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Compact lcK manifolds with parallel vector fields

Abstract: We show that for $n > 2$ a compact locally conformally Kähler manifold (M^{2n}, g, J) carrying a non-trivial parallel vector field is either Vaisman, or globally conformally Kähler, determined in an explicit way by a compact Kähler manifold of dimension $2n - 2$ and a real function.

1 Introduction

A locally conformally Kähler (lcK) manifold is a Hermitian manifold (M, g, J) of real dimension $2n \geq 4$ such that around each point, g is conformal to a Kähler metric relative to J, cf. [3].

The differentials of the (logarithms of the) conformal factors glue up to a well-defined closed 1-form on M – called the Lee form of the lcK structure – which is exact if and only if (M, g, J) is globally conformally Kähler.

Many complex manifolds which for topological reasons do no carry any Kähler metric, have compatible lcK metrics. For example the product metric on the Hopf manifold $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$ (with odd first Betti number) is lcK with respect to the complex structure induced from the identification

$$\mathbb{S}^1 \times \mathbb{S}^{2n-1} \simeq (\mathbb{R}/\mathbb{Z} \times \mathbb{S}^{2n-1}) \simeq (\mathbb{R} \times \mathbb{S}^{2n-1})/\mathbb{Z} \simeq (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}.$$

The Lee form of this structure is easily computed to be the length element of $\mathbb{S}^1$, and is therefore parallel. Compact lcK manifolds with parallel Lee form are called *Vaisman* [8], and their structure is well-understood: they are mapping tori of automorphisms of Sasakian manifolds cf. [7]. Moreover, it was recently proved that every compact *homogeneous* lcK manifold is Vaisman [4].

In real dimension 4 it is well known that a compact complex manifold carries a compatible Kähler metric if and only if its first Betti number is even [2], [6]. It was generally believed that every complex surface with odd first Betti number would carry a compatible lcK structure, until Belgun has shown that some Inoue surfaces do not carry any lcK structure [1]. He also showed that every Hopf surface admits a compatible lcK metric – for primary Hopf surfaces this had been previously proved by Gauduchon and Ornea [5] –, and classified all Vaisman complex surfaces.

In this paper we address the following question: *Are there non-Vaisman compact lcK manifolds which carry a non-trivial parallel 1-form?* It turns out that the answer to this question is positive, and moreover, one can describe the lcK structure of such manifolds in a very explicit way in all dimensions greater than 4 (cf. Theorem 3.5 below). These manifolds are globally conformally Kähler, but the metric is not Kähler in general. In dimension 4 this construction still gives examples of non-Vaisman lcK manifolds carrying a parallel 1-form (cf. Example 3.6), but we do not know whether these are the only examples.

A more general problem, which however will not be considered here, would be to describe all compact lcK manifolds with special holonomy (e.g. with reducible holonomy, or whose holonomy group belongs to

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the Berger list). Note that unlike Kähler manifolds, the Riemannian product of lcK manifolds is no longer lcK (at least not in a canonical way). This somehow indicates that the holonomy reduction of a lcK metric is a strong condition, which might lead in general to classification results in the vein of Theorem 3.5.

2 Some preliminaries on lcK manifolds

As explained in the introduction, a lcK manifold is a Hermitian manifold (M, g, J) of real dimension $2n \geq 4$ carrying an open cover U_α and real maps $f_\alpha : U_\alpha \to \mathbb{R}$ such that $(U_\alpha, e^{-f_\alpha} g, J)$ are Kähler manifolds. Denoting $\Omega(\cdot, \cdot) := g(J\cdot, \cdot)$ the fundamental form of M, the above condition yields

$$0 = d(e^{-f_\alpha} \Omega) = e^{-f_\alpha}(-df_\alpha \wedge \Omega + d\Omega). \tag{2.1}$$

