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A note on Berezin-Toeplitz quantization of the Laplace operator

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Abstract: Given a Hodge manifold, it is introduced a self-adjoint operator on the space of endomorphisms of the global holomorphic sections of the polarization line bundle. Such operator is shown to approximate the Laplace operator on functions when composed with Berezin-Toeplitz quantization map and its adjoint, up to an error which tends to zero when taking higher powers of the polarization line bundle.

1 Introduction

Let M be a n-dimensional projective manifold and let g be a Hodge metric on M. This means that M is equipped with a complex structure J and with a positive Hermitian line bundle (L, h). Denoted by Θ the curvature of the Chern connection, the form $\omega = \frac{i}{2\pi}\Theta$ is positive, and it holds $g(u, v) = \omega(u, Jv)$. This setting is precisely that of quantization of compact Kähler manifolds [2, 8, 10, 11]. See Schlichenmaier [13] for an overview of fundamental results and references therein. In this theory, for any integer $m > 0$ is defined a finite-dimensional Hilbert space V_m together with the Berezin-Toeplitz quantization map and its adjoint (see Section 2)

$$T_m : C^\infty(M) \to V_m, \qquad T_m^\star : V_m \to C^\infty(M).$$

The aim of this note is introducing a self-adjoint positive operator (see Section 3)

$$\Delta_m : V_m \to V_m$$

which approximate in a suitable sense, as m grows, the (positive) Laplacian

$$\Delta : C^\infty(M) \to C^\infty(M)$$

associated with the metric g (recall that it is defined by the identity $\Delta(f)\omega^n = -n\, i\partial\bar\partial f \wedge \omega^{n-1}$ for any complex-valued smooth function f on M). To be a little more specific, we will prove that for any smooth function f on M one has the asymptotic expansion

$$T_m^\star \circ \Delta_m \circ T_m(f) = m^{n-1}\Delta f + O(m^{n-2}) \tag{1}$$

as $m \to \infty$. For this reason the operator Δ_m could be thought of as a *quantized Laplacian*. Interestingly, Δ_m depends just on the projective geometry of the Kodaira embedding of M via L^m and the Fubini-Study metric induced by h^m (see Section 3).

Thanks to results available on asymptotic expansions of Bergman kernels [3, 8, 14, 15] and Toeplitz operators [6, 9], the main result we will prove is indeed the following, which obviously implies (1).

Theorem 1.1. *There is a complete asymptotic expansion*

$$T_m^\star \circ \Delta_m \circ T_m(f) = \sum_{r\geq 0} P_r(f) m^{n-1-r} + O(m^{-\infty}),$$

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where P_r are self-adjoint differential operators on $C^\infty(M)$. More precisely, for any $k, R \geq 0$ there exist constants $C_{k,R,f}$ such that

$$\left\| T_m^\star \circ \Delta_m \circ T_m(f) - \sum_{r=0}^{R} P_r(f) m^{n-1-r} \right\|_{C^k(M)} \leq C_{k,R,f} m^{n-R-2}.$$

Moreover one has

$$P_0(f) = \Delta f, \qquad P_1(f) = -\frac{1}{2\pi}\Delta^2 f.$$

The construction of the quantized Laplacian Δ_m was inspired by a work of J. Fine on the Hessian of the Mabuchi energy [4]. Even though in principle Δ_m is unrelated to the problem of finding canonical metrics on M, when ω is balanced in the sense of Donaldson (see definition recalled at the end of Section 4) the relation between Δ_m and Δ is even more apparent as shown by the following

Theorem 1.2. *If ω is m-balanced then*

$$\Delta_m(A) = C\, T_m \circ \Delta \circ T_m^\star(A)$$

for al $A \in V_m$, where $C = \frac{m^{n-1}\left(\int_M \frac{\omega^n}{n!}\right)^2}{\left(\dim H^0(M, L^m)\right)^2}$.

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2 Preliminaries

Quantization of Kähler manifolds is an extensively studied topic (see [13] and references therein). This section is intended to fix the notation and recall some basic facts of the theory that will be useful in the following.

