega \alpha = \Delta_g^\Omega \nabla_g^{\star_\Omega} \alpha - \nabla_g^{\star_\Omega} \alpha. \tag{8.10}$$

Then the identity (8.8) implies that the map

$$\begin{aligned} \operatorname{Ker} \hat{\Delta}_g^\Omega \quad &\longrightarrow \quad \operatorname{Ker}(\Delta_g^\Omega - 2\mathbb{I}) \subset C^\infty_\Omega(X, \mathbb{R})_0 \\ \alpha \quad &\mapsto \quad \operatorname{div}_g^\Omega \alpha, \end{aligned}$$

is well defined. More precisely there exists the exact sequence of finite dimensional vector spaces

$$\begin{aligned} 0 \longrightarrow \operatorname{Kill}_{g,\Omega} \longrightarrow \quad \operatorname{Ker} \hat{\Delta}_g^\Omega \quad &\longrightarrow \operatorname{Ker}(\Delta_g^\Omega - 2\mathbb{I}) \longrightarrow 0 \\ \xi \mapsto \quad g\xi = \alpha \quad &\mapsto \operatorname{div}_g^\Omega \alpha. \end{aligned}$$

Indeed the surjectivity follows from the isomorphism (8.3). The injectivity follows from the fact that

$$\operatorname{Kill}_{g,\Omega} \quad \cong \quad \left\{ \alpha \in \operatorname{Ker} \hat{\Delta}_g^\Omega \mid \operatorname{div}_g^\Omega \alpha = 0 \right\}.$$

This hold true thanks to the identity

$$\int_X \left| \operatorname{div}_g^\Omega \alpha \right|_g^2 \Omega \quad = \quad \frac{1}{2} \int_X \left| \hat{\nabla}_g \alpha \right|_g^2 \Omega,$$

which follows from the expression

$$\hat{\Delta}_g^\Omega \alpha \quad = \quad \left[\frac{1}{2} \hat{\nabla}_g^{\star_\Omega} \hat{\nabla}_g - d \nabla_g^{\star_\Omega} \right] \alpha.$$

For dimensional reasons we conclude the existence of the required exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Ker}(\Delta_g^\Omega - 2\mathbb{I}) \quad &\longrightarrow \quad \mathbb{A}_g^\Omega \longrightarrow 0 \\ u \quad &\mapsto \quad 2\left(\nabla_g du, -u\Omega\right). \end{aligned}$$

(We observe also that for dimensional reasons (8.4) is an equality.)

9 Properties of the kernel of the Hessian of $\mathcal{W}$

Lemma 8. *In the soliton case $h_{g,\Omega} = 0$ holds the inclusion*

$$\mathbb{A}_g^\Omega \subseteq \mathbb{F}_{g,\Omega} \cap \operatorname{Ker} \nabla_G^2 \mathcal{W}(g, \Omega).$$

We start with a few notations. For any tensor $A \in C^\infty(X, (T_X^\star)^{\otimes p+1} \otimes T_X)$ we define the divergence type operations

$$\begin{aligned} \underline{\operatorname{div}}_g A(u_1, \dots, u_p) \quad &:= \quad \operatorname{Tr}_g \left[\nabla_g A(\cdot, u_1, \dots, u_p, \cdot) \right], \\ \underline{\operatorname{div}}_g^\Omega A(u_1, \dots, u_p) \quad &:= \quad \underline{\operatorname{div}}_g A(u_1, \dots, u_p) - A(u_1, \dots, u_p, \nabla_g f). \end{aligned}$$

The once contracted differential Bianchi identity writes often as $\underline{\operatorname{div}}_g \mathcal{R}_g = -\nabla_{T_X, g} \operatorname{Ric}_g^\star$. This combined with the identity $\nabla_{T_X,g} \nabla_g^2 f = \mathcal{R}_g \cdot \nabla_g f$ implies

$$\underline{\operatorname{div}}_g^\Omega \mathcal{R}_g = -\nabla_{T_X, g} \operatorname{Ric}_g^\star(\Omega). \tag{9.1}$$

We define the Ω-Lichnerowicz Laplacian $\Delta^{\Omega}_{L,g}$ acting on g-symmetric endomorphisms A as

$$\Delta^{\Omega}_{L,g}A := \mathcal{L}^{\Omega}_{g}A + \mathrm{Ric}^{\star}_{g}(\Omega)A + A\,\mathrm{Ric}^{\star}_{g}(\Omega).$$

We fix a point $p \in X$ and we take an arbitrary vector field ξ such that $\nabla_g\xi(p) = 0$. Let also $(e_k)_k$ be a g-orthonormal local frame such that $\nabla_g e_k(p) = 0$. We expand the identity at the point p

$$(\Delta^{\Omega}_{g}\nabla^2_g u)\xi = -\nabla_{g,e_k}\nabla^3_g u(e_k,\xi) + \nabla^3_g u(\nabla_g f,\xi).$$

Commuting derivatives at the point p we obtain

$$\begin{aligned}
\nabla_{g,e_k}\nabla^3_g u(e_k,\xi) &= \nabla_{g,e_k}\left[\nabla^3_g u(e_k,\xi)\right]\\
&= \nabla_{g,e_k}\left[\nabla_{g,e_k}\nabla_{g,\xi}\nabla_g u - \nabla^2_g u\cdot\nabla_{g,e_k}\xi\right]\\
&= \nabla_{g,e_k}\left[\nabla_{g,\xi}\nabla_{g,e_k}\nabla_g u + \mathcal{R}_g(e_k,\xi)\nabla_g u - \nabla^2_g u\cdot\nabla_{g,\xi}e_k\right]\\
&= \nabla_{g,\xi}\nabla_{g,e_k}\nabla_{g,e_k}\nabla_g u + 2\mathcal{R}_g(e_k,\xi)\nabla_{g,e_k}\nabla_g u\\
&+ \nabla_{g,e_k}\mathcal{R}_g(e_k,\xi)\nabla_g u - \nabla^2_g u\cdot\nabla_{g,e_k}\nabla_{g,\xi}e_k,
\end{aligned}$$

since $[e_k,\xi](p) = 0$. Taking a covariant derivative of the identity

$$\Delta_g\nabla_g u = -\nabla_{g,e_k}\nabla_{g,e_k}\nabla_g u + \nabla^2_g u\cdot\nabla_{g,e_k}e_k,$$

we infer

$$\nabla_{g,\xi}\Delta_g\nabla_g u = -\nabla_{g,\xi}\nabla_{g,e_k}\nabla_{g,e_k}\nabla_g u + \nabla^2_g u\cdot\nabla_{g,\xi}\nabla_{g,e_k}e_k,$$

at the point p. Combining with the previous expression we obtain

$$\nabla_{g,e_k}\nabla^3_g u(e_k,\xi) = -2\left(\mathcal{R}_g\star\nabla^2_g u\right)\xi - \nabla_{g,\xi}\Delta_g\nabla_g u + \nabla^2_g u\cdot\mathrm{Ric}^{\star}(g)\xi + \nabla_{g,e_k}\mathcal{R}_g(e_k,\xi)\nabla_g u.$$

On the other hand deriving the identity

$$\Delta^{\Omega}_{g}\nabla_g u = \Delta_g\nabla_g u + \nabla^2_g u\cdot\nabla_g f,$$

we infer

$$\nabla_{g,\xi}\Delta^{\Omega}_{g}\nabla_g u = \nabla_{g,\xi}\Delta_g\nabla_g u + \nabla_{g,\xi}\nabla^2_g u\cdot\nabla_g f + \nabla^2_g u\cdot\nabla^2_{g,\xi}f,$$

and thus

$$\begin{aligned}
\nabla_{g,e_k}\nabla^3_g u(e_k,\xi) &= -2\left(\mathcal{R}_g\star\nabla^2_g u\right)\xi - \nabla_{g,\xi}\Delta^{\Omega}_{g}\nabla_g u + \nabla_{g,\xi}\nabla^2_g u\cdot\nabla_g f\\
&+ \nabla^2_g u\cdot\mathrm{Ric}^{\star}_{g}(\Omega)\xi - \underline{\mathrm{div}}_g\mathcal{R}_g(\xi,\nabla_g u) - \left(\nabla_g u\neg\nabla^{\star}_g\mathcal{R}_g\right)\xi,
\end{aligned}$$

thanks to the algebraic Bianchi identity. We obtain

$$\begin{aligned}
(\Delta^{\Omega}_{g}\nabla^2_g u)\xi &= 2\left(\mathcal{R}_g\star\nabla^2_g u\right)\xi - \nabla_{T_X,g}\nabla^2_g u(\xi,\nabla_g f) + \underline{\mathrm{div}}_g\mathcal{R}_g(\xi,\nabla_g u) + \left(\nabla_g u\neg\nabla^{\star}_g\mathcal{R}_g\right)\xi\\
&+ \nabla_{g,\xi}\Delta^{\Omega}_{g}\nabla_g u - \nabla^2_g u\cdot\mathrm{Ric}^{\star}_{g}(\Omega)\xi.
\end{aligned}$$

The identity $\nabla_{T_X,g}\nabla_g^2 u = \mathcal{R}_g \cdot \nabla_g u$ implies

$$\begin{aligned} -\nabla_{T_X,g}\nabla_g^2 u(\xi, \nabla_g f) &= \mathcal{R}_g(\nabla_g f, \xi)\nabla_g u \\ &= -\mathcal{R}_g(\xi, \nabla_g u)\nabla_g f + \mathcal{R}_g(\nabla_g f, \nabla_g u)\xi, \end{aligned}$$

thanks again to the algebraic Bianchi identity. We infer

$$\begin{aligned} (\mathcal{L}_g^{\Omega}\nabla_g^2 u)\xi &= \left[\nabla_g u \neg \left(\nabla_g^{\star_\Omega}\mathcal{R}_g - \underline{\mathrm{div}}_g^{\Omega}\mathcal{R}_g\right)\right]\xi + \nabla_{g,\xi}\Delta_g^{\Omega}\nabla_g u - \nabla_g^2 u \cdot \mathrm{Ric}_g^{\star}(\Omega)\xi \\ &= \left[\nabla_g u \neg \left(\nabla_g^{\star_\Omega}\mathcal{R}_g + \nabla_{T_X,g}\,\mathrm{Ric}_g^{\star}(\Omega)\right)\right]\xi + (\nabla_g^2\Delta_g^{\Omega}u)\xi - \nabla_{g,\xi}[\mathrm{Ric}_g^{\star}(\Omega)\nabla_g u] - \nabla_g^2 u \cdot \mathrm{Ric}_g^{\star}(\Omega)\xi, \end{aligned}$$

thanks to (9.1) and (8.2). Thus

$$\Delta_{L,g}^{\Omega}\nabla_g^2 u = \nabla_g^2\Delta_g^{\Omega}u + \nabla_g u \neg \left[\nabla_g^{\star_\Omega}\mathcal{R}_g + \nabla_g\,\mathrm{Ric}_g^{\star}(\Omega)\right] - 2\nabla_g\,\mathrm{Ric}_g^{\star}(\Omega)\nabla_g u. \tag{9.2}$$

We observe now that the endomorphism section $\nabla_g u \neg \nabla_g^{\star_\Omega}\mathcal{R}_g$ is g-anti-symmetric thanks to the identity

$$\mathcal{R}_g(\xi, \eta) = -\left(\mathcal{R}_g(\xi, \eta)\right)_g^T,$$

which is a consequence of the alternating property of the $(4, 0)$-Riemann curvature operator. Notice indeed that the previous identity implies

$$\nabla_{g,\mu}\mathcal{R}_g(\xi, \eta) = -\left(\nabla_{g,\mu}\mathcal{R}_g(\xi, \eta)\right)_g^T,$$

for all vector fields ξ, η, μ. Combining the g-symmetric and g-anti-symmetric parts in the identity (9.2) we infer the formulas

$$\begin{aligned} \Delta_{L,g}^{\Omega}\nabla_g^2 u &= \nabla_g^2\Delta_g^{\Omega}u + \nabla_g u \neg \nabla_{T_X,g}\,\mathrm{Ric}_g^{\star}(\Omega) - \left[\nabla_g\,\mathrm{Ric}_g^{\star}(\Omega)\nabla_g u\right]_g^T, \\ \xi \neg \nabla_g^{\star_\Omega}\mathcal{R}_g &= \nabla_g\,\mathrm{Ric}_g^{\star}(\Omega)\xi - \left[\nabla_g\,\mathrm{Ric}_g^{\star}(\Omega)\xi\right]_g^T, \end{aligned}$$

for all $\xi \in T_X$ since the function u is arbitrary. In the case $\nabla_g\,\mathrm{Ric}_g^{\star}(\Omega) = 0$ we deduce the identities $\Delta_{L,g}^{\Omega}\nabla_g^2 u = \nabla_g^2\Delta_g^{\Omega}u$ and $\nabla_g^{\star_\Omega}\mathcal{R}_g = 0$. More in particular in the soliton case $h_{g,\Omega} = 0$ the first formula reduces to the differential identity

$$\mathcal{L}_g^{\Omega}\nabla_g^2 u = \nabla_g^2(\Delta_g^{\Omega} - 2\mathbb{I})u. \tag{9.3}$$

We infer the conclusion of lemma 8. This formula will be also quite crucial for the study of the sign of the second variation of the $\mathcal{W}$ functional at a Kähler-Ricci soliton point.

10 Invariance of $\mathbb{F}$ under the action of the Hessian endomorphism of $\mathcal{W}$

We observe that Perelman's twice contracted second Bianchi type identity (1.2) rewrites as;

$$\nabla_g^{\star_\Omega} h_{g,\Omega} + dH_{g,\Omega} = 0.$$

If we differentiate this over the space $\mathcal{M} \times \mathcal{V}_1$ we obtain

$$\left[(D_{g,\Omega}\nabla_{\bullet}^{\star_\bullet})(v, V)\right] h_{g,\Omega} + \nabla_g^{\star_\Omega}\left[D_{g,\Omega}h(v, V)\right] + d\left[D_{g,\Omega}H(v, V)\right] = 0.$$

We deduce using the fundamental variation formulas (1.4) and (1.5)

$$\nabla_g^{\star_\Omega}\left[\mathcal{L}_g^{\Omega}v + vh_{g,\Omega}^{\star} + h_{g,\Omega}v_g^{\star}\right] + d\left[(\Delta_g^{\Omega} - 2\mathbb{I})V_{\Omega}^{\star} - \langle v, h_{g,\Omega}\rangle_g\right] = -2\left[D_{g,\Omega}\nabla_{\bullet}^{\star_\bullet}(v, V)\right] h_{g,\Omega}$$

in the directions $(v, V) \in \mathbb{F}_{g,\Omega}$. We infer that in the soliton case $h_{g,\Omega} = 0$ the map

$$\nabla^2_G \mathcal{W}(g, \Omega) : \mathbb{F}_{g,\Omega} \longrightarrow \mathbb{F}_{g,\Omega},$$

is well defined. In order to investigate the general case we use a different method which has the advantage to involve less computations. Let $(e_k)_k$ be a g-orthonormal local frame of T_X. For any $u, v \in C^\infty\left(X, S^2 T^\star_X\right)$ we define the real valued 1-form

$$M_g(u, v)(\xi) \quad := \quad 2\nabla_g v(e_k, u^\star_g e_k, \xi) + \nabla_g u(\xi, v^\star_g e_k, e_k) \quad ,$$

for all $\xi \in T_X$. One can show that the operator

$$T_g(u, v) \quad := \quad M_g(u, v) - M_g(v, u),$$

is related with the torsion of the distribution $\mathbb{F}$. We observe now that by lemma 3 holds the identity

$$\Delta^\Omega_g v^\star_g - \nabla_g \nabla^{\star_\Omega}_g v^\star_g \quad = \quad \nabla^{\star_\Omega}_g \nabla_{T_X,g} v^\star_g + \mathcal{R}_g \star v^\star_g - v^\star_g \operatorname{Ric}^\star_g(\Omega).$$

Applying the $\nabla^{\star_\Omega}_g$-operator to both sides of this identity we deduce the commutation formula

$$\left[\nabla^{\star_\Omega}_g, \Delta^\Omega_g\right] v^\star_g \quad = \quad \nabla^{\star_\Omega}_g \left[\nabla^{\star_\Omega}_g \nabla_{T_X,g} v^\star_g + \mathcal{R}_g \star v^\star_g - v^\star_g h^\star_{g,\Omega} - v^\star_g\right].$$

We observe now that for any $\psi \in C^\infty(X, \Lambda^2 T_X \otimes_{\mathbb{R}} T_X)$ and $\xi \in C^\infty(X, T_X)$ hold the equalities

$$\begin{aligned}
\int_X \left\langle (\nabla^{\star_\Omega}_g)^2 \psi, \xi \right\rangle_g \Omega \quad &= \quad \int_X \left\langle \nabla^{\star_\Omega}_g \psi, \nabla_g \xi \right\rangle_g \Omega \\
&= \quad \frac{1}{2} \int_X \left\langle \nabla^{\star_\Omega}_{T_X,g} \psi, \nabla_g \xi \right\rangle_g \Omega \\
&= \quad \frac{1}{2} \int_X \left\langle \psi, \nabla^2_{T_X,g} \xi \right\rangle_g \Omega \\
&= \quad \frac{1}{2} \int_X \langle \psi, \mathcal{R}_g \cdot \xi \rangle_g \Omega,
\end{aligned}$$

and

$$\langle \psi, \mathcal{R}_g \cdot \xi \rangle_g \quad = \quad \left\langle \psi(e_k, e_l), \mathcal{R}_g(e_k, e_l)\xi \right\rangle_g = -\left\langle \mathcal{R}_g(e_k, e_l)\psi(e_k, e_l), \xi \right\rangle_g.$$

We infer

$$\begin{aligned}
(\nabla^{\star_\Omega}_g)^2 \nabla_{T_X,g} v^\star_g \quad &= \quad -\frac{1}{2} \mathcal{R}_g(e_k, e_l) \left[\nabla_g v^\star_g(e_k, e_l) - \nabla_g v^\star_g(e_l, e_k)\right] \\
&= \quad \mathcal{R}_g(e_l, e_k) \nabla_g v^\star_g(e_k, e_l).
\end{aligned}$$

This combined with the expression

$$\nabla^{\star_\Omega}_g (\mathcal{R}_g \star v^\star_g) \quad = \quad \nabla^{\star_\Omega}_g \mathcal{R}_g(e_k) v^\star_g e_k + \mathcal{R}_g(e_l, e_k) \nabla_g v^\star_g(e_k, e_l),$$

implies the identity

$$\nabla^{\star_\Omega}_g \mathcal{L}^\Omega_g v^\star_g = (\Delta^\Omega_g - \mathbb{I}) \nabla^{\star_\Omega}_g v^\star_g + \nabla_g v^\star_g(e_k, h^\star_{g,\Omega} e_k) - v^\star_g \nabla^{\star_\Omega}_g h^\star_{g,\Omega} - \nabla^{\star_\Omega}_g \mathcal{R}_g(e_k) v^\star_g e_k,$$

which rewrites also under the form

$$\nabla^{\star_\Omega}_g \mathcal{L}^\Omega_g v = (\Delta^\Omega_g - \mathbb{I}) \nabla^{\star_\Omega}_g v + \nabla_g v(e_k, h^\star_{g,\Omega} e_k, \bullet) - v \nabla^{\star_\Omega}_g h^\star_{g,\Omega} + v\left(e_k, \nabla^{\star_\Omega}_g \mathcal{R}_g(e_k) \bullet\right),$$

thanks to (3.3) and the anti-symmetry property

$$e_k \neg \nabla_g^{\star_\Omega} \mathcal{R}_g = -\left(e_k \neg \nabla_g^{\star_\Omega} \mathcal{R}_g\right)_g^T.$$

On the other hand the once contracted differential Bianchi type identity (9.1) rewrites as

$$-\nabla_{T_X,g} \operatorname{Ric}_g^\star(\Omega) = \operatorname{Alt}\left(\nabla_g^{\star_\Omega} \mathcal{R}_g\right),$$

thanks to the algebraic Bianchi identity. Therefore for any $\xi \in C^\infty(X, T_X)$ hold the identities

$$\begin{aligned}
\nu\left(e_k, \nabla_g^{\star_\Omega} \mathcal{R}_g(e_k)\xi\right) &= \nu\left(\nabla_g^{\star_\Omega} \mathcal{R}_g(e_k)\xi, e_k\right) \\
&= \nu\left(\nabla_g^{\star_\Omega} \mathcal{R}_g(\xi)e_k, e_k\right) + \nu\left(\left[\xi \neg \nabla_{T_X,g} \operatorname{Ric}_g^\star(\Omega)\right] e_k, e_k\right) \\
&= \operatorname{Tr}_{\mathbb{R}}\left[\nu_g^\star \nabla_g^{\star_\Omega} \mathcal{R}_g(\xi)\right] + \operatorname{Tr}_g\left[\nu\left(\xi \neg \nabla_{T_X,g} \operatorname{Ric}_g^\star(\Omega)\right)\right] \\
&= \operatorname{Tr}_g\left[\nu\left(\xi \neg \nabla_{T_X,g} h_{g,\Omega}^\star\right)\right],
\end{aligned}$$

since the endomorphism section $\nabla_g^{\star_\Omega} \mathcal{R}_g(\xi)$ is g-anti-symmetric. Notice indeed that if $A, B \in C^\infty\left(X, \operatorname{End}(T_X)\right)$ satisfy $A = A_g^T$ and $B = -B_g^T$ then

$$\begin{aligned}
\operatorname{Tr}_{\mathbb{R}}(AB) &= \operatorname{Tr}_{\mathbb{R}}(BA) \\
&= \operatorname{Tr}_{\mathbb{R}}(BA)_g^T \\
&= \operatorname{Tr}_{\mathbb{R}}(A_g^T B_g^T) \\
&= -\operatorname{Tr}_{\mathbb{R}}(AB),
\end{aligned}$$

i.e $\operatorname{Tr}_{\mathbb{R}}(AB) = 0$. We deduce in conclusion the formula

$$\nabla_g^{\star_\Omega} \mathcal{L}_g^\Omega \nu = (\Delta_g^\Omega - \mathbb{I}) \nabla_g^{\star_\Omega} \nu - \nu \nabla_g^{\star_\Omega} h_{g,\Omega}^\star + \nabla_g \nu(e_k, h_{g,\Omega}^\star e_k, \bullet) + \operatorname{Tr}_g\left[\nu\left(\bullet \neg \nabla_{T_X,g} h_{g,\Omega}^\star\right)\right].$$

Using the general formula

$$\nabla_g^{\star_\Omega}(\varphi \nu) = -\nu \nabla_g \varphi + \varphi \nabla_g^{\star_\Omega} \nu,$$

with $\varphi \in C^\infty(X, \mathbb{R})$ we infer

$$\begin{aligned}
\nabla_g^{\star_\Omega}\left[\underline{H}_{g,\Omega} \nu + V_\Omega^\star h_{g,\Omega}\right] &= -\nu \nabla_g \underline{H}_{g,\Omega} + \underline{H}_{g,\Omega} \nabla_g^{\star_\Omega} \nu - h_{g,\Omega} \nabla_g V_\Omega^\star + V_\Omega^\star \nabla_g^{\star_\Omega} h_{g,\Omega} \\
&= \nu \nabla_g^{\star_\Omega} h_{g,\Omega}^\star + \underline{H}_{g,\Omega} \nabla_g^{\star_\Omega} \nu - dV_\Omega^\star \cdot h_{g,\Omega}^\star - V_\Omega^\star d\underline{H}_{g,\Omega},
\end{aligned}$$

thanks to Perelman's twice contracted differential Bianchi type identity (1.2). Using the identity (15.1) we expand the term

$$\begin{aligned}
&d\left[\left(\Delta_g^\Omega + \underline{H}_{g,\Omega} - 2\mathbb{I}\right) V_\Omega^\star - \frac{1}{2}\left\langle \nu, h_{g,\Omega}\right\rangle_g\right] \\
&= \Delta_g^\Omega dV_\Omega^\star + dV_\Omega^\star \cdot \operatorname{Ric}_g^\star(\Omega) + V_\Omega^\star d\underline{H}_{g,\Omega} + \underline{H}_{g,\Omega} dV_\Omega^\star - 2dV_\Omega^\star - \frac{1}{2} d\left\langle \nu, h_{g,\Omega}\right\rangle_g \\
&= (\Delta_g^\Omega + \underline{H}_{g,\Omega} - \mathbb{I}) dV_\Omega^\star + dV_\Omega^\star \cdot h_{g,\Omega}^\star + V_\Omega^\star d\underline{H}_{g,\Omega} - \frac{1}{2} d\left\langle \nu, h_{g,\Omega}\right\rangle_g.
\end{aligned}$$

Summing up we infer

$$\nabla_g^{\star_\Omega}\left[\left(\mathcal{L}_g^\Omega+\underline{H}_{g,\Omega}\right)v+V_\Omega^\star h_{g,\Omega}\right]+d\left[\left(\Delta_g^\Omega+\underline{H}_{g,\Omega}-2\mathbb{I}\right)V_\Omega^\star-\frac{1}{2}\left\langle v,h_{g,\Omega}\right\rangle_g\right]$$

$$=\left(\Delta_g^\Omega+\underline{H}_{g,\Omega}-\mathbb{I}\right)\left(\nabla_g^{\star_\Omega}v+dV_\Omega^\star\right)$$

$$+\nabla_g v(e_k,h_{g,\Omega}^\star e_k,\bullet)+\mathrm{Tr}_g\left[v\left(\bullet\neg\nabla_{T_X,g}h_{g,\Omega}^\star\right)\right]-\frac{1}{2}d\left\langle v,h_{g,\Omega}\right\rangle_g.$$

We observe now the identity

$$\nabla_g v(e_k,h_{g,\Omega}^\star e_k,\bullet)+\mathrm{Tr}_g\left[v\left(\bullet\neg\nabla_{T_X,g}h_{g,\Omega}^\star\right)\right]-\frac{1}{2}d\left\langle v,h_{g,\Omega}\right\rangle_g=\frac{1}{2}T_g(h_{g,\Omega},v).$$

We deduce the formula

$$\nabla_g^{\star_\Omega}\left[\left(\mathcal{L}_g^\Omega+\underline{H}_{g,\Omega}\right)v+V_\Omega^\star h_{g,\Omega}\right]+d\left[\left(\Delta_g^\Omega+\underline{H}_{g,\Omega}-2\mathbb{I}\right)V_\Omega^\star-\frac{1}{2}\left\langle v,h_{g,\Omega}\right\rangle_g\right]$$

$$=\left(\Delta_g^\Omega+\underline{H}_{g,\Omega}-\mathbb{I}\right)\left(\nabla_g^{\star_\Omega}v+dV_\Omega^\star\right)+\frac{1}{2}T_g(h_{g,\Omega},v).$$

Setting $(v,V)=(h_{g,\Omega},\underline{H}_{g,\Omega}\Omega)\in\mathbb{F}_{g,\Omega}$ in the previous identity we infer

$$0=\nabla_g^{\star_\Omega}\left[\left(\mathcal{L}_g^\Omega+2\underline{H}_{g,\Omega}\right)h_{g,\Omega}\right]+d\left[\left(\Delta_g^\Omega+\underline{H}_{g,\Omega}-2\mathbb{I}\right)\underline{H}_{g,\Omega}-\frac{1}{2}\left|h_{g,\Omega}\right|_g^2\right].$$

This shows the fundamental property (1.9) of the Soliton-Ricci flow.

11 The Kähler set up

In this section we introduce a few basic notations needed in sequel. Let (X,J,g) be a compact connected Kähler manifold with symplectic form $\omega:=gJ$. Let $h:=g-igJ=2g\pi_J^{1,0}$ be the hermitian metric over $T_{X,J}$ induced by g. We remind that in the Kähler case the Chern connection

$$D_{T_{X,J}}^g=\partial_{T_{X,J}}^g+\overline{\partial}_{T_{X,J}}:\quad C^\infty(T_{X,J})\longrightarrow C^\infty(T_X^\star\otimes_{\mathbb{R}}T_{X,J}),$$

of the hermitian vector bundle $(T_{X,J},h)$ coincides with the Levi-Civita connection ∇_g. We set $\mathbb{C}T_X:=T_X\otimes_{\mathbb{R}}\mathbb{C}$ and $\mathbb{C}T_X^\star:=T_X^\star\otimes_{\mathbb{R}}\mathbb{C}$. We observe further that the sesquiliner extension of g

$$g_{\mathbb{C}}\in C^\infty(X,\mathbb{C}T_X^\star\otimes_{\mathbb{C}}\overline{\mathbb{C}}T_X^\star),g_{\mathbb{C}}(\xi,\eta):=g(\xi,\bar{\eta}),\forall\xi,\eta\in\mathbb{C}T_X,$$

is a hermitian metric over $\mathbb{C}T_X$ and the $\mathbb{C}$-linear extension of the Levi-Civita connection ∇_g

$$\nabla_g^{\mathbb{C}}:C^\infty(\mathbb{C}T_X)\quad\longrightarrow\quad C^\infty(\mathbb{C}T_X^\star\otimes_{\mathbb{C}}\mathbb{C}T_X),$$

is a $g_{\mathbb{C}}$-hermitian connection over the vector bundle $T_X\otimes_{\mathbb{R}}\mathbb{C}$. We will focus our interest on the sections of the hermitian vector bundle

$$\left((\mathbb{C}T_X^\star)^{\otimes p}\otimes_{\mathbb{C}}T_{X,J},g_{\mathbb{C}}\otimes h\right),$$

and we will denote by abuse of notations $\nabla_g\equiv\nabla_g^{\mathbb{C}}\otimes D_{T_{X,J}}^g$ the $g_{\mathbb{C}}\otimes h$-hermitian connection over this vector bundle. Still by abuse of notations we will use the identification $\langle\cdot,\cdot\rangle_\omega:=g_{\mathbb{C}}\otimes h$. With these notations we define the operators

$$\nabla_{g,J}^{1,0}:C^\infty\left((\mathbb{C}T_X^\star)^{\otimes p}\otimes_{\mathbb{C}}T_{X,J}\right)\quad\longrightarrow\quad C^\infty\left(\Lambda_J^{1,0}T_X^\star\otimes_{\mathbb{C}}(\mathbb{C}T_X^\star)^{\otimes p}\otimes_{\mathbb{C}}T_{X,J}\right),$$

$$\nabla_{g,J}^{0,1}:C^\infty\left((\mathbb{C}T_X^\star)^{\otimes p}\otimes_{\mathbb{C}}T_{X,J}\right)\quad\longrightarrow\quad C^\infty\left(\Lambda_J^{0,1}T_X^\star\otimes_{\mathbb{C}}(\mathbb{C}T_X^\star)^{\otimes p}\otimes_{\mathbb{C}}T_{X,J}\right),$$

by the formulas

$$2\nabla^{1,0}_{g,J} := \nabla_g - J\nabla_{g,J}\bullet,$$

$$2\nabla^{0,1}_{g,J} := \nabla_g + J\nabla_{g,J}\bullet.$$

Then the formal adjoints of the operators $\partial^g_{T_{X,J}}$ and $\overline{\partial}_{T_{X,J}}$ acting on $T_{X,J}$-valued differential forms satisfy the identities (see [26])

$$\partial^{\star_g}_{T_{X,J}}\alpha = -q\,\mathrm{Tr}_g\,\nabla^{0,1}_{g,J}\alpha,$$

$$\overline{\partial}^{\star_g}_{T_{X,J}}\alpha = -q\,\mathrm{Tr}_g\,\nabla^{1,0}_{g,J}\alpha,$$

for any $\alpha \in C^\infty(X, \Lambda^q T^\star_X \otimes_{\mathbb{C}} T_{X,J})$. We remind now that with our conventions (see [23]) the Hodge Laplacian operator acting on T_X-valued q-forms satisfies the identity

$$\Delta_{T_{X,g}} = \frac{1}{q}\nabla_{T_{X,g}}\nabla^\star_{T_{X,g}} + \frac{1}{q+1}\nabla^\star_{T_{X,g}}\nabla_{T_{X,g}}.$$

We define also the holomorphic and antiholomorphic Hodge Laplacian operators acting on T_X-valued q-forms as

$$\Delta^J_{T_{X,g}} := \frac{1}{q}\partial^g_{T_{X,J}}\partial^{\star_g}_{T_{X,J}} + \frac{1}{q+1}\partial^{\star_g}_{T_{X,J}}\partial^g_{T_{X,J}},$$

$$\Delta^{-J}_{T_{X,g}} := \frac{1}{q}\overline{\partial}_{T_{X,J}}\overline{\partial}^{\star_g}_{T_{X,J}} + \frac{1}{q+1}\overline{\partial}^{\star_g}_{T_{X,J}}\overline{\partial}_{T_{X,J}},$$

with the usual convention $\infty \cdot 0 = 0$. This Hodge Laplacian operators coincide with the standard ones used in the literature. We remind that in the Kähler case holds the decomposition identity

$$\Delta_{T_{X,g}} = \Delta^J_{T_{X,g}} + \Delta^{-J}_{T_{X,g}}.$$

We observe now that the formal adjoint of the $\partial^g_{T_{X,J}}$ operator with respect to the hermitian product

$$\langle\cdot,\cdot\rangle_{\omega,\Omega} := \int_X \langle\cdot,\cdot\rangle_\omega\,\Omega, \tag{11.1}$$

is the operator

$$\partial^{\star_{g,\Omega}}_{T_{X,J}} := e^f\partial^{\star_g}_{T_{X,J}}\left(e^{-f}\bullet\right).$$

In a similar way the formal adjoint of the $\overline{\partial}_{T_{X,J}}$ operator with respect to the hermitian product (11.1) is the operator

$$\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} := e^f\overline{\partial}^{\star_g}_{T_{X,J}}\left(e^{-f}\bullet\right).$$

With these notations we define the holomorphic and anti-holomorphic Ω-Hodge Laplacian operators acting on T_X-valued q-forms as

$$\Delta^{\Omega,J}_{T_{X,g}} := \frac{1}{q}\partial^g_{T_{X,J}}\partial^{\star_{g,\Omega}}_{T_{X,J}} + \frac{1}{q+1}\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^g_{T_{X,J}},$$

$$\Delta^{\Omega,-J}_{T_{X,g}} := \frac{1}{q}\overline{\partial}_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} + \frac{1}{q+1}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}.$$

12 The decomposition of the operator $\mathcal{L}_g^{\Omega}$ in the Kähler case

For any $A \in \mathrm{End}(T_X)$ we denote by A_J' and by A_J'' the J-linear, respectively the J-anti-linear parts of A. We observe that the operator

$$\mathcal{L}_g^{\Omega} : C^{\infty}(X, \mathrm{End}(T_X)) \longrightarrow C^{\infty}(X, \mathrm{End}(T_X)),$$

defined by the formula

$$\mathcal{L}_g^{\Omega} A \quad := \quad \Delta_g^{\Omega} A - 2\mathcal{R}_g \star A,$$

restricts as;

$$\mathcal{L}_g^{\Omega} : C^{\infty}\left(X, T_{X,J}^{\star} \otimes T_{X,J}\right) \longrightarrow C^{\infty}(X, T_{X,J}^{\star} \otimes T_{X,J}), \tag{12.1}$$

$$\mathcal{L}_g^{\Omega} : C^{\infty}\left(X, T_{X,-J}^{\star} \otimes T_{X,J}\right) \longrightarrow C^{\infty}(X, T_{X,-J}^{\star} \otimes T_{X,J}), \tag{12.2}$$

Indeed these properties follow from the identities

$$(\mathcal{R}_g \star A)_J' = \mathcal{R}_g \star A_J', \tag{12.3}$$

$$(\mathcal{R}_g \star A)_J'' = \mathcal{R}_g \star A_J'', \tag{12.4}$$

for any $A \in \mathrm{End}(T_X)$. In their turn they are direct consequence of the identities

$$J(\mathcal{R}_g \star A) = \mathcal{R}_g \star (JA), \tag{12.5}$$

$$(\mathcal{R}_g \star A)J = \mathcal{R}_g \star (AJ), \tag{12.6}$$

In order to see (12.5) and (12.6) let $(e_k)_k$ be a g-orthonormal real basis. Using the J-invariant properties of the curvature operator we infer

$$J(\mathcal{R}_g \star A)\xi \quad = \quad J\mathcal{R}_g(\xi, e_k)Ae_k = \mathcal{R}_g(\xi, e_k)JAe_k = \left[\mathcal{R}_g \star (JA)\right]\xi,$$

$$(\mathcal{R}_g \star A)J\xi \quad = \quad \mathcal{R}_g(J\xi, e_k)Ae_k = -\mathcal{R}_g(\xi, Je_k)Ae_k = \mathcal{R}_g(\xi, \eta_k)AJ\eta_k,$$

where $\eta_k := Je_k$. The fact that $(\eta_k)_k$ is also a g-orthonormal real frame implies (12.6). By (12.1) and (12.2) we conclude the decomposition formula

$$\int_X \left\langle \mathcal{L}_g^{\Omega} A, A \right\rangle_g \Omega = \int_X \left\langle \mathcal{L}_g^{\Omega} A_J', A_J' \right\rangle_g \Omega + \int_X \left\langle \mathcal{L}_g^{\Omega} A_J'', A_J'' \right\rangle_g \Omega. \tag{12.7}$$

We observe that the properties (12.1) and (12.2) imply also that $A \in \mathrm{Ker}\,\mathcal{L}_g^{\Omega}$ if and only if $A_J' \in \mathrm{Ker}\,\mathcal{L}_g^{\Omega}$ and $A_J'' \in \mathrm{Ker}\,\mathcal{L}_g^{\Omega}$. We observe further that the identity (9.3) combined with the properties (12.1) and (12.2) implies the formulas

$$\mathcal{L}_g^{\Omega} \partial_{T_{X,J}}^g \nabla_g u = \partial_{T_{X,J}}^g \nabla_g (\Delta_g^{\Omega} - 2\mathbb{I})u, \tag{12.8}$$

$$\mathcal{L}_g^{\Omega} \overline{\partial}_{T_{X,J}} \nabla_g u = \overline{\partial}_{T_{X,J}} \nabla_g (\Delta_g^{\Omega} - 2\mathbb{I})u, \tag{12.9}$$

in the Kähler-Ricci soliton case. The properties (1.8) and (1.9) combined with (12.2) imply (1.12) and (1.13).