Since the linear map $\Lambda^1 M \to \Lambda^3 M$ defined by $\sigma \mapsto \sigma \wedge \Omega$ is injective, (2.1) shows that $df_\alpha = df_\beta$ on $U_\alpha \cap U_\beta$, so the 1-forms df_α glue together to a closed form θ on M – called the *Lee form* – such that $\theta|_{U_\alpha} = \frac{1}{2} df_\alpha$ for all α (the factor $\frac{1}{2}$ is introduced in order to simplify subsequent formulas). The Levi-Civita covariant derivatives ∇ and ∇^α of the conformal metrics g and $e^{-f_\alpha} g$ on U_α are related by the well known formula

$$\nabla_X Y = \nabla_X^\alpha Y + \theta(X)Y + \theta(Y)X - \theta^\sharp g(X, Y),$$

where $\theta^\sharp$ is the vector field dual to θ via the metric g. Using the fact that $\nabla^\alpha J = 0$ on U_α, we thus obtain:

$$
\begin{aligned}
(\nabla_X J)(Y) &= \nabla_X(JY) - J(\nabla_X Y) \\
&= \nabla_X^\alpha(JY) + \theta(X)JY + \theta(JY)X - \theta^\sharp g(X, JY) - J\left(\nabla_X^\alpha Y + \theta(X)Y + \theta(Y)X - \theta^\sharp g(X, Y)\right) \\
&= \theta(JY)X - \theta^\sharp g(X, JY) - \theta(Y)JX + J\theta^\sharp g(X, Y).
\end{aligned}
$$

Identifying 1-forms with vectors using the metric $g =: \langle \cdot, \cdot \rangle$, this relation can be equivalently written as

$$(\nabla_X J)Y = \langle X, Y \rangle J\theta + \theta(JY)X + \langle JX, Y \rangle \theta - \theta(Y)JX \qquad \forall X, Y \in TM, \tag{2.2}$$

or else

$$\nabla_X \Omega = X \wedge J\theta + JX \wedge \theta \qquad \forall X \in TM. \tag{2.3}$$

If e_i denotes a local orthonormal basis of TM we have $\Omega = \frac{1}{2} \sum_i e_i \wedge Je_i$, so by (2.3) we immediately get

$$d\Omega = \sum_i e_i \wedge \nabla_{e_i} \Omega = 2\theta \wedge \Omega, \tag{2.4}$$

which also follows from (2.1).

3 Parallel vector fields on lcK manifolds

Assume throughout this section that that the dimension of M is strictly larger than 4 and that V is a non-trivial parallel vector field on M. We can of course rescale V such that it has unit length. Consider the components of θ along V and JV:

$$a := \theta(V), \qquad b := \theta(JV). \tag{3.1}$$

Since $\nabla V = 0$ we have $\nabla_X(JV) = (\nabla_X J)V$, so using (2.3) we get

$$\nabla_X(JV) = \langle X, V \rangle J\theta + bX + \langle JX, V \rangle \theta - aJX \qquad \forall X \in TM. \tag{3.2}$$

In particular we have

$$\nabla_V(JV) = J\theta + bV - aJV. \tag{3.3}$$

We also infer from (3.2)

$$d(JV) = e_i \wedge \nabla_{e_i}(JV) = V \wedge J\theta - JV \wedge \theta - 2a\Omega, \tag{3.4}$$

whence using (2.4) and (3.4):

$$
\begin{aligned}
0 &= d^2(JV) = -V \wedge d(J\theta) - d(JV) \wedge \theta - 2da \wedge \Omega - 4a\theta \wedge \Omega \\
&= -V \wedge d(J\theta) - V \wedge J\theta \wedge \theta + 2a\Omega \wedge \theta - 2da \wedge \Omega - 4a\theta \wedge \Omega \\
&= -V \wedge \left(d(J\theta) + J\theta \wedge \theta\right) - 2\left(da + a\theta\right) \wedge \Omega.
\end{aligned}
$$

Taking the exterior product with V in this relation yields

$$V \wedge (da + a\theta) \wedge \Omega = 0.$$

The assumption $n > 2$ implies that $V \wedge (da + a\theta) = 0$, so there exists some function f on M such that

$$da + a\theta = fV. \tag{3.5}$$

Since θ is closed and V is parallel, the Cartan formula yields

$$\nabla_V \theta = \mathcal{L}_V \theta = d(V \lrcorner \theta) = da = fV - a\theta, \tag{3.6}$$

and a direct computation using (2.2) gives

$$\nabla_V(J\theta) = \theta + \left(f - |\theta|^2\right) JV. \tag{3.7}$$

Since V is parallel we have $R_{V,X} = 0$ for every vector field X, where $R_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ denotes the curvature tensor of ∇. Consequently, taking X to be ∇-parallel at some point $x \in M$, we obtain