The space V_m mentioned in the introduction is nothing but $\mathrm{End}(H_m)$, being $H_m = H^0(M, L^m)$ the space of the holomorphic sections of L^m. By Riemann-Roch theorem $\dim V_m$ grows like a positive multiple of m^{2n} when $m \to \infty$. The space H_m is equipped with a Hermitian inner product b_m induced by the Hermitian metric h^m on L^m and the Kähler form ω. Explicitly it is given by

$$b_m(s,t) = \int_M h^m(s,t)\frac{\omega^n}{n!} \tag{2}$$

for all $s, t \in H_m$. Thus V_m is a Hermitian vector space with inner product defined by

$$\langle A, B \rangle = \mathrm{tr}(AB^\star), \tag{3}$$

for all $A, B \in V_m$. Here $B^\star$ denotes the adjoint of B with respect to b_m.

The map $T_m : C^\infty(M) \to V_m$ mentioned above is the well known Berezin-Toeplitz quantization operator [2]. Given a smooth function f on M, the operator $T_m(f)$ is the composition $T_m(f) = P \circ M(f)$, where $M(f) : H_m \to \Gamma(M, A^m)$ is the multiplication by f:

$$M(f)(s) = fs,$$

and $P : \Gamma(M, A^m) \to H_m$ is the orthogonal projection with respect to the obvious extension of the inner product b_m to smooth sections. The space of smooth functions $C^\infty(M)$ is equipped with the L^2-product induced by ω, given by

$$\langle f, g\rangle = \int_M f\bar{g}\frac{\omega^n}{n!}, \tag{4}$$

for all $f, g \in C^\infty(M)$. Let $T_m^\star : V_m \to C^\infty(M)$ be the adjoint of T_m. The relationship between $T_m^\star$ and the so-called covariant Berezin symbol has been firstly highlighted by Schlichenmaier [12, Theorem 3.1]. With notation introduced above, his result reads as follows.

Lemma 2.1. *Let $\{s_\alpha\}$ be an orthonormal basis of H_m. For all $A \in V_m$ it holds:*

$$T_m^\star(A) = \sum_\alpha h^m(As_\alpha, s_\alpha).$$

Proof. For convenience of the reader we recall here the Schlichenmaier's argument in our notation. For every $f \in C^\infty(M)$ one has

$$\operatorname{tr}(A\, T_m(f)^\star) = \sum_\alpha b_m\left(As_\alpha, T_m(f)s_\alpha\right).$$

Substituting

$$T_m(f)s_\alpha = \sum_\beta \left(\int_M fh^m(s_\alpha, s_\beta)\frac{\omega^n}{n!}\right)s_\beta,$$

it follows

$$\operatorname{tr}(A\, T_m(f)^\star) = \sum_\alpha \int_M \bar{f}(x)h^m(As_\alpha, s_\alpha)\frac{\omega^n}{n!},$$

which gives the thesis by arbitrariness of f after noting that

$$\int_M T_m^\star(A)\bar{f}\frac{\omega^n}{n!} = \operatorname{tr}(A\, T_m(f)^\star).$$

□

Note that the map $T_m^\star$ takes an endomorphisms $A \in V_m$ to the restriction to the diagonal of its integral kernel. More precisely, given an orthonormal basis $\{s_\alpha\}$ of H_m, the integral kernel of A is the smooth section $K(A)$ of $L^m \boxtimes L^{-m}$ over $M \times M$ given by

$$K(A)(x, y) = \sum_{\alpha,\beta}\int_M h^m(As_\alpha, s_\beta)(z)s_\beta(y) \otimes s_\alpha^\star(x)\frac{\omega_z^n}{n!},$$

where $s_\alpha^\star(x)$ is the metric dual of $s_\alpha(x)$ in the fiber of L^m over the point x. The restriction of the kernel to the diagonal is (naturally identified with) the smooth function $T_m^\star(A)$ thanks to Lemma 2.1. When A is of the form $T_m(f)$ for some smooth function f, the integral kernel is given by

$$K(T_m(f))(x, y) = \sum_{\alpha,\beta}\int_M f(z)h^m(s_\alpha, s_\beta)(z)s_\beta(y) \otimes s_\alpha^\star(x)\frac{\omega_z^n}{n!},$$

whence

$$T_m^\star \circ T_m(f)(x) = \sum_{\alpha,\beta}\int_M f(z)h^m(s_\alpha, s_\beta)(z)h^m(s_\beta, s_\alpha)(x)\frac{\omega_z^n}{n!}.$$

For a constant function $f = c \in \mathbf{R}$, one has

$$T_m^\star \circ T_m(c) = c\,\rho_m,$$

where $\rho_m = \sum_\alpha |s_\alpha|^2_{h^m}$ is the so-called Bergman kernel (along the diagonal) of ω, also known as Rawnsley's ϵ-function (originally introduced as η-function [10] and θ-function [11]).