13 Basic complex Bochner type formulas

We need to review in detail now some fact from [12], (see also [21]). Most of the formulas in this section will be intensively used in the rest of the paper. Let (X, J, g) be a compact connected Kähler manifold with symplectic form $\omega := gJ$. We remind that the hermitian product induced by ω over the bundle $\Lambda^{1,0}_J T^\star_X$ satisfies the identity

$$2\langle \alpha, \beta\rangle_\omega = \mathrm{Tr}_\omega\left(i\alpha \wedge \bar{\beta}\right).$$

Let $\Omega > 0$ be a smooth volume form and set as usual $f := \log \frac{dV_g}{\Omega}$. We define the Ω-weighted complex Laplace type operator acting on functions $u \in C^\infty(X, \mathbb{C})$ as

$$\begin{aligned}\Delta^\Omega_{g,J} u &:= e^f \mathrm{Tr}_\omega\left[i\overline{\partial}_J\left(e^{-f}\partial_J u\right)\right]\\ &= \Delta_g u + 2\left\langle \partial_J u, \partial_J f\right\rangle_\omega\\ &= \Delta_g u + 2\partial_J u \cdot \nabla_g f.\end{aligned}$$

We notice the identities $\Delta^\Omega_{g,J} f = \Delta^\Omega_g f$ and $\Delta^\Omega_g = \mathrm{Re}(\Delta^\Omega_{g,J})$. The complex operator $\Delta^\Omega_{g,J}$ is self-adjoint with respect to the the L^2_Ω-hermitian product

$$\langle u, v\rangle_\Omega := \int_X u\overline{v}\Omega. \tag{13.1}$$

Indeed integrating by parts we obtain

$$\begin{aligned}\int_X i\overline{\partial}_J\left(e^{-f}\partial_J u\right)\overline{v}\wedge\omega^{n-1} &= \int_X \partial_J u \wedge ie^{-f}\overline{\partial}_J\overline{v}\wedge\omega^{n-1}\\ &= -\int_X ui\partial_J\left(e^{-f}\overline{\partial}_J\overline{v}\right)\wedge\omega^{n-1}.\end{aligned}$$

(Notice the equality $\Omega = e^{-f}\omega^n/n!$.) We observe in particular the identity

$$\int_X \Delta^\Omega_{g,J} u\cdot\overline{v}\Omega = \int_X 2\left\langle\partial_J u, \partial_J v\right\rangle_\omega \Omega,$$

which implies that all the eigenvalues satisfy $\lambda_j(\Delta^\Omega_{g,J}) \geqslant 0$. For any function $u \in C^\infty(X, \mathbb{C})$ we define the J-complex g-gradient as the real vector field;

$$\nabla_{g,J} u := \nabla_g \mathrm{Re}\, u + J\nabla_g \mathrm{Im}\, u \in C^\infty(X, T_X).$$

With these notations hold the complex decomposition formula

$$\nabla_{g,J} u \neg g = \partial_J\overline{u} + \overline{\partial}_J u. \tag{13.2}$$

We consider now the linear operator

$$\begin{aligned}B^\Omega_{g,J} : C^\infty(X, \mathbb{R}) &\longrightarrow C^\infty_\Omega(X, \mathbb{R})_0,\\ B^\Omega_{g,J} u &:= \mathrm{div}^\Omega(J\nabla_g u).\end{aligned}$$

This is a first order differential operator. Indeed

$$\begin{aligned}B^\Omega_{g,J} u &= \mathrm{Tr}_{\mathbb{R}}\left(J\nabla^2_g u\right) - df\cdot J\nabla_g u\\ &= g(\nabla_g u, J\nabla_g f),\end{aligned}$$

since J is g-anti-symmetric. We extend $B^{\Omega}_{g,J}$ over $C^{\infty}(X,\mathbb{C})$ by complex linearity. Let also

$$2d^{c}_{J}u \quad := \quad i(\overline{\partial}_{J}-\partial_{J})u=-du\cdot J.$$

Then the identity $2\partial_{J}=d+2id^{c}_{J}$ implies the decomposition

$$\Delta^{\Omega}_{g,J}=\Delta^{\Omega}_{g}+2i\nabla_{g}f\neg d^{c}_{J}.$$

In other terms

$$\Delta^{\Omega}_{g,J} \quad = \quad \Delta^{\Omega}_{g}-iB^{\Omega}_{g,J}.$$

The following lemma is needed for the study of the operator $\Delta^{\Omega}_{g,J}$. (Compare also with [12].)

Lemma 9. *Let (X,J,g) be a Kähler manifold with symplectic form $\omega:=gJ$ and let $\Omega>0$ be a smooth volume form. Then for all $u\in C^{\infty}(X,\mathbb{R})$ and $v\in C^{\infty}(X,\mathbb{C})$ hold the complex Bochner type formulas*

$$2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\nabla_{g}u=\nabla_{g,J}\Delta^{\Omega}_{g,J}u-2\overline{\partial}_{T_{X,J}}\nabla_{g}f\nabla_{g}u, \tag{13.3}$$

$$2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{v}=\nabla_{g,J}\overline{\Delta^{\Omega}_{g,J}v}-2\,\mathrm{Ric}^{\star}_{J}(\Omega)_{\omega}\nabla_{g,J}\overline{v}. \tag{13.4}$$

Proof. Let $\xi\in C^{\infty}(X,T_{X})$ and observe that for bi-degree reasons hold the identity

$$\begin{aligned}
2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\xi &= 2\nabla^{\star_{\Omega}}_{g}\partial^{g}_{T_{X,J}}\xi\\
&= \Delta^{\Omega}_{g}\xi-\nabla^{\star_{\Omega}}_{g}\left(J\nabla_{g,J\bullet}\xi\right)\\
&= \Delta^{\Omega}_{g}\xi-\nabla^{\star}_{g}\left(J\nabla_{g,J\bullet}\xi\right)-J\nabla_{g,J\nabla_{g}f}\xi.
\end{aligned}$$

Let $(e_{k})^{2n}_{k=1}$ be a local g-orthonormal frame over a neighborhood of an arbitrary point p such that $\nabla_{g}e_{k}(p)=0$. Then at the point p hold the equalities

$$\begin{aligned}
-\nabla^{\star}_{g}\left(J\nabla_{g,J\bullet}\xi\right) &= J\nabla_{g,e_{k}}\nabla_{g,Je_{k}}\xi\\
&= \frac{1}{2}\left(J\nabla_{g,e_{k}}\nabla_{g,Je_{k}}\xi-J\nabla_{g,Je_{k}}\nabla_{g,e_{k}}\xi\right),
\end{aligned}$$

since $(Je_{k})^{2n}_{k=1}$ is also a local g-orthonormal frame. Then the fact that $[e_{k},Je_{k}](p)=0$ implies

$$-\nabla^{\star}_{g}\left(J\nabla_{g,J\bullet}\xi\right) \quad = \quad \frac{1}{2}J\mathcal{R}_{g}(e_{k},Je_{k})\xi=\mathrm{Ric}^{\star}(g)\xi.$$

We infer the complex Bochner type formula

$$2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\xi=\Delta^{\Omega}_{g}\xi+\mathrm{Ric}^{\star}(g)\xi-J\nabla_{g,J\nabla_{g}f}\xi. \tag{13.5}$$

In a similar way we obtain

$$2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\xi=\Delta^{\Omega}_{g}\xi-\mathrm{Ric}^{\star}(g)\xi+J\nabla_{g,J\nabla_{g}f}\xi. \tag{13.6}$$

Using formulas (13.5) and (8.2) we deduce the expressions

$$\begin{aligned}
2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\nabla_{g}u &= \nabla_{g}\Delta^{\Omega}_{g}u-\nabla^{2}_{g}f\nabla_{g}u-J\nabla^{2}_{g}uJ\nabla_{g}f\\
&= \nabla_{g}\Delta^{\Omega}_{g}u-\left(\nabla^{2}_{g}f+J\nabla^{2}_{g}fJ\right)\nabla_{g}u-J\left(\nabla^{2}_{g}uJ\nabla_{g}f-\nabla^{2}_{g}fJ\nabla_{g}u\right)\\
&= \nabla_{g}\Delta^{\Omega}_{g}u-2\overline{\partial}_{T_{X,J}}\nabla_{g}f\nabla_{g}u-J\nabla_{g}[g(\nabla_{g}u,J\nabla_{g}f)].
\end{aligned}$$

Using the first order expression of $B^{\Omega}_{J,g}$ we obtain

$$2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\nabla_g u = \left[\nabla_g \Delta^{\Omega}_g - J\nabla_g B^{\Omega}_{g,J}\right] u - 2\overline{\partial}_{T_{X,J}}\nabla_g f\nabla_g u.$$

We infer the complex differential Bochner type formula (13.3). In a similar way using formulas (13.6) and (8.2) we deduce

$$\begin{aligned}
2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g u &= \nabla_g \Delta^{\Omega}_g u - 2\,\mathrm{Ric}^{\star}_g(\Omega)\nabla_g u + \nabla^2_g f\nabla_g u + J\nabla^2_g uJ\nabla_g f \\
&= \nabla_g \Delta^{\Omega}_g u - 2\,\mathrm{Ric}^{\star}_g(\Omega)\nabla_g u + \left(\nabla^2_g f + J\nabla^2_g fJ\right)\nabla_g u + J\left(\nabla^2_g uJ\nabla_g f - \nabla^2_g fJ\nabla_g u\right) \\
&= \nabla_g \Delta^{\Omega}_g u - 2\,\mathrm{Ric}^{\star}_g(\Omega)\nabla_g u + 2\overline{\partial}_{T_{X,J}}\nabla_g f\nabla_g u + J\nabla_g[g(\nabla_g u, J\nabla_g f)].
\end{aligned}$$

Using the first order expression of $B^{\Omega}_{J,g}$ we obtain

$$2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g u = \left[\nabla_g \Delta^{\Omega}_g + J\nabla_g B^{\Omega}_{g,J}\right] u - 2\,\mathrm{Ric}^{\star}_J(\Omega)_{\omega}\nabla_g u.$$

We infer the complex differential Bochner type formula

$$2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g u = \nabla_{g,J}\overline{\Delta^{\Omega}_{g,J}u} - 2\,\mathrm{Ric}^{\star}_J(\Omega)_{\omega}\nabla_g u. \tag{13.7}$$

More in general for all $v \in C^{\infty}(X, \mathbb{C})$ this writes as (13.4). □

Notice that for bi-degree reasons the identity (8.2) decomposes as

$$2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\nabla_g u + 2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g u = \nabla_{g,J}\Delta^{\Omega}_{g,J}u + \nabla_{g,J}\overline{\Delta^{\Omega}_{g,J}u} - 2\overline{\partial}_{T_{X,J}}\nabla_g f\nabla_g u - 2\,\mathrm{Ric}^{\star}_J(\Omega)_{\omega}\nabla_g u.$$

Then we can obtain (13.7) from (13.3) and vice versa. We observe also that the complex Bochner identities (13.3), (13.4) write in the Kähler-Ricci-Soliton case as

$$2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^{g}_{T_{X,J}}\nabla_g u = \nabla_{g,J}\Delta^{\Omega}_{g,J}u, \tag{13.8}$$

$$2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{v} = \nabla_{g,J}\overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})v}, \tag{13.9}$$

for all $u \in C^{\infty}(X, \mathbb{R})$ and $v \in C^{\infty}(X, \mathbb{C})$. Obviously the identity (13.9) still hold in the more general case $\mathrm{Ric}_J(\Omega) = \omega$. We observe now an other integration by parts formula.

Let $\xi \in C^{\infty}(X, T_X)$, $A \in C^{\infty}\left(X, T^{\star}_{X,-J} \otimes T_{X,J}\right)$ and observe that the comparison between Riemannian and hermitian norms of T_X-valued 1-forms (see the appendix in [22]) implies

$$\begin{aligned}
\int_X \left\langle \overline{\partial}_{T_{X,J}}\xi, A\right\rangle_g \Omega &= \frac{1}{2}\int_X \left[\left\langle \overline{\partial}_{T_{X,J}}\xi, A\right\rangle_{\omega} + \left\langle A, \overline{\partial}_{T_{X,J}}\xi\right\rangle_{\omega}\right]\Omega \\
&= \frac{1}{2}\int_X \left[\left\langle \xi, \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}A\right\rangle_{\omega} + \left\langle \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}A, \xi\right\rangle_{\omega}\right]\Omega \\
&= \int_X \left\langle \xi, \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}A\right\rangle_g \Omega.
\end{aligned}$$

Using this and multiplying both sides of (13.9) by $\nabla_{g,J}\overline{v}$ we obtain the identity

$$2\int_X \left|\overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{v}\right|^2_g \Omega = \int_X \left\langle \nabla_{g,J}\overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})v}, \nabla_{g,J}\overline{v}\right\rangle_g \Omega, \tag{13.10}$$

in the case $\mathrm{Ric}_J(\Omega) = \omega$. We consider now the J-anti-linear component of the complex Hessian map;

$$\begin{aligned}
\mathcal{H}^{0,1}_{g,J} : C^{\infty}_{\Omega}(X, \mathbb{C})_0 &\longrightarrow C^{\infty}(X, \Lambda^{0,1}_J T^{\star}_X \otimes_{\mathbb{C}} T_{X,J}) \\
u &\longmapsto \overline{\partial}_{T_{X,J}}\nabla_{g,J}u.
\end{aligned}$$

We observe that $H^1_d(X, \mathbb{C}) = 0$ in the case of Fano manifolds and we remind the following well known fact

Lemma 10. *Let (X, J, g) be a compact connected Kähler manifold such that $H^1_d(X, \mathbb{C}) = 0$. Then the map*

$$\begin{aligned} \operatorname{Ker} \mathcal{H}^{0,1}_{g,J} &\longrightarrow H^0(X, T_{X,J}) \\ u &\longmapsto \nabla_{g,J} u, \end{aligned}$$

is an isomorphism of complex vector spaces.

Proof. We observe first the injectivity. Using the complex decomposition (13.2) we infer the formula

$$d(\nabla_{g,J} u \neg g) = \partial_J \overline{\partial}_J (u - \overline{u}),$$

which in the case $\nabla_{g,J} u = 0$ implies $\operatorname{Im} u = 0$ and thus $\operatorname{Re} u = 0$. In order to show the surjectivity we consider an arbitrary $\xi \in H^0(X, T_{X,J})$. Then the identity (13.11) below implies

$$\overline{\partial}_J (\xi^{1,0}_J \neg \omega) = 0.$$

By Hodge decomposition hold the identity $H^{0,1}_{\overline{\partial}}(X, \mathbb{C}) = 0$. We deduce the existence of a unique function $u \in C^\infty_\Omega(X, \mathbb{C})_0$ such that

$$i\overline{\partial}_J u = \xi^{1,0}_J \neg \omega = i\xi^{1,0}_J \neg g.$$

Thus $\xi = \nabla_{g,J} u$ thanks to the complex decomposition (13.2). □

Lemma 11. *Let (X, J) be a complex manifold and let $\omega \in C^\infty(X, \Lambda^{1,1}_J T^\star_X)$, $\xi \in C^\infty(X, T^{1,0}_X)$. Then hold the identity*

$$\overline{\partial}_J (\xi \neg \omega) = \overline{\partial}_{T^{1,0}_{X,J}} \xi \neg \omega - \xi \neg \overline{\partial}_J \omega. \tag{13.11}$$

Proof. Let $\eta, \mu \in C^\infty(X, T^{0,1}_X)$ and observe the identities (see [20])

$$\begin{aligned} \overline{\partial}_J (\xi \neg \omega)(\eta, \mu) &= \eta.\omega(\xi, \mu) - \mu.\omega(\xi, \eta) - \omega(\xi, [\eta, \mu]), \\ \overline{\partial}_J \omega(\eta, \xi, \mu) &= \eta.\omega(\xi, \mu) + \mu.\omega(\eta, \xi) - \omega([\eta, \xi,]^{1,0}, \mu) + \omega([\eta, \mu], \xi) - \omega([\xi, \mu]^{1,0}, \eta) \\ &= \overline{\partial}_J (\xi \neg \omega)(\eta, \mu) - \omega\left(\overline{\partial}_{T^{1,0}_{X,J}} \xi(\eta), \mu\right) + \omega\left(\overline{\partial}_{T^{1,0}_{X,J}} \xi(\mu), \eta\right) \\ &= \overline{\partial}_J (\xi \neg \omega)(\eta, \mu) - \omega\left(\overline{\partial}_{T^{1,0}_{X,J}} \xi(\eta), \mu\right) - \omega\left(\eta, \overline{\partial}_{T^{1,0}_{X,J}} \xi(\mu)\right) \\ &= \left[\overline{\partial}_J (\xi \neg \omega) - \overline{\partial}_{T^{1,0}_{X,J}} \xi \neg \omega\right](\eta, \mu), \end{aligned}$$

which implies the required identity. □

On the other hand the identities (13.9) and (13.10) show that in the case $\operatorname{Ric}_J(\Omega) = \omega$ hold the identity

$$\overline{\operatorname{Ker}(\Delta^\Omega_{g,J} - 2\mathbb{I})} = \operatorname{Ker} \mathcal{H}^{0,1}_{g,J}. \tag{13.12}$$

We infer the following well known result due to Futaki [12]. (See also [14] and the sub-section 21.2 in appendix B for a more more complete statement.)

Corollary 1. *Let (X, J) be a Fano manifold and let g be a J-invariant Kähler metric such that $\omega := gJ \in 2\pi c_1(X, [J])$. Let also $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\operatorname{Ric}_J(\Omega) = \omega$. Then the map*

$$\begin{aligned} \overline{\operatorname{Ker}(\Delta^\Omega_{g,J} - 2\mathbb{I})} &\longrightarrow H^0(X, T_{X,J}) \\ u &\longmapsto \nabla_{g,J} u, \end{aligned}$$

is well defined and it represents an isomorphism of complex vector spaces. The first eigenvalue $\lambda_1(\Delta^{\Omega}_{g,J})$ of the operator $\Delta^{\Omega}_{g,J}$ satisfies the estimate $\lambda_1(\Delta^{\Omega}_{g,J}) \geqslant 2$, with equality in the case $H^0(X, T_{X,J}) \neq 0$. Moreover if we set $\mathrm{Kill}_g := \mathrm{Lie}(\mathrm{Isom}^0_g)$ *and*

$$\mathrm{Ker}_{\mathbb{R}}(\Delta^{\Omega}_{g,J} - 2\mathbb{I}) \quad := \quad \mathrm{Ker}(\Delta^{\Omega}_{g,J} - 2\mathbb{I}) \cap C^{\infty}_{\Omega}(X, \mathbb{R})_0,$$

then the map

$$J\nabla_g : \mathrm{Ker}_{\mathbb{R}}(\Delta^{\Omega}_{g,J} - 2\mathbb{I}) \longrightarrow \mathrm{Kill}_g, \tag{13.13}$$

is well defined and it represents an isomorphism of real vector spaces.

Proof. We only need to show the statement concerning the map (13.13). Let $\xi \in \mathrm{Kill}_g$ and let $(\varphi_t)_{t\in\mathbb{R}} \subset \mathrm{Isom}^0_g$ be the corresponding 1-parameter sub-group. The Kähler condition $\nabla_g J = 0$ implies $\Delta_{d,g}\omega = 0$ and thus $\Delta_{d,g}(\varphi_t^{\star}\omega) = 0$. Time deriving the latter at $t = 0$ we infer

$$\Delta_{d,g} L_{\xi}\omega = 0. \tag{13.14}$$

But $L_{\xi}\omega = d(\xi\neg\omega)$ and (13.14) rewrites as $d^{\star_g} d(\xi\neg\omega) = 0$. We infer

$$0 \quad = \quad L_{\xi}\omega = gL_{\xi}J = 2\omega\overline{\partial}_{T_{X,J}}\xi,$$

and thus

$$\begin{aligned}
\mathrm{Kill}_g &= \left\{\xi \in H^0(X, T_{X,J}) \mid L_{\xi}\omega = 0\right\} \\
&= \left\{\xi \in H^0(X, T_{X,J}) \mid d(\xi\neg\omega) = 0\right\} \\
&= \left\{\xi \in H^0(X, T_{X,J}) \mid \exists u \in C^{\infty}_{\Omega}(X, \mathbb{R})_0 : \xi\neg\omega = du\right\},
\end{aligned}$$

thanks to the fact that $H^1_d(X, \mathbb{R}) = 0$. But the latter identity rewrites as

$$\mathrm{Kill}_g \quad = \quad \left\{\xi \in H^0(X, T_{X,J}) \mid \exists u \in C^{\infty}_{\Omega}(X, \mathbb{R})_0 : \xi = J\nabla_g u\right\},$$

which shows that the map (13.13) is well defined thanks to the first statement of corollary 1. The surjectivity of the map (13.13) follows from the identity (13.9) applied to the function $v := -iu$, with $u \in C^{\infty}_{\Omega}(X, \mathbb{R})_0$ such that $J\nabla_g u \in \mathrm{Kill}_g$. The injectivity of the map (13.13) is obvious. □

Using the variational characterization of the first eigenvalue we observe;

$$\begin{aligned}
\lambda_1(\Delta^{\Omega}_{g,J}) &= \inf\left\{\frac{\int_X \Delta^{\Omega}_{g,J} u\overline{u}\Omega}{\int_X |u|^2\,\Omega} \mid u \in C^{\infty}_{\Omega}(X, \mathbb{C})_0 \smallsetminus \{0\}\right\} \\
&\leqslant \inf\left\{\frac{\int_X 2\left|\partial_J u\right|^2_{\omega}\,\Omega}{\int_X u^2\Omega} \mid u \in C^{\infty}_{\Omega}(X, \mathbb{R})_0 \smallsetminus \{0\}\right\} \\
&= \inf\left\{\frac{\int_X \Delta^{\Omega}_g uu\Omega}{\int_X u^2\Omega} \mid u \in C^{\infty}_{\Omega}(X, \mathbb{R})_0 \smallsetminus \{0\}\right\} \\
&= \lambda_1(\Delta^{\Omega}_g),
\end{aligned}$$

thanks to the identity $2\left|\partial_J u\right|^2_{\omega} = |\nabla_g u|^2_g$. We deduce that in the set-up of corollary 1 hold the estimate

$$\lambda_1(\Delta^{\Omega}_g) \geqslant 2. \tag{13.15}$$

14 Symmetric variations of Kähler structures

We show a few fundamental facts about the space of symmetric variations of Kähler structures $\mathbb{KV}_g^J$ given by the elements $v \in C^\infty\left(X, S^2_{\mathbb{R}} T_X^\star\right)$ such that there exists a smooth family $(J_t, g_t)_t \subset \mathcal{KS}$ with $(J_0, g_0) = (J, g)$, $\dot{g}_0 = v$ and $\dot{J}_0 = (\dot{J}_0)_g^T$. One can observe (see [23]) that $\mathbb{KV}_g^J \subseteq \mathbb{D}_g^J$ with

$$\mathbb{D}_g^J := \left\{ v \in C^\infty\left(X, S^2_{\mathbb{R}} T_X^\star\right) \mid \partial^g_{T_{X,J}}(v_g^\star)_J^{1,0} = 0, \overline{\partial}_{T_{X,J}}(v_g^\star)_J^{0,1} = 0 \right\}, \tag{14.1}$$

where $(v_g^\star)_J^{1,0}$ and $(v_g^\star)_J^{0,1}$ denote respectively the J-linear and J-anti-linear parts of the endomorphism $v_g^\star$. We remind here some lines of this basic fact. We define

$$2\nabla^{1,0}_{g,i,J} := \nabla_g - i\nabla_{g,J}\bullet$$

$$2\nabla^{0,1}_{g,i,J} := \nabla_g + i\nabla_{g,J}\bullet$$

Let v_J' and v_J'' be respectively the J-invariant and J-anti-invariant parts of v and set for notation simplicity

$$A' := (v_g^\star)_J^{1,0} = (v_J')_g^\star,$$

$$A'' := (v_g^\star)_J^{0,1} = (v_J'')_g^\star.$$

The identity $A'' = -J\dot{J}_0$ (see [23]) implies directly $\dot{\omega}_0 = v_J' J = \omega A'$. We infer

$$0 = d\dot{\omega}_0 = d(v_J' J).$$

The fact that the $(1,1)$-form $v_J' J$ is real implies that the identity $d(v_J' J) = 0$ is equivalent to the identity $\partial_J(v_J' J) = 0$. In its turn this is equivalent to the identity $\partial^g_{T_{X,J}} A' = 0$. We observe indeed that for all $\xi, \eta, \mu \in C^\infty(X, T_X \otimes_{\mathbb{R}} \mathbb{C})$ hold the equalities

$$\begin{aligned}
\partial_J(\omega A')(\xi', \eta', \mu'') &= \nabla^{1,0}_{g,i,J}(\omega A')(\xi', \eta', \mu'') - \nabla^{1,0}_{g,i,J}(\omega A')(\eta', \xi', \mu'') + \nabla^{1,0}_{g,i,J}(\omega A')(\mu'', \xi', \eta') \\
&= \omega\left(\left[\nabla^{1,0}_{g,i,J} A'(\xi', \eta') - \nabla^{1,0}_{g,i,J} A'(\eta', \xi')\right], \mu''\right) \\
&- \omega\left(\nabla^{1,0}_{g,i,J} A'(\mu'', \xi'), \eta'\right) \\
&= \omega\left(\left[\nabla^{1,0}_{g,J} A'(\xi', \eta') - \nabla^{1,0}_{g,J} A'(\eta', \xi')\right], \mu''\right) \\
&= \omega\left(\partial^g_{T_{X,J}} A'(\xi', \eta'), \mu''\right).
\end{aligned}$$

In order to continue the study of the space $\mathbb{D}_g^J$ we need to show a few general and fundamental facts. We start with the following weighted complex Weitzenböck type formula obtained in [28]. (We include the proof for readres convenience).

Lemma 12. *Let (X, J, g) be a Kähler manifold, let $\Omega > 0$ be a smooth volume form and let $A \in C^\infty\left(X, T^\star_{X,-J} \otimes T_{X,J}\right)$. Then hold the identity*

$$\Delta^{\Omega,-J}_{T_{X,g}} A = \nabla_g^{\star_\Omega} \nabla^{0,1}_{g,J} A - \mathcal{R}_g \star A + A\,\mathrm{Ric}_J^\star(\Omega)_\omega \tag{14.2}$$

Proof. We observe that for bi-degree reasons hold the identities

$$\begin{aligned}\Delta^{\Omega,-J}_{T_{X,g}}A &:= \overline{\partial}_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}A + \frac{1}{2}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}A\\ &= \overline{\partial}_{T_{X,J}}\nabla^{\star_\Omega}_{T_X,g}A + \frac{1}{2}\nabla^{\star_\Omega}_{T_X,g}\overline{\partial}_{T_{X,J}}A\\ &= \overline{\partial}_{T_{X,J}}\nabla^{\star_\Omega}_{g}A + \nabla^{\star_\Omega}_{g}\overline{\partial}_{T_{X,J}}A.\end{aligned}$$

Let

$$\widehat{\nabla^{0,1}_{g,J}A}(\xi,\eta) := \nabla^{0,1}_{g,J}A(\eta,\xi).$$

Then

$$2\Delta^{\Omega,-J}_{T_{X,g}}A = \nabla_g\nabla^{\star_\Omega}_g A + J\nabla_{g,J\bullet}\nabla^{\star_\Omega}_g A + 2\nabla^{\star_\Omega}_g\nabla^{0,1}_{g,J}A - 2\nabla^{\star_\Omega}_g\widehat{\nabla^{0,1}_{g,J}A}.$$

We fix an arbitrary point p and we choose an arbitrary vector field ξ such that $\nabla_g\xi(p) = 0$. Let $(e_k)_k$ be a g-orthonormal local frame such that $\nabla_g e_k(p) = 0$. We observe the local expression

$$\nabla^{\star_\Omega}_g\widehat{\nabla^{0,1}_{g,J}A}\cdot\xi = -\nabla_{g,e_k}\widehat{\nabla^{0,1}_{g,J}A}(e_k,\xi) + \nabla^{0,1}_{g,J}A(\xi,\nabla_g f).$$

At the point p hold the identities

$$\begin{aligned}2\nabla_{g,e_k}\widehat{\nabla^{0,1}_{g,J}A}(e_k,\xi) &= 2\nabla_{g,e_k}\left[\nabla^{0,1}_{g,J}A(\xi,e_k)\right]\\ &= \nabla_{g,e_k}\nabla_{g,\xi}A\cdot e_k + J\nabla_{g,e_k}\nabla_{g,J\xi}A\cdot e_k,\end{aligned}$$

and thus

$$2\nabla^{\star_\Omega}_g\widehat{\nabla^{0,1}_{g,J}A}\cdot\xi = -\nabla_{g,e_k}\nabla_{g,\xi}A\cdot e_k - J\nabla_{g,e_k}\nabla_{g,J\xi}A\cdot e_k + \nabla_{g,\xi}A\cdot\nabla_g f + J\nabla_{g,J\xi}A\cdot\nabla_g f.$$

We obtain the identity at the point p,

$$\begin{aligned}2\Delta^{\Omega,-J}_{T_{X,g}}A\cdot\xi &= -\nabla_{g,\xi}\nabla_{g,e_k}A\cdot e_k + \nabla_{g,\xi}\left(A\cdot\nabla_g f\right) - J\nabla_{g,J\xi}\nabla_{g,e_k}A\cdot e_k + J\nabla_{g,J\xi}\left(A\cdot\nabla_g f\right)\\ &+ 2\nabla^{\star_\Omega}_g\nabla^{0,1}_{g,J}A\cdot\xi + \nabla_{g,e_k}\nabla_{g,\xi}A\cdot e_k + J\nabla_{g,e_k}\nabla_{g,J\xi}A\cdot e_k\\ &- \nabla_{g,\xi}A\cdot\nabla_g f - J\nabla_{g,J\xi}A\cdot\nabla_g f.\end{aligned}$$

We remind that for any $A\in C^\infty\left(X,\mathrm{End}(T_X)\right)$ and $\xi,\eta\in C^\infty(X,T_X)$ hold the general formula

$$\nabla_{g,\xi}\nabla_{g,\eta}A - \nabla_{g,\eta}\nabla_{g,\xi}A = \left[\mathcal{R}_g(\xi,\eta),A\right] + \nabla_{g,[\xi,\eta]}A. \tag{14.3}$$

Using (14.3) and the fact that in our case $[e_k,\xi](p) = [e_k,J\xi](p) = 0$ we obtain

$$\begin{aligned}2\Delta^{\Omega,-J}_{T_{X,g}}A\cdot\xi &= \mathcal{R}_g(e_k,\xi)A\cdot e_k - A\mathcal{R}_g(e_k,\xi)\cdot e_k + J\left[\mathcal{R}_g(e_k,J\xi)A\cdot e_k - A\mathcal{R}_g(e_k,J\xi)\cdot e_k\right]\\ &+ 2\nabla^{\star_\Omega}_g\nabla^{0,1}_{g,J}A\cdot\xi + A\nabla^2_{g,\xi}f + JA\nabla_{g,J\xi}\nabla_g f\\ &= -(\mathcal{R}_g\star A)\cdot\xi + A\,\mathrm{Ric}^*(g)\cdot\xi - J(\mathcal{R}_g\star A)\cdot J\xi + JA\,\mathrm{Ric}^*(g)\cdot J\xi\\ &+ 2\nabla^{\star_\Omega}_g\nabla^{0,1}_{g,J}A\cdot\xi + A\partial^g_{T_{X,J}}\nabla_g f\cdot\xi\\ &= 2\left[\nabla^{\star_\Omega}_g\nabla^{0,1}_{g,J}A - \mathcal{R}_g\star A + A\,\mathrm{Ric}^*_J(\Omega)_\omega\right]\cdot\xi,\end{aligned}$$

thanks to (12.4). □

Multiplying both sides of (14.2) by A and integrating by parts we infer

$$\int_X \left\langle \Delta^{\Omega,-J}_{T_{X,g}} A, A \right\rangle_g \Omega = \int_X \left[\left\langle \nabla^{0,1}_{g,J} A, \nabla_g A \right\rangle_g + \left\langle A \operatorname{Ric}^\star_J(\Omega)_\omega - \mathcal{R}_g \star A, A \right\rangle_g \right] \Omega.$$

Using the fact that $\left\langle \nabla^{1,0}_{g,J} A''_J, \nabla^{0,1}_{g,J} A''_J \right\rangle_g = 0$ we obtain the integral identity

$$\int_X \left\langle \Delta^{\Omega,-J}_{T_{X,g}} A, A \right\rangle_g \Omega = \int_X \left[|\nabla^{0,1}_{g,J} A|^2_g + \left\langle A \operatorname{Ric}^\star_J(\Omega)_\omega - \mathcal{R}_g \star A, A \right\rangle_g \right] \Omega. \tag{14.4}$$

We observe also the following corollary.

Corollary 2. *Let (X, J, g) be a Kähler manifold, let $\Omega > 0$ be a smooth volume form and let $A \in C^\infty\left(X, T^\star_{X,-J} \otimes T_{X,J}\right)$. Then hold the identities*

$$\mathcal{L}^\Omega_g A = 2\Delta^{\Omega,-J}_{T_{X,g}} A + \operatorname{div}^\Omega_g \nabla_{g,J\bullet}(JA) - 2A \operatorname{Ric}^\star_J(\Omega)_\omega, \tag{14.5}$$

$$\operatorname{div}^\Omega_g \nabla_{g,J\bullet}(JA) = \operatorname{Ric}^\star(g)A + A \operatorname{Ric}^\star(g) - (J\nabla_g f)\neg(J\nabla_g A). \tag{14.6}$$

Proof. It is obvious that the identity (14.2) rewrites as (14.5). In order to show (14.6) let $(\eta_k)_{k=1}^n$ be a local complex frame of $T_{X,J}$ in a neighborhood of a point p with $\nabla_g \eta_k(p) = 0$ such that the real frame $(e_l)_{l=1}^{2n}$, $e_l = \eta_l$, $l = 1, \ldots, n$ and $e_{n+k} = J\eta_k$, $k = 1, \ldots, n$ is g-orthonormal. Then at the point p hold the equalities

$$\begin{aligned}
\operatorname{div}_g \nabla_{g,J\bullet}(JA''_J) &= \sum_{l=1}^{2n} \nabla_{g,e_l} \nabla_{g,Je_l}(JA''_J) \\
&= \sum_{k=1}^{n} \left[\nabla_{g,\eta_k} \nabla_{g,J\eta_k}(JA''_J) - \nabla_{g,J\eta_k} \nabla_{g,\eta_k}(JA''_J) \right] \\
&= \sum_{k=1}^{n} \left[\mathcal{R}_g(\eta_k, J\eta_k), JA''_J \right],
\end{aligned}$$

thanks to the general formula (14.3) and thanks to the fact that $[\eta_k, J\eta_k](p) = 0$. Using the J-linear and J-anti-linear properties of the tensors involved in the previous equality we obtain

$$\begin{aligned}
\operatorname{div}_g \nabla_{g,J\bullet}(JA''_J) &= \sum_{k=1}^{n} \left[J\mathcal{R}_g(\eta_k, J\eta_k)A''_J + A''_J J\mathcal{R}_g(\eta_k, J\eta_k) \right] \\
&= \operatorname{Ric}^\star(g)A''_J + A''_J \operatorname{Ric}^\star(g).
\end{aligned}$$

Notice indeed the identities

$$\begin{aligned}
2\operatorname{Ric}^\star(g) &= \sum_{l=1}^{2n} J\mathcal{R}_g(e_l, Je_l) \\
&= \sum_{k=1}^{n} \left[J\mathcal{R}_g(\eta_k, J\eta_k) - J\mathcal{R}_g(J\eta_k, \eta_k) \right] \\
&= 2\sum_{k=1}^{n} J\mathcal{R}_g(\eta_k, J\eta_k).
\end{aligned}$$

We conclude the required formula (14.6). □

We define now the vector spaces

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) := \mathrm{Ker}\,\Delta^{\Omega,-J}_{T_{X,g}} \cap C^\infty\left(X, T^\star_{X,-J}\otimes T_{X,J}\right),$$

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_{\mathrm{sm}} := \left\{A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \mid A = A^T_g\right\}.$$

The follwing result as also been obtained in [28]. We include the proof for readres convenience.

Lemma 13. *Let (X, J) be a Fano manifold, let g be a J-invariant Kähler metric with symplectic form $\omega := gJ \in 2\pi c_1(X, [J])$ and let $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\omega = \mathrm{Ric}_J(\Omega)$. Then hold the identity*

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) = \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)_{\mathrm{sm}}.$$

Proof. We consider the decomposition $A = A_{\mathrm{sm}} + A_{\mathrm{as}}$, where A_{as} and A_{as} are respectively the g-symmetric and g-anti-symmetric parts of A. We observe the symmetries

$$\mathcal{R}_g \star A_{\mathrm{sm}} = (\mathcal{R}_g \star A_{\mathrm{sm}})^T_g,$$

$$\mathcal{R}_g \star A_{\mathrm{as}} = -(\mathcal{R}_g \star A_{\mathrm{as}})^T_g.$$

The fact that $A \in C^\infty\left(X, T^\star_{X,-J}\otimes T_{X,J}\right)$ implies $A_{\mathrm{sm}}, A_{\mathrm{as}} \in C^\infty\left(X, T^\star_{X,-J}\otimes T_{X,J}\right)$ and thus

$$\nabla^{\star_\Omega}_g \nabla^{0,1}_{g,J} A_{\mathrm{sm}} = \left(\nabla^{\star_\Omega}_g \nabla^{0,1}_{g,J} A_{\mathrm{sm}}\right)^T_g,$$

$$\nabla^{\star_\Omega}_g \nabla^{0,1}_{g,J} A_{\mathrm{as}} = -\left(\nabla^{\star_\Omega}_g \nabla^{0,1}_{g,J} A_{\mathrm{as}}\right)^T_g,$$

Then the identity (14.2) implies the equalities

$$\left(\Delta^{\Omega,-J}_{T_{X,g}} A_{\mathrm{sm}}\right)^T_g - \Delta^{\Omega,-J}_{T_{X,g}} A_{\mathrm{sm}} = \left[\mathrm{Ric}^\star_J(\Omega)\omega, A_{\mathrm{sm}}\right], \tag{14.7}$$

$$\left(\Delta^{\Omega,-J}_{T_{X,g}} A_{\mathrm{as}}\right)^T_g + \Delta^{\Omega,-J}_{T_{X,g}} A_{\mathrm{as}} = \left[A_{\mathrm{as}}, \mathrm{Ric}^\star_J(\Omega)\omega\right]. \tag{14.8}$$

We deduce that in the case $\mathrm{Ric}_J(\Omega) = \lambda\omega$, with $\lambda = \pm 1, 0$, the condition $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$ is equivalent to the conditions $A_{\mathrm{sm}}, A_{\mathrm{as}} \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$. We focus now on the Fano case $\lambda = 1$. We remind the identity $\mathcal{R}_g \star A_{\mathrm{as}} = 0$. (See (20.9) in the appendix.) Thus if $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$ and $\mathrm{Ric}_J(\Omega) = \omega$ then the integral formula (14.4) reduces to

$$0 = \int_X \left[|\nabla^{0,1}_{g,J} A_{\mathrm{as}}|^2_g + |A_{\mathrm{as}}|^2_g\right]\Omega,$$

which shows $A_{\mathrm{as}} = 0$ and thus the required conclusion of the lemma. □

We obtain also the following statement (the case $c_1 < 0$ has been proved in [9]).