$$\nabla_V \nabla_X \Omega - \nabla_X \nabla_V \Omega = R_{V,X}\Omega = [R_{V,X}, \Omega] = 0 \tag{3.8}$$

at x, where the bracket in the last formula means the commutator of endomorphisms of TM. Using (2.2), (2.3), (3.6) and (3.7) we compute at x:

$$
\begin{aligned}
\nabla_V \nabla_X \Omega &= \nabla_V(X \wedge J\theta + JX \wedge \theta) = X \wedge \nabla_V(J\theta) + JX \wedge \nabla_V \theta + (\nabla_V J)X \wedge \theta \\
&= X \wedge \left(b\theta + \left(f - |\theta|^2\right) JV\right) + JX \wedge (fV - a\theta) + \left(\langle V, X\rangle J\theta + \theta(JX)V + \langle JV, X\rangle\theta - \theta(X)JV\right) \wedge \theta
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_X \nabla_V \Omega &= \nabla_X(V \wedge J\theta + JV \wedge \theta) = V \wedge \nabla_X(J\theta) + JV \wedge \nabla_X \theta + (\nabla_X J)V \wedge \theta \\
&= V \wedge \nabla_X(J\theta) + JV \wedge \nabla_X \theta + (\langle X, V\rangle J\theta + bX + \langle JX, V\rangle\theta - aJX) \wedge \theta
\end{aligned}
$$

After straightforward simplifications we get from (3.8):

$$
\begin{aligned}
0 &= \nabla_V \nabla_X \Omega - \nabla_X \nabla_V \Omega \\
&= \left(\nabla_X(J\theta) + fJX - \theta(JX)\theta\right) \wedge V + \left(\nabla_X \theta + \left(f - |\theta|^2\right) X + \theta(X)\theta\right) \wedge JV.
\end{aligned}
$$

This relation is tensorial in X, so it actually holds at every point of M.

Remark now that if $A \wedge V + B \wedge JV = 0$ for some vectors A and B, then both vectors belong to the plane generated by V and JV. The previous relation thus shows that there exist some 1-forms μ and v such that

$$\nabla_X \theta + \left(f - |\theta|^2\right) X + \theta(X)\theta = \mu(X)V + v(X)JV, \qquad \forall X \in TM. \tag{3.9}$$

We take the exterior product with X in this relation and sum over some local orthonormal basis $X = e_i$. As $d\theta = 0$, we get $\mu \wedge V + v \wedge JV = 0$, hence by the previous remark there exist smooth functions α, β, γ on M such that $\mu = \alpha V - \gamma JV$ and $v = \gamma V + \beta JV$. Taking $X = V$ in Equation (3.9) and using (3.6) yields

$$fV - a\theta + \left(f - |\theta|^2\right) V + a\theta = \alpha V + \gamma JV,$$

whence $\gamma = 0$ and

$$\alpha = 2f - |\theta|^2. \tag{3.10}$$

Equation (3.9) thus becomes

$$\nabla_X \theta = (f - \alpha) X - \theta(X)\theta + \alpha\langle X, V\rangle V + \beta\langle X, JV\rangle JV, \qquad \forall X \in TM. \tag{3.11}$$

Using this relation together with (2.2) we readily obtain

$$\nabla_X(J\theta) = -fJX + \theta(JX)\theta + \alpha\langle X, V\rangle JV - \beta\langle X, JV\rangle V, \qquad \forall X \in TM. \tag{3.12}$$

In particular the exterior derivative of $J\theta$ reads

$$d(J\theta) = e_i \wedge \nabla_{e_i}(J\theta) = -2f\Omega + \theta \wedge J\theta + (\alpha + \beta)V \wedge JV. \tag{3.13}$$

We now take the scalar product with V in (3.12) and obtain

$$\langle\nabla_X(J\theta), V\rangle = f\langle X, JV\rangle + a\theta(JX) - \beta\langle X, JV\rangle, \qquad \forall X \in TM. \tag{3.14}$$