3 The operator Δ_m

In this section we introduce the quantized Laplacian and we describe some of its fundamental properties.

The quantized Laplacian $\Delta_m : V_m \to V_m$ is a self-adjoint operator which depends just on projective geometry of M in $\mathbf{P}(H_m)$. Consider the embedding

$$\iota_m : M \to \mathbf{P}(H_m),$$

given by the Kodaira map of M in $\mathbf{P}(H_m^\star)$ induced by L^m, followed by the isomorphism $\mathbf{P}(H_m^\star) \simeq \mathbf{P}(H_m)$ induced by the Hermitian product b_m. Every endomorphism A of H_m induces a (holomorphic) vector field $v(A)$ on $\mathbf{P}(H_m)$ whose flow is given by

$$\Phi^t_{v(A)}(z) = e^{tA}z.$$

Let Λ_m be the hyperplane bundle on $\mathbf{P}(H_m)$, endowed with the Hermitian metric induced by b_m, and let g_m be the pull-back to M of the associated Fubini-Study metric on $\mathbf{P}(H_m)$. One can restrict $v(A)$ to M as a section of $\iota_m^\star T\mathbf{P}(H_m)$, and then project orthogonally to $TM \subset \iota_m^\star T\mathbf{P}(H_m)$ to get a smooth vector field $e_m(A)$ on M. This defines a map

$$e_m : V_m \to \Gamma(TM).$$

Recall that V_m has an inner product defined by (3). On the other hand, $\Gamma(TM)$ is equipped with the L^2-inner product induced by the Kähler metric g_m:

$$(\eta, \xi)_m = \int_M g_m(\eta, \xi)\frac{\omega_m^n}{n!},$$

for all $\eta, \xi \in \Gamma(TM)$ (here ω_m is the Käler form of g_m, i.e. the pull-back of the Fubini-study form to M). Thus one can form the adjoint operator

$$e_m^\star : \Gamma(TM) \to V_m,$$

and finally define

$$\Delta_m = e_m^\star \circ e_m. \tag{5}$$

The next lemma shows that the vector field $e_m(A)$ and the function $T_m^\star(A)$ are related through the projectively induced Kähler metric g_m.

Lemma 3.1. *For all $A \in V_m$ one has*

$$e_m(A) = \mathrm{grad}_m \frac{T_m^\star(A)}{\rho_m},$$

where the gradient is taken with respect to the Riemannian metric g_m.

Proof. We have to show that $g_m\left(e_m(A), v\right) = v\left(T_m^\star(A)/\rho_m\right)$ for all vector field $v \in \Gamma(TM)$. In order to do this, consider a smooth extension $\tilde{v}$ of v to a smooth vector field of $\mathbf{P}(H_m)$. Since g_m is induced by the Fubini-Study metric g_{FS} on $\mathbf{P}(H_m)$, and $e_m(A)$ is the orthogonal projection of $v(A)$ on TM, one has

$$g_m(e_m(A), v) = \iota_m^\star\, g_{FS}(v(A), \tilde{v}). \tag{6}$$

The right hand side of the equation above can be related to a function on $\mathbf{P}(H_m)$ naturally associated to A. Indeed we claim that $v(A)$ is the gradient of the function μ_A defined by

$$\mu_A(s) = \frac{b_m\left(As, s\right)}{b_m(s, s)}.$$

For the reader's convenience a proof of this (standard) fact is given below. Now we go ahead taking the claim for grant. From (6) one gets

$$g_m(e_m(A), v) = v(\iota_m^\star \mu_A),$$

thus it remains to prove the identity

$$\iota_m^\star \mu_A = T_m^\star(A)/\rho_m. \tag{7}$$

To this end, let $\{s_\alpha\}$ be an orthonormal basis of H_m, so that the pull-back of μ_A to M is given by