Lemma 14. *Let (X, J, g) be a compact non Ricci flat Kähler-Einstein manifold. Then hold the identity*

$$\mathcal{H}^{0,1}_{g}\left(T_{X,J}\right) = \mathcal{H}^{0,1}_{g}\left(T_{X,J}\right)_{\mathrm{sm}}.$$

Proof. Using the identities (14.7) and (14.8) with $\Omega = CdV_g$ we deduce that in the Kähler-Einstein case $\mathrm{Ric}(g) = \lambda g$, with $\lambda = \pm 1, 0$, the condition $A \in \mathcal{H}^{0,1}_{g}\left(T_{X,J}\right)$ is equivalent to the conditions $A_{\mathrm{sm}}, A_{\mathrm{as}} \in \mathcal{H}^{0,1}_{g}\left(T_{X,J}\right)$. On the other hand the identities (14.5) and (14.6) imply in the case $\Omega = CdV_g$ the formula

$$\mathcal{L}_g A = 2\Delta^{-J}_{T_{X,g}} A + [\mathrm{Ric}^*(g), A], \tag{14.9}$$

for any $A \in C^{\infty}\left(X, T^{\star}_{X,-J} \otimes T_{X,J}\right)$. The fact that $\mathcal{R}_g \star A_{\rm as} = 0$ implies the formula

$$\Delta_g A_{\rm as} \quad = \quad 2\Delta^{-J}_{T_{X,g}} A_{\rm as} + [{\rm Ric}^{\star}(g), A_{\rm as}].$$

We conclude that in the Kähler-Einstein case ${\rm Ric}(g) = \lambda g$, with $\lambda = \pm 1, 0$, any $A \in \mathcal{H}^{0,1}_g\left(T_{X,J}\right)$ satisfies $\nabla_g A_{\rm as} = 0$. Then the formula (14.6) with $\Omega = CdV_g$ implies

$$0 = {\rm div}_g \nabla_{g,J\bullet}(JA_{\rm as}) \quad = \quad {\rm Ric}^{\star}(g)A_{\rm as} + A_{\rm as}\,{\rm Ric}^{\star}(g) = 2\lambda A_{\rm as}.$$

We deduce $A_{\rm as} = 0$ in the case $\lambda = \pm 1$. This shows the required conclusion. □

We denote by

$$\Lambda^{\Omega}_{g,J} := {\rm Ker}(\Delta^{\Omega}_{g,J} - 2\mathbb{I}) \quad \subset \quad C^{\infty}_{\Omega}(X, \mathbb{C})_0,$$

and by

$$\Lambda^{\Omega,\perp}_{g,J} \quad := \quad \left[{\rm Ker}(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\right]^{\perp_{\Omega}} \subseteq C^{\infty}_{\Omega}(X, \mathbb{C})_0,$$

its L^2_{Ω}-orthogonal inside $C^{\infty}_{\Omega}(X, \mathbb{C})_0$. We obtain as corollary of lemma (13) the following fundamental fact obtained in [28]. (We include the proof for readres convenience).

Corollary 3. (***Decomposition of the variation of the complex structure***)

Let (X, J) be a Fano manifold, let g be a J-invariant Kähler metric with symplectic form $\omega := gJ \in 2\pi c_1(X, [J])$ and let $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\omega = {\rm Ric}_J(\Omega)$. Then for all $v \in \mathbb{D}^J_g$ there exists a unique $\psi \in \Lambda^{\Omega,\perp}_{g,J}$ and a unique $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$ such that

$$(v^{\star}_g)^{0,1}_J \quad = \quad \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A.$$

Proof. We observe that the identity

$$\overline{\partial}_{T_{X,J}} (v^{\star}_g)^{0,1}_J \quad = \quad 0,$$

combined with the Ω-Hodge isomorphism

$$\mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \quad \simeq \quad H^{0,1}(X, T_{X,J})$$

$$:= \quad \frac{\left\{B \in C^{\infty}(X, \Lambda^{0,1}_J T^{\star}_X \otimes_{\mathbb{C}} T_{X,J}) \mid \overline{\partial}_{T_{X,J}} B = 0\right\}}{\left\{\overline{\partial}_{T_{X,J}} \xi \mid \xi \in C^{\infty}(X, T_X)\right\}},$$

implies the decomposition

$$(v^{\star}_g)^{0,1}_J \quad = \quad \overline{\partial}_{T_{X,J}} \xi + A,$$

with $\xi \in C^{\infty}(X, T_X)$ and unique $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$. Then the fact that the endomorphism $(v^{\star}_g)^{0,1}_J$ is g-symmetric combined with lemma 13 implies that $\overline{\partial}_{T_{X,J}} \xi$ is also g-symmetric. Then formula (13.11) implies that for all $\eta, \mu \in C^{\infty}(X, T^{0,1}_X)$ holds the identity

$$\begin{aligned}
\overline{\partial}_J(\xi^{1,0}_J \lrcorner \omega)(\eta, \mu) \quad &= \quad \omega\left(\overline{\partial}_{T_{X,J}} \xi \cdot \eta, \mu\right) + \omega\left(\eta, \overline{\partial}_{T_{X,J}} \xi \cdot \mu\right) \\
&= \quad g\left(J\overline{\partial}_{T_{X,J}} \xi \cdot \eta, \mu\right) + g\left(J\eta, \overline{\partial}_{T_{X,J}} \xi \cdot \mu\right) \\
&= \quad g\left(\left[\left(\overline{\partial}_{T_{X,J}} \xi\right)^T_g - \overline{\partial}_{T_{X,J}} \xi\right] \cdot J\eta, \mu\right) \\
&= \quad 0.
\end{aligned}$$

Then the argument showing the surjectivity of the map in lemma 10 in the section 13 implies the existence of a function $\Psi \in C^{\infty}_{\Omega}(X, \mathbb{C})_0$ such that
$\xi = \nabla_{g,J}\overline{\Psi}$. This combined with the identity (13.12) implies the existence and uniqueness of $\psi \in \Lambda^{\Omega,\perp}_{g,J}$ such that

$$\overline{\partial}_{T_{X,J}}\xi \quad = \quad \overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{\psi}.$$

We infer the required conclusion. □

We show now the inclusion (1.23). Time deriving the condition $\omega_t := g_t J_t \in 2\pi c_1$ we infer $\{\dot{\omega}_0\}_d = 0$. Then (1.23) follows from the complex decomposition identity

$$v^{\star}_g \quad = \quad g^{-1}\dot{g}_0 = \omega^{-1}\dot{\omega}_0 - J\dot{J}_0 = (v'_J)^{\star}_g + (v''_J)^{\star}_g.$$

15 The decomposition of the space $\mathbb{F}_{g,\Omega}$ in the soliton case

Lemma 15. *Let (X, g, Ω) be a compact shrinking Ricci soliton. Then the linear map*

$$T_{g,\Omega} : C^{\infty}_{\Omega}(X, \mathbb{R})_0 \oplus \left[\mathrm{Ker}\,\nabla^{\star_{\Omega}}_g \cap C^{\infty}(X, S^2T^{\star}_X)\right] \quad \longrightarrow \quad \mathbb{F}_{g,\Omega}$$

$$(\varphi, \theta) \quad \longmapsto \quad \left(\nabla_g d\varphi + \theta, (\varphi - \Delta^{\Omega}_g\varphi)\Omega\right),$$

is an isomorphism of vector spaces.

Proof. **STEP I.** We observe first that in the compact shrinking Ricci soliton case the first eigenvalue $\lambda_1(\Delta^{\Omega}_g)$ of Δ^{Ω}_g satisfies the inequality $\lambda_1(\Delta^{\Omega}_g) > 1$. Indeed multiplying both sides of the identity (8.2) with $\nabla_g u$ and integrating we infer

$$\begin{aligned}\int\limits_X \left\langle \nabla_g\Delta^{\Omega}_g u, \nabla_g u\right\rangle_g \Omega \quad &= \quad \int\limits_X \left[\left\langle \Delta^{\Omega}_g\nabla_g u, \nabla_g u\right\rangle_g + \mathrm{Ric}_g(\Omega)(\nabla_g u, \nabla_g u)\right]\Omega \\ &= \quad \int\limits_X \left[\left|\nabla^2_g u\right|^2_g + \mathrm{Ric}_g(\Omega)(\nabla_g u, \nabla_g u)\right]\Omega.\end{aligned}$$

Let now $u \in C^{\infty}_{\Omega}(X, \mathbb{R})_0$ be an eigen-function corresponding to $\lambda_1(\Delta^{\Omega}_g) > 0$. By definition $u \not\equiv 0$. Thus by the previous integral identity we deduce

$$\begin{aligned}\lambda_1(\Delta^{\Omega}_g)\int\limits_X |\nabla_g u|^2_g\,\Omega \quad &= \quad \int\limits_X \left\langle \nabla_g\Delta^{\Omega}_g u, \nabla_g u\right\rangle_g \Omega \\ &= \quad \int\limits_X \left[\left|\nabla^2_g u\right|^2_g + |\nabla_g u|^2_g\right]\Omega \\ &> \quad \int\limits_X |\nabla_g u|^2_g\,\Omega > 0,\end{aligned}$$

which implies the required estimate.

STEP II. Multiplying both sides of the the identity (8.2) with g we obtain

$$d\Delta^{\Omega}_g u = \Delta^{\Omega}_g du + du \cdot \mathrm{Ric}^{\star}_g(\Omega). \tag{15.1}$$

Let now $(v, V) := T_{g,\Omega}(\varphi, \theta)$ and observe the equalities

$$\nabla^{\star_{\Omega}}_g v \quad = \quad \nabla^{\star_{\Omega}}_g\nabla_g d\varphi = \Delta^{\Omega}_g d\varphi = d(\Delta^{\Omega}_g\varphi - \varphi).$$

The last one follows from (15.1). We infer that the linear map $T_{g,\Omega}$ is well defined. The fact that in the soliton case $h_{g,\Omega} = 0$ the differential operator $\Delta_g^\Omega - \mathbb{I}$ is invertible over $C^\infty_\Omega(X,\mathbb{R})_0$ implies the injectivity of the map $T_{g,\Omega}$.

In order to show the surjectivity of the map $T_{g,\Omega}$ let $(v,V) \in \mathbb{F}_{g,\Omega}$ and define the function

$$\varphi \quad := \quad (\mathbb{I} - \Delta_g^\Omega)^{-1} V_\Omega^\star \in C^\infty_\Omega(X,\mathbb{R})_0.$$

Then the identity

$$\nabla_g^{\star_\Omega} \nabla_g d\varphi \quad = \quad d(\Delta_g^\Omega \varphi - \varphi),$$

implies that the tensor $\theta := v - \nabla_g d\varphi$ satisfies $\nabla_g^{\star_\Omega}\theta = 0$. We deduce the orthogonal decomposition with respect to the scalar product (1.1)

$$v = \nabla_g d\varphi + \theta, \tag{15.2}$$

with $\nabla_g^{\star_\Omega}\theta = 0$. We deduce the required surjectivity statement. □

We need to introduce a few notations. From now on we assume $H^1_d(X,\mathbb{R}) = 0$ (this is the case of any Fano manifold) and we observe that the first projection map

$$p_1 : \mathbb{F}_{g,\Omega} \longrightarrow \mathbb{S}_{g,\Omega} := \left\{ v \in C^\infty(X, S^2 T_X) \mid d\nabla_g^{\star_\Omega} v = 0 \right\},$$

is an isomorphism. Over a compact Kähler manifold we define the real vector spaces

$$\mathbb{S}^J_{g,\Omega} \quad := \quad \mathbb{S}_{g,\Omega} \cap \mathbb{D}^J_g,$$

$$\mathbb{S}^J_{g,\Omega}(0) \quad := \quad \mathbb{S}_{g,\Omega} \cap \mathbb{D}^J_{g,0},$$

$$\mathbb{S}^J_{g,\Omega}[0] \quad := \quad \mathbb{S}_{g,\Omega} \cap \mathbb{D}^J_{g,[0]},$$

and

$$\mathbb{E}^\Omega_{g,J} \quad := \quad \left\{ \psi \in \Lambda^{\Omega,\perp}_{g,J} \mid \Delta^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi = \overline{\Delta^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi} \right\}.$$

With the notations introduced so far we can state the following decomposition result.

Lemma 16. *Let (J,g) be a Kähler-Ricci-Soliton and let $\Omega > 0$ be the unique smooth volume form such that $gJ = \mathrm{Ric}_J(\Omega)$ and $\int_X \Omega = 1$. Then the linear map*

$$C^\infty_\Omega(X,\mathbb{R})_0 \oplus \mathbb{E}^\Omega_{g,J} \oplus \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \quad \longrightarrow \quad \mathbb{S}^J_{g,\Omega}(0)$$

$$(\varphi,\psi,A) \quad \longmapsto \quad v,$$

$$(v_g^\star)^{1,0}_J \quad := \quad \partial^g_{T_{X,J}} \nabla_g(\varphi + \tau),$$

$$(v_g^\star)^{0,1}_J \quad := \quad \overline{\partial}_{T_{X,J}} \nabla_{g,J}(\varphi + \overline{\psi}) + A,$$

with $\tau \in C^\infty_\Omega(X,\mathbb{R})_0$ the unique solution of the equation

$$-\overline{\Delta^\Omega_{g,J}\tau} = (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi, \tag{15.3}$$

is an isomorphism of real vector spaces. In particular the linear map

$$\mathbb{E}^\Omega_{g,J} \oplus \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right) \quad \longrightarrow \quad \mathbb{S}^J_{g,\Omega}[0]$$

$$(\psi,A) \quad \longmapsto \quad v,$$

$$(v_g^\star)^{0,1}_J \quad := \quad \overline{\partial}_{T_{X,J}} \nabla_{g,J}(\varphi + \overline{\psi}) + A,$$

with $\varphi \in C^\infty_\Omega(X, \mathbb{R})_0$ the unique solution of the equation

$$\overline{\Delta^\Omega_{g,J}\varphi} = (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi, \tag{15.4}$$

is also an isomorphism of real vector spaces.

Proof. Let first $v \in \mathbb{S}_{g,\Omega}$ and observe that the decomposition formula (15.2) rewrites as

$$v^\star_g = \partial^g_{T_{X,J}}\nabla_g\varphi + \overline{\partial}_{T_{X,J}}\nabla_g\varphi + \theta^\star_g.$$

This implies that $v \in \mathbb{D}^J_g$ if and only if $\theta \in \mathbb{D}^J_g$, and also $v \in \mathbb{D}^J_{g,0}$ if and only if $\theta \in \mathbb{D}^J_{g,0}$.

Let now $v \in \mathbb{S}^J_{g,\Omega}(0)$. Then the decomposition of the variation of the complex structure in corollary 3 implies the existence of unique $\tau \in C^\infty_\Omega(X, \mathbb{R})_0$, $\psi \in \Lambda^{\Omega,\perp}_{g,J}$ and $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$ such that

$$\theta^\star_g = \partial^g_{T_{X,J}}\nabla_g\tau + \overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{\psi} + A. \tag{15.5}$$

For bi-degree reasons the condition $\nabla^{\star_\Omega}_g\theta = 0$ is equivalent to the identity

$$0 = 2\partial^{\star_{g,\Omega}}_{T_{X,J}}\partial^g_{T_{X,J}}\nabla_g\tau + 2\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{\psi}.$$

The latter is equivalent to the equation

$$0 = \nabla_{g,J}\left[\Delta^\Omega_{g,J}\tau + \overline{(\Delta^\Omega_g - 2\mathbb{I})\psi}\right],$$

thanks to the complex Bochner identities (13.8) and (13.9). We remind that if $u \in C^\infty_\Omega(X, \mathbb{C})_0$ satisfies $\nabla_{g,J}u = 0$ then $u = 0$. (See the proof of the injectivity statement in lemma 10 in the section 13.) We conclude that the condition $\nabla^{\star_\Omega}_g\theta = 0$ is equivalent to the equation (15.3) via the decomposition (15.5) of θ.

Then the required decomposition statement concerning the space $\mathbb{S}^J_{g,\Omega}(0)$ follows from the fact that the condition τ real valued is equivalent to the equation defining $\psi \in \mathbb{E}^\Omega_{g,J}$. In order to see this we show first the commutation identity

$$\left[\Delta^\Omega_g, B^\Omega_{g,J}\right] = 0. \tag{15.6}$$

Indeed using an arbitrary g-orthonormal local frame $(e_k)_k$ we obtain

$$\begin{aligned}
\Delta^\Omega_g B^\Omega_{g,J}u &= \Delta^\Omega_g\left[g(\nabla_g u, J\nabla_g f)\right] \\
&= g(\Delta^\Omega_g\nabla_g u, J\nabla_g f) - 2g(\nabla^2_g u\cdot e_k, J\nabla^2_g f\cdot e_k) + g(\nabla_g u, J\Delta^\Omega_g\nabla_g f) \\
&= g(\Delta^\Omega_g\nabla_g u + \nabla_g u, J\nabla_g f) - 2\,\mathrm{Tr}_{\mathbb{R}}\left(\nabla^2_g u J\nabla^2_g f\right)
\end{aligned}$$

thanks to formula (8.2) applied to f and thanks to the fact that $(\Delta^\Omega_g - 2\mathbb{I})f = 0$. Moreover the endomorphism $J\nabla^2_g f$ is g-anti-symmetric since in the soliton case $\left[J, \nabla^2_g f\right] = 0$. We deduce

$$\Delta^\Omega_g B^\Omega_{g,J}u = g(\nabla_g\Delta^\Omega_g u, J\nabla_g f) = B^\Omega_{g,J}\Delta^\Omega_g u,$$

thanks to formula (8.2) applied to u. We infer the identity (15.6) which implies

$$\left[\Delta^\Omega_{g,J}, \overline{\Delta^\Omega_{g,J}}\right] = 2i\left[B^\Omega_{g,J}, \Delta^\Omega_g\right] = 0. \tag{15.7}$$

Multiplying both sides of (15.3) with $\Delta^\Omega_{g,J}$ we obtain

$$-\left(\Delta^\Omega_{g,J}\overline{\Delta^\Omega_{g,J}}\right)\tau = \Delta^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi. \tag{15.8}$$

The invertible operator

$$\Delta^\Omega_{g,J}\overline{\Delta^\Omega_{g,J}} : C^\infty_\Omega(X, \mathbb{C})_0 \longrightarrow C^\infty_\Omega(X, \mathbb{C})_0,$$

is real thanks to (15.7). We deduce that the condition τ real valued is equivalent to the left hand side of (15.8) being real valued, thus equivalent to the equation defining $\psi \in \mathbb{E}^{\Omega}_{g,J}$.

We observe finally that a variation $v \in \mathbb{S}^{J}_{g,\Omega}[0] \subset \mathbb{S}^{J}_{g,\Omega}(0)$ corresponds to $\varphi = -\tau$, i.e. to (φ, ψ) solution of the equation (15.4). □

Remark 1. If we write $\psi = \psi_1 + i\psi_2$, with $\psi_1, \psi_2 \in C^{\infty}_{\Omega}(X, \mathbb{R})_0$, then (15.3) is equivalent to the system

$$\begin{cases} -\Delta^{\Omega}_{g}\tau = (\Delta^{\Omega}_{g} - 2\mathbb{I})\psi_1 + B^{\Omega}_{g,J}\psi_2, \\ -B^{\Omega}_{g,J}\tau = (\Delta^{\Omega}_{g} - 2\mathbb{I})\psi_2 - B^{\Omega}_{g,J}\psi_1. \end{cases} \tag{15.9}$$

Moreover separating real and imaginary parts in the equation defining $\psi \in \mathbb{E}^{\Omega}_{g,J}$ and using the commutation identity (15.6) we obtain

$$\mathbb{E}^{\Omega}_{g,J} = \left\{ \psi \in \Lambda^{\Omega,\perp}_{g,J} \mid \left[\Delta^{\Omega}_{g}(\Delta^{\Omega}_{g} - 2\mathbb{I}) - (B^{\Omega}_{g,J})^2 \right] \psi_2 = 2(\Delta^{\Omega}_{g} - \mathbb{I})B^{\Omega}_{g,J}\psi_1 \right\}. \tag{15.10}$$

Using (15.10) and the complex Bochner formula (13.9) we obtain also the identity

$$\mathbb{E}^{\Omega}_{g,J} = \left\{ \psi \in \Lambda^{\Omega,\perp}_{g,J} \mid -\operatorname{div}^{\Omega} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \overline{\partial}_{T_{X,J}} \nabla_g \psi_2 = (\Delta^{\Omega}_{g} - \mathbb{I})B^{\Omega}_{g,J}\psi_1 \right\}.$$

Remark 2. We observe that the linear map

$$\Delta^{\Omega}_{g,J} : \Lambda^{\Omega,\perp}_{g,J} \longrightarrow \Lambda^{\Omega,\perp}_{g,J}, \tag{15.11}$$

is well defined and it represents an isomorphism of complex vector spaces. In fact this follows from the identity

$$2\int_X u\,\overline{v}\,\Omega \quad = \quad \int_X \Delta^{\Omega}_{g,J} u\,\overline{v}\,\Omega,$$

for all $v \in \Lambda^{\Omega}_{g,J}$. Thus the linear map

$$\Delta^{\Omega}_{g,J} - 2\mathbb{I} : \Lambda^{\Omega,\perp}_{g,J} \longrightarrow \Lambda^{\Omega,\perp}_{g,J}, \tag{15.12}$$

is also well defined and represents an isomorphisms of complex vector spaces. The surjectivity of the latter follows from the finiteness theorem for elliptic operators. By definition of $\mathbb{E}^{\Omega}_{g,J}$ we deduce the existence of the isomorphism of real vector spaces

$$\Delta^{\Omega}_{g,J}(\Delta^{\Omega}_{g,J} - 2\mathbb{I}) : \mathbb{E}^{\Omega}_{g,J} \longrightarrow \Lambda^{\Omega,\perp}_{g,J} \cap C^{\infty}_{\Omega}(X, \mathbb{R})_0.$$

We notice also the inclusion

$$\Lambda^{\Omega,\perp}_{g,J} \cap C^{\infty}_{\Omega}(X, \mathbb{R})_0 \quad \supseteq \quad (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})} C^{\infty}_{\Omega}(X, \mathbb{R})_0.$$

16 The geometric meaning of the space $\mathbb{F}^{J}_{g,\Omega}[0]$

We define the subspaces

$$\mathbb{F}^{J}_{g,\Omega}(0) \quad := \quad \left\{ (v, V) \in \mathbb{F}_{g,\Omega} \mid v \in \mathbb{S}^{J}_{g,\Omega}(0) \right\},$$

$$\mathbb{F}^{J}_{g,\Omega}[0] \quad := \quad \left\{ (v, V) \in \mathbb{F}_{g,\Omega} \mid v \in \mathbb{S}^{J}_{g,\Omega}[0] \right\}.$$

In the previous section we gave a parametrization of the space $\mathbb{S}^{J}_{g,\Omega}(0)$, and thus of $\mathbb{F}^{J}_{g,\Omega}(0)$, which is fundamental for the computation of a general second variation formula for the $\mathcal{W}$ functional at a Kähler-Ricci soliton point. In this section we give a simpler parametrization of the sub-space $\mathbb{F}^{J}_{g,\Omega}[0]$ and a useful geometric interpretation of it. We show first a quite general variation formula for the Chern-Ricci form.

Lemma 17. *Let $(g_t, J_t)_t \subset \mathcal{KS}$, $(\Omega_t)_t \subset \mathcal{V}_1$ be two smooth families such that $\dot{J}_t = (\dot{J}_t)^T_{g_t}$. Then hold the first variation formula*

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t) = -d\left(g_t\nabla^{\star_{\Omega_t}}_{g_t}\dot{J}_t + 2d^c_{J_t}\dot{\Omega}^\star_t\right). \tag{16.1}$$

Proof. In the case of a fixed volume form $\Omega > 0$ we have the variation formula (see [26])

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega) = -d\left(g_t\nabla^{\star_\Omega}_{g_t}\dot{J}_t\right).$$

For an arbitrary family $(\Omega_t)_t \subset \mathcal{V}_1$ we fix an arbitrary time τ and we time derive at $t = \tau$ the decomposition

$$\mathrm{Ric}_{J_t}(\Omega_t) = \mathrm{Ric}_{J_t}(\Omega_\tau) - dd^c_{J_t}\log\frac{\Omega_t}{\Omega_\tau}.$$

We obtain the required variation formula. □

We show now that for any point $(g, \Omega) \in \mathcal{S}_\omega$ hold the inclusion (1.11). Indeed for any smooth curve $(g_t, \Omega_t)_t \subset \mathcal{S}_\omega$, with $(g_0, \Omega_0) = (g, \Omega)$ we have $\dot{g}^\star_t = -J_t\dot{J}_t$ and thus

$$0 = 2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t) = -d\left[\left(\nabla^{\star_{\Omega_t}}_{g_t}\dot{g}^\star_t + \nabla_{g_t}\dot{\Omega}^\star_t\right)\neg\,\omega\right],$$

thanks to the variation formula (16.1). Then the inclusion (1.11) follows from (1.10) and Cartan's formula for the Lie derivative of differential forms.

We can provide at this point the geometric interpretation of the sub-space $\mathbb{F}^J_{g,\Omega}[0]$.

Lemma 18. *For any point $(g, \Omega) \in \mathcal{S}_\omega$ hold the identities (1.14) and (1.15).*

Proof. We remind that by the orthogonal decomposition in corollary 3 any element $v \in \mathbb{D}^J_{g,[0]}$ decomposes as

$$v^\star_g = \overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{\psi}_v + A_v,$$

with unique $\psi_v \in \Lambda^{\Omega,\perp}_{g,J}$ and $A_v \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J})$. Moreover the weighted complex Bochner identity (13.9) implies the equality

$$\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}v^\star_g + \nabla_g V^\star_\Omega = \frac{1}{2}\nabla_{g,J}\left[\overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi_v} + 2V^\star_\Omega\right], \tag{16.2}$$

for any $(v, V) \in \mathbb{D}^J_{g,[0]} \times T_{\mathcal{V}_1}$. Thus

$$\mathbb{F}^J_{g,\Omega}[0] = \left\{(v, V) \in \mathbb{D}^J_{g,[0]} \times T_{\mathcal{V}_1} \mid (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi_v = -2V^\star_\Omega\right\}. \tag{16.3}$$

Let

$$R_\psi := \mathrm{Re}\left[(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\right], \tag{16.4}$$

$$I_\psi := \mathrm{Im}\left[(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\right], \tag{16.5}$$

(for any $z \in \mathbb{C}$ we write $z = \mathrm{Re}\,z + i\,\mathrm{Im}\,z$) and observe that (16.2) implies the identity

$$\left(\nabla^{\star_\Omega}_g v^\star_g + \nabla_g V^\star_\Omega\right)\neg\,\omega = \frac{1}{2}dI_{\psi_v} + d^c_J\left(R_{\psi_v} + 2V^\star_\Omega\right),$$

for any $(v, V) \in \mathbb{D}^J_{g,[0]} \times T_{\mathcal{V}_1}$. Thus

$$\mathbb{T}^J_{g,\Omega} = \left\{(v, V) \in \mathbb{D}^J_{g,[0]} \times T_{\mathcal{V}_1} \mid R_{\psi_v} = -2V^\star_\Omega\right\}. \tag{16.6}$$

We notice now the equalities

$$\begin{aligned} T_{[g,\Omega]_\omega,(g,\Omega)} &= \left\{(L_\xi g, L_\xi\Omega) \mid \xi \in C^\infty(X, T_X) : L_\xi\omega = 0\right\} \\ &= \left\{(L_{J\nabla_g u}g, L_{J\nabla_g u}\Omega) \mid u \in C^\infty_\Omega(X, \mathbb{R})_0\right\} \\ &= \left\{(2gJ\overline{\partial}_{T_{X,J}}\nabla_g u, \mathrm{div}^\Omega(J\nabla_g u)\Omega) \mid u \in C^\infty_\Omega(X, \mathbb{R})_0\right\}, \end{aligned}$$

indeed

$$(L_{J\nabla_g u}g)^\star_g \quad = \quad J\nabla^2_g u - \nabla^2_g u J = 2J\overline{\partial}_{T_{X,J}}\nabla_g u.$$

We deduce that $(v, V) \in T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)}$ if and only if for all $u \in C^\infty_\Omega(X, \mathbb{R})_0$ hold the equalities

$$\begin{aligned} 0 &= 2\int_X \left[\left\langle J\overline{\partial}_{T_{X,J}}\nabla_g u, v^\star_g\right\rangle_g - \mathrm{div}^\Omega(J\nabla_g u)\cdot V^\star_\Omega\right]\Omega \\ &= -2\int_X \left\langle \nabla_g u, J\left(\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} v^\star_g + \nabla_g V^\star_\Omega\right)\right\rangle_g \Omega \\ &= 2\int_X u\cdot \mathrm{div}^\Omega\left[J\left(\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} v^\star_g + \nabla_g V^\star_\Omega\right)\right]\Omega. \end{aligned}$$

If we assume $(v, V) \in \mathbb{T}^J_{g,\Omega}$ then

$$\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} v^\star_g + \nabla_g V^\star_\Omega = -\frac{1}{2}J\nabla_g I_{\psi_v}, \tag{16.7}$$

thanks to (16.2) and (16.6). Thus if $(v, V) \in T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} \cap \mathbb{T}^J_{g,\Omega}$ then

$$0 \quad = \quad -\int_X u\cdot \Delta^\Omega_g I_{\psi_v}\Omega,$$

for all $u \in C^\infty_\Omega(X, \mathbb{R})_0$, i.e. $\Delta^\Omega_g I_{\psi_v} = 0$, which is equivalent to the condition $I_{\psi_v} = 0$. We infer

$$T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} \cap \mathbb{T}^J_{g,\Omega} \quad \subseteq \quad \mathbb{F}^J_{g,\Omega}[0].$$

The reverse inclusion is obvious. We deduce the identity (1.14). Then the identity (1.15) follows from the inclusion (1.11). □

17 The sign of the second variation of the $\mathcal{W}$ functional at a Kähler-Ricci soliton point

Proposition 1. *Let (X, J, g) be a compact Kähler-Ricci-Soliton and let $\Omega > 0$ be the unique smooth volume form such that $gJ = \mathrm{Ric}_J(\Omega)$ and $\int_X \Omega = 1$. Let also $(g_t, \Omega_t)_{t\in\mathbb{R}} \subset \mathcal{M}\times\mathcal{V}_1$ be a smooth curve with $(g_0, \Omega_0) = (g, \Omega)$ and with $(\dot g_0, \dot\Omega_0) = (v, V) \in \mathbb{F}^J_{g,\Omega}(0)$. Then with the notations of lemma 16 hold the second variation formula*

$$\begin{aligned} \frac{d^2}{dt^2}_{|t=0}\mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) \\ &= \frac{1}{2}\int_X (\Delta^\Omega_g - \mathbb{I})(\Delta^\Omega_g - 2\mathbb{I})\varphi\cdot(\Delta^\Omega_g - 2\mathbb{I})\varphi\Omega \\ &- \frac{1}{2}\int_X \left[(\Delta^\Omega_g - \mathbb{I})\,\Delta^\Omega_g\tau\cdot\Delta^\Omega_g\tau + P^\Omega_{g,J}\,\mathrm{Im}\,\psi\cdot\mathrm{Im}\,\psi + |A|^2_g F\right]\Omega, \end{aligned}$$

where

$$P^\Omega_{g,J} \quad := \quad (\Delta^\Omega_{g,J} - 2\mathbb{I})\overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})},$$

is a non-negative self-adjoint real elliptic operator with respect to the L^2_Ω-hermitian product. In particular if $(v, V) \in \mathbb{F}^J_{g,\Omega}[0]$ then

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V)\\ &= -\frac{1}{2}\int_X \left[4\left|(\Delta^\Omega_g - \mathbb{I})\varphi\right|^2 + P^\Omega_{g,J}\,\mathrm{Im}\,\psi\cdot\mathrm{Im}\,\psi + |A|^2_g F\right]\Omega.\end{aligned}$$

Proof. **STEP I.** Let (X, g, Ω) be a compact shrinking Ricci soliton point and let $(g_t, \Omega_t)_{t\in\mathbb{R}} \subset \mathcal{M}\times\mathcal{V}_1$ be a smooth curve with $(g_0, \Omega_0) = (g, \Omega)$ and with arbitrary speed $(\dot g_0, \dot\Omega_0) = (v, V) \in \mathbb{F}_{g,\Omega}$. We know from lemma 7

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V)\\ &= -\frac{1}{2}\int_X \left[\left\langle \mathcal{L}^\Omega_g v, v\right\rangle_g - 2(\Delta^\Omega_g - 2\mathbb{I})V^\star_\Omega\cdot V^\star_\Omega\right]\Omega.\end{aligned}$$

By the considerations in the beginning of section 10 we deduce that in the soliton case $h_{g,\Omega} = 0$ holds the identity

$$\nabla^{\star_\Omega}_g \mathcal{L}^\Omega_g v + d(\Delta^\Omega_g V^\star_\Omega - 2V^\star_\Omega) = 0, \tag{17.1}$$

for all $(v, V) \in \mathbb{F}_{g,\Omega}$. Applying the operator $\nabla^{\star_\Omega}_g$ to both sides of this identity we infer

$$(\nabla^{\star_\Omega}_g)^2 \mathcal{L}^\Omega_g v + \Delta^\Omega_g(\Delta^\Omega_g - 2\mathbb{I})V^\star_\Omega = 0. \tag{17.2}$$

For any function $\varphi \in C^\infty_\Omega(X, \mathbb{R})_0$ let $(v, V) := T_{g,\Omega}(\varphi, 0)$. Integrating by parts and using the identity (17.2) we infer the equalities

$$\begin{aligned}\nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) &= -\frac{1}{2}\int_X \left[(\nabla^{\star_\Omega}_g)^2\mathcal{L}^\Omega_g v\cdot\varphi + 2(\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_g - \mathbb{I})\varphi\cdot(\mathbb{I} - \Delta^\Omega_g)\varphi\right]\Omega\\ &= -\frac{1}{2}\int_X \Delta^\Omega_g(\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_g - \mathbb{I})\varphi\cdot\varphi\Omega\\ &- \frac{1}{2}\int_X 2(\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_g - \mathbb{I})\varphi\cdot(\mathbb{I} - \Delta^\Omega_g)\varphi\Omega\\ &= \frac{1}{2}\int_X (\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_g - \mathbb{I})\varphi\cdot(\Delta^\Omega_g - 2\mathbb{I})\varphi\Omega\\ &= \frac{1}{2}\int_X (\Delta^\Omega_g - \mathbb{I})(\Delta^\Omega_g - 2\mathbb{I})\varphi\cdot(\Delta^\Omega_g - 2\mathbb{I})\varphi\Omega.\end{aligned}$$

Remark 1. We can also compute the integral

$$\int_X \left\langle \mathcal{L}^\Omega_g\nabla^2_g\varphi, \nabla^2_g\varphi\right\rangle_g \Omega,$$

in the previous expansion via the formula (9.3). Indeed

$$\begin{aligned}\int_X \left\langle \mathcal{L}^\Omega_g\nabla^2_g\varphi, \nabla^2_g\varphi\right\rangle_g \Omega &= \int_X \left\langle \nabla^2_g(\Delta^\Omega_g - 2\mathbb{I})\varphi, \nabla^2_g\varphi\right\rangle_g \Omega\\ &= \int_X \left\langle \Delta^\Omega_g\nabla_g(\Delta^\Omega_g - 2\mathbb{I})\varphi, \nabla_g\varphi\right\rangle_g \Omega\\ &= \int_X \left\langle \nabla_g(\Delta^\Omega_g - \mathbb{I})(\Delta^\Omega_g - 2\mathbb{I})\varphi, \nabla_g\varphi\right\rangle_g \Omega,\end{aligned}$$

thanks to the identity (8.2). We conclude integrating by parts

$$\int_X \left\langle \mathcal{L}_g^\Omega \nabla_g^2 \varphi, \nabla_g^2 \varphi \right\rangle_g \Omega = \int_X \Delta_g^\Omega (\Delta_g^\Omega - 2\mathbb{I})(\Delta_g^\Omega - \mathbb{I})\varphi \cdot \varphi \Omega.$$

Remark 2. We set $\Phi := (\Delta_g^\Omega - 2\mathbb{I})\varphi \in C^\infty_\Omega(X, \mathbb{R})_0$. Then the previous variation formula rewrites also as

$$\begin{aligned} \frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) \\ &= \frac{1}{2}\int_X \left[|\nabla_g \Phi|_g^2 - \Phi^2 \right] \Omega \geqslant 0. \end{aligned}$$

the last inequality follows from the variational characterization of the first eigenvalue of Δ_g^Ω,

$$\lambda_1(\Delta_g^\Omega) = \inf \left\{ \frac{\int_X |\nabla_g u|_g^2 \, \Omega}{\int_X u^2 \Omega} \mid u \in C^\infty_\Omega(X, \mathbb{R})_0 \setminus \{0\} \right\},$$

which satisfies the inequality $\lambda_1(\Delta_g^\Omega) > 1$.