On the other hand

$$\langle\nabla_X(J\theta), V\rangle = \nabla_X\langle J\theta, V\rangle = -X(b),$$

so from (3.14) we obtain

$$db = (\beta - f)JV + aJ\theta. \tag{3.15}$$

Taking the exterior derivative in this equation and using (3.4), (3.5) and (3.13) yields

$$\begin{aligned} 0 \quad &= \quad d^2 b = d(\beta - f) \wedge JV + (\beta - f)(V \wedge J\theta - JV \wedge \theta - 2a\Omega) \\ &\quad + (fV - a\theta) \wedge J\theta + a\left(-2f\Omega + \theta \wedge J\theta + (\alpha + \beta)V \wedge JV\right) \\ &= \quad d(\beta - f) \wedge JV + \beta V \wedge J\theta - (\beta - f)JV \wedge \theta + a(\alpha + \beta)V \wedge JV - 2a\beta\Omega. \end{aligned}$$

This shows in particular that $V \wedge JV \wedge (a\beta\Omega) = 0$, whence

$$a\beta = 0. \tag{3.16}$$

Reinjecting in the previous equation gives

$$d(\beta - f) \wedge JV + \beta V \wedge J\theta - (\beta - f)JV \wedge \theta + a\alpha V \wedge JV = 0. \tag{3.17}$$

We now use (3.13) together with (2.4) and (3.4):

$$\begin{aligned} 0 \quad &= \quad d^2(J\theta) = -2df \wedge \Omega - 4f\theta \wedge \Omega - \theta \wedge \left(-2f\Omega + \theta \wedge J\theta + (\alpha + \beta)V \wedge JV\right) \\ &\quad + d(\alpha + \beta) \wedge V \wedge JV - (\alpha + \beta)V \wedge (V \wedge J\theta - JV \wedge \theta - 2a\Omega) \\ &= \quad 2\left(-df - f\theta + a(\alpha + \beta)V\right) \wedge \Omega + d(\alpha + \beta) \wedge V \wedge JV. \end{aligned}$$

As $n > 2$, this shows that

$$df + f\theta = a\alpha V. \tag{3.18}$$

Using this relation together with (3.17) yields

$$(d\beta + \beta\theta) \wedge JV + \beta V \wedge J\theta = 0.$$

We take the interior product with V in this relation and obtain

$$V(\beta)JV + \beta J\theta + b\beta V = 0.$$

Since by (3.16), JV is orthogonal to $\beta(J\theta + bV)$, this implies that $V(\beta) = 0$ and $\beta(J\theta + bV) = 0$.

We now use (3.11) in order to express the differential of the square norm $|\theta|^2$. For every tangent vector X we have

$$
\begin{aligned}
X(|\theta|^2) &= 2\langle\nabla_X\theta, \theta\rangle = 2\langle(f - \alpha)X - \theta(X)\theta + \alpha\langle X, V\rangle V + \beta\langle X, JV\rangle JV, \theta\rangle \\
&= 2\langle X, (f - \alpha)\theta - |\theta|^2\theta + a\alpha V + b\beta JV\rangle,
\end{aligned}
$$

so from (3.10) we get

$$
d|\theta|^2 = 2\left((f - \alpha)\theta - |\theta|^2\theta + a\alpha V + b\beta JV\right) = 2\left(-f\theta + a\alpha V + b\beta JV\right),
$$

whence using (3.18):

$$
d\alpha = d\left(2f - |\theta|^2\right) = 2(a\alpha V - f\theta) - 2(-f\theta + a\alpha V + b\beta JV) = -2b\beta JV. \tag{3.19}
$$

We are now ready to prove the key result of this section

Lemma 3.1. *If M is compact, the Lee form θ of the lcK structure belongs to the space generated by V and JV. Equivalently,*

$$
\theta = aV + bJV. \tag{3.20}
$$

Proof. Let $d\mu_g$ denote the volume form of M. Taking the trace in (3.2) we get

$$
\delta(JV) = (2 - n)b \tag{3.21}
$$

and from (3.11) together with (3.10) we readily compute

$$
\delta\theta = n(\alpha - f) + |\theta|^2 - \alpha - \beta = (n - 2)(\alpha - f) - \beta. \tag{3.22}
$$