$$\iota_m^\star \mu_A(x) = \frac{\sum_{\alpha,\beta} s_\alpha(x)\overline{s_\beta(x)} b_m(As_\alpha, s_\beta)}{\sum_\gamma |s_\gamma(x)|^2},$$

where the ratio $\frac{s_\alpha(x)\overline{s_\beta(x)}}{\sum_\gamma |s_\gamma(x)|^2}$ is well defined and can be computed choosing an arbitrary Hermitian metric on the line bundle L^m. In particular, taking h^m it becomes $\frac{h^m(s_\alpha,s_\beta)(x)}{\sum_\gamma |s_\gamma|^2_{h^m}(x)}$, whence

$$\iota_m^\star \mu_A(x) = \frac{\sum_{\alpha,\beta} \int_M h^m(As_\alpha, s_\beta)(z) h^m(s_\alpha, s_\beta)(x) \frac{\omega_z^n}{n!}}{\sum_\gamma |s_\gamma|^2_{h^m}(x)},$$

and the identity (7) follows by definition of ρ_m and Lemma 2.1.

Finally, in order to prove the claim above, let (z_α) be homogeneous coordinates on $\mathbf{P}(H_m)$ corresponding to the basis $\{s_\alpha\}$. The function μ_A then takes the form

$$\mu_A(z) = \frac{\bar{z} A z^t}{|z|^2},$$

where now $A = (A_{\alpha\beta})$ denotes the matrix that represents the endomorphism A with respect the chosen basis. The equality between $v(A)$ and the gradient of μ_A can be proved in local affine coordinates, but here we consider the projection of $H_m \setminus \{0\}$ on $\mathbf{P}(H_m)$, and the fact that $v(A)$, g_{FS} and μ^A lift to $\mathbf{C}^\star$-invariant objects (which will be denotes with the same symbols). In particular one has

$$v(A) = \sum_{\alpha,\beta} A_{\alpha\beta} \left(z_\alpha \frac{\partial}{\partial z_\beta} + \bar{z}_\beta \frac{\partial}{\partial \bar{z}_\alpha} \right),$$

and

$$g_{FS} = \sum_i \frac{dz_i d\bar{z}_i}{|z|^2} - \sum_{i,j} \frac{\bar{z}_i z_j dz_i d\bar{z}_j}{|z|^4},$$

whence

$$i_{v(A)} g_{FS} = \sum_{\alpha,\beta} A_{\alpha\beta} \left(\frac{z_\alpha d\bar{z}_\beta + \bar{z}_\beta dz_\alpha}{|z|^2} - \frac{z_\alpha \bar{z}_\beta d|z|^2}{|z|^4} \right) = d\mu_A,$$

which proves the claim. □

Next lemma characterizes the kernel of Δ_m.

Lemma 3.2. $\Delta_m(A) = 0$ *if and only if* A *is a multiple of the identity.*

Proof. By definition $\Delta_m = e_m^\star \circ e_m$, and by Lemma 3.1 and its proof it follows $e_m(A) = \mathrm{grad}_m \frac{T_m^\star(A)}{\rho_m}$. Thus $\Delta_m(A) = 0$ if and only if $\frac{T_m^\star(A)}{\rho_m} = c$ for some $c \in \mathbf{C}$. Let $I \in V_m$ be the identity. The identity $T_m^\star(I) = \rho_m$ implies $\Delta_m(A) = 0$ if and only if $A - cI \in \ker T_m^\star$, thus the thesis follows by injectivity of $T_m^\star$ [2, Proposition 4.1].