STEP II. Let $(v, V) \in \mathbb{F}_{g,\Omega}$. Using the L^2-orthogonal decomposition (15.2) in the proof of lemma 15, we expand the integral term

$$\int_X \left\langle \mathcal{L}_g^\Omega v, v \right\rangle_g \Omega = \int_X \left[\left\langle \mathcal{L}_g^\Omega \nabla_g d\varphi + \mathcal{L}_g^\Omega \theta, \nabla_g d\varphi \right\rangle_g + \left\langle \mathcal{L}_g^\Omega \nabla_g d\varphi + \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \right] \Omega.$$

We observe that

$$\int_X \left\langle \mathcal{L}_g^\Omega \theta, \nabla_g d\varphi \right\rangle_g \Omega = \int_X \left\langle \nabla_g^{\star_\Omega} \mathcal{L}_g^\Omega \theta, d\varphi \right\rangle_g \Omega = 0,$$

since $\nabla_g^{\star_\Omega} \mathcal{L}_g^\Omega \theta = 0$ thanks to the identity (17.1) applied to $(\theta, 0) \in \mathbb{F}_{g,\Omega}$. On the other hand formula (9.3) implies

$$\int_X \left\langle \mathcal{L}_g^\Omega \nabla_g d\varphi, \theta \right\rangle_g \Omega = \int_X \left\langle d(\Delta_g^\Omega - 2\mathbb{I})\varphi, \nabla_g^{\star_\Omega} \theta \right\rangle_g \Omega = 0.$$

We conclude the decomposition identity

$$\int_X \left\langle \mathcal{L}_g^\Omega v, v \right\rangle_g \Omega = \int_X \left[\left\langle \mathcal{L}_g^\Omega \nabla_g d\varphi, \nabla_g d\varphi \right\rangle_g + \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \right] \Omega.$$

Then step I implies

$$\begin{aligned} \frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) \\ &= \frac{1}{2}\int_X (\Delta_g^\Omega - \mathbb{I})(\Delta_g^\Omega - 2\mathbb{I})\varphi \cdot (\Delta_g^\Omega - 2\mathbb{I})\varphi \Omega - \frac{1}{2}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega. \end{aligned}$$

On the other hand using the decomposition (15.5) of θ and the decomposition formula (12.7) we infer

$$\begin{aligned} \int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \int_X \left\langle \mathcal{L}_g^\Omega \partial^g_{T_{X,J}} \nabla_g \tau, \partial^g_{T_{X,J}} \nabla_g \tau \right\rangle_g \Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi}, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A \right\rangle_g \Omega + \int_X \left\langle \mathcal{L}_g^\Omega A, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A \right\rangle_g \Omega. \end{aligned}$$

Using the identities (12.8), (12.9) and the property (12.5) we deduce

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \int_X \left\langle \partial^g_{T_{X,J}} \nabla_g (\Delta_g^\Omega - 2\mathbb{I})\tau, \partial^g_{T_{X,J}} \nabla_g \tau \right\rangle_g \Omega + \int_X \left\langle \overline{\partial}_{T_{X,J}} \nabla_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\overline{\psi}, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A \right\rangle_g \Omega \\ &+ \int_X \left[\left\langle \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \mathcal{L}_g^\Omega A, \nabla_{g,J} \overline{\psi} \right\rangle_g + \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \right] \Omega.\end{aligned}$$

By bi-degree reasons $\nabla_g^{\star_\Omega} A = 0$, which means $(gA, 0) \in \mathbb{F}_{g,\Omega}$. We infer $\nabla_g^{\star_\Omega} \mathcal{L}_g^\Omega A = 0$ thanks to the identity (17.1). Then the property (12.2) implies

$$\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \mathcal{L}_g^\Omega A = 0, \tag{17.3}$$

by bi-degree reasons. Integrating by parts further and using the weighted complex Bochner identities (13.8), (13.9) we obtain

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \int_X \left\langle \nabla_g (\Delta_g^\Omega - 2\mathbb{I})\tau, \partial^{\star_{g,\Omega}}_{T_{X,J}} \partial^g_{T_{X,J}} \nabla_g \tau \right\rangle_g \Omega + \int_X \left\langle \nabla_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\overline{\psi}, \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} \right\rangle_g \Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \Omega \\ &= \frac{1}{2} \int_X \left\langle \nabla_g (\Delta_g^\Omega - 2\mathbb{I})\tau, \nabla_{g,J} \Delta^\Omega_{g,J} \tau \right\rangle_g \Omega + \frac{1}{2} \int_X \left\langle \nabla_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\overline{\psi}, \nabla_{g,J} \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi} \right\rangle_g \Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \Omega.\end{aligned}$$

Using the integration by parts formulas (20.4) and (20.3) in the subsection 20.2 of the appendix A, we deduce

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \frac{1}{4} \int_X \Delta^\Omega_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\tau \cdot \Delta^\Omega_{g,J} \tau \Omega + \frac{1}{4} \int_X \overline{\Delta^\Omega_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\tau} \cdot \overline{\Delta^\Omega_{g,J} \tau} \Omega \\ &+ \frac{1}{4} \int_X \Delta^\Omega_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\psi \cdot \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi} \Omega + \frac{1}{4} \int_X \overline{\Delta^\Omega_{g,J} (\Delta_g^\Omega - 2\mathbb{I})\psi} \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi \Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \Omega.\end{aligned}$$

We observe now that the commutation identity (15.6) implies

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \frac{1}{4} \int_X (\Delta_g^\Omega - 2\mathbb{I}) \Delta^\Omega_{g,J} \tau \cdot \Delta^\Omega_{g,J} \tau \Omega + \frac{1}{4} \int_X (\Delta_g^\Omega - 2\mathbb{I}) \overline{\Delta^\Omega_{g,J} \tau} \cdot \overline{\Delta^\Omega_{g,J} \tau} \Omega \\ &+ \frac{1}{4} \int_X (\Delta_g^\Omega - 2\mathbb{I}) \Delta^\Omega_{g,J} \psi \cdot \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi} \Omega + \frac{1}{4} \int_X (\Delta_g^\Omega - 2\mathbb{I}) \overline{\Delta^\Omega_{g,J} \psi} \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi \Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \Omega.\end{aligned}$$

Completing the square we obtain

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \frac{1}{4}\int_X (\Delta_g^\Omega - 2\mathbb{I})\Delta_{g,J}^\Omega \tau \cdot \Delta_{g,J}^\Omega \tau \Omega + \frac{1}{4}\int_X (\Delta_g^\Omega - 2\mathbb{I})\overline{\Delta_{g,J}^\Omega \tau} \cdot \overline{\Delta_{g,J}^\Omega \tau}\Omega \\ &+ \frac{1}{4}\int_X (\Delta_g^\Omega - 2\mathbb{I})(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi \cdot \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi}\Omega + \frac{1}{4}\int_X (\Delta_g^\Omega - 2\mathbb{I})\overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi} \cdot (\Delta_{g,J}^\Omega - 2\mathbb{I})\psi\Omega \\ &+ \frac{1}{2}\int_X (\Delta_g^\Omega - 2\mathbb{I})\psi \cdot \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi}\Omega + \frac{1}{2}\int_X (\Delta_g^\Omega - 2\mathbb{I})\overline{\psi} \cdot (\Delta_{g,J}^\Omega - 2\mathbb{I})\psi\Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \Omega.\end{aligned}$$

Using the equation (15.3) we infer

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \frac{1}{4}\int_X (\Delta_g^\Omega - 2\mathbb{I})\left(\Delta_{g,J}^\Omega \tau + \overline{\Delta_{g,J}^\Omega \tau}\right) \cdot \left(\Delta_{g,J}^\Omega \tau + \overline{\Delta_{g,J}^\Omega \tau}\right)\Omega \\ &+ \int_X |(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi|^2 \Omega + \frac{i}{2}\int_X \left[B_{g,J}^\Omega \psi \cdot \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi} - B_{g,J}^\Omega \overline{\psi} \cdot (\Delta_{g,J}^\Omega - 2\mathbb{I})\psi\right]\Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g \Omega \\ &= \int_X \left[(\Delta_g^\Omega - 2\mathbb{I})\Delta_g^\Omega \tau \cdot \Delta_g^\Omega \tau + |\Delta_{g,J}^\Omega \tau|^2 + \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g\right]\Omega \\ &- \frac{i}{2}\int_X \left[B_{g,J}^\Omega \psi \cdot \Delta_{g,J}^\Omega \tau - B_{g,J}^\Omega \overline{\psi} \cdot \overline{\Delta_{g,J}^\Omega \tau}\right]\Omega.\end{aligned}$$

We observe now that the operator $B_{g,J}^\Omega$ is L_Ω^2-anti-adjoint. This implies in particular

$$\int_X \Delta_g^\Omega \tau \cdot B_{g,J}^\Omega \tau \, \Omega = 0,$$

and

$$\begin{aligned}-\frac{i}{2}\int_X \left[B_{g,J}^\Omega \psi \cdot \Delta_{g,J}^\Omega \tau - B_{g,J}^\Omega \overline{\psi} \cdot \overline{\Delta_{g,J}^\Omega \tau}\right]\Omega &= \frac{i}{2}\int_X \left[\psi \cdot \Delta_{g,J}^\Omega B_{g,J}^\Omega \tau - \overline{\psi} \cdot \overline{\Delta_{g,J}^\Omega B_{g,J}^\Omega \tau}\right]\Omega \\ &= \frac{i}{2}\int_X \left[\overline{\Delta_{g,J}^\Omega \psi} - \Delta_{g,J}^\Omega \overline{\psi}\right] B_{g,J}^\Omega \tau \Omega \\ &= -\int_X \left(\Delta_g^\Omega \psi_2 + B_{g,J}^\Omega \psi_1\right) B_{g,J}^\Omega \tau \Omega.\end{aligned}$$

thanks to the commutation identity (15.6). Thus

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega \theta, \theta \right\rangle_g \Omega &= \int_X \left[(\Delta_g^\Omega - \mathbb{I})\Delta_g^\Omega \tau \cdot \Delta_g^\Omega \tau + |B_{g,J}^\Omega \tau|^2 + \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g\right]\Omega \\ &- \int_X \left(\Delta_g^\Omega \psi_2 + B_{g,J}^\Omega \psi_1\right) B_{g,J}^\Omega \tau \Omega \\ &= \int_X \left[(\Delta_g^\Omega - \mathbb{I})\Delta_g^\Omega \tau \cdot \Delta_g^\Omega \tau + \left\langle \mathcal{L}_g^\Omega A, A \right\rangle_g\right]\Omega - 2\int_X (\Delta_g^\Omega - \mathbb{I})\psi_2 \cdot B_{g,J}^\Omega \tau \Omega,\end{aligned}$$

thanks to the second equation in (15.9). Using again the second equation in (15.9) we expand the term

$$\begin{aligned}
-2\int_X(\Delta_g^\Omega-\mathbb{I})\psi_2\cdot B_{g,J}^\Omega\tau\Omega &= 2\int_X(\Delta_g^\Omega-\mathbb{I})\psi_2\cdot(\Delta_g^\Omega-2\mathbb{I})\psi_2\Omega-2\int_X(\Delta_g^\Omega-\mathbb{I})\psi_2\cdot B_{g,J}^\Omega\psi_1\Omega\\
&= \int_X\left[\left|(\Delta_g^\Omega-2\mathbb{I})\psi_2\right|^2+\Delta_g^\Omega\psi_2\cdot(\Delta_g^\Omega-2\mathbb{I})\psi_2\right]\Omega-2\int_X\psi_2\cdot(\Delta_g^\Omega-\mathbb{I})B_{g,J}^\Omega\psi_1\Omega\\
&= \int_X\left[(\Delta_g^\Omega-2\mathbb{I})^2+(B_{g,J}^\Omega)^2\right]\psi_2\cdot\psi_2\Omega,
\end{aligned}$$

thanks to the expression (15.10). We observe further that the formula

$$P_{g,J}^\Omega = (\Delta_g^\Omega-2\mathbb{I})^2+(B_{g,J}^\Omega)^2, \tag{17.4}$$

hold thanks to the commutation identity (15.6). We conclude

$$\int_X\left\langle\mathcal{L}_g^\Omega\theta,\theta\right\rangle_g\Omega = \int_X\left[(\Delta_g^\Omega-\mathbb{I})\Delta_g^\Omega\tau\cdot\Delta_g^\Omega\tau+P_{g,J}^\Omega\psi_2\cdot\psi_2+\left\langle\mathcal{L}_g^\Omega A,A\right\rangle_g\right]\Omega,$$

which implies the required formula for the variations $(v,V)\in\mathbb{F}_{g,\Omega}^J(0)$.

STEP III. We compute now the stability integral involving A. The trivial identity

$$\mathcal{L}_g^\Omega A = \mathcal{L}_gA+\nabla_gf\neg\nabla_gA,$$

combined with the formula (14.9) implies

$$\mathcal{L}_g^\Omega A = 2\overline{\partial}_{T_{X,J}}\overline{\partial}_{T_{X,J}}^{\star_g}A+\left[\mathrm{Ric}^\star(g),A\right]+\nabla_gf\neg\nabla_gA,$$

since $\overline{\partial}_{T_{X,J}}A=0$. Integrating by parts we deduce

$$\begin{aligned}
\int_X\left\langle\mathcal{L}_g^\Omega A,A\right\rangle_g\Omega &= \int_X\left[2\left\langle\overline{\partial}_{T_{X,J}}\overline{\partial}_{T_{X,J}}^{\star_g}A,A\right\rangle_g+\langle\nabla_gf\neg\nabla_gA,A\rangle_g\right]\Omega\\
&= \int_X\left[2\left\langle\overline{\partial}_{T_{X,J}}^{\star_g}A,\overline{\partial}_{T_{X,J}}^{\star_{g,\Omega}}A\right\rangle_g+\frac{1}{2}\nabla_gf.|A|_g^2\right]\Omega\\
&= \frac{1}{2}\int_X\Delta_g^\Omega f\,|A|_g^2\,\Omega\\
&= \int_X F\,|A|_g^2\,\Omega,
\end{aligned}$$

since $\overline{\partial}_{T_{X,J}}^{\star_{g,\Omega}}A=0$ and (J,g) is a Kähler-Ricci-Soliton. (This last identity has been obtained by Hall-Murphy [16] using a different integration by parts method.)

We show now the second variation formula corresponding to the particular case $(v,V)\in\mathbb{F}_{g,\Omega}^J[0]$. With this assumption hold the relation $\varphi=-\tau$. Thus we rearrange the expression

$$\begin{aligned}
E &:= \frac{1}{2}\int_X(\Delta_g^\Omega-\mathbb{I})(\Delta_g^\Omega-2\mathbb{I})\varphi\cdot(\Delta_g^\Omega-2\mathbb{I})\varphi\Omega-\frac{1}{2}\int_X(\Delta_g^\Omega-\mathbb{I})\Delta_g^\Omega\varphi\cdot\Delta_g^\Omega\varphi\Omega\\
&= \frac{1}{2}\int_X\left[-4(\Delta_g^\Omega-\mathbb{I})\Delta_g^\Omega\varphi\cdot\varphi+4(\Delta_g^\Omega-\mathbb{I})\varphi\cdot\varphi\right]\Omega\\
&= -2\int_X(\Delta_g^\Omega-\mathbb{I})^2\varphi\cdot\varphi\Omega\\
&= -2\int_X\left|(\Delta_g^\Omega-\mathbb{I})\varphi\right|^2\Omega,
\end{aligned}$$

which implies the required formula in the particular case $(v, V) \in \mathbb{F}^J_{g,\Omega}[0]$.

Let $A := \Delta^{\Omega}_{g,J} - 2\mathbb{I}$ and observe that $[A, \bar{A}] = -2i[\Delta^{\Omega}_g, B^{\Omega}_{g,J}] = 0$, thanks to (15.6). Then the statement concerning the operator $P^{\Omega}_{g,J}$ follows from the elementary lemma below. □

Lemma 19. *Let $H := L^2_{\Omega}(X, \mathbb{C})_0$ and $A, B : D \subset H \longrightarrow H$ be closed densely defined linear operators such that $0 \leqslant A = A^{\star}$, $0 \leqslant B = B^{\star}$, $[A, B] = 0$. If A and B are differential operators of same order with A elliptic then $AB \geqslant 0$. In particular if $[A, \bar{A}] = 0$ then $A\bar{A} \geqslant 0$.*

Proof. Let $E_{\lambda_k}(A) \subset H$ be the eigenspace of A corresponding to an eigenvalue $\lambda_k \in \mathbb{R}_{\geqslant 0}$. Then the identity $[A, B] = 0$ implies that the restriction $B : E_{\lambda_k}(A) \longrightarrow E_{\lambda_k}(A)$ is well defined and represents a non-negative self-adjoint operator. We deduce by the spectral theorem in finite dimensions the existence of an orthonormal basis $(e_{k,l})_{l \in I_k} \subset E_{\lambda_k}(A)$ such that $Be_{k,l} = \mu_{k,l} e_{k,l}$, with $\mu_{k,l} \in \mathbb{R}_{\geqslant 0}$. Moreover $Ae_{k,l} = \lambda_k e_{k,l}$. We consider a strictly monotone increasing parametrization $(\lambda_k)_k$. Then any $u \in H$ writes as

$$u = \sum_{k \geqslant 0} \sum_{l \in I_k} c_{k,l} e_{k,l},$$

$c_{k,l} \in \mathbb{C}$. In particular for $u \in C^{\infty}(X, \mathbb{C})_0$ hold the expressions

$$Au = \sum_{k \geqslant 0} \sum_{l \in I_k} \lambda_k c_{k,l} e_{k,l},$$

$$Bu = \sum_{k \geqslant 0} \sum_{l \in I_k} \mu_{k,l} c_{k,l} e_{k,l},$$

and

$$(ABu, u)_{\Omega} = (Bu, Au)_{\Omega} = \sum_{k \geqslant 0} \sum_{l \in I_k} \lambda_k \mu_{k,l} \left| c_{k,l} \right|^2 \geqslant 0.$$

The inequality in the general case $u \in D$ follows from the density of the smooth functions in the graph topology of A. In order to see that $\bar{A} \geqslant 0$ we observe the trivial equalities

$$0 \leqslant \int_X Au \cdot \overline{u} \Omega = \int_X u \cdot \overline{Au} \Omega = \int_X \bar{A}\overline{u} \cdot u \Omega = \int_X \bar{A} v \cdot \overline{v} \Omega,$$

with $v := \overline{u}$. In order to show its self-adjointness we observe also the trivial equalities

$$\int_X \bar{A} u \cdot \overline{v} \Omega = \int_X \overline{v} \cdot \overline{A\overline{u}} \Omega = \int_X A\overline{v} \cdot u \Omega = \int_X u \cdot \overline{\bar{A} v} \Omega.$$

□

We deduce the following corollary of proposition 1.

Corollary 4. *In the setting of proposition 1 assume $(v, V) \in \mathbb{F}^J_{g,\Omega}[0]$ with $A_v = 0$. Then*

$$\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) \leqslant 0,$$

with equality if and only if $(v, V) = (0, 0)$.

We notice indeed that the equality hold if and only if $\varphi = 0$.

17.1 The Kähler-Einstein case

In the Kähler-Einstein case the complex operator $\Delta^{\Omega}_{g,J}$ reduces to the real operator Δ^{Ω}_g. Let

$$\Lambda_g := \mathrm{Ker}_{\mathbb{R}}(\Delta_g - 2\mathbb{I}) \subset C^{\infty}(X, \mathbb{R})_0,$$

and let $\Lambda_g^{\perp} \subset C^{\infty}(X, \mathbb{R})_0$ be its L^2-orthogonal with respect to the measure dV_g. We observe the decomposition $\Lambda_{g,J}^{\Omega} = \Lambda_g \oplus i\Lambda_g$, which implies the decomposition

$$\Lambda_{g,J}^{\Omega,\perp} = \Lambda_g^{\perp} \oplus i\Lambda_g^{\perp},$$

and thus the identity $\mathbb{E}_{g,J}^{\Omega} = \Lambda_g^{\perp}$. With the notations of lemma 16 let $\Phi := (\Delta_g - 2\mathbb{I})\varphi$, and $\Psi := (\Delta_g - 2\mathbb{I})\psi$. Then the second variation formulas in proposition 1 reduces to

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V)\\ &= \frac{1}{2\,\mathrm{Vol}_g(X)} \int_X [(\Delta_g - \mathbb{I})\Phi \cdot \Phi - (\Delta_g - \mathbb{I})\Psi \cdot \Psi] dV_g,\end{aligned}$$

in the case $(v, V) \in \mathbb{F}_g^J(0)$ and to

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V)\\ &= -\frac{2}{\mathrm{Vol}_g(X)} \int_X \left|\Delta_g^{-1}(\Delta_g - \mathbb{I})\Psi\right|^2 dV_g \geqslant 0,\end{aligned}$$

in the case $(v, V) \in \mathbb{F}_g^J[0]$, with equality if and only if $v_g^{\star} \in \mathcal{H}_g^{0,1}\left(T_{X,J}\right)$.

Proof of step II in the Kähler-Einstein case.

The most difficult part in the proof of proposition 1 is the computation of the stability integral

$$\int_X \left\langle \mathcal{L}_g^{\Omega}\theta, \theta \right\rangle_g \Omega,$$

in step II of the proof. In the Kähler-Einstein case the argument is much more simple. We include the details for readers convenience.

We remind first the isomorphism $g^{-1} : S_{\mathbb{R}}^2 T_X^{\star} \simeq \mathrm{End}_g(T_X)$. We have the g-orthogonal spiting

$$\begin{aligned}\mathrm{End}_g(T_X) &= E'_{g,J} \oplus_g E''_{g,J},\\ E'_{g,J} &:= \mathrm{End}_g(T_X) \cap C^{\infty}\left(X, T_{X,J}^{\star} \otimes T_{X,J}\right),\\ E''_{g,J} &:= \mathrm{End}_g(T_X) \cap C^{\infty}\left(X, T_{X,-J}^{\star} \otimes T_{X,J}\right).\end{aligned}$$

We observe that if $\alpha \in \Lambda_{\mathbb{R}}^2 T_X^{\star}$ then holds the identity $\alpha_{\omega}^{\star} = -J\alpha_g^{\star}$, where $\alpha_{\omega}^{\star} := \omega^{-1}\alpha$. We define also the vector bundle

$$\Lambda_{J,\mathbb{R}}^{1,1} := \Lambda_J^{1,1} T_X^{\star} \cap \Lambda_{\mathbb{R}}^2 T_X^{\star},$$

and we notice the isomorphism $\omega^{-1} : \Lambda_{J,\mathbb{R}}^{1,1} \simeq E'_{g,J}$. Moreover the identity (3.4) combined with the properties (12.1) and (12.2) implies that the maps

$$\mathcal{L}_g^{\Omega} : C^{\infty}\left(X, E'_{g,J}\right) \longrightarrow C^{\infty}(X, E'_{g,J}), \tag{17.5}$$

$$\mathcal{L}_g^{\Omega} : C^{\infty}\left(X, E''_{g,J}\right) \longrightarrow C^{\infty}(X, E''_{g,J}), \tag{17.6}$$

are well defined. We observe also that by (12.5) and (20.7) we deduce the formula

$$\omega\mathcal{L}_g^{\Omega}\alpha_{\omega}^{\star} = \Delta_g^{\Omega}\alpha + 2\mathcal{R}_g \star \alpha, \tag{17.7}$$

for all $\alpha \in C^\infty(X, \Lambda^{1,1}_{J,\mathbb{R}})$. Notice indeed that the endomorphism $J\alpha^\star_\omega$ is g-anti-symmetric thanks to the J-linearity of $\alpha^\star_\omega$. We deduce using (17.7) and the identity $\mathrm{Tr}_\omega\, \alpha = \mathrm{Tr}_{\mathbb{R}}\, \alpha^\star_\omega$,

$$\mathrm{Tr}_\omega \left(\Delta^\Omega_g \alpha + 2\mathcal{R}_g \star \alpha\right) \quad = \quad \mathrm{Tr}_{\mathbb{R}} \left(\mathcal{L}^\Omega_g \alpha^\star_\omega\right).$$

This combined with the identity (3.10), which in our case rewrites as

$$\mathrm{Tr}_{\mathbb{R}} \left(\mathcal{R}_g \star \alpha^\star_\omega\right) \quad = \quad \mathrm{Tr}_{\mathbb{R}} \left[\mathrm{Ric}^\star(g)\alpha^\star_\omega\right],$$

implies that in the Einstein case hold the trace formula

$$\mathrm{Tr}_\omega\, (\Delta_g \alpha + 2\mathcal{R}_g \star \alpha) = (\Delta_g - 2\mathbb{I})\, \mathrm{Tr}_\omega\, \alpha. \tag{17.8}$$

We observe also the identity

$$\langle \alpha, \beta \rangle_g = \left\langle \alpha^\star_\omega, \beta^\star_\omega \right\rangle_g, \tag{17.9}$$

for all $\alpha, \beta \in \Lambda^{1,1}_{J,\mathbb{R}}$. Indeed we consider the equalities

$$\begin{aligned}
\langle \alpha, \beta \rangle_g &= \left\langle \alpha^\star_g, \beta^\star_g \right\rangle_g \\
&= \mathrm{Tr}_{\mathbb{R}} \left[\alpha^\star_g (\beta^\star_g)^T_g\right] \\
&= -\mathrm{Tr}_{\mathbb{R}} \left[\alpha^\star_g \beta^\star_g\right] \\
&= \mathrm{Tr}_{\mathbb{R}} \left[J\alpha^\star_g J\beta^\star_g\right] \\
&= \mathrm{Tr}_{\mathbb{R}} \left[\alpha^\star_\omega \beta^\star_\omega\right] \\
&= \left\langle \alpha^\star_\omega, \beta^\star_\omega \right\rangle_g.
\end{aligned}$$

We deduce by the identity (20.6) in the appendix and by the Stokes theorem that over a compact Kähler manifold if $\alpha, \beta \in C^\infty(X, \Lambda^{1,1}_{J,\mathbb{R}})$, $d\alpha = d\beta = 0$ satisfy $\{\alpha\}_d = 0$, or $\{\beta\}_d = 0$ then

$$2 \int_X \langle \alpha, \beta \rangle_g\, dV_g = \int_X \mathrm{Tr}_\omega\, \alpha\, \mathrm{Tr}_\omega\, \beta dV_g. \tag{17.10}$$

(Notice indeed the identity $\langle \alpha, \beta \rangle_g = \langle \alpha, \beta \rangle_\omega$ for all $\alpha, \beta \in \Lambda^{1,1}_{J,\mathbb{R}}$.) We decompose now the endomorphism section $\theta^\star_g = A'_J + A''_J$ and we estimate the integral

$$\begin{aligned}
\int_X \left\langle \mathcal{L}^\Omega_g \theta, \theta \right\rangle_g \Omega &= \int_X \left\langle \mathcal{L}^\Omega_g \theta^\star_g, \theta^\star_g \right\rangle_g \Omega \\
&= \int_X \left[\left\langle \mathcal{L}^\Omega_g A'_J, A'_J \right\rangle_g + \left\langle \mathcal{L}^\Omega_g A''_J, A''_J \right\rangle_g\right] \Omega.
\end{aligned}$$

The last equality hold thanks to the identity (12.7). Let $\alpha := \omega A'_J$ and assume $\{\alpha\}_d = 0$. Using the identity (20.10) we obtain

$$\Delta^\Omega_g \alpha + 2\mathcal{R}_g \star \alpha \quad = \quad (\Delta^\Omega_{d,g} - 2\mathbb{I})\alpha = d\nabla^{\star_\Omega}_g \alpha - 2\alpha,$$

and thus $\left\{\Delta_g^{\Omega}\alpha + 2\mathcal{R}_g \star \alpha\right\}_d = 0$. Then the identities (17.9), (17.7), (17.10) and (17.8) imply

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g A'_J, A'_J\right\rangle_g dV_g &= \int_X \left\langle \Delta_g\alpha + 2\mathcal{R}_g \star \alpha, \alpha\right\rangle_g dV_g \\ &= \frac{1}{2}\int_X (\Delta_g - 2\mathbb{I})\,\mathrm{Tr}_\omega\,\alpha \cdot \mathrm{Tr}_\omega\,\alpha dV_g \geqslant 0,\end{aligned}$$

since $\lambda_1(\Delta_g) \geqslant 2$ in the Kähler-Einstein case. (Notice that the condition $\int_X \mathrm{Tr}_\omega\,\alpha dV_g = 0$ hold thanks to the assumption $\{\alpha\}_d = 0$.) In the set up of lemma 16 we have $\alpha = i\partial_J\overline{\partial}_J\tau$ and

$$A''_J = \overline{\partial}_{T_{X,J}}\nabla_g\psi + A,$$

with $\psi \in \Lambda_g^{\perp}$ and $A \in \mathcal{H}_g^{0,1}\left(T_{X,J}\right)$. Thus by the previous computation

$$\int_X \left\langle \mathcal{L}_g A'_J, A'_J\right\rangle_g dV_g = \frac{1}{2}\int_X (\Delta_g - 2\mathbb{I})\Delta_g\tau \cdot \Delta_g\tau dV_g.$$

On the other hand formula (14.9) implies in the Kähler-Einstein case

$$\mathcal{L}_g A''_J = 2\Delta^{-J}_{T_{X,g}} A''_J = 2\overline{\partial}_{T_{X,J}}\overline{\partial}^{\star_g}_{T_{X,J}} A''_J,$$

since $\overline{\partial}_{T_{X,J}} A''_J = 0$. Integrating by parts we deduce

$$\int_X \left\langle \mathcal{L}_g A''_J, A''_J\right\rangle_g dV_g = 2\int_X \left|\overline{\partial}^{\star_g}_{T_{X,J}} A''_J\right|_g^2 dV_g = 2\int_X \left|\overline{\partial}^{\star_g}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g\psi\right|_g^2 dV_g.$$

In the Kähler-Einstein case the complex Bochner type formula (13.9) combined with the equation (15.3) implies

$$2\overline{\partial}^{\star_g}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g\psi = \nabla_g(\Delta_g - 2\mathbb{I})\psi = -\nabla_g\Delta_g\tau.$$

We obtain

$$\int_X \left\langle \mathcal{L}_g A''_J, A''_J\right\rangle_g dV_g = \frac{1}{2}\int_X |\nabla_g\Delta_g\tau|_g^2 dV_g = \frac{1}{2}\int_X \Delta_g^2\tau \cdot \Delta_g\tau dV_g,$$

and thus the required formula

$$\int_X \left\langle \mathcal{L}_g\theta, \theta\right\rangle_g dV_g = \int_X (\Delta_g - \mathbb{I})\Delta_g\tau \cdot \Delta_g\tau dV_g \geqslant 0.$$

We notice also that the latter implies the statement of theorem 2. Indeed in the equality case holds $\Delta_g\tau = 0$ since $\lambda_1(\Delta_g) \geqslant 2$. Then the equation (15.3) implies $\psi = 0$. The conclusion follows from the decomposition identity (15.5).

Remark 3. *We consider the particular case of a smooth curve $(g_t, \Omega_t)_t \subset \mathcal{S}_\omega$ with g_0 Kähler-Einstein metric. Time deriving twice the expression*

$$\mathcal{W}(g_t, \Omega_t) = 2\int_X \log\left(\frac{\omega^n}{\Omega_t}\right)\Omega_t - 2\log n!,$$

we infer

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}}\mathcal{W}(g_t, \Omega_t) &= -2\int_X \left|\dot{\Omega}_0^{\star}\right|^2\Omega_0 - 2\int_X \log\left(\frac{\Omega_0}{\omega^n}\right)\ddot{\Omega}_0 \\ &= -2\int_X \left|\dot{\Omega}_0^{\star}\right|^2\Omega_0,\end{aligned}$$

thanks to the Kähler-Einstein condition. Then a trivial change of variables allows to deduce our previous second variation formula in the particular case $(\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}^{J_0}_{g_0,\Omega_0}[0]$.

17.2 The case of variations in the direction $\mathbb{T}^J_{g,\Omega}$

The first part of the following result has been obtained in [28]. We provide here a new proof of the crucial case of the variations inside $\mathbb{F}^J_{g,\Omega}[0]$. For readers convenience we include also the general proof explained in [28].

Proposition 2. *Let (X, J, g) be a compact Kähler-Ricci-Soliton and let $\Omega > 0$ be the unique smooth volume form such that $gJ = \mathrm{Ric}_J(\Omega)$ and $\int_X \Omega = 1$. Let also $(g_t, \Omega_t)_{t\in\mathbb{R}} \subset \mathcal{M} \times \mathcal{V}_1$ be a smooth curve with $(g_0, \Omega_0) = (g, \Omega)$ and with $(\dot{g}_0, \dot{\Omega}_0) = (v, V) \in \mathbb{T}^J_{g,\Omega}$. Then with respect to the decomposition*

$$v^\star_g \quad = \quad \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A,$$

with unique $\psi \in \Lambda^{\Omega,\perp}_{g,J}$ and $A \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$, hold the second variation formula

$$\begin{aligned} \frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) \\ &= -\frac{1}{2} \int_X \left[P^\Omega_{g,J} \operatorname{Re}\psi \cdot \operatorname{Re}\psi \ + |A|^2_g F \right] \Omega, \end{aligned}$$

where

$$P^\Omega_{g,J} \quad := \quad (\Delta^\Omega_{g,J} - 2\mathbb{I})\overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})},$$

is a non-negative self-adjoint real elliptic operator with respect to the L^2_Ω-hermitian product. Moreover if $(v, V) \in \mathbb{F}^J_{g,\Omega}[0]$ then the previous formula writes as

$$\begin{aligned} \frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g, \Omega)(v, V; v, V) \\ &= -\frac{1}{2} \int_X \left[4 \left| V^\star_\Omega \right|^2 \ + \ P^\Omega_{g,J} \operatorname{Im}\psi \cdot \operatorname{Im}\psi \ + |A|^2_g F \right] \Omega. \end{aligned}$$

Proof. **Step I.** Reconsidering a computation in the poof of step II of the proposition 1 we see that for all variations $v \in \mathbb{D}^J_{g,[0]}$ holds the identity

$$\int_X \left\langle \mathcal{L}^\Omega_g v, v \right\rangle_g \Omega = \int_X \left\langle \mathcal{L}^\Omega_g \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi}, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A \right\rangle_g \Omega + \int_X \left\langle \mathcal{L}^\Omega_g A, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A \right\rangle_g \Omega.$$

Using the identity (12.9) and the property (12.5) we deduce

$$\int_X \left\langle \mathcal{L}^\Omega_g v, v \right\rangle_g \Omega = \int_X \left\langle \overline{\partial}_{T_{X,J}} \nabla_{g,J} (\Delta^\Omega_g - 2\mathbb{I})\overline{\psi}, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + A \right\rangle_g \Omega + \int_X \left[\left\langle \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \mathcal{L}^\Omega_g A, \nabla_{g,J} \overline{\psi} \right\rangle_g + \left\langle \mathcal{L}^\Omega_g A, A \right\rangle_g \right] \Omega.$$

Using the identity (17.3), integrating by parts further and using the weighted complex Bochner identity (13.9) we obtain

$$\begin{aligned} \int_X \left\langle \mathcal{L}^\Omega_g v, v \right\rangle_g \Omega &= \int_X \left\langle \nabla_{g,J} (\Delta^\Omega_g - 2\mathbb{I}) \, \overline{\psi}, \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} \right\rangle_g \Omega + \int_X \left\langle \mathcal{L}^\Omega_g A, A \right\rangle_g \Omega \\ &= \frac{1}{2} \int_X \left\langle \nabla_{g,J} (\Delta^\Omega_g - 2\mathbb{I}) \, \overline{\psi}, \nabla_{g,J} \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi} \right\rangle_g \Omega \\ &+ \int_X \left\langle \mathcal{L}^\Omega_g A, A \right\rangle_g \Omega. \end{aligned}$$

Using the integration by parts formula (20.3) in the subsection 20.2 of the appendix A we infer

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega v, v\right\rangle_g \Omega &= \frac{1}{4}\int_X \Delta_{g,J}^\Omega(\Delta_g^\Omega - 2\mathbb{I})\psi \cdot \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi}\Omega + \frac{1}{4}\int_X \overline{\Delta_{g,J}^\Omega(\Delta_g^\Omega - 2\mathbb{I})\psi}\cdot(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi\Omega \\ &+ \int_X \left\langle \mathcal{L}_g^\Omega A, A\right\rangle_g \Omega \end{aligned} \tag{17.11}$$

for all $v \in \mathbb{D}^J_{g,[0]}$.

STEP II. We show first the variation formula in the case $(v, V) \in \mathbb{F}^J_{g,\Omega}[0]$ since the proof is simpler. Using the expression (16.3) for the space $\mathbb{F}^J_{g,\Omega}[0]$ inside the identity (17.11) we deduce the equalities

$$\begin{aligned}\int_X \left\langle \mathcal{L}_g^\Omega v, v\right\rangle_g \Omega &= -\int_X (\Delta_g^\Omega - 2\mathbb{I})\,\mathrm{Re}(\Delta_{g,J}^\Omega\psi)\cdot V_\Omega^\star\,\Omega + \int_X \left\langle \mathcal{L}_g^\Omega A, A\right\rangle_g \Omega \\ &= -\int_X (\Delta_g^\Omega - 2\mathbb{I})(\Delta_g^\Omega\psi_1 + B_{g,J}^\Omega\psi_2)\cdot V_\Omega^\star\,\Omega + \int_X \left\langle \mathcal{L}_g^\Omega A, A\right\rangle_g \Omega.\end{aligned}$$

Let write $\psi = \psi_1 + i\psi_2$, with ψ_j real valued functions. Then the condition in the expression (16.3) rewrites as

$$(\Delta_g^\Omega - 2\mathbb{I})\,\psi_1 + B_{g,J}^\Omega\,\psi_2 = -2V_\Omega^\star, \tag{17.12}$$

$$(\Delta_g^\Omega - 2\mathbb{I})\,\psi_2 - B_{g,J}^\Omega\,\psi_1 = 0. \tag{17.13}$$

We use now the condition (17.12) in the formula

$$\begin{aligned}-2\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= -2\nabla_G D\mathcal{W}(g,\Omega)(v, V; v, V) \\ &= \int_X \left[\left\langle \mathcal{L}_g^\Omega v, v\right\rangle_g - 2(\Delta_g^\Omega - 2\mathbb{I})V_\Omega^\star\cdot V_\Omega^\star\right]\Omega \\ &= -2\int_X (\Delta_g^\Omega - 2\mathbb{I})\psi_1\cdot V_\Omega^\star\,\Omega + \int_X \left\langle \mathcal{L}_g^\Omega A, A\right\rangle_g \Omega.\end{aligned}$$

Using again the condition (17.12), we expand the integral

$$\begin{aligned}-2\int_X (\Delta_g^\Omega - 2\mathbb{I})\psi_1\cdot V_\Omega^\star\,\Omega &= \int_X (\Delta_g^\Omega - 2\mathbb{I})\psi_1\cdot\left[(\Delta_g^\Omega - 2\mathbb{I})\,\psi_1 + B_{g,J}^\Omega\,\psi_2\right]\,\Omega \\ &= \int_X \psi_1\left[(\Delta_g^\Omega - 2\mathbb{I})^2\psi_1 + B_{g,J}^\Omega(\Delta_g^\Omega - 2\mathbb{I})\,\psi_2\right]\,\Omega \\ &= \int_X \psi_1\left[(\Delta_g^\Omega - 2\mathbb{I})^2\psi_1 + \left(B_{g,J}^\Omega\right)^2\,\psi_1\right]\,\Omega,\end{aligned}$$

thanks to the identities (15.6) and (17.13). Using the formula (17.4), we infer

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}} \mathcal{W}(g_t, \Omega_t) &= \nabla_G D\mathcal{W}(g,\Omega)(v, V; v, V) \\ &= -\frac{1}{2}\int_X \left[P_{g,J}^\Omega\psi_1\cdot\psi_1 + \left\langle \mathcal{L}_g^\Omega A, A\right\rangle_g\right]\Omega.\end{aligned}$$

Using again the condition (17.12) and the commutation identity (15.6) we expand the integral

$$\begin{aligned}\int_X 4\left|V_\Omega^\star\right|^2\Omega &= \int_X\left[\left|(\Delta_g^\Omega-2\mathbb{I})\psi_1\right|^2+\left|B_{g,J}^\Omega\psi_2\right|^2+2(\Delta_g^\Omega-2\mathbb{I})\psi_1\cdot B_{g,J}^\Omega\psi_2\right]\Omega\\ &= \int_X\left[(\Delta_g^\Omega-2\mathbb{I})^2\psi_1\cdot\psi_1-(B_{g,J}^\Omega)^2\psi_2\cdot\psi_2+2\psi_1\cdot B_{g,J}^\Omega(\Delta_g^\Omega-2\mathbb{I})\psi_2\right]\Omega,\end{aligned}$$

thanks to the fact that the operator $B_{g,J}^\Omega$ is L_Ω^2-anti-adjoint. Using again this fact and the condition (17.13) we deduce

$$\begin{aligned}\int_X 2\psi_1\cdot B_{g,J}^\Omega(\Delta_g^\Omega-2\mathbb{I})\psi_2\Omega &= \int_X\psi_1\cdot(B_{g,J}^\Omega)^2\psi_1\Omega-\int_X B_{g,J}^\Omega\,\psi_1\cdot(\Delta_g^\Omega-2\mathbb{I})\psi_2\Omega\\ &= \int_X\left[(B_{g,J}^\Omega)^2\psi_1\cdot\psi_1-(\Delta_g^\Omega-2\mathbb{I})\psi_2\cdot(\Delta_g^\Omega-2\mathbb{I})\,\psi_2\right]\Omega,\end{aligned}$$

and thus

$$\int_X 4\left|V_\Omega^\star\right|^2\Omega = \int_X\left[P_{g,J}^\Omega\psi_1\cdot\psi_1\ -P_{g,J}^\Omega\psi_2\cdot\psi_2\ \right]\Omega.$$

We infer the second variation formula

$$\begin{aligned}\frac{d^2}{dt^2}_{|_{t=0}}\mathcal{W}(g_t,\Omega_t) &= \nabla_G D\mathcal{W}(g,\Omega)(v,V;v,V)\\ &= -\frac{1}{2}\int_X\left[4\left|V_\Omega^\star\right|^2\ +\ P_{g,J}^\Omega\psi_2\cdot\psi_2\ +\left\langle\mathcal{L}_g^\Omega A,A\right\rangle_g\right]\Omega.\end{aligned}$$

The conclusion follows from the computation in the beginning of step III in the proof of the proposition 1.