Moreover, taking the scalar product with V in (3.12) and choosing $X = JV$ gives

$$
-JV(b) = \langle\nabla_{JV}J\theta, V\rangle = f - a^2 - \beta,
$$

which together with (3.5) yields $\beta = JV(b) + V(a)$. Using the Stokes Theorem several times we obtain

$$
\int_M f d\mu_g = \int_M \left(V(a) + a^2\right) d\mu_g = \int_M \left(a\delta V + a^2\right) d\mu_g = \int_M a^2 d\mu_g,
$$

and from (3.21)–(3.22)

$$
\begin{aligned}
\int_M (\alpha - f) d\mu_g &= \frac{1}{n-2}\int_M (\beta + \delta\theta)\, d\mu_g = \frac{1}{n-2}\int_M \beta d\mu_g \\
&= \frac{1}{n-2}\int_M \left(V(a) + JV(b)\right) d\mu_g = \frac{1}{n-2}\int_M (a\delta V + b\delta(JV))d\mu_g \\
&= \frac{1}{n-2}\int_M (2 - n)b^2 d\mu_g = -\int_M b^2 d\mu_g,
\end{aligned}
$$

so finally

$$
\begin{aligned}
\int_M |\theta - aV - bJV|^2 d\mu_g &= \int_M (|\theta|^2 - a^2 - b^2)d\mu_g = \int_M (2f - \alpha - a^2 - b^2)d\mu_g \\
&= \int_M (f - a^2 + (f - \alpha) - b^2)d\mu_g = 0.
\end{aligned}
$$

$\square$

From now on M will be assumed compact.

Lemma 3.2. *The following relations hold: $ab = 0, f = a^2, \alpha = a^2 - b^2$.*

Proof. Taking the covariant derivative in (3.20) with respect to some arbitrary vector X and using (3.2), (3.5) and (3.15) yields:

$$
\begin{aligned}
\nabla_X \theta &= X(a)V + X(b)JV + b\nabla_X JV \\
&= f\langle X, V\rangle V - a\langle X, aV + bJV\rangle V + (\beta - f)\langle X, JV\rangle JV + a\langle X, aJV - bV\rangle JV \\
&\quad + b\left(\langle X, V\rangle(aJV - bV) + bX + \langle JX, V\rangle(aV + bJV) - aJX\right) \\
&= b^2 X - abJX + \left(f - a^2 - b^2\right)\langle X, V\rangle V - 2ab\langle X, JV\rangle V \\
&\quad + \left(\beta - f + a^2 - b^2\right)\langle X, JV\rangle JV.
\end{aligned}
$$

Comparing with (3.11) we thus get:

$$
\begin{aligned}
\nabla_X \theta &= (f - \alpha)X - \theta(X)\theta + \alpha\langle X, V\rangle V + \beta\langle X, JV\rangle JV \\
&= (f - \alpha)X + \langle X, aV + bJV\rangle(aV + bJV) + \alpha\langle X, V\rangle V + \beta\langle X, JV\rangle JV
\end{aligned}
$$

and identifying the corresponding terms yields the result. $\qquad\square$

Using Lemma 3.2 we now get from (3.5):

$$
\mathrm{d}a = fV - a\theta = fV - a^2 V - abJV = 0,
$$

thus showing that a is constant on M. We distinguish two cases:

    Case 1: $a \neq 0$. From Lemma 3.2 we must have $b = 0$, whence $\theta = aV$ is parallel, so (M, g, J) is Vaisman and the parallel vector field V is proportional to the Lee form.

    Case 2: $a = 0$. From Lemma 3.2 again we get $f = 0$, $\alpha = -b^2$ and $\theta = bJV$. Equation (3.2) now reads

$$
\nabla_X(JV) = b\left(X - \langle X, V\rangle V - \langle X, JV\rangle JV\right) \qquad \forall X \in TM. \tag{3.23}
$$

Remark that M is never Vaisman in this case (unless it is Kähler). Indeed, from (3.15) we see that $X(b) = 0$ for all X orthogonal to V and JV. If θ were parallel, we would have by (3.23): $0 = \nabla_X\theta = n\nabla_X JV = b^2 X$ for every vector X orthogonal to V and JV, and thus $b = 0$, so $\theta = 0$.