Alternatively, one can argue more geometrically as follows. In the proof of Lemma 3.1 has been introduced a smooth function μ_A on $\mathbf{P}(H_m)$ satisfying $\frac{T_m^\star(A)}{\rho_m} = \iota_m^\star \mu_A$. Thus by Lemma 3.1 one has $\Delta_m(A) = 0$ if and only if $\iota_m^\star d\mu_A = 0$. Then the thesis follows by showing that the locus where $d\mu_A = 0$ contains no positive dimensional holomorphic submanifolds (or, in other words, $\ker d\mu_A$ is totally real), unless μ_A is constant. □

Now we pass to give a more explicit description of the operator Δ_m. To this end fix an orthonormal basis $\{s_\alpha\}$ of H_m and let (z_i) be the corresponding homogeneous coordinates on $\mathbf{P}(H_m)$. Moreover this identifies V_m with the space of $\dim H_m \times \dim H_m$ complex matrices. Consider the map

$$\Psi_m : \mathbf{P}(H_m) \to V_m$$

defined by $\Psi_m(z) = \frac{z\bar{z}^t}{|z|^2}$. Note that by Lemma 2.1 it follows that

$$\iota_m^\star \operatorname{tr}(A\Psi_m) = \frac{T_m^\star(A)}{\rho_m} \tag{8}$$

for all $A \in V_m$. On the other hand, by definition of Δ_m one has

$$\operatorname{tr}(\Delta_m(A)B^\star) = \int_M g_m(e_m(A), e_m(B))\frac{\omega_m^n}{n!},$$

thus by Lemma 3.1 together with (8) one gets

$$\int_M g_m(e_m(A), e_m(B))\frac{\omega_m^n}{n!} = \int_M i\partial \operatorname{tr}(\Psi_m A) \wedge \bar\partial \operatorname{tr}(\Psi_m B^\star) \wedge \frac{\omega_m^{n-1}}{(n-1)!}$$

whence

$$\operatorname{tr}(\Delta_m(A)B^\star) = \int_M i\partial \operatorname{tr}(\Psi_m A) \wedge \bar\partial \operatorname{tr}(\Psi_m B^\star) \wedge \frac{\omega_m^{n-1}}{(n-1)!}.$$

Let

$$\Phi_m : \mathbf{P}(H_m) \to V_m^\star$$

be the map obtained by composing Φ with the dual paring induced by the Hermitian metric b_m on V_m. More explicitiely one has

$$\Phi_m(z)(A) = \operatorname{tr}(\Psi_m(z)A)$$

for all $A \in V_m$ and $z \in \mathbf{P}(H_m)$. The computation above yields the following

Proposition 3.3. *Consider the* $\operatorname{End}(V_m)$*-valued differential form on* $\mathbf{P}(H_m)$ *defined by*

$$\Xi_m = i\partial\Phi_m \wedge \bar\partial\Psi_m \wedge e^{\omega_{FS}}.$$

Then it holds

$$\Delta_m = \int_M \Xi_m.$$

Here $e^{\omega_{FS}}$ is a mixed-degree form defined by the exponential series. Since $\omega_{FS}^k = 0$ for all $k \geq \dim H_m$, one has

$$e^{\omega_{FS}} = 1 + \omega_{FS} + \frac{\omega_{FS}^2}{2} + \cdots + \frac{\omega_{FS}^{\dim H_m - 1}}{(\dim H_m - 1)!}.$$

This implies that Ξ_m has mixed degree. More interestingly it depends just on the dimension of $\mathbf{P}(H_m)$ (and on a choice of homogeneous coordinates) and it is independent of M.

Corollary 3.4.

$$\operatorname{tr}(\Delta_m) = 2\pi n\, m^n \int_M \frac{\omega^n}{n!}.$$

Proof. Recall that we identified V_m with the space of $\dim H_m \times \dim H_m$ matrices by choosing an orthonormal basis $\{s_\alpha\}$ of H_m. The set of canonical matrices E_{ij}, then form an orthonormal basis of V_m. Thus by Proposition

3.3 one has

$$\begin{aligned}
\operatorname{tr}(\Delta_m) &= \sum_{\alpha,\beta} \langle \Delta_m(E_{\alpha\beta}), E_{\alpha\beta}\rangle \\
&= \sum_{\alpha,\beta} \int_M i\partial\left(\frac{z_\alpha \bar z_\beta}{|z|^2}\right) \wedge \bar\partial\left(\frac{z_\beta \bar z_\alpha}{|z|^2}\right) \wedge e^{\omega_{FS}} \\
&= \int_M \left(\frac{i\partial\bar\partial|z|^2}{|z|^2} - \frac{i\partial|z|^2 \wedge \bar\partial|z|^2}{|z|^4}\right) \wedge e^{\omega_{FS}} \\
&= 2\pi \int_M \omega_{FS} \wedge e^{\omega_{FS}} \\
&= 2\pi n \int_M \frac{\omega_m^n}{n!},
\end{aligned}$$

whence the thesis follows since ω_m is cohomologous to $m\omega$. □

4 Proof of Theorems 1.1 and 1.2

First of all we recall a fundamental result on asymptotic expansion of Berezin-Toeplitz quantization map which is originally due to Karabegov and Schlichenmaier [6].