STEP III. We show now the second variation formula in the more general case of variations $(v,V)\in\mathbb{T}_{g,\Omega}^J$. We observe first that the general expression of $\nabla_G^2\mathcal{W}(g,\Omega)$ obtained at the end of the proof of lemma 7 implies that over a shrinking-Ricci-Soliton point holds the variation formula

$$\begin{aligned}-2\frac{d^2}{dt^2}_{|_{t=0}}\mathcal{W}(g_t,\Omega_t) &= -2\nabla_G D\mathcal{W}(g,\Omega)(v,V;v,V)\\ &= \int_X\left\langle\mathcal{L}_g^\Omega v-L_{\nabla_g^{\star_\Omega}v_g^\star+\nabla_g V_\Omega^\star}g,v\right\rangle_g\Omega\\ &- 2\int_X\left[(\Delta_g^\Omega-2\mathbb{I})V_\Omega^\star-\mathrm{div}^\Omega\left(\nabla_g^{\star_\Omega}v_g^\star+\nabla_g V_\Omega^\star\right)\right]V_\Omega^\star\,\Omega\\ &= \int_X\left[\left\langle\mathcal{L}_g^\Omega v,v\right\rangle_g-2(\Delta_g^\Omega-2\mathbb{I})V_\Omega^\star\cdot V_\Omega^\star\right]\Omega\\ &- 2\int_X\left\langle\nabla_g\left(\nabla_g^{\star_\Omega}v_g^\star+\nabla_g V_\Omega^\star\right),v_g^\star\right\rangle_g\Omega\\ &- 2\int_X\left\langle\nabla_g^{\star_\Omega}v_g^\star+\nabla_g V_\Omega^\star,\nabla_g V_\Omega^\star\right\rangle_g\Omega\\ &= \int_X\left[\left\langle\mathcal{L}_g^\Omega v,v\right\rangle_g-2(\Delta_g^\Omega-2\mathbb{I})V_\Omega^\star\cdot V_\Omega^\star-2\left|\nabla_g^{\star_\Omega}v_g^\star+\ \nabla_g V_\Omega^\star\right|_g^2\right]\Omega,\end{aligned}$$

for arbitrary directions $(v, V) \in T_{\mathcal{M}\times\mathcal{V}_1}$. Using now the fact that in the case $(v, V) \in \mathbb{T}^J_{g,\Omega}$ hold the expressions $R_\psi = -2V^\star_\Omega$, (we use here the definitions (16.4), (16.5)) and (16.7) we obtain

$$-2\nabla_G DW(g,\Omega)(v,V;v,V) = \int_X \left[\left\langle \mathcal{L}^\Omega_g v, v\right\rangle_g - \frac{1}{2}(\Delta^\Omega_g - 2\mathbb{I})R_\psi \cdot R_\psi - \frac{1}{2}\left|\nabla_g I_\psi\right|^2_g\right]\Omega,$$

for all $(v, V) \in \mathbb{T}^J_{g,\Omega}$. Thanks to the commutation identity (15.6) we can rewrite the identity (17.11) as

$$\begin{aligned}\int_X \left[\left\langle \mathcal{L}^\Omega_g v, v\right\rangle_g - \left\langle \mathcal{L}^\Omega_g A, A\right\rangle_g\right]\Omega &= \frac{1}{4}\int_X (\Delta^\Omega_g - 2\mathbb{I})\Delta^\Omega_{g,J}\psi \cdot \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi}\Omega \\ &+ \frac{1}{4}\int_X (\Delta^\Omega_g - 2\mathbb{I})\overline{\Delta^\Omega_{g,J}\psi} \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\Omega.\end{aligned}$$

Adding and subtracting 2ψ to the factor $\Delta^\Omega_{g,J}\psi$ and respectively $2\overline{\psi}$ to the factor $\overline{\Delta^\Omega_{g,J}\psi}$, we infer

$$\begin{aligned}\int_X \left[\left\langle \mathcal{L}^\Omega_g v, v\right\rangle_g - \left\langle \mathcal{L}^\Omega_g A, A\right\rangle_g\right]\Omega &= \frac{1}{4}\int_X (\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi \cdot \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi}\Omega \\ &+ \frac{1}{4}\int_X (\Delta^\Omega_g - 2\mathbb{I})\overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi} \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\Omega \\ &+ \frac{1}{2}\int_X (\Delta^\Omega_g - 2\mathbb{I})\psi \cdot \overline{(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi}\Omega \\ &+ \frac{1}{2}\int_X (\Delta^\Omega_g - 2\mathbb{I})\overline{\psi} \cdot (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\Omega.\end{aligned}$$

We deduce the equalities

$$\begin{aligned}-2\nabla_G DW(g,\Omega)(v,V;v,V) &= \int_X \left[(\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi \cdot \overline{\psi} + \frac{1}{2}(\Delta^\Omega_g - 2\mathbb{I})I_\psi \cdot I_\psi - \frac{1}{2}\left|\nabla_g I_\psi\right|^2_g\right]\Omega \\ &+ \int_X \left\langle \mathcal{L}^\Omega_g A, A\right\rangle_g \Omega \\ &= \int_X \left\{\left[P^\Omega_{g,J} - iB^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})\right]\psi \cdot \overline{\psi} - I_\psi \cdot I_\psi + \left\langle \mathcal{L}^\Omega_g A, A\right\rangle_g\right\}\Omega.\end{aligned}$$

Using the expression

$$I_\psi = (\Delta^\Omega_g - 2\mathbb{I})\,\psi_2 - B^\Omega_{g,J}\,\psi_1,$$

we find the formula

$$\begin{aligned}-2\nabla_G DW(g,\Omega)(v,V;v,V) &= \int_X \left[P^\Omega_{g,J} - iB^\Omega_{g,J}(\Delta^\Omega_g - 2\mathbb{I}) - (B^\Omega_{g,J})^2\right]\psi \cdot \overline{\psi}\,\Omega \\ &- \int_X \left[\left|(\Delta^\Omega_g - 2\mathbb{I})\psi_2\right|^2 + \left|B^\Omega_{g,J}\psi_1\right|^2 - 2(\Delta^\Omega_g - 2\mathbb{I})\,\psi_2 \cdot B^\Omega_{g,J}\,\psi_1\right]\Omega \\ &+ \int_X \left\langle \mathcal{L}^\Omega_g A, A\right\rangle_g \Omega.\end{aligned}$$

The fact that the operator $B^{\Omega}_{g,J}$ is L^2_{Ω}-anti-adjoint combined with the commutation identity (15.6) implies that $B^{\Omega}_{g,J}(\Delta^{\Omega}_g - 2\mathbb{I})$ is also L^2_{Ω}-anti-adjoint. We deduce in particular the identity

$$\int_X B^{\Omega}_{g,J}(\Delta^{\Omega}_g - 2\mathbb{I})\psi_j \cdot \psi_j \, \Omega \quad = \quad 0,$$

and thus the equality

$$\begin{aligned}
-2\nabla_G D\mathcal{W}(g,\Omega)(v,V;v,V) \quad &= \quad \int_X \left[P^{\Omega}_{g,J}\psi_1 \cdot \psi_1 \; + \; P^{\Omega}_{g,J}\psi_2 \cdot \psi_2 \right] \Omega \\
&+ \quad \int_X \left[B^{\Omega}_{g,J}(\Delta^{\Omega}_g - 2\mathbb{I}) \; \psi_2 \cdot \psi_1 - B^{\Omega}_{g,J}(\Delta^{\Omega}_g - 2\mathbb{I})\psi_1 \cdot \psi_2 \right] \Omega \\
&- \quad \int_X \left[(B^{\Omega}_{g,J})^2\psi_1 \cdot \psi_1 + (B^{\Omega}_{g,J})^2\psi_2 \cdot \psi_2 \right] \Omega \\
&- \quad \int_X \left[\left| (\Delta^{\Omega}_g - 2\mathbb{I})\psi_2 \right|^2 + \left| B^{\Omega}_{g,J}\psi_1 \right|^2 - 2(\Delta^{\Omega}_g - 2\mathbb{I}) \; \psi_2 \cdot B^{\Omega}_{g,J}\,\psi_1 \right] \Omega \\
&+ \quad \int_X \left\langle \mathcal{L}^{\Omega}_g A, A \right\rangle_g \Omega.
\end{aligned}$$

Using the fact that the operator $B^{\Omega}_{g,J}$ is L^2_{Ω}-anti-adjoint and the commutation identity (15.6) we can simplify in order to obtain the required variation formula. □

18 Positivity of the metric $G_{g,\Omega}$ over the space $\mathbb{T}^J_{g,\Omega}$

Lemma 20. *For any $(g,\Omega) \in \mathcal{S}_{\omega}$ the restriction of the symmetric form $G_{g,\Omega}$ to the vector space $\mathbb{T}^J_{g,\Omega}$, with $J := g^{-1}\omega$, is positive definite.*

Proof. Let $(u,U), (v,V) \in \mathbb{T}^J_{g,\Omega}$. Using the expression (16.6) for the space $\mathbb{T}^J_{g,\Omega}$ we have

$$\begin{aligned}
u^{\star}_g \quad &= \quad \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\varphi} + A, \\
-2U^{\star}_{\Omega} \quad &= \quad \mathrm{Re}\left[\, (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \right],
\end{aligned}$$

and

$$\begin{aligned}
v^{\star}_g \quad &= \quad \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} + B, \\
-2V^{\star}_{\Omega} \quad &= \quad \mathrm{Re}\left[\, (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \right],
\end{aligned}$$

with unique $\varphi, \psi \in \Lambda^{\Omega,\perp}_{g,J}$ and $A, B \in \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right)$. We decompose now the term

$$\begin{aligned}
G_{g,\Omega}(u,U;v,V) \quad &= \quad \int_X \left[\langle u, v \rangle_g \, - 2U^{\star}_{\Omega} \cdot V^{\star}_{\Omega} \right] \Omega \\
&= \quad \int_X \left[\left\langle \overline{\partial}_{T_{X,J}} \nabla_{g,J} \, \overline{\varphi}, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} \right\rangle_g + \langle A, B \rangle_g \right] \Omega \\
&- \quad \frac{1}{2} \int_X \mathrm{Re}\left[\, (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \right] \mathrm{Re}\left[\, (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \right] \Omega.
\end{aligned}$$

Integrating by parts and using the weighted complex Bochner formula (13.9) we transform the integral

$$\begin{aligned} I_1 &:= \int\limits_X \left\langle \overline{\partial}_{T_{X,J}} \nabla_{g,J}\, \overline{\varphi}, \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi} \right\rangle_g \Omega \\ &= \int\limits_X \left\langle \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \overline{\partial}_{T_{X,J}} \nabla_{g,J}\, \overline{\varphi}, \nabla_{g,J} \overline{\psi} \right\rangle_g \Omega \\ &= \frac{1}{2} \int\limits_X \left\langle \nabla_{g,J} \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi}\, , \nabla_{g,J} \overline{\psi} \right\rangle_g \Omega. \end{aligned}$$

Using the integration by parts formula (20.4) in the subsection 20.2 of the appendix we deduce

$$\begin{aligned} I_1 &= \frac{1}{4} \int\limits_X \left[\Delta^{\Omega}_{g,J} (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\psi} + \overline{\Delta^{\Omega}_{g,J} (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi} \cdot \psi \right] \Omega \\ &= \frac{1}{4} \int\limits_X \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\Delta^{\Omega}_{g,J} \psi} + \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi} \cdot \Delta^{\Omega}_{g,J} \psi \right] \Omega. \end{aligned}$$

Adding and subtracting $2\overline{\psi}$ to the factor $\overline{\Delta^{\Omega}_{g,J}\psi}$ and respectively 2ψ to the factor $\Delta^{\Omega}_{g,J}\psi$, we infer

$$\begin{aligned} I_1 &= \frac{1}{2} \int\limits_X \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\psi} + \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi} \cdot \psi \right] \Omega \\ &+ \frac{1}{4} \int\limits_X \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi} + \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi} \cdot (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \right] \Omega \\ &= \frac{1}{2} \int\limits_X \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\psi} + \overline{\varphi} \cdot (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \right] \Omega \\ &+ \frac{1}{4} \int\limits_X \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi} + \overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi} \cdot (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \right] \Omega. \end{aligned}$$

We deduce the general formula

$$\begin{aligned} G_{g,\Omega}(u, U; v, V) &= \int\limits_X \left\{ \frac{1}{2} \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\psi} + (\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \cdot \overline{\varphi} \right] + \langle A, B \rangle_g \right\} \Omega \\ &+ \frac{1}{2} \int\limits_X \mathrm{Im} \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \right] \mathrm{Im} \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi \right] \Omega. \end{aligned}$$

In particular

$$G_{g,\Omega}(u, U; u, U) = \int\limits_X \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\varphi} + |A|^2_g \right] \Omega + \frac{1}{2} \int\limits_X \left\{ \mathrm{Im} \left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\varphi \right] \right\}^2 \Omega \geqslant 0,$$

with equality if and only if $\varphi = 0$ and $A = 0$, i.e. $(u, U) = (0, 0)$, thanks to the variational characterization of the first eigenvalue $\lambda_1 \left(\Delta^{\Omega}_{g,J} \right) \geqslant 2$ of the elliptic operator $\Delta^{\Omega}_{g,J}$. □

Corollary 5. *For any* $(g, \Omega) \in \mathcal{S}_{\omega}$ *hold the identity*

$$\mathrm{Ker}_{\mathbb{R}} (\Delta^{\Omega}_{g} - 2\mathbb{I}) = \mathrm{Ker}_{\mathbb{R}} (\Delta^{\Omega}_{g,J} - 2\mathbb{I}), \tag{18.1}$$

with $J := g^{-1}\omega$.

Proof. Let $u \in C^\infty_\Omega(X,\mathbb{R})_0$ and $(\varphi_t)_{t\in\mathbb{R}} \subset \mathrm{Symp}^0(X,\omega)$ the 1-parameter sub-group generated by the symplectic vector field $\xi := (du)^\star_\omega = -J\nabla_g u$. We set $J_t := \varphi_t^\star J$, $g_t := \varphi_t^\star g$, $\Omega_t := \varphi_t^\star \Omega$ and we compute $\dot g_0 = L_\xi g$ and $\dot\Omega_0 = L_\xi \Omega$. The expression of the tangent space to the symplectic orbit $[g,\Omega]_\omega$ in the proof of lemma 18 implies

$$\begin{aligned}\dot g_0^\star &= -2J\overline{\partial}_{T_{X,J}}\nabla_g u,\\ \dot\Omega_0^\star &= -B^\Omega_{g,J}u.\end{aligned}$$

Then the weighted complex Bochner formula (13.9) implies

$$\begin{aligned}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\dot g_0^\star + \nabla_g\dot\Omega_0^\star &= -2J\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla_g u + \nabla_g\dot\Omega_0^\star\\ &= -J\nabla_g(\Delta^\Omega_g - 2\mathbb{I})u + \nabla_g B^\Omega_{g,J}u + \nabla_g\dot\Omega_0^\star\\ &= -J\nabla_g(\Delta^\Omega_g - 2\mathbb{I})u.\end{aligned}$$

We deduce that $(\dot g_0,\dot\Omega_0)\in \mathbb{F}^J_{g,\Omega}[0]$ if and only if $(\Delta^\Omega_g - 2\mathbb{I})u = 0$. On the other hand the (strict) positivity of the metric $G_{g,\Omega}$ over $\mathbb{T}^J_{g,\Omega} \supset T_{[g,\Omega]_\omega,(g,\Omega)}$ implies

$$T_{[g,\Omega]_\omega,(g,\Omega)} \cap T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} = 0.$$

Then lemma 18 implies

$$T_{[g,\Omega]_\omega,(g,\Omega)} \cap T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} = T_{[g,\Omega]_\omega,(g,\Omega)} \cap \mathbb{F}^J_{g,\Omega}[0] = \{0\},$$

So if $(\dot g_0,\dot\Omega_0)\in\mathbb{F}^J_{g,\Omega}[0]$ then $(\dot g_0,\dot\Omega_0) = (0,0)$. We infer the inclusion

$$\mathrm{Ker}_{\mathbb{R}}(\Delta^\Omega_g - 2\mathbb{I}) \subseteq \mathrm{Ker}_{\mathbb{R}} B^\Omega_{g,J},$$

and thus the required identity (18.1). □

18.1 Double splitting of the space $\mathbb{T}^J_{g,\Omega}$

Let H^k, with $H^0 = L^2$, be a Sobolev space of sections over X. For any subset S of smooth sections over X we denote with H^kS its closure with respect to the H^k-topology. The pseudo-Riemannian metric $G_{g,\Omega}$ is obviously continuous with respect to the L^2-topology. At the moment we are unable to say if the topology induced by $G_{g,\Omega}$ over $L^2\mathbb{T}^J_{g,\Omega}$ is equivalent with the L^2-topology. Nevertheless we can show the following basic decomposition result

Corollary 6. *For any $(g,\Omega)\in\mathcal{S}_\omega$ holds the decomposition identity*

$$L^2\mathbb{T}^J_{g,\Omega} = L^2T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G L^2\mathbb{F}^J_{g,\Omega}[0],$$

with $J := g^{-1}\omega$.

Proof. We set

$$\Lambda^\Omega_{g,\mathbb{R}} := \mathrm{Ker}_{\mathbb{R}}(\Delta^\Omega_g - 2\mathbb{I}),$$

and let $\Lambda^{\Omega,\perp}_{g,\mathbb{R}} \subset L^2_\Omega(X,\mathbb{R})_0$ be its L^2-orthogonal with respect to the measure Ω. Then corollary 5 and its proof shows that the map

$$\begin{aligned}\chi : \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap C^\infty_\Omega(X,\mathbb{R})_0 &\longrightarrow T_{[g,\Omega]_\omega,(g,\Omega)},\\ \varphi &\longmapsto \left(2\omega\,\overline{\partial}_{T_{X,J}}\nabla_g\varphi, \left(B^\Omega_{g,J}\varphi\right)\Omega\right),\end{aligned}$$

is an isomorphism. We notice also that the expression of the metric $G_{g,\Omega}$ obtained at the end of the proof of lemma 20 hold true for arbitrary functions Φ and Ψ. So we put $(u, U) := \chi(\varphi)$ and $\Phi = \Psi = -2i\varphi$ in this formula. Using the fact that the operator $B^{\Omega}_{g,J}$ is L^2_{Ω}-anti-adjoint and the expression

$$\mathrm{Im}\left[(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\Phi\right] \quad = \quad -2(\Delta^{\Omega}_{g} - 2\mathbb{I})\varphi,$$

we infer

$$\begin{aligned} G_{g,\Omega}(u, U; u, U) \quad &= \quad 2\int\limits_X \left[\left|(\Delta^{\Omega}_{g} - 2\mathbb{I})\varphi\right|^2 + 2(\Delta^{\Omega}_{g} - 2\mathbb{I})\varphi \ \cdot \varphi\right] \Omega \\ &=: \quad \gamma_{g,\Omega}(\varphi, \varphi) \ \geqslant \ 0, \end{aligned}$$

(with equality if and only if $\varphi = 0$). We remind now that the proof of the weighted Bochner formula (13.9) shows the identity

$$-2\,\mathrm{div}^{\Omega}\,\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla g \quad = \quad \Delta^{\Omega}_{g}(\Delta^{\Omega}_{g} - 2\mathbb{I}) - (B^{\Omega}_{g,J})^2.$$

Thus the operator

$$\left(\overline{\partial}_{T_{X,J}}\nabla g\right)^{\star_{g,\Omega}}\overline{\partial}_{T_{X,J}}\nabla g \quad = \quad -\,\mathrm{div}^{\Omega}\,\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\nabla g,$$

is elliptic. This implies (see for example [11]) that the immage

$$\overline{\partial}_{T_{X,J}}\nabla g\left[H^{k+2}(X, \mathbb{R})\right] \subset H^k,$$

is closed in the H^k-topology, for all integers $k \geqslant 0$. We infer that the map

$$\overline{\partial}_{T_{X,J}}\nabla g : \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap H^{k+2}(X, \mathbb{R}) \quad \longrightarrow \quad \overline{\partial}_{T_{X,J}}\nabla g\left[H^{k+2}(X, \mathbb{R})\right] \subset H^k, \tag{18.2}$$

is a topological isomorphism. We deduce that the extension in the sense of distributions

$$\chi : \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap H^2(X, \mathbb{R}) \quad \longrightarrow \quad L^2T_{[g,\Omega]_{\omega},(g,\Omega)},$$

of the map χ is also a topological isomorphism and a $(\gamma_{g,\Omega}, G_{g,\Omega})$-isometry. The fact that the map

$$\Delta^{\Omega}_{g} - 2\mathbb{I} : \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap H^2(X, \mathbb{R}) \quad \longrightarrow \quad \Lambda^{\Omega,\perp}_{g,\mathbb{R}},$$

is a topological isomorphism provides the estimate

$$\begin{aligned} \gamma_{g,\Omega}(\varphi, \varphi) \quad &\geqslant \quad 2\int\limits_X \left|(\Delta^{\Omega}_{g} - 2\mathbb{I})\varphi\right|^2 \Omega \\ &\geqslant \quad 2\|(\Delta^{\Omega}_{g} - 2\mathbb{I})^{-1}\|^{-2} \cdot \|\varphi\|^2_{H^2}. \end{aligned}$$

Then the Lax-Milgram theorem implies that the map

$$\gamma_{g,\Omega} : \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap H^2(X, \mathbb{R}) \quad \longrightarrow \quad \left[\Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap H^2(X, \mathbb{R})\right]^{\star},$$

is a topological isomorphism. (The sign $\star$ here denotes the topological dual). We infer that the restricted map

$$G_{g,\Omega} : L^2T_{[g,\Omega]_{\omega},(g,\Omega)} \quad \longrightarrow \quad \left[L^2T_{[g,\Omega]_{\omega},(g,\Omega)}\right]^{\star},$$

is also a topological isomorphism thanks to the fact that the extended map χ is a $(\gamma_{g,\Omega}, G_{g,\Omega})$-isometry. Applying the elementary lemma 21 below to the spaces $E := L^2\left(X, S^2T^{\star}_X\right) \oplus L^2_{\Omega}(X, \mathbb{R})_0$ and $V := L^2T_{[g,\Omega]_{\omega},(g,\Omega)}$ we deduce the G-orthogonal decomposition

$$L^2\left(X, S^2T^{\star}_X\right) \oplus L^2_{\Omega}(X, \mathbb{R})_0 \quad = \quad L^2T_{[g,\Omega]_{\omega},(g,\Omega)} \oplus L^2T^{\perp_G}_{[g,\Omega]_{\omega},(g,\Omega)},$$

and thus

$$\begin{aligned} L^2\mathbb{T}^J_{g,\Omega} &= L^2T_{[g,\Omega]_\omega,(g,\Omega)} \oplus \left[L^2T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} \cap\ L^2\mathbb{T}^J_{g,\Omega}\right] \\ &= L^2T_{[g,\Omega]_\omega,(g,\Omega)} \oplus L^2\left[T^{\perp_G}_{[g,\Omega]_\omega,(g,\Omega)} \cap\ \mathbb{T}^J_{g,\Omega}\right]. \end{aligned}$$

Then the conclusion follows from the identity (1.14). □

Lemma 21. *Let E be a real Banach space, E^* its topological dual and $G : E \times E \longrightarrow \mathbb{R}$ be a topologically non degenerate bilinear form, i.e. $G : E \longrightarrow E^*$ is an isomorphism. If there exists a closed subspace $V \subset E$ such that the restriction $G : V \times V \longrightarrow \mathbb{R}$ is also topologically non degenerate then $E = V \oplus V^{\perp_G}$.*

Proof. Let $j : V \longhookrightarrow E$ be the canonical inclusion and notice the trivial identity

$$V^\perp := \{\alpha \in E^* \mid \alpha \cdot v = 0, \forall v \in V\} = \mathrm{Ker}\, j^*.$$

By assumption for any element $e \in E$ there exists a unique $v \in V$ such that $j^*(e\neg G) = j^*(v\neg G)$. Thus $(e-v)\neg G \in V^\perp$. By definition the restriction $G : V^{\perp_G} \longrightarrow V^\perp$ provides an isomorphism. We conclude $e - v \in V^{\perp_G}$. □

We notice that the condition $V \cap V^{\perp_G} = \{0\}$ is equivalent to $\mathrm{Ker}(G : V \longrightarrow V^*) = \{0\}$ but in general not sufficient to insure the surjectivity of $G : V \longrightarrow V^*$.

18.2 Triple splitting of the space $\mathbb{T}^J_{g,\Omega}$

By abuse of notations we will denote by $G_{g,\Omega}$ the scalar product over $\Lambda^{\Omega,\perp}_{g,J} \subset C^\infty$ induced by the isomorphism

$$\begin{aligned} \eta : \Lambda^{\Omega,\perp}_{g,J} \oplus \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) &\longrightarrow \mathbb{T}^J_{g,\Omega} \\ (\psi, A) &\longmapsto \left(g\left(\overline{\partial}_{T_{X,J}}\nabla_{g,J}\overline{\psi} + A\right), -\frac{1}{2}\,\mathrm{Re}\left[(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\right]\Omega\right). \end{aligned}$$

Explicitly

$$\begin{aligned} G_{g,\Omega}(\varphi,\psi) &= \frac{1}{2}\int_X \left[(\Delta^\Omega_{g,J} - 2\mathbb{I})\varphi \cdot \overline{\psi} + (\Delta^\Omega_{g,J} - 2\mathbb{I})\psi \cdot \overline{\varphi}\right]\Omega \\ &+ \frac{1}{2}\int_X \mathrm{Im}\left[(\Delta^\Omega_{g,J} - 2\mathbb{I})\varphi\right]\mathrm{Im}\left[(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi\right]\Omega. \end{aligned}$$

We introduce the vector space

$$\mathbb{E}^J_{g,\Omega} := \left\{u \in \Lambda^{\Omega,\perp}_{g,J} \mid (\Delta^\Omega_{g,J} - 2\mathbb{I})u \in \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X,\mathbb{R})_0\right\},$$

and we observe that the expression (16.3) for the space $\mathbb{F}^J_{g,\Omega}[0]$ shows that the map η restricts to the isomorphism

$$\eta : \mathbb{E}^J_{g,\Omega} \oplus \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \longrightarrow \mathbb{F}^J_{g,\Omega}[0].$$

The subspaces $\mathbb{E}^J_{g,\Omega}[0] := \eta\mathbb{E}^J_{g,\Omega}$ and $\mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \subset \mathbb{F}^J_{g,\Omega}[0]$ (embedded via the previous isomorphism) are G-orthogonal thanks to the expression of the restriction of G over $\mathbb{F}^J_{g,\Omega}[0]$ computed in the proof of lemma 20. We deduce the G-orthogonal decomposition

$$\mathbb{F}^J_{g,\Omega}[0] = \mathbb{E}^J_{g,\Omega}[0] \oplus_G \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}).$$

Let now

$$\mathbb{O}^J_{g,\Omega} := \left(\mathbb{E}^J_{g,\Omega}\right)^{\perp_G} \cap \Lambda^{\Omega,\perp}_{g,J},$$

and observe that the decomposition in corollary 6 implies

$$\begin{aligned} L^2 T_{[g,\Omega]_\omega,(g,\Omega)} &= \left[L^2\mathbb{F}^J_{g,\Omega}[0]\right]^{\perp_G} \cap L^2\mathbb{T}^J_{g,\Omega} \\ &= \mathbb{F}^J_{g,\Omega}[0]^{\perp_G} \cap L^2\mathbb{T}^J_{g,\Omega}, \end{aligned}$$

and thus

$$T_{[g,\Omega]_\omega,(g,\Omega)} = \mathbb{F}^J_{g,\Omega}[0]^{\perp_G} \cap \mathbb{T}^J_{g,\Omega}.$$

We deduce that the map η restricts to a G-isometry

$$\eta : \mathbb{O}^J_{g,\Omega} \longrightarrow T_{[g,\Omega]_\omega,(g,\Omega)}.$$

We show now the smooth G-orthogonal decomposition (1.16).

Proof. . The decomposition in the statement of corollary 6 implies that for any $(u, U) \in \mathbb{T}^J_{g,\Omega}$ there exists $(\theta, \Theta) \in L^2 T_{[g,\Omega]_\omega,(g,\Omega)}$ and $(v, V) \in L^2\mathbb{F}^J_{g,\Omega}[0]$ such that $(u, U) = (\theta, \Theta) + (v, V)$. On the other hand, the fact that the map (18.2) is a topological isomorphism implies the existence of $\rho \in \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \cap H^2(X, \mathbb{R})$ such that

$$\theta^\star_g = -2J\overline{\partial}_{T_{X,J}}\nabla_g\rho.$$

Thus $\Theta^\star_\Omega = -B^\Omega_{g,J}\rho \in H^1(X, \mathbb{R})$ and $V^\star_\Omega \in H^1(X, \mathbb{R})$. Then the identity $(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi_v = -2V^\star_\Omega$ implies $\psi_v \in \Lambda^{\Omega,\perp}_{g,J} \cap H^3(X, \mathbb{C})$ by elliptic regularity. We deduce $(v, V) \in H^1\mathbb{F}^J_{g,\Omega}[0]$ and thus $(\theta, \Theta) \in H^1 T_{[g,\Omega]_\omega,(g,\Omega)}$. The conclusion follws by induction.

We provide also a second argument. Combining formula (16.7) with a computation in the proof of corollary 5 we deduce the identity $I_{\psi_\theta} = 2(\Delta^\Omega_g - 2\mathbb{I})\rho$. On the other hand we observe that $\psi_\theta - 2i\rho \in \Lambda^\Omega_{g,J}$. In more explicit terms

$$(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi_\theta = 2B^\Omega_{g,J}\rho + 2i(\Delta^\Omega_g - 2\mathbb{I})\rho \in \Lambda^{\Omega,\perp}_{g,J} \cap L^2(X, \mathbb{C}).$$

We infer

$$(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi_\theta = \left[B^\Omega_{g,J}(\Delta^\Omega_g - 2\mathbb{I})^{-1} - i\mathbb{I}\right] I_{\psi_\theta}.$$

(Notice that $I_\psi \in \Lambda^{\Omega,\perp}_{g,\mathbb{R}}$ for any $\psi \in H^2(X, \mathbb{C})$ thanks to corollary 5). The fact that $I_{\psi_\theta} = I_{\psi_u}$ is smooth implies that

$$\psi_\theta = (\Delta^\Omega_{g,J} - 2\mathbb{I})^{-1}\left[B^\Omega_{g,J}(\Delta^\Omega_g - 2\mathbb{I})^{-1} - i\mathbb{I}\right] I_{\psi_u},$$

is also smooth. We infer the required smooth decomposition. □

We deduce the G-orthogonal triple splitting

$$\mathbb{T}^J_{g,\Omega} = T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G \mathbb{E}^J_{g,\Omega}[0] \oplus_G \mathcal{H}^{0,1}_{g,\Omega}\left(T_{X,J}\right). \tag{18.3}$$

We infer in particular the G-orthogonal decomposition

$$\Lambda^{\Omega,\perp}_{g,J} = \mathbb{O}^J_{g,\Omega} \oplus_G \mathbb{E}^J_{g,\Omega}, \tag{18.4}$$

thanks to the fact that the map η is a topological isomorphism. We observe now the following elementary lemma.

Lemma 22. *Let $T : D \subset L^2(X, \mathbb{C}) \longrightarrow L^2(X, \mathbb{C})$ be a closed densely defined L^2_Ω-self-adjoint operator such that $[T, \overline{T}] = 0$. Then*

$$\mathrm{Ker}(T\overline{T}) \cap L^2(X, \mathbb{R}) \quad = \quad \{\mathrm{Re}\, u \mid u \in \mathrm{Ker}\, T\}\,.$$

Proof. The assumption $[T, \overline{T}] = 0$ implies that the restriction

$$\overline{T} : \mathrm{Ker}\, T \longrightarrow \mathrm{Ker}\, T,$$

is well defined. This combined with the fact that $\overline{T}$ is also L^2_Ω-self-adjoint implies that the restriction

$$\overline{T} : D \cap (\mathrm{Ker}\, T)^\perp \quad \longrightarrow \quad (\mathrm{Ker}\, T)^\perp,$$

is also well defined. The inclusion $\mathrm{Ker}(T\overline{T}) \supseteq \mathrm{Ker}\, T + \mathrm{Ker}\, \overline{T}$ is obvious. In order to show the reverse inclusion let $u \in \mathrm{Ker}(T\overline{T})$, i.e. $\overline{T}u \in \mathrm{Ker}\, T$, and consider the decomposition $u = u_1 + u_2$ with $u_1 \in \mathrm{Ker}\, T$ and $u_2 \in (\mathrm{Ker}\, T)^\perp$. Then $\overline{T}u \in \mathrm{Ker}\, T$ if and only if $\overline{T}u_2 \in \mathrm{Ker}\, T$ since $\overline{T}u_1 \in \mathrm{Ker}\, T$. But $\overline{T}u_2 \in \mathrm{Ker}\, T$ if and only if $\overline{T}u_2 = 0$ since $\overline{T}u_2 \in (\mathrm{Ker}\, T)^\perp$. We infer the reverse inclusion. Thus

$$\mathrm{Ker}(T\overline{T}) \quad = \quad \{u + \overline{v} \mid u, v \in \mathrm{Ker}\, T\}\,,$$

which implies the required conclusion. □

We remind that if $(g, \Omega) \in \mathcal{S}_\omega$ is a Kähler-Ricci-Soliton with $J := g^{-1}\omega$ then

$$\left[\Delta^\Omega_{g,J} - 2\mathbb{I}, \overline{\Delta^\Omega_{g,J} - 2\mathbb{I}}\right] \quad = \quad 0,$$

which allows to apply the previous lemma to the L^2_Ω-self-adjoint operator $P^\Omega_{g,J}$. Thus

$$\mathrm{Ker}\, P^\Omega_{g,J} \cap C^\infty_\Omega(X, \mathbb{R})_0 \quad = \quad \left\{\mathrm{Re}\, u \mid u \in \Lambda^\Omega_{g,J}\right\} =: \mathrm{Re}\, \Lambda^\Omega_{g,J}.$$

The finiteness theorem for elliptic operators implies

$$P^\Omega_{g,J} C^\infty_\Omega(X, \mathbb{R})_0 \quad = \quad \left(\mathrm{Re}\, \Lambda^\Omega_{g,J}\right)^\perp \cap C^\infty_\Omega(X, \mathbb{R})_0 \supseteq \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X, \mathbb{R})_0\,.$$

The last inclusion is obvious. The inclusion $P^\Omega_{g,J} C^\infty_\Omega(X, \mathbb{R})_0 \subseteq \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X, \mathbb{R})_0$ is also obvious. We conclude

$$P^\Omega_{g,J} C^\infty_\Omega(X, \mathbb{R})_0 = \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X, \mathbb{R})_0 = \left(\mathrm{Re}\, \Lambda^\Omega_{g,J}\right)^\perp \cap C^\infty_\Omega(X, \mathbb{R})_0\,. \tag{18.5}$$

Lemma 23. *If $(g, \Omega) \in \mathcal{S}_\omega$ is a Kähler-Ricci-Soliton then holds the identity*

$$\mathbb{O}^J_{g,\Omega} \quad = \quad \left\{\psi \in \Lambda^{\Omega,\perp}_{g,J} \mid P^\Omega_{g,J}\, \mathrm{Re}\, \psi = 0\right\},$$

with $J := g^{-1}\omega$.

Proof. We notice that for any $\varphi \in \mathbb{E}^J_{g,\Omega}$ and $\psi \in \mathbb{O}^J_{g,\Omega}$ holds the identity

$$0 \quad = \quad G_{g,\Omega}(\varphi, \psi) = \int_X (\Delta^\Omega_{g,J} - 2\mathbb{I})\varphi \cdot \mathrm{Re}\, \psi \Omega.$$

We infer

$$\begin{aligned}\mathbb{O}^J_{g,\Omega} \quad &= \quad \left\{\psi \in \Lambda^{\Omega,\perp}_{g,J} \mid \mathrm{Re}\, \psi \in \left[\Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X, \mathbb{R})_0\right]^\perp\right\} \\ &= \quad \left\{\psi \in \Lambda^{\Omega,\perp}_{g,J} \mid \mathrm{Re}\, \psi \in \mathrm{Re}\, \Lambda^\Omega_{g,J}\right\},\end{aligned}$$

thanks to (18.5). □

19 Infinitesimal properties of the function H

By abuse of notations also we will consider from now on $\Lambda^{\Omega,\perp}_{g,\mathbb{R}} \subset C^\infty$. We observe that lemma 5 implies; $(g,\Omega)\in\mathcal{S}_\omega$ is a Kähler-Ricci-Soliton if and only if $\underline{H}_{g,\Omega}=0$. Furthermore the identity (4.18) rewrites as

$$2\underline{H}_{g,\Omega} = -(\Delta^\Omega_{g,J} - 2\mathbb{I})F \in \Lambda^{\Omega,\perp}_{g,J}\cap C^\infty_\Omega(X,\mathbb{R})_0\,,$$

for all $(g,\Omega)\in\mathcal{S}_\omega$. We show now the following fact.

Lemma 24. *If $(g,\Omega)\in\mathcal{S}_\omega$ is a Kähler-Ricci-Soliton then the linear map*

$$D_{g,\Omega}\underline{H} : \mathbb{E}^J_{g,\Omega}[0] \longrightarrow \Lambda^{\Omega,\perp}_{g,J}\cap C^\infty_\Omega(X,\mathbb{R})_0\,, \tag{19.1}$$

with $J := g^{-1}\omega$, is well defined and represents an isomorphism of real vector spaces.

Proof. The identity $2\underline{H}_{g,\Omega} = 2H_{g,\Omega} - \mathcal{W}(g,\Omega)$ combined with the basic variation formula (1.5) implies

$$2D_{g,\Omega}\underline{H}(v,V) = (\Delta^\Omega_g - 2\mathbb{I})V^\star_\Omega,$$

for all $(v,V)\in\mathbb{F}_{g,\Omega}$ over a shrinking Ricci soliton point (g,Ω). In our Kähler-Ricci soliton set up the latter rewrites as

$$2D_{g,\Omega}\underline{H}(v,V) = -\frac{1}{2}(\Delta^\Omega_g - 2\mathbb{I})(\Delta^\Omega_{g,J} - 2\mathbb{I})\psi_v, \tag{19.2}$$

for all $(v,V)\in\mathbb{F}^J_{g,\Omega}[0]$. The commutation identity

$$\left[\Delta^\Omega_g - 2\mathbb{I}\,,\Delta^\Omega_{g,J} - 2\mathbb{I}\right] = 0,$$

implies the inclusion

$$(\Delta^\Omega_g - 2\mathbb{I})\Lambda^\Omega_{g,J}\subseteq\Lambda^\Omega_{g,J}, \tag{19.3}$$

and thus

$$(\Delta^\Omega_g - 2\mathbb{I})\Lambda^{\Omega,\perp}_{g,J}\subseteq\Lambda^{\Omega,\perp}_{g,J}. \tag{19.4}$$

Then the identity (19.2) shows that the map (19.1) is well defined. We will deduce that it is an isomorphism if we show that the map

$$\Delta^\Omega_g - 2\mathbb{I} : \Lambda^{\Omega,\perp}_{g,J}\cap C^\infty_\Omega(X,\mathbb{R})_0 \longrightarrow \Lambda^{\Omega,\perp}_{g,J}\cap C^\infty_\Omega(X,\mathbb{R})_0\,, \tag{19.5}$$

is an isomorphism. Indeed this is the case. The injectivity of (19.5) follows from the inclusion

$$\mathbb{C}\Lambda^\Omega_{g,\mathbb{R}} \subseteq \Lambda^\Omega_{g,J},$$

which holds thanks to the identity (18.1). This inclusion implies also

$$\mathbb{C}\Lambda^{\Omega,\perp}_{g,\mathbb{R}} = (\mathbb{C}\Lambda^\Omega_{g,\mathbb{R}})^\perp \supseteq \Lambda^{\Omega,\perp}_{g,J}, \tag{19.6}$$

and thus

$$\Lambda^{\Omega,\perp}_{g,\mathbb{R}}\supseteq\Lambda^{\Omega,\perp}_{g,J}\cap C^\infty_\Omega(X,\mathbb{R})_0\,.$$

We use now the obvious fact that

$$\Delta^\Omega_g - 2\mathbb{I} : \Lambda^{\Omega,\perp}_{g,\mathbb{R}} \longrightarrow \Lambda^{\Omega,\perp}_{g,\mathbb{R}},$$

is an isomorphism. Thus for any $f\in\Lambda^{\Omega,\perp}_{g,J}\cap C^\infty_\Omega(X,\mathbb{R})_0$ there exists a unique $u\in\Lambda^{\Omega,\perp}_{g,\mathbb{R}}$ such that

$$(\Delta^\Omega_g - 2\mathbb{I})u = f.$$

We decompose $u = u_1 + u_2$, with $u_1 \in \Lambda^{\Omega}_{g,J}$ and $u_2 \in \Lambda^{\Omega,\perp}_{g,J}$. Then the inclusions (19.3) and (19.4) imply the L^2_{Ω}-orthogonal decomposition

$$(\Delta^{\Omega}_g - 2\mathbb{I})u_1 + (\Delta^{\Omega}_g - 2\mathbb{I})u_2 = f.$$

We deduce $u_1 \in \mathbb{C}\Lambda^{\Omega}_{g,\mathbb{R}}$. But $u_2 \in \mathbb{C}\Lambda^{\Omega,\perp}_{g,\mathbb{R}}$ thanks to the inclusion (19.6). We infer $u_1 = 0$ since $u \in \Lambda^{\Omega,\perp}_{g,\mathbb{R}}$. Thus

$$u = u_2 \in \Lambda^{\Omega,\perp}_{g,J} \cap C^{\infty}_{\Omega}(X, \mathbb{R})_0 .$$

We obtain the surjectivity of the map (19.5) and thus the required conclusion. □

Lemma 25. *If $(g, \Omega) \in \mathcal{S}_{\omega}$ is a Kähler-Ricci-Soliton then hold the identity*

$$\operatorname{Ker} D_{g,\Omega}\underline{H} \cap \mathbb{T}^J_{g,\Omega} = T_{[g,\Omega]_{\omega},(g,\Omega)} \oplus_G \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}), \tag{19.7}$$

with $J := g^{-1}\omega$.

Proof. With the notations in the proof of lemma 18, the basic variation formula (1.5) combined with the identities (16.6) and (16.7) implies that for all $(v, V) \in \mathbb{T}^J_{g,\Omega}$ over a Kähler-Ricci-Soliton point (J, g, Ω) hold the equalities

$$\begin{aligned}
2D_{g,\Omega}\underline{H}(v, V) &= -\frac{1}{2}(\Delta^{\Omega}_g - 2\mathbb{I})R_{\psi} + \frac{1}{2}\left(L_{J\nabla_g I_{\psi}}\Omega\right)^{*}_{\Omega} \\
&= -\frac{1}{2}(\Delta^{\Omega}_g - 2\mathbb{I})R_{\psi} + \frac{1}{2}B^{\Omega}_{g,J}I_{\psi} \\
&= -\frac{1}{2}\operatorname{Re}\left[(\Delta^{\Omega}_g - 2\mathbb{I})\overline{(\Delta^{\Omega}_{g,J} - 2\mathbb{I})\psi}\right] \\
&= -\frac{1}{2}\operatorname{Re}\left[P^{\Omega}_{g,J}\overline{\psi}\right] \\
&= -\frac{1}{2}P^{\Omega}_{g,J}\operatorname{Re}\psi,
\end{aligned}$$

since $P^{\Omega}_{g,J}$ is a real operator in our Kähler-Ricci-Soliton case. Then lemma 23 implies

$$\operatorname{Ker} D_{g,\Omega}\underline{H} \cap \mathbb{T}^J_{g,\Omega} \simeq \mathbb{O}^J_{g,\Omega} \oplus_G \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}),$$

i.e. the required conclusion. □

Proof of the main theorem 1

Proof. The inequality in the statement follows immediately from proposition 2. If equality holds then obviously $A \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J})_0$ and

$$\int_X P^{\Omega}_{g,J}\operatorname{Re}\psi \cdot \operatorname{Re}\psi \ \Omega = 0.$$

Then the spectral theorem applied to the non-negative L^2_{Ω}-self-adjoint real elliptic operator $P^{\Omega}_{g,J}$ implies $P^{\Omega}_{g,J}\operatorname{Re}\psi = 0$. Thus the conclusion

$$(v, V) \in T_{[g,\Omega]_{\omega},(g,\Omega)} \oplus_G \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J})_0 = \operatorname{Ker} D_{g,\Omega}\underline{H} \cap \mathbb{T}^{J,0}_{g,\Omega},$$

follows from lemma 23 and the identity (19.7). In order to show the inclusion (1.22) we observe that if $(g_t, \Omega_t)_{t\in\mathbb{R}} \subset \mathrm{KRS}_{\omega}$ is a smooth curve with $(g_0, \Omega_0) = (g, \Omega)$ and with $(\dot{g}_0, \dot{\Omega}_0) = (v, V)$ then holds the identity $\underline{H}_{g_t,\Omega_t} \equiv 0$ and thus

$$(v, V) \in \operatorname{Ker} D_{g,\Omega}\underline{H} \cap \mathbb{T}^J_{g,\Omega} = T_{[g,\Omega]_{\omega},(g,\Omega)} \oplus_G \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}),$$

thanks to the identity (19.7). On the other side if we set $\mathcal{W}_t := \mathcal{W}(g_t, \Omega_t)$ then $\dot{\mathcal{W}}_t \equiv 0$ and thus

$$0 = \ddot{\mathcal{W}}_0 = \int_X |A|_g^2 F \Omega,$$

thanks to proposition 2 and lemma 23. We conclude the required inclusion. □

20 Appendix A

20.1 The first variation of Perelman's $\mathcal{W}$ functional

We give a short proof of Perelman's first variation formula [29] for the $\mathcal{W}$ functional based on the identity (3.2). Let $(g_t, \Omega_t)_t \subset \mathcal{M} \times \mathcal{V}_1$ be a smooth family and set $f_t := \log \frac{dV_{g_t}}{\Omega_t}$. Then

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{W}(g_t, \Omega_t) &= \frac{d}{dt}\int_X \left[\mathrm{Tr}_{\mathbb{R}}(g_t^{-1}h_t) \quad + \quad 2f_t\right]\Omega_t \\
&= \int_X \left[\mathrm{Tr}_{\mathbb{R}}\left(-\dot{g}_t^{\star}h_t^{\star} \quad + \quad \dot{h}_t^{\star}\right) \quad + \quad \mathrm{Tr}_{g_t}\dot{g} - 2\dot{\Omega}_t^{\star}\right]\Omega_t \\
&+ \int_X \left[\mathrm{Tr}_{g_t} h_t \quad + \quad 2f_t\right]\dot{\Omega}_t \\
&= \int_X \left[-\langle \dot{g}_t,\ h_t\rangle_{g_t} + \left\langle g_t, \frac{d}{dt}\ \mathrm{Ric}_{g_t}(\Omega_t)\right\rangle_{g_t} - 2\dot{\Omega}_t^{\star}\right]\Omega_t \\
&+ \int_X \left[\mathrm{Tr}_{g_t} h_t \quad + \quad 2f_t\right]\dot{\Omega}_t.
\end{aligned}$$

Using the variation formula (3.2) and integrating by parts we infer

$$\begin{aligned}
\int_X \left\langle g_t, \frac{d}{dt}\ \mathrm{Ric}_{g_t}(\Omega_t)\right\rangle_{g_t}\Omega_t &= \int_X \left[-\ \frac{1}{2}\left\langle g_t, \nabla_{g_t}^{\star_\Omega}\mathcal{D}_{g_t}\dot{g}_t\right\rangle_{g_t} + \Delta_{g_t}\dot{\Omega}_t^{\star}\right]\Omega_t \\
&= -\int_X \left[\frac{1}{2}\left\langle \nabla g_t g_t, \mathcal{D}_{g_t}\dot{g}_t\right\rangle_{g_t} + \left\langle \nabla_{g_t}\dot{\Omega}_t^{\star}, \nabla_{g_t} f_t\right\rangle_{g_t}\right]\Omega_t \\
&= -\int_X \dot{\Omega}_t^{\star}\Delta_{g_t}^{\Omega_t} f_t \Omega_t,
\end{aligned}$$

which implies Perelman's first variation formula

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{W}(g_t, \Omega_t) &= -\int_X \left[\langle \dot{g}_t, h_t\rangle_{g_t} - 2\dot{\Omega}_t^{\star}(H_t - 1)\right]\Omega_t \\
&= -\int_X \left[\langle \dot{g}_t, h_t\rangle_{g_t} - 2\dot{\Omega}_t^{\star}\underline{H}_t\right]\Omega_t,
\end{aligned}$$

since $\int_X \dot{\Omega}_t = 0$.

20.2 Basic complex identities

We provide a useful expression of the hermitian product $\langle \cdot, \cdot\rangle_\omega$ on $T_X^{\star} \otimes_{\mathbb{R}} \mathbb{C}$, which is the sesquilinear extension of the dual of g. We observe first that for any $\xi \in T_X \otimes_{\mathbb{R}} \mathbb{C}$ and any $\alpha \in T_X^{\star} \otimes_{\mathbb{R}} \mathbb{C}$ hold the elementary

equalities

$$0 = \xi\neg(\alpha\wedge\omega^n) = (\alpha\cdot\xi)\omega^n - \alpha\wedge(\xi\neg\omega^n).$$

We obtain the formula

$$(\alpha\cdot\xi)\omega^n = n\alpha\wedge(\xi\neg\omega)\wedge\omega^{n-1}, \tag{20.1}$$

and thus

$$2\alpha\cdot\xi = \mathrm{Tr}_\omega\left[\alpha\wedge(\xi\neg\omega)\right]. \tag{20.2}$$

Using (20.2) we deduce that for all $\alpha,\beta\in T^\star_X\otimes_{\mathbb{R}}\mathbb{C}$ hold the equalities

$$2\langle\alpha,\beta\rangle_\omega = 2\alpha\cdot\bar\beta^\star_g = \mathrm{Tr}_\omega\left[\alpha\wedge(\bar\beta^\star_g\neg\omega)\right],$$

where $\bar\beta^\star_g := g^{-1}\bar\beta$. Then the identity $\bar\beta J = -\bar\beta^\star_g\neg\omega$ implies the formula

$$2\langle\alpha,\beta\rangle_\omega = -\mathrm{Tr}_\omega\left[\alpha\wedge(\bar\beta J)\right].$$

Thus in the case $\alpha,\beta\in\Lambda^{1,0}_J T^\star_X$ we deduce the identities

$$2\langle\alpha,\beta\rangle_\omega = \mathrm{Tr}_\omega\left(i\alpha\wedge\bar\beta\right),$$

$$\overline{\langle\alpha,\beta\rangle_\omega} = \langle\overline{\alpha},\bar\beta\rangle_\omega.$$

We show now the following integration by parts formulas

Lemma 26. *For any $u,v\in C^\infty(X,\mathbb{C})$ holds the integration by parts identity*

$$\int_X\left[\Delta^\Omega_{g,J}u\cdot\overline{v}+\overline{\Delta^\Omega_{g,J}u}\cdot v\right]\Omega = 2\int_X g(\nabla_{g,J}\overline{u},\nabla_{g,J}\overline{v})\Omega. \tag{20.3}$$

If $u\in C^\infty(X,\mathbb{R})$ then holds also the integration by parts identity

$$\int_X\left[\Delta^\Omega_{g,J}u\cdot v+\overline{\Delta^\Omega_{g,J}u}\cdot\overline{v}\right]\Omega = 2\int_X g(\nabla_g u,\nabla_{g,J}v)\Omega. \tag{20.4}$$

Proof. Using the complex decomposition (13.2) and the fact that hermitian product $\langle\cdot,\cdot\rangle_\omega$ on $T^\star_X\otimes_{\mathbb{R}}\mathbb{C}$ is the sesquilinear extension of the dual of g, we deduce

$$\begin{aligned} g(\nabla_{g,J}u,\nabla_{g,J}v) &= \left\langle\partial_J\overline{u}+\overline{\partial}_J u,\partial_J\overline{v}+\overline{\partial}_J v\right\rangle_g \\ &= \left\langle\partial_J\overline{u}+\overline{\partial}_J u,\partial_J\overline{v}+\overline{\partial}_J v\right\rangle_\omega \\ &= \left\langle\partial_J\overline{u},\partial_J\overline{v}\right\rangle_\omega+\left\langle\overline{\partial}_J u,\overline{\partial}_J v\right\rangle_\omega \\ &= \left\langle\partial_J\overline{u},\partial_J\overline{v}\right\rangle_\omega+\overline{\left\langle\partial_J\overline{u},\partial_J\overline{v}\right\rangle_\omega}. \end{aligned}$$

Integrating by parts and taking the conjugate we infer the identity

$$2\int_X g(\nabla_{g,J}u,\nabla_{g,J}v)\Omega = \int_X\left[\Delta^\Omega_{g,J}\overline{u}\cdot v+\overline{\Delta^\Omega_{g,J}\overline{u}}\cdot\overline{v}\right]\Omega. \tag{20.5}$$

Replacing u with $\overline{u}$, v with $\overline{v}$ in (20.5) we obtain (20.3). In the case $u\in C^\infty(X,\mathbb{R})$ formula (20.5) implies directly (20.4). □

We show now that for all $\alpha, \beta \in \Lambda^{1,1}_J T^\star_X \cap \Lambda^2_{\mathbb{R}} T^\star_X$ holds the identity

$$4n(n-1)\frac{\alpha \wedge \beta \wedge \omega^{n-2}}{\omega^n} = \mathrm{Tr}_\omega\, \alpha\, \mathrm{Tr}_\omega\, \beta - 2\,\langle \alpha, \beta \rangle_\omega\,. \tag{20.6}$$

Indeed we consider the local expressions

$$\omega = \frac{i}{2}\sum_k \zeta^\star_k \wedge \bar\zeta^\star_k, \qquad \alpha = i\sum_{k,l} \alpha_{k\bar l}\, \zeta^\star_k \wedge \bar\zeta^\star_l, \qquad \beta = i\sum_{k,l} \beta_{k\bar l}\, \zeta^\star_k \wedge \bar\zeta^\star_l,$$

and we set

$$\begin{aligned} \Psi := \alpha \wedge \beta &= \sum_{k_1,k_2,l_1,l_2} \alpha_{k_1\bar l_1}\beta_{k_2\bar l_2}\, \zeta^\star_{k_1} \wedge \zeta^\star_{k_2} \wedge \bar\zeta^\star_{l_1} \wedge \bar\zeta^\star_{l_2} \\ &= \sum_{|K|=|L|=2} \Psi_{K,L}\, \zeta^\star_K \wedge \bar\zeta^\star_L, \end{aligned}$$

where $K = (k_1, k_2)$, $1 \le k_1 < k_2 \le n$ and the same holds for L. Explicitly the coefficients $\Psi_{K,L}$ are given by the expression

$$\Psi_{K,L} = \alpha_{k_1\bar l_1}\beta_{k_2\bar l_2} + \alpha_{k_2\bar l_2}\beta_{k_1\bar l_1} - \alpha_{k_1\bar l_2}\beta_{k_2\bar l_1} - \alpha_{k_2\bar l_1}\beta_{k_1\bar l_2}.$$

We conclude the identity

$$\begin{aligned} 4n(n-1)\frac{\Psi \wedge \omega^{n-2}}{\omega^n} &= 16\sum_{|L|=2} \Psi_{L,L} = 16\sum_{k,l} \alpha_{k\bar k}\beta_{l\bar l} - 16\sum_{k,l} \alpha_{k\bar l}\beta_{l\bar k} \\ &= \mathrm{Tr}_\omega\, \alpha\, \mathrm{Tr}_\omega\, \beta - 2\,\langle \alpha, \beta \rangle_\omega\,. \end{aligned}$$

20.3 Action of the curvature on alternating 2-forms

We observe that as in the symmetric case we can define an action of the curvature operator over alternating 2-forms as follows

$$(\mathcal{R}_g \star \alpha)(\xi, \eta) := -\mathrm{Tr}_g\left[\alpha\left(\mathcal{R}_g(\xi, \cdot)\eta, \cdot\right)\right],$$

for any $\alpha \in \Lambda^2 T^\star_X$. The tensor $\mathcal{R}_g \star \alpha$ is anti-symmetric. In fact let $(e_k)_k$ be a $g(x)$-orthonormal base of $T_{X,x}$ and consider the local expression $\alpha^\star_g = A_{l,k} e^\star_k \otimes e_l$, with $A_{l,k} = -A_{k,l}$. Then

$$\begin{aligned} (\mathcal{R}_g \star \alpha)(\xi, \eta) &= -g\left(\alpha^\star_g \mathcal{R}_g(\xi, e_k)\eta, e_k\right) \\ &= g\left(\mathcal{R}_g(\xi, e_k)\eta, \alpha^\star_g e_k\right) \\ &= R_g(\xi, e_k, \alpha^\star_g e_k, \eta) \\ &= R_g(\xi, e_k, A_{l,k} e_l, \eta) \\ &= -R_g(\xi, A_{k,l} e_k, e_l, \eta) \\ &= -R_g(\xi, \alpha^\star_g e_l, e_l, \eta) \\ &= -R_g(\eta, e_l, \alpha^\star_g e_l, \xi) \\ &= -(\mathcal{R}_g \star \alpha)(\eta, \xi), \end{aligned}$$

thanks to the symmetry properties of the curvature form. We observe also that the previous computation shows the identity

$$\begin{aligned}(\mathcal{R}_g \star \alpha)(\xi,\eta) &= -R_g(\xi, e_k, \eta, \alpha_g^\star e_k)\\ &= -g(\mathcal{R}_g(\xi, e_k)\alpha_g^\star e_k, \eta)\\ &= -g\left(\left(\mathcal{R}_g \star \alpha_g^\star\right)\xi, \eta\right),\end{aligned}$$

i.e.

$$(\mathcal{R}_g \star \alpha)_g^\star = -\mathcal{R}_g \star \alpha_g^\star. \tag{20.7}$$

On the other hand using the algebraic Bianchi identity we obtain the equalities

$$\begin{aligned}(\mathcal{R}_g \star \alpha)(\xi,\eta) &= -g\left(\alpha_g^\star \mathcal{R}_g(\xi, e_k)\eta, e_k\right)\\ &= g\left(\alpha_g^\star \mathcal{R}_g(e_k,\eta)\xi, e_k\right) + g\left(\alpha_g^\star \mathcal{R}_g(\eta,\xi)e_k, e_k\right)\\ &= -g\left(\alpha_g^\star \mathcal{R}_g(\eta, e_k)\xi, e_k\right) - g\left(\mathcal{R}_g(\eta,\xi)e_k, \alpha_g^\star e_k\right)\\ &= (\mathcal{R}_g \star \alpha)(\eta,\xi) + g\left(\mathcal{R}_g(\eta,\xi)\alpha_g^\star e_k, e_k\right)\\ &= -(\mathcal{R}_g \star \alpha)(\xi,\eta) - \mathrm{Tr}_{\mathbb{R}}\left[\mathcal{R}_g(\xi,\eta)\alpha_g^\star\right],\end{aligned}$$

and thus the formula

$$(\mathcal{R}_g \star \alpha)(\xi,\eta) = -\frac{1}{2}\,\mathrm{Tr}_{\mathbb{R}}\left[\mathcal{R}_g(\xi,\eta)\alpha_g^\star\right]. \tag{20.8}$$

We assume further that (X, J, g) is Kähler and α is J-anti-invariant. In this case $\alpha_g^\star = A$ is J-anti-linear and so is the endomorphism $\mathcal{R}_g(\xi,\eta)\alpha_g^\star$. We deduce

$$\mathcal{R}_g \star \alpha = 0, \quad \text{i.e.} \quad \mathcal{R}_g \star A = 0. \tag{20.9}$$

thanks to the identity (20.7).

20.4 Weighted Weitzenböck formula for alternating 2-forms

We show the weighted Weitzenböck type formula

$$\Delta_{d,g}^{\Omega}\alpha = \Delta_g^{\Omega}\alpha + 2\mathcal{R}_g \star \alpha + \alpha\,\mathrm{Ric}_g^\star(\Omega) + \mathrm{Ric}_g(\Omega)\alpha_g^\star, \tag{20.10}$$

for any alternating 2-form α over a Riemannian manifold. For this purpose we fix an arbitrary point x_0 and we choose the vector fields ξ and η such that $0 = \nabla_g\xi(x_0) = \nabla_g\eta(x_0)$. Let $(e_k)_k$ be a g-orthonormal local frame such that $\nabla_g e_k(x_0) = 0$. Then at the point x_0 holds the identities

$$\begin{aligned}d\nabla_g^\star\alpha(\xi,\eta) &= \nabla_{g,\xi}\nabla_g^\star\alpha\cdot\eta - \nabla_{g,\eta}\nabla_g^\star\alpha\cdot\xi\\ &= \nabla_{g,\xi}\left[\nabla_g^\star\alpha\cdot\eta\right] - \nabla_{g,\eta}\left[\nabla_g^\star\alpha\cdot\xi\right]\\ &= -\nabla_{g,\xi}\left[\nabla_{g,e_k}\alpha(e_k,\eta)\right] + \nabla_{g,\eta}\left[\nabla_{g,e_k}\alpha(e_k,\xi)\right]\\ &= -\nabla_{g,\xi}\nabla_{g,e_k}\alpha(e_k,\eta) + \nabla_{g,\eta}\nabla_{g,e_k}\alpha(e_k,\xi),\end{aligned}$$

and

$$\begin{aligned}
\nabla_g^{\star} d\alpha(\xi,\eta) &= -\nabla_{g,e_k} d\alpha(e_k,\xi,\eta)\\
&= -\nabla_{g,e_k}\left[d\alpha(e_k,\xi,\eta)\right]\\
&= -\nabla_{g,e_k}\left[\nabla_{g,e_k}\alpha(\xi,\eta)-\nabla_{g,\xi}\alpha(e_k,\eta)+\nabla_{g,\eta}\alpha(e_k,\xi)\right]\\
&= -\nabla_{g,e_k}\nabla_{g,e_k}\alpha(\xi,\eta)+\nabla_{g,e_k}\nabla_{g,\xi}\alpha(e_k,\eta)-\nabla_{g,e_k}\nabla_{g,\eta}\alpha(e_k,\xi).
\end{aligned}$$

We remind now that for any vector fields μ,ζ such that $[\mu,\zeta](x_0)=0$ holds the identity at the point x_0

$$\nabla_{g,\mu}\nabla_{g,\zeta}\alpha-\nabla_{g,\zeta}\nabla_{g,\mu}\alpha = \mathcal{R}_g(\zeta,\mu)\neg\alpha,$$

where the contraction operation $T\neg:\Lambda^2T_X^{\star}\longrightarrow\Lambda^2T_X^{\star}$ associated to an endomorphism $T\in\mathrm{End}(T_X)$ is defined by the formula

$$(T\neg\alpha)(\xi,\eta) := \alpha(T\xi,\eta)+\alpha(\xi,T\eta).$$

We deduce

$$\begin{aligned}
\left(\nabla_{g,e_k}\nabla_{g,\xi}\alpha-\nabla_{g,\xi}\nabla_{g,e_k}\right)(e_k,\eta) &= \alpha\left(\mathcal{R}_g(\xi,e_k)e_k,\eta\right)+\alpha\left(e_k,\mathcal{R}_g(\xi,e_k)\eta\right)\\
&= \left[\alpha\,\mathrm{Ric}^{\star}(g)+(\mathcal{R}_g\star\alpha)\right](\xi,\eta),
\end{aligned}$$

and also

$$\begin{aligned}
(\nabla_{g,\eta}\nabla_{g,e_k}\alpha-\nabla_{g,e_k}\nabla_{g,\eta}\alpha)(e_k,\xi) &= -\left[\alpha\,\mathrm{Ric}^{\star}(g)+(\mathcal{R}_g\star\alpha)\right](\eta,\xi)\\
&= -g\left(\alpha_g^{\star}\,\mathrm{Ric}^{\star}(g)\eta,\xi\right)+(\mathcal{R}_g\star\alpha)(\xi,\eta)\\
&= g(\eta,\mathrm{Ric}^{\star}(g)\alpha_g^{\star}\xi)+(\mathcal{R}_g\star\alpha)(\xi,\eta)\\
&= \left[\mathrm{Ric}(g)\alpha_g^{\star}+(\mathcal{R}_g\star\alpha)\right](\xi,\eta).
\end{aligned}$$

Summing up the terms $d\nabla_g^{\star}\alpha(\xi,\eta)$ and $\nabla_g^{\star}d\alpha(\xi,\eta)$ and using these last identities we infer the formula (20.10) in the case $\Omega=CdV_g$. In order to obtain the general case we observe the decompositions

$$\begin{aligned}
d\nabla_g^{\star_\Omega}\alpha &= d\nabla_g^{\star}\alpha+d(\nabla_g f\neg\alpha),\\
\nabla_g^{\star_\Omega}d\alpha &= \nabla_g^{\star}d\alpha+\nabla_g f\neg d\alpha,
\end{aligned}$$

and the identities at the point x_0,

$$\begin{aligned}
d(\nabla_g f\neg\alpha)(\xi,\eta) &= \nabla_{g,\xi}(\nabla_g f\neg\alpha)\cdot\eta-\nabla_{g,\eta}(\nabla_g f\neg\alpha)\cdot\xi\\
&= \nabla_{g,\xi}\left[\alpha(\nabla_g f,\eta)\right]-\nabla_{g,\eta}\left[\alpha(\nabla_g f,\xi)\right]\\
&= \nabla_{g,\xi}\alpha(\nabla_g f,\eta)+\alpha(\nabla_{g,\xi}^2 f,\eta)
\end{aligned}$$

$$
\begin{aligned}
&- \nabla_{g,\eta}\alpha(\nabla_g f,\xi) - \alpha(\nabla^2_{g,\eta}f,\xi)\\
&= \nabla_{g,\xi}\alpha(\nabla_g f,\eta) + \alpha(\nabla^2_{g,\xi}f,\eta)\\
&- \nabla_{g,\eta}\alpha(\nabla_g f,\xi) - \nabla_g df\alpha^{\star}_g(\xi,\eta),\\
(\nabla_g f\neg d\alpha)(\xi,\eta) &= (\nabla_g f\neg\nabla_g\alpha)(\xi,\eta) - \nabla_{g,\xi}\alpha(\nabla_g f,\eta) + \nabla_{g,\eta}\alpha(\nabla_g f,\xi).
\end{aligned}
$$

Summing up we infer the required formula (20.10).

21 Appendix B

21.1 Reformulation of the weighted complex Bochner identity (13.9)

We define the complex operator

$$\Delta^{\Omega}_{g,-J} := \overline{\Delta^{\Omega}_{g,J}}.$$

With this notation the weighted complex Bochner type identity (13.9) rewrites also as

$$2\Delta^{\Omega,-J}_{T_{X,g}}\nabla_{g,J}u = \nabla_{g,J}(\Delta^{\Omega}_{g,-J} - 2\mathbb{I})u,$$

for all $u\in C^{\infty}(X,\mathbb{C})$. We show now that the fundamental identity (13.9) implies an other important formula. We need a few preliminaries.

Lemma 27. *For any $u,v\in C^{\infty}(X,\mathbb{C})$ holds the integration by parts identity*

$$\int_X \Delta^{\Omega}_{g,-J}u\cdot\overline{v}\Omega = 2\int_X \left\langle\nabla_{g,J}u,\nabla_{g,J}v\right\rangle_{\omega}\Omega.$$

Proof. We define the complex components of the g-gradient as

$$\nabla^{1,0}_{g,J}u := (\nabla_g u)^{1,0}_J \in C^{\infty}(X,T^{1,0}_{X,J}),$$

$$\nabla^{0,1}_{g,J}u := (\nabla_g u)^{0,1}_J \in C^{\infty}(X,T^{0,1}_{X,J}).$$

With these notations holds the decomposition formula

$$\nabla_{g,J}u = \nabla^{1,0}_{g,J}u + \nabla^{0,1}_{g,J}\overline{u}. \tag{21.1}$$

We observe that for all $\xi,\eta\in T_X$ holds the identity

$$\langle\xi,\eta\rangle_{\omega}\equiv h(\xi,\eta) = 2i\omega(\eta^{0,1}_J,\xi^{1,0}_J).$$

This combined with (21.1) implies

$$\left\langle\nabla_{g,J}u,\nabla_{g,J}v\right\rangle_{\omega} = 2i\omega(\nabla^{0,1}_{g,J}\overline{v},\nabla^{1,0}_{g,J}u).$$

We observe now that the complex spiting of the g-gradient

$$\nabla_g u = \nabla^{1,0}_{g,J}u + \nabla^{0,1}_{g,J}u$$

implies the identities

$$\nabla^{1,0}_{g,J} u \neg \omega = i\overline{\partial}_J u,$$

$$\nabla^{0,1}_{g,J} u \neg \omega = -i\partial_J u.$$

Using this and the identity (20.2) we deduce

$$\begin{aligned}\left\langle \nabla_{g,J} u, \nabla_{g,J} v \right\rangle_\omega &= 2\partial_J \overline{v} \cdot \nabla^{1,0}_{g,J} u \\ &= \mathrm{Tr}_\omega \left[\partial_J \overline{v} \wedge (\nabla^{1,0}_{g,J} u \neg \omega) \right] \\ &= \mathrm{Tr}_\omega \left[i\partial_J \overline{v} \wedge \overline{\partial}_J u \right] \\ &= 2 \left\langle \partial_J \overline{v}, \partial_J \overline{u} \right\rangle_\omega .\end{aligned}$$

We infer the equalities

$$\begin{aligned}\int_X \left\langle \nabla_{g,J} u, \nabla_{g,J} v \right\rangle_\omega \Omega &= 2\overline{\int_X \left\langle \partial_J \overline{u}, \partial_J \overline{v} \right\rangle_\omega \Omega} \\ &= \overline{\int_X \Delta^\Omega_{g,J} \overline{u} \cdot v\Omega} \\ &= \int_X \Delta^\Omega_{g,-J} u \cdot \overline{v}\Omega.\end{aligned}$$

□

We equip $C^\infty_\Omega(X, \mathbb{C})_0$ with the L^2_Ω-product (13.1) and the space $C^\infty(X, \Lambda^{0,1}_J T^\star_X \otimes_{\mathbb{C}} T_{X,J})$ with the $L^2_{\omega,\Omega}$-hermitian product (11.1). Then the formal adjoint of $\mathcal{H}^{0,1}_{g,J}$ with respect to such products

$$(\mathcal{H}^{0,1}_{g,J})^{\star_{\omega,\Omega}} : C^\infty(X, \Lambda^{0,1}_J T^\star_X \otimes_{\mathbb{C}} T_{X,J}) \longrightarrow C^\infty_\Omega(X, \mathbb{C})_0,$$

$$\int_X \left\langle \mathcal{H}^{0,1}_{g,J} u, A \right\rangle_\omega \Omega = \int_X u \cdot \overline{(\mathcal{H}^{0,1}_{g,J})^{\star_{\omega,\Omega}} A}\Omega,$$

satisfies the identity

$$(\mathcal{H}^{0,1}_{g,J})^{\star_{\omega,\Omega}} = \nabla^{\star_{\omega,\Omega}}_{g,J} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}.$$

Moreover lemma implies the identity

$$\Delta^\Omega_{g,-J} = \nabla^{\star_{\omega,\Omega}}_{g,J} \nabla_{g,J}.$$

Then the complex Bochner type identity (13.9) implies

$$\begin{aligned}2(\mathcal{H}^{0,1}_{g,J})^{\star_{\omega,\Omega}} \mathcal{H}^{0,1}_{g,J} \overline{v} &= 2\nabla^{\star_{\omega,\Omega}}_{g,J} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{v} \\ &= \overline{\Delta^\Omega_{g,J}(\Delta^\Omega_{g,J} - 2\mathbb{I})v},\end{aligned}$$

or in other terms

$$2(\mathcal{H}^{0,1}_{g,J})^{\star_{\omega,\Omega}} \mathcal{H}^{0,1}_{g,J} u = \Delta^\Omega_{g,-J}(\Delta^\Omega_{g,-J} - 2\mathbb{I})u,$$

for all $u \in C^\infty(X, \mathbb{C})$.

21.2 A Poisson structure on the first eigenspace of $\Delta^{\Omega}_{g,J}$

For any complex valued function u and symplectic form ω we define the complex vector field

$$(du)^{\star}_{\omega} \quad := \quad \omega^{-1}du = -J\nabla_g u,$$

and the Poisson bracket

$$\{u, v\}_{\omega} \quad := \quad dv \cdot (du)^{\star}_{\omega} = -\omega((du)^{\star}_{\omega}, (dv)^{\star}_{\omega}).$$

We define the Poisson bracket over the space $C^{\infty}_{\Omega}(X, \mathbb{C})$ as

$$\{u, v\}_{\omega,\Omega} \quad := \quad \{u, v\}_{\omega} - \int_X \{u, v\}_{\omega}\, \Omega.$$

With these notations holds the following lemma (see also [12], [14]).

Lemma 28. *Let (X, J) be a Fano manifold and let g be a J-invariant Kähler metric such that $\omega := gJ \in 2\pi c_1(X, [J])$. Let also $\Omega > 0$ be the unique smooth volume form with $\int_X \Omega = 1$ such that $\mathrm{Ric}_J(\Omega) = \omega$.*

A) Then the map

$$\chi : \left(\overline{\mathrm{Ker}(\Delta^{\Omega}_{g,J} - 2\mathbb{I})}, i\{\cdot,\cdot\}_{\omega,\Omega}\right) \quad \longrightarrow \quad \left(H^0(X, T_{X,J}), [\cdot,\cdot]\right)$$

$$u \quad \longmapsto \quad \nabla_{g,J} u,$$

is well defined and it represents an isomorphism of complex lie algebras.

B) The first eigenvalue $\lambda_1(\Delta^{\Omega}_{g,J})$ of the operator $\Delta^{\Omega}_{g,J}$ satisfies the estimate $\lambda_1(\Delta^{\Omega}_{g,J}) \geqslant 2$, with equality in the case $H^0(X, T_{X,J}) \neq 0$.

C) If we set $\mathrm{Kill}_g := \mathrm{Lie}(\mathrm{Isom}^0_g)$ then the map

$$J\nabla_g : \mathrm{Ker}_{\mathbb{R}}(\Delta^{\Omega}_g - 2\mathbb{I}) \longrightarrow \mathrm{Kill}_g, \tag{21.2}$$

is well defined and it represents an isomorphism of real vector spaces.