Theorem 4.1. *There is a sequence $\{b_r\}$ of self-adjoint differential operators acting on $C^\infty(M)$ such that for any smooth function $f \in C^\infty(M)$ one has the asymptotic expansion*

$$T_m^\star \circ T_m(f) = \sum_{r\geq 0} b_r(f) m^{n-r} + O(m^{-\infty}), \tag{9}$$

and for any $k, R \geq 0$ there exist constants $C_{k,R,f}$ such that

$$\left\| T_m^\star \circ T_m(f) - \sum_{r=0}^{R} b_r(f) m^{n-r} \right\|_{C^k(M)} \leq C_{k,R,f}\, m^{n-R-1}.$$

Moreover one has

$$\begin{aligned}
b_0(f) &= f, \\
b_1(f) &= \frac{\operatorname{scal}(g)}{8\pi} f - \frac{1}{4\pi}\Delta f.
\end{aligned}$$

Proof. The original proof is due to Karabegov and Schlichenmaier [6]. Actually they prove the existence of an asymptotic expansion for the Berezin transform $T_m^\star \circ T_m(f)/\rho_m$, but the statement above follows directly from their result together with the well-known expansion for ρ_m [3, 14, 15]. For a proof with normalization used above we refer to Ma and Marinescu [9, Theorem 0.1]. The only fact one still needs to show is self-adjointness of the operator b_r. It follows readily by self-adjointness of $T_m^\star \circ T_m$ and expansion (9). Indeed one has

$$0 = \sum_{r=0}^{R} m^{n-r} \int_M \left(b_r(f)\, \bar g - f\, \overline{b_r(g)} \right) \frac{\omega^n}{n!} + O(m^{n-R-1}),$$

as $m \to +\infty$, for all $f, g \in C^\infty(M)$. □

Since the Bergman kernel satisfies $\rho_m = T_m^\star \circ T_m(1)$, one recovers the well known asymptotic expansion [3, 5, 8, 9, 14, 15]

$$\rho_m = \sum_{r\geq 0} a_r m^{n-r} + O(m^{-\infty}), \tag{10}$$

where $a_r = b_r(1) \in C^\infty(M)$ depends polynomially in the curvature of g and its covariant derivatives. In particular

$$a_0 = 1, \qquad a_1 = \frac{\mathrm{scal}(g)}{8\pi}. \tag{11}$$

The next lemma express the quantized Laplacians in terms of objects for which an asymptotic expansion is known.

Lemma 4.2. *For any $A \in V_m$ one has*

$$\Delta_m(A) = T_m\left(\frac{\omega_m^n}{\rho_m\,\omega^n}\Delta_{g_m}\left(\frac{T_m^\star(A)}{\rho_m}\right)\right),$$

where Δ_{g_m} denotes the Laplacian of the metric g_m.

Proof. By definition of Δ_m, for any $B \in V_m$ it holds

$$\mathrm{tr}(\Delta_m(A)B^\star) = \int_M g_m(e_m(A), e_m(B))\frac{\omega_m^n}{n!},$$

whence, by Lemma 3.1 and integration by parts it follows

$$\mathrm{tr}(\Delta_m(A)B^\star) = \int_M \Delta_{g_m}\left(\frac{T_m^\star(A)}{\rho_m}\right)\frac{\overline{T_m^\star(B)}}{\rho_m}\frac{\omega_m^n}{n!}.$$