D) The hermitian form

$$(u, v) \quad \longmapsto \quad \int_X i\{u, \bar{v}\}_{\omega}\, \Omega,$$

over $\overline{\mathrm{Ker}(\Delta^{\Omega}_{g,J} - 2\mathbb{I})}$ is non-negative and let $(\mu_j)_{j=0}^{N} \subset \mathbb{R}_{\geqslant 0}$, $\mu_0 = 0$, be its spectrum with respect to the L^2_{Ω}-product. If g is a J-invariant Kähler-Ricci soliton then holds the decomposition

$$H^0(X, T_{X,J}) \quad = \quad \bigoplus_{j=0}^{N} V_{\mu_j},$$

$$V_{\mu_j} \quad := \quad \left\{\xi \in H^0(X, T_{X,J}) \mid [\nabla_g f, \xi] = \mu_j \xi\right\},$$

$$V_0 \quad = \quad \mathrm{Kill}_g \oplus J\,\mathrm{Kill}_g\,.$$

Proof. **Step A.** In this step we show the statement **A.** The fact that χ is an isomorphism follows from corollary 1. We show now that χ is also a morphism of complex Lie algebras. Let

$$\mathbb{K}_{\pm} \quad := \quad \mathrm{Ker}(\Delta^{\Omega}_{g,\pm J} - 2\mathbb{I}).$$

For any $\xi \in H^0(X, T_{X,J})$ we denote $u^\xi := \chi^{-1}(\xi) \in \mathbb{K}_-$ and we decompose $u^\xi = u_1^\xi + iu_2^\xi$, with $u_j^\xi \in C^\infty_\Omega(X, \mathbb{R})_0$. For any $u, v \in \mathbb{K}_- = \overline{\mathbb{K}_+}$ we set $\xi := \nabla_{g,J} u$, $\eta := \nabla_{g,J} v$ and as in [14] we observe the identities

$$\begin{aligned} L_{[\xi,\eta]}\omega &= L_\xi L_\eta \omega - L_\eta L_\xi \omega \\ &= 2L_\xi i\partial_J\overline{\partial}_J u_1^\eta - 2L_\eta i\partial_J\overline{\partial}_J u_1^\xi \\ &= 2i\partial_J\overline{\partial}_J\left(\xi.u_1^\eta - \eta.u_1^\xi\right), \end{aligned}$$

since ξ, η are holomorphic. We infer that for some constant $C_1 \in \mathbb{R}$ holds the identities

$$\begin{aligned} u_1^{[\xi,\eta]} + C_1 &= \xi.u_1^\eta - \eta.u_1^\xi \\ &= \xi.v_1 - \eta.u_1 \\ &= g(\nabla_g v_1, \nabla_g u_1 + J\nabla_g u_2) \\ &- g(\nabla_g u_1, \nabla_g v_1 + J\nabla_g v_2) \\ &= g(\nabla_g v_1, J\nabla_g u_2) - g(\nabla_g u_1, J\nabla_g v_2) \\ &= \omega(\nabla_g u_2, \nabla_g v_1) - \omega(\nabla_g v_2, \nabla_g u_1) \\ &= \omega(J\nabla_g u_2, J\nabla_g v_1) + \omega(J\nabla_g u_1, J\nabla_g v_2) \\ &= \omega\left((du_2)^\star_\omega, (dv_1)^\star_\omega\right) + \omega((du_1)^\star_\omega, (dv_2)^\star_\omega) \\ &= -\{u_1, v_2\}_\omega - \{u_2, v_1\}_\omega . \end{aligned}$$

On the other hand

$$u_2^{[\xi,\eta]} = -u_1^{J[\xi,\eta]} = -u_1^{[\xi,J\eta]},$$

since ξ is holomorphic. We infer that for some constant $C_2 \in \mathbb{R}$ holds the identities

$$\begin{aligned} u_2^{[\xi,\eta]} + C_2 &= -\xi.u_1^{J\eta} + J\eta.u_1^\xi \\ &= \xi.u_2^\eta + J\eta.u_1^\xi \\ &= \xi.v_2 + J\eta.u_1 \\ &= g(\nabla_g v_2, \nabla_g u_1 + J\nabla_g u_2) \\ &+ g(\nabla_g u_1, J\nabla_g v_1 - \nabla_g v_2) \\ &= g(\nabla_g v_2, J\nabla_g u_2) + g(\nabla_g u_1, J\nabla_g v_1) \end{aligned}$$

$$
\begin{aligned}
&= \omega\left(\nabla_g u_2, \nabla_g v_2\right) + \omega\left(\nabla_g v_1, \nabla_g u_1\right)\\
&= \omega\left(J\nabla_g u_2, J\nabla_g v_2\right) - \omega\left(J\nabla_g u_1, J\nabla_g v_1\right)\\
&= \omega\left((du_2)^{\star}_{\omega}, (dv_2)^{\star}_{\omega}\right) - \omega\left((du_1)^{\star}_{\omega}, (dv_1)^{\star}_{\omega}\right)\\
&= \{u_1, v_1\}_{\omega} - \{u_2, v_2\}_{\omega}.
\end{aligned}
$$

We conclude that for all $u, v \in \mathbb{K}_-$ holds the identity

$$\nabla_{g,J} i\{u, v\}_{\omega,\Omega} = \left[\nabla_{g,J} u, \nabla_{g,J} v\right],$$

which shows that $i\{\cdot,\cdot\}_{\omega,\Omega}$ is a complex Lie algebra product over $\mathbb{K}_-$ and that the map χ is a morphism of complex Lie algebras.

Step B,C. The statements **B** and **C** follow from corollary 1 and the remarkable identity (18.1).

Step D. We show now the statement **D.** We observe first that for all $u, v \in C^{\infty}(X, \mathbb{C})$ holds the identity

$$\int_X i\{u, v\}_{\omega}\Omega = -\int_X iB^{\Omega}_{g,J} u \cdot v\Omega.$$

Indeed thanks to the computations in step **A** we deduce

$$
\begin{aligned}
\int_X i\{u, v\}_{\omega}\Omega &= -\int_X \left[\{u_1, v_2\}_{\omega} + \{u_2, v_1\}_{\omega}\right]\Omega\\
&+ i\int_X \left[\{u_1, v_1\}_{\omega} - \{u_2, v_2\}_{\omega}\right]\Omega\\
&= \int_X \left[\langle\nabla_g v_1, J\nabla_g u_2\rangle_g + \langle\nabla_g u_1, J\nabla_g v_2\rangle_g\right]\Omega\\
&+ i\int_X \left[\langle\nabla_g v_2, J\nabla_g u_2\rangle_g + \langle\nabla_g u_1, J\nabla_g v_1\rangle_g\right]\Omega.
\end{aligned}
$$

Integrating by parts we infer

$$
\begin{aligned}
\int_X i\{u, v\}_{\omega}\Omega &= \int_X \left[v_1 \cdot B^{\Omega}_{g,J} u_2 - u_1 \cdot B^{\Omega}_{g,J} v_2\right]\Omega\\
&+ i\int_X \left[v_2 \cdot B^{\Omega}_{g,J} u_2 + u_1 \cdot B^{\Omega}_{g,J} v_1\right]\Omega\\
&= \int_X \left[B^{\Omega}_{g,J} u_1 \cdot v_2 + B^{\Omega}_{g,J} u_2 \cdot v_1\right]\Omega\\
&- i\int_X \left[B^{\Omega}_{g,J} u_1 \cdot v_1 - B^{\Omega}_{g,J} u_2 \cdot v_2\right]\Omega\\
&= -\int_X iB^{\Omega}_{g,J} u \cdot v\Omega,
\end{aligned}
$$

thanks to the fact that $B^{\Omega}_{g,J}$ is L^2_{Ω}-anti-adjoint. Thus if $u \in \mathbb{K}_-$

$$\int_X i\{u, \bar{u}\}_{\omega}\Omega = \int_X (\Delta^{\Omega}_g - 2\mathbb{I})u \cdot \bar{u}\Omega \geqslant 0.$$

The Kähler-Ricci-Soliton assumption implies the commutation identity

$$\left[\Delta^{\Omega}_{g,J}, \Delta^{\Omega}_{g,-J}\right] \quad = \quad 0.$$

We infer that

$$\Delta^{\Omega}_{g,J} - 2\mathbb{I} : \mathbb{K}_{-} \longrightarrow \mathbb{K}_{-}, \tag{21.3}$$

is a well defined non-negative L^2_{Ω}-self-adjoint operator and let $(\lambda_j)_{j=0}^{N} \subset \mathbb{R}_{\geqslant 0}$, $\lambda_0 = 0$ be it's spectrum. Notice also that by definition of $\mathbb{K}_{-}$ this operator coincides with the operator

$$-2iB^{\Omega}_{g,J} : \mathbb{K}_{-} \quad \longrightarrow \quad \mathbb{K}_{-}.$$

Thus $u \in \mathbb{K}_{-}$ is an eigen-vector corresponding to the eigenvalue λ_j if and only if $u \in \mathbb{K}_{-}$ satisfies

$$(J\nabla_g f).u \quad = \quad \frac{\lambda_j}{2} iu.$$

This rewrites as

$$i\{f, u\}_{\omega,\Omega} \quad = \quad \frac{\lambda_j}{2} u,$$

and is equivalent to the equation

$$\left[\nabla_g f, \nabla_{g,J} u\right] \quad = \quad \frac{\lambda_j}{2} \nabla_{g,J} u.$$

Notice also that the kernel of (21.3) is given by the identity

$$\mathbb{K}_{+} \cap \mathbb{K}_{-} \quad = \quad \mathbb{K}_{\mathbb{R}} \oplus J\mathbb{K}_{\mathbb{R}},$$

$$\mathbb{K}_{\mathbb{R}} \quad := \quad \mathrm{Ker}_{\mathbb{R}}(\Delta^{\Omega}_{g,\pm J} - 2\mathbb{I}).$$

We deduce the required conclusion with $\mu_j = \lambda_j/2$. □

21.3 Consequences of the Bochner-Kodaira-Nakano formula

The holomorphic and antiholomorphic Hodge Laplacian operators are related by the Bochner-Kodaira-Nakano identity. At the level of T_X-valued 1-forms it reduces to the identity

$$\Delta^{-J}_{T_X,g} A = \Delta^{J}_{T_X,g} A + \frac{1}{6} (J\mathcal{R}_g \wedge A)(\omega^{\star} \wedge \bullet), \tag{21.4}$$

where $\omega^{\star} \equiv \omega^{-1} \in C^{\infty}(X, \Lambda^{1,1}_J T_X \cap \Lambda^2_{\mathbb{R}} T_X)$ is the dual element associated to ω. If in holomorphic coordinates ω writes as

$$\omega \quad = \quad \frac{i}{2} \omega_{k,\bar{l}} dz_k \wedge d\bar{z}_l,$$

then

$$\omega^{\star} \quad = \quad 2i\omega^{l,\bar{k}} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial \bar{z}_l}.$$

The factor $1/6$ in front of the last term on the right hand side of (21.4) is due to the convention

$$v_1 \wedge \ldots \wedge v_p \quad := \quad \sum_{\sigma \in S_p} \varepsilon_{\sigma} v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_p}.$$

We explicit the latter term. For this purpose we observe first that for any $\alpha \in \Lambda^{1,1}_J T^{\star}_X \otimes_{\mathbb{C}} E$ holds the identity

$$\mathrm{Tr}_{\omega} \alpha \quad = \quad -\mathrm{Tr}_g \left[\alpha(J\cdot, \cdot)\right].$$

We infer the expressions

$$\begin{aligned}\frac{1}{6}(J\mathcal{R}_g \wedge A)(\omega^\star \wedge \xi) &= (J\mathcal{R}_g \wedge A)\left(2i\omega^{l,\bar{k}}\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}, \xi\right)\\ &= -\frac{1}{2}\operatorname{Tr}_\omega[(J\mathcal{R}_g \wedge A)(\cdot,\cdot,\xi)]\\ &= \frac{1}{2}(J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi),\end{aligned}$$

for an arbitrary g-orthonormal real frame $(e_k)_k$. We explicit the exterior product using the J-invariant properties of the curvature operator. We obtain

$$\begin{aligned}(J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi) &= J\mathcal{R}_g(Je_k, e_k)A\xi - J\mathcal{R}_g(Je_k, \xi)Ae_k + J\mathcal{R}_g(e_k, \xi)AJe_k\\ &= -\operatorname{Tr}_\omega(J\mathcal{R}_g)A\xi - \mathcal{R}_g(Je_k, \xi)JAe_k + \mathcal{R}_g(e_k, \xi)JAJe_k\\ &= -2\operatorname{Ric}^\star(g)A\xi + \mathcal{R}_g(\xi, Je_k)JAe_k - \left[\mathcal{R}_g \star (JAJ)\right]\xi\\ &= -2\operatorname{Ric}^\star(g)A\xi - \mathcal{R}_g(\xi, \eta_k)JAJ\eta_k - \left[\mathcal{R}_g \star (JAJ)\right]\xi,\end{aligned}$$

where $\eta_k := Je_k$. But $(\eta_k)_k$ is also a g-orthonormal real frame. We infer

$$(J\mathcal{R}_g \wedge A)(Je_k, e_k, \xi) = -2\operatorname{Ric}^\star(g)A\xi - 2\left[\mathcal{R}_g \star (JAJ)\right]\xi.$$

We deduce that the Bochner-Kodaira-Nakano identity rewrites at the level of T_X-valued 1-forms as

$$\Delta^{-J}_{T_{X,g}}A = \Delta^{J}_{T_{X,g}}A - \operatorname{Ric}^\star(g)A - \mathcal{R}_g \star \left(A''_J - A'_J\right), \tag{21.5}$$

where A'_J and A''_J are respectively the J-linear and J-anti-linear parts of A. Using the Weitzenböck type formula in lemma 3 with $\Omega = CdV_g$ we infer

$$\begin{aligned}\mathcal{L}^\Omega_g A &= \Delta_g A + \nabla_g f \neg \nabla_g A - 2\mathcal{R}_g \star A\\ &= \Delta_{T_{X,g}}A - \mathcal{R}_g \star A - A\operatorname{Ric}^\star(g) + \nabla_g f \neg \nabla_g A\\ &= \left(\Delta^J_{T_{X,g}} + \Delta^{-J}_{T_{X,g}}\right)A - \mathcal{R}_g \star A - A\operatorname{Ric}^\star(g) + \nabla_g f \neg \nabla_g A.\end{aligned}$$

Using the Bochner-Kodaira-Nakano identity (21.5) we deduce the formulas

$$\begin{aligned}\mathcal{L}^\Omega_g A &= 2\Delta^J_{T_{X,g}}A - \operatorname{Ric}^\star(g)A - A\operatorname{Ric}^\star(g) - 2\mathcal{R}_g \star A''_J + \nabla_g f \neg \nabla_g A,\\ \mathcal{L}^\Omega_g A &= 2\Delta^{-J}_{T_{X,g}}A + \operatorname{Ric}^\star(g)A - A\operatorname{Ric}^\star(g) - 2\mathcal{R}_g \star A'_J + \nabla_g f \neg \nabla_g A,\end{aligned}$$

and thus the identities

$$\mathcal{L}^\Omega_g A'_J = 2\Delta^J_{T_{X,g}}A'_J - \operatorname{Ric}^\star(g)A'_J - A'_J\operatorname{Ric}^\star(g) + \nabla_g f \neg \nabla_g A'_J, \tag{21.6}$$

$$\mathcal{L}^\Omega_g A''_J = 2\Delta^{-J}_{T_{X,g}}A''_J + \operatorname{Ric}^\star(g)A''_J - A''_J\operatorname{Ric}^\star(g) + \nabla_g f \neg \nabla_g A''_J. \tag{21.7}$$

We point out that one can obtain directly these formulas by using the methods in the proof of identities (14.2), (14.5) and (14.6). We remind now that the properties (12.1) and (12.2) imply that $A \in \operatorname{Ker}\mathcal{L}_g$ if and only if

$A'_J \in \mathrm{Ker}\,\mathcal{L}_g$ and $A''_J \in \mathrm{Ker}\,\mathcal{L}_g$. Thus if $A \in \mathrm{Ker}\,\mathcal{L}_g$ we infer thanks to the identity (21.6) with $\Omega = CdV_g$,

$$\begin{aligned} 0 &= \int_X \left\langle \mathcal{L}_g A'_J, A'_J \right\rangle_g dV_g \\ &= 2\int_X \left[\left\langle \Delta^J_{T_{X,g}} A'_J, A'_J \right\rangle_g - \left\langle \mathrm{Ric}^\star(g) A'_J, A'_J \right\rangle_g \right] dV_g. \end{aligned}$$

Using the identity between Riemannian and hermitian norms of T_X-valued forms we obtain

$$\begin{aligned} 2\int_X \left\langle \Delta^J_{T_{X,g}} A'_J, A'_J \right\rangle_g dV_g &= 2\int_X \left\langle \Delta^J_{T_{X,g}} A'_J, A'_J \right\rangle_\omega dV_g \\ &= \int_X \left[2\left| \partial^{\star_g}_{T_{X,J}} A'_J \right|^2_\omega + \left| \partial^g_{T_{X,J}} A'_J \right|^2_\omega \right] dV_g \\ &= \int_X \left[2\left| \partial^{\star_g}_{T_{X,J}} A'_J \right|^2_g + \left| \partial^g_{T_{X,J}} A'_J \right|^2_g \right] dV_g. \end{aligned}$$

We deduce

$$\int_X \left[2\left| \partial^{\star_g}_{T_{X,J}} A'_J \right|^2_g + \left| \partial^g_{T_{X,J}} A'_J \right|^2_g - 2\left\langle \mathrm{Ric}^\star(g) A'_J, A'_J \right\rangle_g \right] dV_g = 0. \tag{21.8}$$

Assume from now on the Kähler-Einstein condition $\mathrm{Ric}(g) = \lambda g$, $\lambda = \pm 1, 0$. The identity (21.7) with $\Omega = CdV_g$ implies in this case

$$\mathcal{L}_g A''_J = 2\Delta^{-J}_{T_{X,g}} A''_J,$$

and thus

$$\mathrm{Ker}\,\mathcal{L}_g \cap C^\infty\left(X, T^\star_{X,-J} \otimes T_{X,J}\right) = \mathcal{H}^{0,1}_g\left(T_{X,J}\right).$$

Let now $A \in \mathrm{Ker}\,\nabla^\star_g$ and observe that for be-degree reasons holds the decomposition

$$\begin{aligned} 0 = \nabla^\star_g A &= \nabla^\star_{T_{X,g}} A'_J + \nabla^\star_{T_{X,g}} A''_J \\ &= \partial^{\star_g}_{T_{X,J}} A'_J + \overline{\partial}^{\star_g}_{T_{X,J}} A''_J. \end{aligned}$$

Thus if $A \in \mathrm{Ker}\,\nabla^\star_g \cap \mathrm{Ker}\,\mathcal{L}_g$ then $\overline{\partial}^{\star_g}_{T_{X,J}} A''_J = 0$ and thus $\partial^{\star_g}_{T_{X,J}} A'_J = 0$ which implies

$$\int_X \left[\left| \partial^g_{T_{X,J}} A'_J \right|^2_g - 2\left\langle \mathrm{Ric}^\star(g) A'_J, A'_J \right\rangle_g \right] dV_g = 0,$$

thanks to (21.8). We will still denote by $\mathcal{L}_g$ the analogue operator over $C^\infty\left(X, S^2_{\mathbb{R}} T^\star_X\right)$. We infer that if $\lambda \neq 0$ then holds the identity

$$\mathrm{Ker}\,\nabla^\star_g \cap \mathrm{Ker}\,\mathcal{L}_g \cap \mathbb{D}^J_g = \left\{ v \in C^\infty\left(X, S^2_{\mathbb{R}} T^\star_X\right) \mid v = v''_J,\ v^\star_g \in \mathcal{H}^{0,1}_g\left(T_{X,J}\right) \right\},$$

i.e. there exists an isomorphism

$$\begin{aligned} \mathrm{Ker}\,\nabla^\star_g \cap \mathrm{Ker}\,\mathcal{L}_g \cap \mathbb{D}^J_g &\longrightarrow \mathcal{H}^{0,1}_g\left(T_{X,J}\right)_{\mathrm{sm}} \\ v &\longmapsto v^\star_g. \end{aligned}$$

But $\mathcal{H}^{0,1}_g\left(T_{X,J}\right)_{\mathrm{sm}} = \mathcal{H}^{0,1}_g\left(T_{X,J}\right)$, thanks to lemma 14. We conclude the following fact.

Lemma 29. *Over any compact non Ricci flat Kähler-Einstein manifold* (X, J, g) *there exists the canonical isomorphism*

$$\begin{aligned} \operatorname{Ker}\nabla_g^\star \cap \operatorname{Ker}\mathcal{L}_g \cap \mathbb{D}_g^J &\longrightarrow H^{0,1}(X, T_{X,J}) \simeq H^1(X, \mathcal{O}(T_{X,J})) \\ v &\longmapsto \left\{v_g^\star\right\}. \end{aligned}$$

This result was proved by [9] in the negative Kähler-Einstein case $\mathrm{Ric}(g) = -g$.

21.4 Polarized deformations of Fano manifolds

In this subsection we review a few basic facts on deformation theory which clarifies the Fano set up in the paper. In particular we wish the readers would avoid the frequent inaccuracies we found in the application of this theory to Kähler geometry.

21.4.1 The Maurer-Cartan equation

Let (V, J_0) be a complex vector space of dimension n. We remind that the data of a complex structure J over V is equivalent with a n-dimensional complex subspace data $\Gamma \subset \mathbb{C}V := V \otimes_{\mathbb{R}} \mathbb{C}$ such that $\Gamma \cap \overline{\Gamma} = \{0\}$. A complex structure J over V is called J_0-compatible if the projection map

$$\pi_{J_0}^{0,1} : V_J^{0,1} \longrightarrow V_{J_0}^{0,1},$$

is surjective, i.e. a $\mathbb{C}$-isomorphism. This is equivalent to the condition $V_J^{0,1} \cap V_{J_0}^{1,0} = \{0\}$, which in its turn is equivalent to the existence of a $\mathbb{C}$-linear map $\theta : V_{J_0}^{0,1} \longrightarrow V_{J_0}^{1,0}$ such that

$$V_J^{0,1} = (\mathbb{I} + \theta)V_{J_0}^{0,1}.$$

If we set $\mu := (\theta + \bar{\theta})_{|V} \in \mathrm{End}_{-J_0}(V)$ then the condition $V_J^{0,1} \cap \overline{V_J^{0,1}} = \{0\}$ is equivalent to say $\mathbb{I} + \mu \in \mathrm{GL}_{\mathbb{R}}(V)$. Notice that we can obtain θ by the formula $\theta = \mu_{\mathbb{C}} \cdot \pi_{J_0}^{0,1}$, with $\mu_{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}V)$ the natural complexification of μ. If we denote by $\mathcal{J}(V, J_0)$ the set of J_0-compatible complex structures over V and if we set

$$C(V, J_0) := \left\{\mu \in \mathrm{End}_{-J_0}(V) \mid \mathbb{I} + \mu \in \mathrm{GL}_{\mathbb{R}}(V)\right\},$$

we infer the existence of a bijection, called the Caley transform (see [14])

$$\begin{aligned} \chi : C(V, J_0) &\longrightarrow \mathcal{J}(V, J_0) \\ \mu &\longleftrightarrow J := (\mathbb{I} + \mu)J_0(\mathbb{I} + \mu)^{-1} \\ \mu := (J_0 + J)^{-1}(J_0 - J) &\longleftrightarrow J. \end{aligned}$$

Notice indeed that $\mathcal{J}(V, J_0)$ is the sub-set of the complex structures such that $J_0 + J \in \mathrm{GL}_{\mathbb{R}}(V)$. We observe that for any $\mu \in C(V, J_0)$ as above $\mathbb{I} - \mu \in \mathrm{GL}_{\mathbb{R}}(V)$. Indeed

$$-J_0 J = (\mathbb{I} - \mu)(\mathbb{I} + \mu)^{-1}.$$

Thus $\mu \in \mathrm{End}_{-J_0}(V)$ satisfies $\mathbb{I} + \mu \in \mathrm{GL}_{\mathbb{R}}(V)$ if and only if

$$(\mathbb{I} - \mu^2) = (\mathbb{I} - \mu)(\mathbb{I} + \mu) \in \mathrm{GL}_{J_0}(V)$$

This last condition is equivalent with

$$\pi_{J_0}^{1,0} \cdot (\mathbb{I} - \mu^2) = (\mathbb{I}_{V_{J_0}^{1,0}} - \theta\overline{\theta}) \in \mathrm{GL}_{\mathbb{C}}(V_{J_0}^{1,0}).$$

We assume from now on that (X, J_0) is a compact complex manifold and let $\mathcal{J}(X, J_0)$ be the set of J_0-compatible smooth almost complex structures. For any $J \in \mathcal{J}(X, J_0)$ let

$$\theta \equiv \theta_J \in C^\infty(X, \Lambda^{0,1}_{J_0} T^\star_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0}), \quad (\mathbb{I}_{T^{1,0}_{X,J_0}} - \theta\overline{\theta}) \in \mathrm{GL}_{\mathbb{C}}(T^{1,0}_{X,J_0}),$$

be the corresponding inverse Caley transform. We show that the subset $\mathcal{J}_{\mathrm{int}}(X, J_0)$ of integrable almost complex structures is given by the Maurer-Cartan equation

$$\overline{\partial}_{T^{1,0}_{X,J_0}} \theta + \frac{1}{2}[\theta, \theta] = 0, \tag{21.9}$$

where for any $\alpha, \beta \in C^\infty(X, \Lambda^{0,\bullet}_{J_0} T^\star_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0})$ we define the exterior differential Lie product

$$[\alpha, \beta] \quad \in \quad C^\infty(X, \Lambda^{0,\bullet}_{J_0} T^\star_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0}),$$

of degree $d = \deg\alpha + \deg\beta$ by the formula

$$[\alpha, \beta](\xi) \quad := \quad \sum_{|I| = \deg\alpha} \varepsilon_I \left[\alpha(\xi_I), \beta(\xi_{\complement I})\right],$$

for all $\xi \in \bar{\mathcal{O}}(T^{0,1}_{X,J_0})^{\times d}$. Notice that this formula defines a priori only an element

$$[\alpha, \beta] \quad \in \quad \mathrm{Alt}^d_{\bar{\mathcal{O}}}(\bar{\mathcal{O}}(T^{0,1}_{X,J_0}); C^\infty(T^{1,0}_{X,J_0})).$$

However we can define pointwise the section $[\alpha, \beta]$ as follows. For any $v \in (T^{0,1}_{X,J_0,x})^{\times d}$

$$[\alpha, \beta](v) \quad := \quad [\alpha, \beta](\xi)_{|x},$$

with $\xi \in \bar{\mathcal{O}}(T^{0,1}_{X,J_0})^{\times d}$ such that $\xi_x = v$. This is well defined by the $\bar{\mathcal{O}}$-linearity of $[\alpha, \beta]$. Indeed the coefficients of ξ with respect to the local frame $(\bar{\zeta}_k)^n_{k=1} \subset \bar{\mathcal{O}}(U, T^{0,1}_{X,J_0})$, with $\zeta_j := \frac{\partial}{\partial z_j}$, and J_0-holomorphic coordinates $(z_1, \ldots, z_n)$, are J_0-anti-holomorphic functions which value at the point x is uniquely determined by v. The section $[\alpha, \beta]$ is smooth since its coefficients with respect to the frame $(\bar{\zeta}_k)^n_{k=1}$ are smooth functions.

Notice now that $(\mathbb{I} + \theta)(\bar{\zeta}_k)$, $k = 1, \ldots, n$, is a local frame of the bundle $T^{0,1}_{X,J}$ over an open set U. Then the integrability of J is equivalent to the condition

$$\left[(\mathbb{I} + \theta)(\bar{\zeta}_k), (\mathbb{I} + \theta)(\bar{\zeta}_l)\right] \in C^\infty(U, T^{0,1}_{X,J}), \tag{21.10}$$

since the torsion form $\tau_J \in C^\infty(X, \Lambda^{0,2}_{J_0} T^\star_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0})$ of J satisfies

$$\tau_J\left((\mathbb{I} + \theta)(\bar{\zeta}_k), (\mathbb{I} + \theta)(\bar{\zeta}_l)\right) \quad = \quad \left[(\mathbb{I} + \theta)(\bar{\zeta}_k), (\mathbb{I} + \theta)(\bar{\zeta}_l)\right]^{1,0}_J.$$

We observe also the identities

$$\begin{aligned}
\left[(\mathbb{I} + \theta)(\bar{\zeta}_k), (\mathbb{I} + \theta)(\bar{\zeta}_l)\right] &= \left[\bar{\zeta}_k, \theta(\bar{\zeta}_l)\right] + \left[\theta(\bar{\zeta}_k), \bar{\zeta}_l\right] + \left[\theta(\bar{\zeta}_k), \theta(\bar{\zeta}_l)\right] \\
&= \left[\bar{\zeta}_k, \theta(\bar{\zeta}_l)\right]^{1,0}_{J_0} - \left[\bar{\zeta}_l, \theta(\bar{\zeta}_k)\right]^{1,0}_{J_0} + \frac{1}{2}[\theta, \theta](\bar{\zeta}_k, \bar{\zeta}_l) \\
&= \left(\overline{\partial}_{T^{1,0}_{X,J_0}} \theta + \frac{1}{2}[\theta, \theta]\right)(\bar{\zeta}_k, \bar{\zeta}_l) \in C^\infty(U, T^{1,0}_{X,J_0}).
\end{aligned}$$

We have $T^{0,1}_{X,J} \cap T^{1,0}_{X,J_0} = 0_X$ by the J_0-compatibility of J. We infer that if (21.10) holds then also (21.9) holds true and

$$\left[(\mathbb{I} + \theta)(\bar{\zeta}_k), (\mathbb{I} + \theta)(\bar{\zeta}_l)\right] \quad = \quad 0.$$

On the other hand if (21.9) is satisfied then the previous identity is satisfied and thus (21.10) holds true.

Remark 4. *For any $\alpha, \beta \in C^\infty(X, \Lambda^{0,\bullet}_{J_0} T^\star_X \otimes_{\mathbb{C}} T_{X,J_0})$ we define the exterior differential Lie product*

$$[\alpha, \beta] \quad \in \quad C^\infty(X, \Lambda^{0,\bullet}_{J_0} T^\star_X \otimes_{\mathbb{C}} T_{X,J_0}),$$

of degree $d = \deg\alpha + \deg\beta$ by the formula

$$[\alpha, \beta] \quad := \quad \Big[\pi^{1,0}_J \cdot \alpha_{\mathbb{C}}\,, \pi^{1,0}_J \cdot \beta_{\mathbb{C}}\Big] + \overline{\Big[\pi^{1,0}_J \cdot \alpha_{\mathbb{C}}\,, \pi^{1,0}_J \cdot \beta_{\mathbb{C}}\Big]}.$$

Then the Maurer-Cartan equation (21.9) can be rewritten in the equivalent form

$$\overline{\partial}_{T_{X,J_0}} \mu + \frac{1}{2}\,[\mu, \mu] \quad = \quad 0,$$

since

$$\overline{\partial}_{T_{X,J_0}} \mu \quad = \quad \overline{\partial}_{T^{1,0}_{X,J_0}} \theta + \overline{\overline{\partial}_{T^{1,0}_{X,J_0}} \theta},$$

$$[\mu, \mu] \quad = \quad [\theta, \theta] + \overline{[\theta, \theta]}.$$

Let now $B \subset \mathbb{C}^p$ be the unitary open ball and observe that, by a refinement of Ehresmann theorem for any proper holomorphic submersion $\pi : \mathfrak{X} \longrightarrow B$ of a complex manifold $\mathfrak{X}$ onto B with central fiber $(X, J_0) = \pi^{-1}(0)$ there exists a smooth map $\varphi : \mathfrak{X} \longrightarrow X$ such that the map

$$(\varphi, \pi) : \mathfrak{X} \quad \longrightarrow \quad X \times B,$$

is a diffeomorphism with $\varphi_{|_X} = \mathbb{I}_X$ and with $\varphi^{-1}(x) \subset \mathfrak{X}$ complex sub-variety for all $x \in X$.

Let now $\theta := (\theta_t)_{t\in B} \subset C^\infty(X, \Lambda^{0,1}_{J_0} T^\star_X \otimes_{\mathbb{C}} T^{1,0}_{X,J_0})$ with $\theta_0 = 0$ and

$$\det(\mathbb{I}_{T^{1,0}_{X,J_0}} - \theta_t\overline{\theta}_t) \quad \neq \quad 0,$$

be a smooth family of J_0-compatible complex structures. We observe that the almost complex manifold

$$\mathfrak{X} \quad \equiv \quad (\mathfrak{X}, \theta) \quad := \quad \bigsqcup_{t\in B} (X, \theta_t),$$

is a complex one if and only if θ_t satisfies the Maurer-Cartan equation (21.9) for all $t \in B$ and the map

$$t \in B \quad \longmapsto \quad \theta_t(x) \in \Lambda^{0,1}_{J_0} T^\star_{X,x} \otimes_{\mathbb{C}} T^{1,0}_{X,J_0,x},$$

is holomorphic for all $x \in X$. Indeed the distribution $T^{0,1}_{\mathfrak{X},\theta}$ is integrable if and only if its local generators $\bar\tau_r := \frac{\partial}{\partial\bar t_r}$, $r = 1, \ldots, p$, $(\mathbb{I} + \theta_t)(\bar\zeta_k)$, $k = 1, \ldots, n$, satisfy the conditions

$$\big[\bar\tau_r, (\mathbb{I} + \theta_t)(\bar\zeta_k)\big] \in C^\infty(U, T^{0,1}_{X,\theta_t}), \tag{21.11}$$

and

$$\big[(\mathbb{I} + \theta_t)(\bar\zeta_k), (\mathbb{I} + \theta_t)(\bar\zeta_l)\big] \in C^\infty(U, T^{0,1}_{X,\theta_t}). \tag{21.12}$$

The latter is equivalent with the Maurer-Cartan equation (21.9). Let $\theta_t = \theta^{k,l}_t \bar\zeta^\star_k \otimes \zeta_l$ be the local expression of θ_t. Then the identity

$$\big[\bar\tau_r, (\mathbb{I} + \theta_t)(\bar\zeta_k)\big] \quad = \quad \bar\tau_r.\theta^{k,l}_t \zeta_l \in C^\infty(U, T^{0,1}_{X,J_0}),$$

combined with the property $T^{0,1}_{X,\theta_t} \cap T^{1,0}_{X,J_0} = 0_X$, shows that (21.11) holds true if and only if the map $t \longmapsto \theta_t$ is holomorphic.

For any $p \in X$ a coordinate chart of $\mathfrak{X}$ in a open neighborhood $U_p \times B$ of $(p, 0)$ is given by a smooth function $f : U_p \times B \longrightarrow \mathbb{C}^n \times \mathbb{C}^p$ such that

$$\begin{cases} \overline{\partial}_{J_0} f \quad + \quad \partial_{J_0} f \cdot \theta_t \;= 0, \\ \\ \overline{\partial}_B f \quad = \quad 0 \\ \\ \det(df) \neq 0. \end{cases}$$

In order to produce such family θ we need to remind a few basic facts about Hodge theory.

21.4.2 Basic facts about Hodge theory and $\overline{\partial}$-equations

Let ω be a hermitian metric over X and let $(E, \overline{\partial}_E, h)$ be a hermitian holomorphic vector bundle over it. We define the anti-holomorphic Hodge Laplacian

$$\Delta''_E \quad := \quad \overline{\partial}_E \overline{\partial}^\star_E + \overline{\partial}^\star_E \overline{\partial}_E,$$

acting on the sections of $\Lambda^{p,q}_J T^\star_X \otimes_{\mathbb{C}} E$. Let $\mathcal{E}^{p,q}(E) := C^\infty(X, \Lambda^{p,q}_J T^\star_X \otimes_{\mathbb{C}} E)$ and set

$$\mathcal{H}^{p,q}(E) \quad := \quad \mathrm{Ker}\, \Delta''_E \cap \mathcal{E}^{p,q}(E)\,.$$

We remind the L^2-Hodge decomposition

$$\mathcal{E}^{p,q}(E) \quad = \quad \mathcal{H}^{p,q}(E) \oplus \overline{\partial}_E \mathcal{E}^{p,q-1}(E) \oplus \overline{\partial}^\star_E \mathcal{E}^{p,q+1}(E).$$

We observe that if there exists two subspaces $L, V \subset C^\infty(X, E)$ such that the L^2-decomposition

$$C^\infty(X, E) \quad = \quad L \oplus V,$$

holds then L and V are closed subspaces of $C^\infty(X, E)$. Indeed $L = V^\perp$ and $V = L^\perp$ by the L^2-decomposition. The same consideration holds for the Sobolev spaces $W^k(X, E)$. Thus the L^2-Hodge decomposition implies that the spaces $\overline{\partial}_E \mathcal{E}^{p,q-1}(E)$ and $\overline{\partial}^\star_E \mathcal{E}^{p,q+1}(E)$ are closed in the smooth topology. We infer the L^2-decomposition

$$\mathcal{E}^{p,q}(E) \quad = \quad [\ \mathrm{Ker}\, \overline{\partial}_E \cap \mathcal{E}^{p,q}(E)\ ] \oplus \overline{\partial}^\star_E \mathcal{E}^{p,q+1}(E),$$

and thus

$$\mathrm{Ker}\, \overline{\partial}_E \cap \mathcal{E}^{p,q}(E) \quad = \quad \mathcal{H}^{p,q}(E) \oplus \overline{\partial}_E \mathcal{E}^{p,q-1}(E).$$

An other way to see this decomposition is the following. Let

$$H_E : \mathcal{E}^{p,q}(E) \quad \longrightarrow \quad \mathcal{H}^{p,q}(E),$$

be the L^2-projection operator over $\mathcal{H}^{p,q}(E)$. For any $\alpha \in \mathcal{E}^{p,q}(E)$ there exists $\beta \in \mathcal{E}^{p,q-1}(E)$ and $\gamma \in \mathcal{E}^{p,q+1}(E)$ such that

$$\alpha \quad = \quad H_E \alpha + \overline{\partial}_E \beta + \overline{\partial}^\star_E \gamma.$$

Now if $\overline{\partial}_E \alpha = 0$ then $\overline{\partial}_E \overline{\partial}^\star_E \gamma = 0$, i.e $\overline{\partial}^\star_E \gamma = 0$. Let

$$W^{p,q}_k(E) := W^k(X, \Lambda^{p,q}_J T^\star_X \otimes_{\mathbb{C}} E).$$

We remind that the Green operator

$$G_E : W^{p,q}_k(E) \quad \longrightarrow \quad \Delta''_E W^{p,q}_{k+4}(E),$$

is defined by the identity $\mathbb{I} = H_E + \Delta''_E G_E$. The latter implies $\mathrm{Ker}\, G_E = \mathrm{Ker}\, \Delta''_E$ and the L^2-orthogonal decomposition

$$\alpha \quad = \quad H_E \alpha + \overline{\partial}_E \overline{\partial}^\star_E G_E \alpha + \overline{\partial}^\star_E \overline{\partial}_E G_E \alpha.$$

We show now the identity $\overline{\partial}_E G_E = G_E \overline{\partial}_E$. Indeed $\overline{\partial}_E$-differentiating the identity defining G_E we infer

$$\overline{\partial}_E \alpha \quad = \quad \overline{\partial}_E \Delta''_E G_E \alpha = \Delta''_E \overline{\partial}_E G_E \alpha.$$

Applying the same identity to $\overline{\partial}_E \alpha$ we obtain $\overline{\partial}_E \alpha = \Delta''_E G_E \overline{\partial}_E \alpha$, since $H_E \overline{\partial}_E = 0$ by orthogonality. Thus

$$\Delta''_E \left(\overline{\partial}_E G_E \alpha - G_E \overline{\partial}_E \alpha \right) \quad = \quad 0.$$

The fact that by definition $G_E W_k^{p,q}(E) = \Delta_E'' W_{k+4}^{p,q}(E)$ implies the existence of $\beta \in W_{k+4}^{p,q}(E)$ and $\gamma \in W_{K+3}^{p,q+1}(E)$ such that

$$G_E\alpha = \Delta_E''\beta,$$

$$G_E\overline{\partial}_E\alpha = \Delta_E''\gamma.$$

We deduce

$$\overline{\partial}_E G_E\alpha - G_E\overline{\partial}_E\alpha = \Delta_E''\left(\overline{\partial}_E\beta - \gamma\right) = 0,$$

thanks to the orthogonality of the Kernel and image of Δ_E''.