The right hand side can be rewritten as

$$\int_M \frac{\omega_m^n}{\rho_m\,\omega^n}\Delta_{g_m}\left(\frac{T_m^\star(A)}{\rho_m}\right)\overline{T_m^\star(B)}\frac{\omega^n}{n!} = \mathrm{tr}\left(T_m\left(\frac{\omega_m^n}{\rho_m\,\omega^n}\Delta_{g_m}\left(\frac{T_m^\star(A)}{\rho_m}\right)\right)B^\star\right),$$

whence the statement follows by arbitrariness of B. □

Now we can prove the Theorem 1.1 as follows. For any $f \in C^\infty(M)$, by Lemma above one has

$$T_m^\star \circ \Delta_m \circ T_m(f) = T_m^\star \circ T_m\left(\frac{\omega_m^n}{\rho_m\,\omega^n}\Delta_{g_m}\left(\frac{T_m^\star \circ T_m(f)}{\rho_m}\right)\right), \tag{12}$$

thus the statement of Theorem 1.1 follows readily by Theorem 9 and asymptotic expansion (10). In particular one has

$$T_m^\star \circ T_m(f) = m^n f + m^{n-1}b_1(f) + O(m^{n-2})$$

whence

$$\begin{aligned}
\rho_m &= m^n + m^{n-1}a_1 + O(m^{n-2}),\\
\frac{T_m^\star \circ T_m(f)}{\rho_m} &= f + m^{-1}(b_1(f) - a_1 f) + O(m^{-2})\\
\omega_m &= m\omega + O(m^{-1}),\\
\frac{\omega_m^n}{\rho_m\omega^n} &= 1 - m^{-1}a_1 + O(m^{-2}),\\
\Delta_{g_m}(f) &= m^{-1}\Delta(f) + O(m^{-3}).
\end{aligned}$$

Substituting in (12) finally gives

$$\begin{aligned}
T_m^\star \circ \Delta_m \circ T_m(f) &= T_m^\star \circ T_m\left(\left(1 - m^{-1}a_1\right)m^{-1}\Delta\left(f + m^{-1}(b_1(f) - a_1 f)\right) + O(m^{-3})\right)\\
&= m^{-1}T_m^\star \circ T_m\left(\Delta(f) + m^{-1}\left(\Delta(b_1(f)) - \Delta(a_1 f) - a_1\Delta(f)\right) + O(m^{-2})\right)\\
&= m^{n-1}\Delta f + m^{n-2}\left(\Delta(b_1(f)) + b_1(\Delta(f)) - \Delta(a_1 f) - a_1\Delta(f)\right) + O(m^{n-3}).
\end{aligned}$$

This obviously proves $P_0 = \Delta$ and, recalling that $b_1(f) = \frac{\mathrm{scal}(g)}{8\pi} f - \frac{1}{4\pi}\Delta(f)$ and $a_1 = \frac{\mathrm{scal}(g)}{8\pi}$, it gives

$$\begin{aligned} 8\pi P_1(f) &= \Delta(\mathrm{scal}(g)f - 2\Delta(f)) + \mathrm{scal}(g)\Delta(f) - 2\Delta^2(f) - \Delta(\mathrm{scal}(g)f) - \mathrm{scal}(g)\Delta(f) \\ &= -4\Delta^2(f), \end{aligned}$$

which concludes the proof of Theorem 1.1.

Now we turn to the proof of Theorem 1.2. Balanced metrics have been introduced by Donaldson in connection with the existence problem of constant scalar curvature Kähler metric on polarized manifolds [1]. Recall that a metric is called *m-balanced* if the Bergman kernel ρ_m is constant. Note that the value of such a constant is not arbitrary for ρ_m satisfies $\int_M \rho_m \frac{\omega^n}{n!} = \dim H^0(M, L^m)$. Moreover, since in general one has $\omega_m = m\omega + \frac{i}{2\pi}\partial\bar{\partial}\log\rho_m$, ω is m-balanced if and only if $\omega_m = m\omega$. Thus, assuming that ω is m-balanced, by Lemma 4.2 for any $A \in V_m$ one has

$$\begin{aligned} \Delta_m(A) &= \rho_m^{-2}\, T_m\left(\frac{\omega_m^n}{\omega^n}\Delta_{g_m}\left(T_m^\star(A)\right)\right) \\ &= \frac{m^{n-1}\left(\int_M \frac{\omega^n}{n!}\right)^2}{\left(\dim H^0(M, L^m)\right)^2}\, T_m \circ \Delta \circ T_m^\star(A), \end{aligned}$$

which proves Theorem 1.2.

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