We observe finally that the equation $\overline{\partial}_E\alpha = \beta$ admits a solution if and only if $\overline{\partial}_E\beta = 0$ and $H_E\beta = 0$. In this case the unique solution of minimal L^2-norm is given by $\alpha = \overline{\partial}_E^{\star} G_E\beta$.

21.4.3 The equation of holomorphic maps

For any smooth map $f : (X, J_X) \longrightarrow (Y, J_Y)$ we define the operators

$$2\partial_{J_X,J_Y} f := df - (J_Y \circ f)\cdot df \cdot J_X \in C^{\infty}\left(X, \Lambda_{J_X}^{1,0} T_X^{\star} \otimes_{\mathbb{C}} f^{\star} T_{Y,J_Y}\right),$$

$$2\overline{\partial}_{J_X,J_Y} f := df + (J_Y \circ f)\cdot df \cdot J_X \in C^{\infty}\left(X, \Lambda_{J_X}^{0,1} T_X^{\star} \otimes_{\mathbb{C}} f^{\star} T_{Y,J_Y}\right),$$

and we notice the elementary identities

$$\partial_{J_X,J_Y} f = \pi_Y^{1,0}\cdot df \cdot \pi_X^{1,0} + \pi_Y^{0,1}\cdot df \cdot \pi_X^{0,1},$$

$$\overline{\partial}_{J_X,J_Y} f = \pi_Y^{1,0}\cdot df \cdot \pi_X^{0,1} + \pi_Y^{0,1}\cdot df \cdot \pi_X^{1,0}.$$

The map f is called holomorphic if $(J_Y \circ f)\cdot df = df \cdot J_X$. We deduce that the map f is holomorphic if and only if $\overline{\partial}_{J_X,J_Y} f = 0$, thus if and only if

$$\pi_Y^{1,0}\cdot df \cdot \pi_X^{0,1} = 0.$$

We infer that a map $f : (X, J_\varphi) \longrightarrow (Y, J_Y)$ is holomorphic if and only if

$$\pi_Y^{1,0}\cdot df_{|T_{X,J_\varphi}^{0,1}} = 0.$$

The identity

$$T_{X,J_\varphi}^{0,1} = \left(\pi_{J_0}^{0,1} + \varphi\right)\mathbb{C}T_X,$$

implies that $f : (X, J_\varphi) \longrightarrow (Y, J_Y)$ is holomorphic if and only if

$$\pi_Y^{1,0}\cdot df \cdot \left(\pi_{J_0}^{0,1} + \varphi\right) = 0.$$

This last condition rewrites as

$$0 = \overline{\partial}_{J_0,J_Y} f\cdot \pi_{J_0}^{0,1} + \pi_Y^{1,0}\cdot df \cdot \pi_{J_0}^{1,0}\cdot\varphi$$

$$= \overline{\partial}_{J_0,J_Y} f\cdot \pi_{J_0}^{0,1} + \partial_{J_0,J_Y} f\cdot\varphi.$$

We explicit the latter condition in the case of a smooth map
$f : (X, J_\varphi) \longrightarrow (X, J_\theta)$. Indeed

$$\begin{aligned}
0 &= 2\overline{\partial}_{J_0,J_\theta} f \cdot \pi^{0,1}_{J_0} + 2\, \partial_{J_0,J_\theta} f \cdot \varphi \\
&= [\mathbb{I} - i\,(J_\theta \circ f)] \cdot (df \cdot \pi^{0,1}_{J_0} + df \cdot \varphi) \\
&= [\mathbb{I} - i\,(J_\theta \circ f)] \cdot \pi^{1,0}_{J_0} \cdot (df \cdot \pi^{0,1}_{J_0} + df \cdot \varphi) \\
&+ [\mathbb{I} - i\,(J_\theta \circ f)] \cdot \pi^{0,1}_{J_0} \cdot (df \cdot \pi^{0,1}_{J_0} + df \cdot \varphi) \\
&= [\mathbb{I} - i\,(J_\theta \circ f)] \cdot \pi^{1,0}_{J_0} \cdot (\overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi) \\
&+ [\mathbb{I} - i\,(J_\theta \circ f)] \cdot \pi^{0,1}_{J_0} \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi) \quad .
\end{aligned}$$

We explicit at this point the expression of J_θ. For this purpose let $\mu := \theta + \overline{\theta}$ and decompose the identity

$$\begin{aligned}
J_\theta &:= J_0\,(\mathbb{I} - \mu)\,(\mathbb{I} + \mu)^{-1} \\
&= J_0\,(\mathbb{I} - \mu)^2 \left(\mathbb{I} - \mu^2\right)^{-1} \\
&= J_0 \left(\mathbb{I} - 2\mu + \mu^2\right) \left(\mathbb{I} - \mu^2\right)^{-1} \\
&= J_0 \left(\mathbb{I} - 2\theta - 2\overline{\theta} + \theta\overline{\theta} + \overline{\theta}\theta\right) \left(\mathbb{I} - \theta\overline{\theta} - \overline{\theta}\theta\right)^{-1} .
\end{aligned}$$

Decomposing in types we infer

$$\begin{aligned}
J_\theta &= i \left(\mathbb{I}_{1,0} + \theta\overline{\theta}\right) (\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} + 2i\overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} \\
&- 2i\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} - i(\mathbb{I}_{0,1} + \overline{\theta}\theta)(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1}.
\end{aligned}$$

Let $A := \theta\overline{\theta}$. Using the trivial identity

$$(\mathbb{I} + A)\,(\mathbb{I} - A)^{-1} = \mathbb{I} + 2A\,(\mathbb{I} - A)^{-1}\,,$$

we conclude the expression

$$\begin{aligned}
J_\theta &= \underbrace{i\mathbb{I}_{1,0} + 2i\theta\overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1}}_{\in \mathcal{E}^{1,0}\left(T^{1,0}_{X,J_0}\right)} + \underbrace{2i\overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1}}_{\in \mathcal{E}^{1,0}\left(T^{0,1}_{X,J_0}\right)} \\
&- \underbrace{2i\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1}}_{\in \mathcal{E}^{0,1}\left(T^{1,0}_{X,J_0}\right)} - \underbrace{i\mathbb{I}_{0,1} - 2i\overline{\theta}\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1}}_{\in \mathcal{E}^{0,1}\left(T^{0,1}_{X,J_0}\right)} .
\end{aligned}$$

For notation simplicity we identify $J_\theta \equiv J_\theta \circ f$ and thus $\theta \equiv \theta \circ f$. Using the previous expression we infer the equalities

$$\begin{aligned}
\frac{1}{2}\,[\mathbb{I} - i\,(J_\theta \circ f)] \cdot \pi^{1,0}_{J_0} &= \mathbb{I}_{1,0} + \theta\overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} + \overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} \\
&= (\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} + \overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1}, \\
\frac{1}{2}\,[\mathbb{I} - i\,(J_\theta \circ f)] \cdot \pi^{0,1}_{J_0} &= -\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} - \overline{\theta}\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1}.
\end{aligned}$$

The second equality follows from the trivial identity

$$\mathbb{I} + A(\mathbb{I} - A)^{-1} = (\mathbb{I} - A)^{-1}.$$

We deduce that the holomorphy condition for f writes in the form

$$\begin{aligned} 0 &= \overline{\partial}_{J_0,J_\theta} f \cdot \pi_{J_0}^{0,1} + \partial_{J_0,J_\theta} f \cdot \varphi \\ &= (\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} \cdot (\overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi) - \theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi) \\ &+ \overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} \cdot (\overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi) - \overline{\theta}\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi). \end{aligned}$$

The fact that the second line is composed by elements in $\mathcal{E}^{0,1}\left(T^{1,0}_{X,J_0}\right)$ and the third by elements in $\mathcal{E}^{0,1}\left(T^{0,1}_{X,J_0}\right)$ implies that the holomorphy condition for f is equivalent to the equations

$$\begin{aligned} (\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} \cdot (\overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi) &= \theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi), \\ \overline{\theta}(\mathbb{I}_{1,0} - \theta\overline{\theta})^{-1} \cdot (\overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi) &= \overline{\theta}\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi). \end{aligned}$$

But the last one is obtained multiplying both sides of the first with $\overline{\theta}$. We infer that the holomorphy condition for f writes as

$$\pi_{J_0}^{1,0} \cdot \overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi = (\mathbb{I}_{1,0} - \theta\overline{\theta})\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1} \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi).$$

We notice now the identity

$$\theta = (\mathbb{I}_{1,0} - \theta\overline{\theta})\theta(\mathbb{I}_{0,1} - \overline{\theta}\theta)^{-1}.$$

The latter follows decomposing the trivial identity

$$\mu = \left(\mathbb{I} - \mu^2\right)\mu\left(\mathbb{I} - \mu^2\right)^{-1}.$$

We conclude finally that the map $f : (X, J_\varphi) \longrightarrow (X, J_\theta)$ is holomorphic if and only if

$$\pi_{J_0}^{1,0} \cdot \overline{\partial}_{J_0} f + \partial_{J_0} f \cdot \varphi = (\theta \circ f) \cdot (\partial_{J_0} f + \overline{\partial}_{J_0} f \cdot \varphi).$$

For any $f \in \mathrm{Diff}(X)$ sufficiently close to the identity in C^1-norm, the almost complex structure $f^* J_\theta$ is J_0-compatible, i.e. $\det\left(J_0 + f^* J_\theta\right) \neq 0$. Thus there exists a unique form θ_f such that $f^* J_\theta = J_{\theta_f}$.

By definition the map $f : \left(X, J_{\theta_f}\right) \longrightarrow (X, J_\theta)$ is holomorphic. We conclude that θ_f is given by the formula

$$\pi_{J_0}^{1,0} \cdot \overline{\partial}_{J_0} f - (\theta \circ f) \cdot \partial_{J_0} f = -\left[\partial_{J_0} f - (\theta \circ f) \cdot \overline{\partial}_{J_0} f\right] \cdot \theta_f, \tag{21.13}$$

and thus

$$\theta_f = -\left[\partial_{J_0} f - (\theta \circ f) \cdot \overline{\partial}_{J_0} f\right]^{-1}_{|T^{1,0}_{X,J_0}} \left[\pi_{J_0}^{1,0} \cdot \overline{\partial}_{J_0} f - (\theta \circ f) \cdot \partial_{J_0} f\right],$$

as long as

$$\left[\partial_{J_0} f - (\theta \circ f) \cdot \overline{\partial}_{J_0} f\right]_{|T^{1,0}_{X,J_0}} \in \mathrm{GL}_{\mathbb{C}}\left(T^{1,0}_{X,J_0}\right).$$

Adding the complex conjugate we infer

$$\overline{\partial}_{J_0} f - (\mu \circ f) \cdot \partial_{J_0} f = -\left[\partial_{J_0} f - (\mu \circ f) \cdot \overline{\partial}_{J_0} f\right] \cdot \mu_f, \tag{21.14}$$

and thus

$$\mu_f = -\left[\partial_{J_0} f - (\mu \circ f) \cdot \overline{\partial}_{J_0} f\right]^{-1} \left[\overline{\partial}_{J_0} f - (\mu \circ f) \cdot \partial_{J_0} f\right],$$

as long as

$$\partial_{J_0} f - (\mu \circ f) \cdot \overline{\partial}_{J_0} f \in \mathrm{GL}(T_X).$$

21.4.4 The Kuranishi space of a compact complex manifold

Let (X, J) be a complex manifold and consider

$$E_J'' := T^{\star}_{X,-J} \otimes T_{X,J},$$

$$E''_{g,J} := \mathrm{End}_g(T_X) \cap E_J'',$$

$$\mathcal{C}_J := \left\{ \mu \in \mathcal{E}(E_J'') \mid (1+\mu) \in \mathrm{GL}_{\mathbb{R}}(T_X), \overline{\partial}_{T_{X,J}} \mu + \frac{1}{2}[\mu, \mu] = 0 \right\}.$$

Then the Caley transform (see [14]) provides a bijection

$$\mathrm{Cal}_{J_0} : \mathcal{C}_{J_0} \longrightarrow \mathcal{J}_{J_0} := \left\{ J \in \mathcal{J}_{\mathrm{int}} \mid \pi^{0,1}_{J_0}(T^{0,1}_{X,J}) = T^{0,1}_{X,J_0} \right\}$$

$$\mu \longleftrightarrow J := (\mathbb{I} + \mu) J_0 (\mathbb{I} + \mu)^{-1}$$

$$\mu := (J_0 + J)^{-1}(J_0 - J) \longleftrightarrow J.$$

For notations convenience we will restrict our considerations to the Fano case even if the result that will follow and its argument holds for a general compact complex manifold. For any polarized Fano manifold (X, J, ω) we define also the sub-set of Ω-divergence free tensors in $\mathcal{C}_J$

$$\mathcal{C}^{\mathrm{div}}_{J,g} := \left\{ \mu \in \mathcal{C}_J \mid \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \mu = 0 \right\}.$$

We denote by $H^0(T_{X,J})^{\perp} \cap W_k(T_{X,J})$ the $L^2_{\omega,\Omega}$-orthogonal space to the space of holomorphic vector fields inside $W_k(T_{X,J})$. For any $\xi \in \mathcal{E}(T_X)$ of sufficiently small norm the map $e(\xi) : X \longrightarrow X$ defined by

$$e(\xi)_x := \exp_{g,x}(\xi_x),$$

is a smooth diffeomorphism. For readers convenience we provide a proof (in the Fano case) of the following fundamental result due to Kuranishi [18].

Theorem 3. (***The Kuranishi space*** $\mathcal{K}_{J,g}$**.**) *For any polarized Fano manifold (X, J, ω) and any integer $k \geqslant 2(n+1)$ with $n := \dim_{\mathbb{C}} X$, there exists;*

(A) *$\varepsilon, \delta \in \mathbb{R}_{>0}$, a complex analytic subset $\mathcal{K}_{J,g} \subseteq \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \cap B^g_{\delta}(0)$, $0 \in \mathcal{K}_{J,g}$ and a holomorphic embedding*

$$\mu : \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \cap B^g_{\delta}(0) \longrightarrow B_{\varepsilon}^{W^{0,1}_k(T_{X,J})}(0),$$

with $\mu_0 = 0$, which restricts to a bijection

$$\mu : \mathcal{K}_{J,g} \longrightarrow \mathcal{C}^{\mathrm{div}}_{J,g} \cap B_{\varepsilon}^{W^{0,1}_k(T_{X,J})}(0),$$

with the property $d_0\mu(v) = v$, for all $v \in \mathrm{TC}_{\mathcal{K}_{J,g},0}$:=the tangent cone of $\mathcal{K}_{J,g}$ at the origin.

(B) *$\varepsilon_0 \in \mathbb{R}_{>0}$, $\varepsilon_0 < \varepsilon$, and a smooth map*

$$B_{\varepsilon_0}^{W^{0,1}_k(T_{X,J})}(0) \longrightarrow H^0(T_{X,J})^{\perp} \cap W_k(T_{X,J})$$

$$\varphi \longmapsto \xi_{\varphi},$$

with $\xi_0 = 0$, such that $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \varphi_{e(\xi_{\varphi})} = 0$ which restricts to an application

$$B_{\varepsilon_0}^{W^{0,1}_k(T_{X,J})}(0) \cap \mathcal{E}^{0,1}(T_{X,J}) \longrightarrow H^0(T_{X,J})^{\perp} \cap \mathcal{E}(T_{X,J}),$$

and such that the map

$$\begin{aligned} \mathcal{C}_J \cap B_{\varepsilon_0}^{W_k^{0,1}(T_{X,J})}(0) &\longrightarrow \mathcal{C}_{J,g}^{\mathrm{div}} \cap B_{\varepsilon}^{W_k^{0,1}(T_{X,J})}(0) \\ \varphi &\longmapsto \mu(\varphi) := \varphi_{e(\xi_\varphi)}, \end{aligned}$$

is well defined.

Proof. We divide Kuranishi's proof in a few steps.

STEP A1. We show first that the system

$$(S_1)\begin{cases} \overline{\partial}_{T_{X,J}}\mu + \frac{1}{2}[\mu,\mu] = 0, \\ \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\mu = 0. \end{cases}$$

is equivalent to the system

$$(S_2)\begin{cases} \mu + \frac{1}{2}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}}[\mu,\mu] = H_{T_{X,J}}\mu, \\ H_{T_{X,J}}[\mu,\mu] = 0, \end{cases}$$

provided that μ is sufficiently close to 0. Indeed let μ be a solution of (S_1). Then the considerations about the resolution of the $\overline{\partial}$-equation imply the second equation in (S_2). Moreover if we set

$$\varphi := -\frac{1}{2}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}}[\mu,\mu],$$

then $\alpha := \mu - \varphi$ satisfies $\overline{\partial}_{T_{X,J}}\alpha = 0$ and $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\alpha = 0$. Thus $\alpha \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J})$ and $H_{T_{X,J}}\mu = \alpha$ since

$$H_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} = 0,$$

by orthogonality. This shows that also the first equation in (S_2) holds. Assume now that μ is a solution of (S_2). It is clear that the second equation in (S_1) holds true. We set

$$\psi := \overline{\partial}_{T_{X,J}}\mu + \frac{1}{2}[\mu,\mu],$$

and we observe the equalities

$$\begin{aligned} \psi &= -\frac{1}{2}\overline{\partial}_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}}[\mu,\mu] + \frac{1}{2}[\mu,\mu] \\ &= \frac{1}{2}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}} G_{T_{X,J}}[\mu,\mu] \\ &= \frac{1}{2}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}}\overline{\partial}_{T_{X,J}}[\mu,\mu] \\ &= \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}}\left[\overline{\partial}_{T_{X,J}}\mu,\mu\right]. \end{aligned}$$

We deduce the identity

$$\psi = \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}}[\psi,\mu].$$

The assumption $k > n+1$ implies that the Sobolev embedding $W^k \subset C^{k-n-1}$ holds true. Using the standard estimates on the Sobolev norms of $W_k^{\bullet,\bullet}$:

$$\begin{aligned} \|G_{T_{X,J}}\varphi\|_{k+2} &\leqslant C_0\|\varphi\|_k, \\ \|[\varphi,\psi]\|_{k-1} &\leqslant C_2\|\varphi\|_k\|\psi\|_k, \end{aligned}$$

we obtain

$$\|\psi\|_k \quad \leqslant \quad C_1 \| [\psi, \mu] \|_{k-1} \leqslant C_1 C_2 \|\psi\|_k \|\mu\|_k.$$

Thus if $\|\mu\|_k \leqslant \varepsilon/(C_1 C_2)$ for some $\varepsilon \in (0,1)$ then $(1-\varepsilon)\|\psi\|_k \leqslant 0$, which holds true if and only if $\psi = 0$.

STEP A2. We remind that the previous discussion shows that the first equation in (S_2) is equivalent to the condition

$$F(\mu) := \mu + \frac{1}{2} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}} [\mu, \mu] \quad \in \quad \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}).$$

Let $\Xi_k \subset W^{0,1}_k(T_{X,J})$ be the subset of the elements satisfying this condition. We notice that the map

$$F : W^{0,1}_k(T_{X,J}) \quad \longrightarrow \quad W^{0,1}_k(T_{X,J}),$$

is well defined and continuous thanks to the estimate

$$\| \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}} [\mu, \mu] \|_k \quad \leqslant \quad C_1 \| [\mu, \mu] \|_{k-1} \leqslant C_1 C_2 \|\mu\|^2_k.$$

We infer that F is also holomorphic since $F - \mathbb{I}$ is a continuous quadratic form. The fact that the differential of F at the origin is the identity implies the existence of an inverse holomorphic map F^{-1} in a neighborhood $B^{W_k}_\varepsilon(0)$ of the origin. Restricting this to $\mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \cap B^{W_k}_\varepsilon(0)$ we deduce the existence of a holomorphic map

$$\alpha \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \cap B^{W_k}_\varepsilon(0) \quad \longmapsto \quad \mu_\alpha \in W^{0,1}_k(T_{X,J}),$$

such that

$$\mu_\alpha + \frac{1}{2} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}} [\mu_\alpha, \mu_\alpha] \quad = \quad \alpha.$$

By construction $\mathrm{Im}\,(\alpha \longmapsto \mu_\alpha)$ represents a neighborhood of the origin inside Ξ_k. It is clear that μ_α is of class C^{k-n} by the Sobolev embedding. We show further that μ_α is smooth for a sufficiently small choice of ε. Indeed applying the Hodge Laplacian $\Delta^{\Omega,-J}_{T_{X,g}}$ to both sides of the previous identity and using the equalities

$$\Delta^{\Omega,-J}_{T_{X,g}} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} G_{T_{X,J}} = \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \Delta^{\Omega,-J}_{T_{X,g}} G_{T_{X,J}} = \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}},$$

(notice that $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} H_{T_{X,J}} = 0$) we obtain the equation

$$\Delta^{\Omega,-J}_{T_{X,g}} \mu_\alpha \; + \; \frac{1}{2} \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} [\mu_\alpha, \mu_\alpha] \quad = \quad 0 \; ,$$

which rewrites also as

$$\Delta^{\Omega,-J}_{T_{X,g}} \mu_\alpha \; + \; \frac{1}{2} \; \mu_\alpha \star \nabla^2_g \mu_\alpha \quad = \quad \frac{1}{2} \nabla_g \mu_\alpha \star \nabla_g \mu_\alpha + \frac{1}{2} \; \mu_\alpha \star \nabla_g \mu_\alpha \star \nabla_g f \; ,$$

where $\star$ denotes adequate contraction operators. The fact that the C^0-norm of μ_α can be made arbitrary small for sufficiently small ε implies that the operator

$$\Delta^{\Omega,-J}_{T_{X,g}} \; + \; \frac{1}{2} \; \mu_\alpha \star \nabla^2_g \; ,$$

is elliptic. Then the smoothness of μ_α follows by standard elliptic bootstrapping. We denote by $\mathcal{K}_{J,g}$ the zero set of the holomorphic map

$$\chi : \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \cap B^{W_k}_\varepsilon(0) \quad \longrightarrow \quad \mathcal{H}^{0,2}_{g,\Omega}(T_{X,J})$$

$$\alpha \quad \longmapsto \quad H_{T_{X,J}} [\mu_\alpha, \mu_\alpha] \; .$$

Then the set $\{\mu_\alpha \mid \alpha \in \mathcal{K}_{J,g}\}$ covers the set of the solutions of the system (S_2) in a neighborhood of the origin.

STEP B. We observe first that $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\varphi = 0$ if and only if $G_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\varphi = 0$. Indeed

$$\operatorname{Im}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}} \perp \operatorname{Ker} G_{T_{X,J}},$$

since $\operatorname{Ker} G_{T_{X,J}} = \operatorname{Ker}\Delta^{\Omega,-J}_{T_{X,g}}$. Thus in order to construct the application $\varphi \longmapsto \xi_\varphi$ we need to find the zeros of the map

$$\begin{aligned} R : W^{0,1}_k\left(T_{X,J}\right) \times \left[H^0\left(T_{X,J}\right)^\perp \cap B^{W_k(T_{X,J})}_{\varepsilon_0}(0)\right] &\longrightarrow H^0\left(T_{X,J}\right)^\perp \cap W_k\left(T_{X,J}\right) \\ (\varphi,\xi) &\longmapsto G_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\varphi_{e(\xi)}, \\ (0,0) &\longmapsto 0. \end{aligned}$$

For notations simplicity we denote $\Psi : (\mu, f) \longmapsto \mu_f$. With these notations the formula (21.14) writes as

$$\overline{\partial}_J e\left(t\xi\right) = -\partial_J e\left(t\xi\right)\cdot\Psi\left(0, e\left(t\xi\right)\right).$$

Time deriving this identity at $t = 0$ and using the fact that $\frac{d}{dt}_{|t=0} e\left(t\xi\right) = \xi$, $\Psi\left(0, \mathrm{Id}_X\right) = 0$ and $e\left(0\right) = \mathrm{Id}_X$ we obtain

$$\overline{\partial}_{T_{X,J}}\xi = -D_f\Psi\left(0, \mathrm{Id}_X\right)\cdot\xi,$$

where $D_f\Psi$ denotes the partial Frechet derivative of Ψ in the variable f. We observe now that for any $\xi \in W_k\left(T_{X,J}\right)$ holds the decomposition formula

$$\xi = H_{T_{X,J}}\xi + G_{T_{X,J}}\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\xi.$$

Thus if $\xi \in H^0\left(T_{X,J}\right)^\perp \cap W_k\left(T_{X,J}\right)$ then holds the identity

$$\xi = G_{T_{X,J}}\overline{\partial}^{\star_g}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\xi.$$

We conclude the identity $D_\xi R\left(0,0\right) = \mathbb{I}$ and the existence of the map $\varphi \longmapsto \xi_\varphi$ by the implicit function theorem. In local coordinates we can consider the expansion

$$e\left(\xi\right) = \mathrm{Id}_X + \xi + O(|\xi|^2).$$

Then the formula (21.14) implies the local identity

$$\varphi_{e(\xi)} = -\overline{\partial}_{T_{X,J}}\xi + \varphi + Q\left(\varphi,\xi\right),$$

with Q an analytic function (depending on the local coordinates). Then the condition $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\varphi_{e(\xi_\varphi)} = 0$ implies

$$-\Delta^{\Omega,-J}_{T_{X,g}}\xi_\varphi + \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\varphi + \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}Q\left(\varphi,\xi_\varphi\right) = 0.$$

Thus ξ_φ is smooth if φ is smooth by elliptic regularity. □

21.4.5 Parametrization of a sub-space of the ω-compatible complex structures

Let (X, J, ω) be a polarized Fano manifold and consider the set

$$\mathcal{C}_{\omega,J} := \left\{\mu \in \mathcal{E}(E''_{g,J}) \mid g\left(1 \pm \mu\right) > 0, \overline{\partial}_{T_{X,J}}\mu + \frac{1}{2}\left[\mu,\mu\right] = 0\right\},$$

with $g := -\omega J$. Then the Caley transform restricts to a bijection (see [14])

$$\mathrm{Cal}_J : \mathcal{C}_{\omega,J} \longrightarrow \mathcal{J}_\omega.$$

We define also the sub-set of Ω-divergence free tensors in $\mathcal{C}_{\omega,J}$

$$\mathcal{C}^{\mathrm{div}}_{\omega,J} := \left\{\mu \in \mathcal{C}_{\omega,J} \mid \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\mu = 0\right\}.$$

Definition 2. (***The Kuranishi space of polarized deformations***) *For any polarized Fano manifold (X, J, ω) we define the Kuranishi space of ω-polarized complex deformations as the complex analytic subset*

$$\mathcal{K}_J^\omega \quad := \quad \left\{ \alpha \in \mathcal{K}_{g,J} \mid \mu_\alpha = (\mu_\alpha)_g^T \right\}.$$

With these notations the map μ in theorem 3 restricts to a bijection

$$\mu : \mathcal{K}_J^\omega \quad \longrightarrow \quad \mathcal{C}_{\omega,J}^{\mathrm{div}} \cap B_\varepsilon^{W_k^{0,1}(T_{X,J})}(0).$$

For any $\alpha \in \mathcal{K}_J^\omega$ we define $J_\alpha := \mathrm{Cal}_J \mu_\alpha$. Let also $\mathcal{U}_\omega \subset C_\Omega^\infty(X, \mathbb{C})_0$ be an open neighborhood of the origin such that $\omega + dd_{J_\alpha}^c u_1 > 0$ for all $\alpha \in \mathcal{K}_J^\omega$ and $u = u_1 + iu_2 \in \mathcal{U}_\omega$, with u_j real valued. We define the real vector field

$$\xi_t^{\alpha,u} \quad := \quad -\left(\omega + tdd_{J_\alpha}^c u_1\right)^{-1} \left(d_{J_\alpha}^c u_1 + \frac{1}{2} du_2 \right),$$

for all $t \in (-\varepsilon, 1+\varepsilon)$, for some small $\varepsilon > 0$. We define also the family of diffeomorphisms $\left(\Phi_t^{\alpha,u}\right)_{t\in(-\varepsilon,1+\varepsilon)}$ over X given by $\partial_t \Phi_t^{\alpha,u} = \xi_t^{\alpha,u} \circ \Phi_t^{\alpha,u}$, with $\Phi_0^{\alpha,u} = \mathrm{Id}_X$. We set finally

$$J_{\alpha,u} \quad := \quad \left(\Phi_1^{\alpha,u}\right)^\star J_\alpha.$$

With these notations holds the following lemma.

Lemma 30. *The map*

$$\mathcal{K}_J^\omega \times \mathcal{U}_\omega \quad \longrightarrow \quad \mathcal{J}_\omega,$$

$$(\alpha, u) \quad \longmapsto \quad J_{\alpha,u},$$

is well defined and its differential at the origin is given by the fiberwise injection

$$\mathrm{TC}_{\mathcal{K}_J^\omega,0} \oplus \Lambda_{g,J}^{\Omega,\perp} \quad \longrightarrow \quad \mathrm{TC}_{\mathcal{J}_\omega,J}$$

$$(A, v) \quad \longmapsto \quad -J\left[\overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{v} + 2A\right].$$

Proof. Let denote for simplicity $\omega_t := \omega + tdd_{J_\alpha}^c u_1$ and we observe the elementary identities

$$\dot{\omega}_t \quad = \quad dd_{J_\alpha}^c u_1 = -d(\xi_t^{\alpha,u} \neg \omega_t) = -L_{\xi_t^{\alpha,u}} \omega_t.$$

We infer

$$\frac{d}{dt}\left[\left(\Phi_t^{\alpha,u}\right)^\star \omega_t \right] \quad = \quad \left(\Phi_t^{\alpha,u}\right)^\star \left(\dot{\omega}_t + L_{\xi_t^{\alpha,u}} \omega_t \right) = 0,$$

and thus $\left(\Phi_1^{\alpha,u}\right)^\star \omega_1 = \left(\Phi_0^{\alpha,u}\right)^\star \omega_0 = \omega$, i.e.

$$\left(\Phi_1^{\alpha,u}\right)^\star \left(\omega + dd_{J_\alpha}^c u_1\right) \quad = \quad \omega.$$

The fact that the complex structure J_α is integrable implies that the form ω_1 is J_α-invariant. (This is no longer true in the non-integrable case!) We conclude $J_{\alpha,u} \in \mathcal{J}_\omega$.

We compute now the differential at the origin. We consider for this purpose a smooth family $(u(s))_s \subset \mathcal{U}_\omega$ such that $u(0) = 0$ and $\dot{u}(0) = v$. We denote for simplicity $\xi_{t,s} := \xi_t^{0,u(s)}$ and $\Phi_{t,s} := \Phi_t^{0,u(s)}$. Then deriving with respect to s at $s = 0$ the identity

$$\frac{\partial}{\partial t} \Phi_{t,s} \quad = \quad \xi_{t,s} \circ \Phi_{t,s},$$

and using the fact that $\xi_{t,0} = 0$, (which implies in particular $\Phi_{t,0} = \mathrm{Id}_X$) we obtain

$$\begin{aligned}\frac{\partial}{\partial s}_{|_{s=0}} \frac{\partial}{\partial t} \Phi_{t,s} &= \frac{\partial}{\partial s}_{|_{s=0}} \xi_{t,s} + d_x \xi_{t,0} \cdot \frac{\partial}{\partial s}_{|_{s=0}} \Phi_{t,s} \\ &= \frac{\partial}{\partial s}_{|_{s=0}} \xi_{t,s}.\end{aligned}$$

On the other hand deriving with respect to s at $s = 0$ the identity

$$\xi_{t,s} \neg \left(\omega + t d d_J^c u_1 (s) \right) = - \left(d_J^c u_1 (s) + \frac{1}{2} d u_2 (s) \right),$$

we obtain

$$\left(\frac{\partial}{\partial s}_{|_{s=0}} \xi_{t,s} \right) \neg \omega = - \left(d_J^c v_1 + \frac{1}{2} d v_2 \right),$$

and thus

$$\frac{\partial}{\partial s}_{|_{s=0}} \xi_{t,s} = -\frac{1}{2} \nabla_{g,J} \overline{v}.$$

Commuting the derivatives in s and t we infer the identity

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s}_{|_{s=0}} \Phi_{t,s} = -\frac{1}{2} \nabla_{g,J} \overline{v}.$$

Integrating in t from 0 to 1 we deduce

$$\eta := \frac{\partial}{\partial s}_{|_{s=0}} \Phi_{1,s} = -\frac{1}{2} \nabla_{g,J} \overline{v},$$

since $\Phi_{0,s} = \mathrm{Id}_X$. We infer

$$\frac{d}{ds}_{|_{s=0}} J_{0,u(s)} = L_\eta J = -J \overline{\partial}_{T_{X,J}} \nabla_{g,J} \overline{v}.$$

Assume now $(\alpha (s))_s \subset \mathcal{K}_{J,\omega}$ is a smooth curve with $\alpha (0) = 0$ and $\dot{\alpha} (0) = A$. Then

$$\frac{d}{ds}_{|_{s=0}} J_{\alpha(s),u(s)} = \frac{d}{ds}_{|_{s=0}} J_{\alpha(s)} + \frac{d}{ds}_{|_{s=0}} J_{0,u(s)},$$

with

$$\frac{d}{ds}_{|_{s=0}} J_{\alpha(s)} = -2JA,$$

thanks to the properties of the differential of the Caley transform (see [14]). □

Lemma 31. *For any point $J \in \mathcal{J}_\omega$ holds the inclusions*

$$\begin{aligned}&\left[\overline{\partial}_{T_{X,J}} \nabla_{g,J} C^\infty (X, \mathbb{C}) \right] \oplus_\Omega \mathrm{TC}_{\mathcal{K}_J^\omega, 0} \\ \subseteq\ & \mathrm{TC}_{\mathcal{J}_\omega, J} \\ \subseteq\ & \left[\overline{\partial}_{T_{X,J}} \nabla_{g,J} C^\infty (X, \mathbb{C}) \right] \oplus_\Omega \mathrm{TC}_{\mathcal{K}_{J,g}, 0}.\end{aligned}$$

Proof. The first inclusion is a direct consequence of lemma 30. In order to show the second one let $(\varphi_t)_t \subset \mathcal{C}_J \cap B_{\varepsilon_0}^{W_k^{0,1}(T_{X,J})} (0)$ with $\varphi_0 = 0$ and set for notation simplicity $e_t := e (\xi_{\varphi_t})$. With these notations, the identity (21.14) writes as

$$\overline{\partial}_J e_t - (\varphi_t \circ e_t) \cdot \partial_J e_t = - \left[\partial_J e_t - (\varphi_t \circ e_t) \cdot \overline{\partial}_J e_t \right] \cdot \mu (\varphi_t).$$

Time deriving this at $t = 0$ and using the obvious equality $\dot{e}_0 = D\xi(0)\,\dot{\varphi}_0$ we deduce the equality

$$\overline{\partial}_{T_{X,J}}\left[D\,\xi(0)\,\dot{\varphi}_0\right] - \dot{\varphi}_0 \;=\; -\frac{d}{dt}_{|_{t=0}}\mu(\varphi_t)\,.$$

This combined with the identity $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\mu(\varphi_t) \equiv 0$ implies

$$\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\overline{\partial}_{T_{X,J}}\left[D\,\xi(0)\,\dot{\varphi}_0\right] - \overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\dot{\varphi}_0 \;=\; 0.$$

Thus if $\overline{\partial}^{\star_{g,\Omega}}_{T_{X,J}}\dot{\varphi}_0 = 0$ then $D\,\xi(0)\,\dot{\varphi}_0 = 0$ and

$$\dot{\varphi}_0 \;=\; \frac{d}{dt}_{|_{t=0}}\mu(\varphi_t)\,.$$

We infer the equality

$$\left\{A \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \mid \exists\,(J_t)_t \subset \mathcal{J}_{\mathrm{int}} : J_0 = J,\ \dot{J}_0 = A\right\}$$

$$= \quad \left\{A \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \mid \exists\,(\varphi_t)_t \subset \mathcal{C}^{\mathrm{div}}_{J,g} : \varphi_0 = 0,\ \dot{\varphi}_0 = A\right\} \;=\; \mathrm{TC}_{\mathcal{K}_{J,g},0}\,.$$

By gauge transformation we deduce

$$\left\{A \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \mid \exists\,(J_t)_t \subset \mathcal{J}_{\omega} : J_0 = J,\ H_{T_{X,J}}\dot{J}_0 = A\right\}$$

$$\subseteq \quad \left\{A \in \mathcal{H}^{0,1}_{g,\Omega}(T_{X,J}) \mid \exists\,(J_t)_t \subset \mathcal{J}_{\mathrm{int}} : J_0 = J,\ \dot{J}_0 = A\right\},$$

and thus the required inclusion. □

This result combined with the existence of the isomorphism η and with the triple decomposition identity (18.3) implies the inclusions (1.17) and (1.18) in the introduction of the paper.

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