    By symmetrization (3.23) gives:

$$
\mathcal{L}_{JV}g = 2b(g - V^\flat \otimes V^\flat - JV^\flat \otimes JV^\flat). \tag{3.24}
$$

This relation allows us to make more explicit the metric structure of M:

Lemma 3.3. *The universal cover $(\tilde{M}, \tilde{g}, \tilde{\Omega})$ of (M, g, Ω) is (holomorphically) isometric to $\mathbb{R}^2 \times (N, g_N, \Omega_N)$ endowed with the metric $\mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N$ and the Kähler form $\mathrm{d}s \wedge \mathrm{d}t + e^{2c(t)}\Omega_N$, for some complete simply connected Kähler manifold (N, g_N, Ω_N) of real dimension $2n - 2$ and some smooth real function c.*

Proof. The tangent bundle of M is the direct sum of three orthogonal distributions: $TM = \langle V\rangle \oplus \langle JV\rangle \oplus D$, where $D := \langle V, JV\rangle^\perp$. Since $\mathrm{d}V = 0$ (as V is parallel) and $\mathrm{d}(JV) = 0$ by (3.23), the distribution D is involutive. From (3.23) again we easily check that $[V, JV] = 0$, $[V, D] \subset D$ and $[JV, D] \subset D$. The Frobenius theorem shows that there exist local coordinates $(s, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ around each point of M, such that $V = \partial_s$, $JV = \partial_t$ and the metric g has the form $g = \mathrm{d}s^2 + \mathrm{d}t^2 + h(s, t)$, for some family of metrics $h(s, t)$ on $\mathbb{R}^{n-2}$.

    We first note that for each s, t the metric $h(s, t)$ is Kähler. Indeed, J defines by restriction to D an integrable complex structure on each local leaf $\mathbb{R}^{n-2}$, whose Kähler form $\Omega(s, t)$ is just the restriction of Ω. Consequently, $\mathrm{d}\Omega(s, t)$ is the restriction to the leaves of $\mathrm{d}\Omega = \theta \wedge \Omega$, which vanishes since $\theta|_D = 0$.

Now, since V is parallel on M, its flow preserves g, whence $h(s, t) = h(t)$ is independent on s. From (3.15) we see that $b = b(t)$ depends on t alone. Moreover, (3.24) yields

$$\frac{\partial h}{\partial t} = 2b(t)h,$$

whence

$$h(s, t) = e^{\int_0^t 2b(\tau)d\tau} h(0).$$

This proves the local version of the lemma, by defining $c(t) := \int_0^t b(\tau)d\tau$ and $g_N := h(0)$. The global statement follows from the Frobenius theorem applied to the universal cover of M. $\qquad\square$

The fundamental group of M induces a co-compact group of isometries of the globally conformally Kähler manifold $(\tilde{M}, \tilde{g}) := (\mathbb{R}^2 \times N, \mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N)$. Our next aim is to show that the Lee form of M is exact. Note that the Kähler form of $\tilde{M}$ is $\tilde{\Omega} = \mathrm{d}s \wedge \mathrm{d}t + e^{2c(t)}\Omega_N$, which satisfies

$$\mathrm{d}\tilde{\Omega} = 2c'(t)\mathrm{d}t \wedge e^{2c(t)}\Omega_N = 2c'(t)\mathrm{d}t \wedge \tilde{\Omega} = 2\mathrm{d}c \wedge \tilde{\Omega},$$

showing that the Lee form of $\tilde{M}$ is $\tilde{\theta} = \mathrm{d}c$. It suffices to check that the function c is Γ-invariant. This follows from a more general statement:

Lemma 3.4. *Assume that (N^d, g_N) is a complete simply connected Riemannian manifold of dimension $d \geq 1$, $c : \mathbb{R} \to \mathbb{R}$ is a smooth function and Γ is a co-compact group acting totally discontinuously by isometries on the Riemannian manifold $(\mathbb{R}^2 \times N, \mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N)$. Assume moreover that Γ preserves the vector fields ∂_s and ∂_t. Then the function c is invariant by Γ.*

Proof. The last assumption shows that every element $\gamma \in \Gamma$ has the form $\gamma(s, t, x) = (s + s_\gamma, t + t_\gamma, \psi_\gamma(x))$, where s_γ and t_γ are real numbers and ψ_γ is a diffeomorphism of N. The condition that γ is an isometry of the metric $\mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N$ reads

$$e^{2c(t)}g_N(X, Y) = e^{2c(t+t_\gamma)}g_N((\psi_\gamma)_\star(X), (\psi_\gamma)_\star(Y)), \qquad \forall\, t \in \mathbb{R},\ X, Y \in \mathrm{T}N.$$

Thus ψ_γ is a homothety of (N, g_N) with ratio

$$\rho_\gamma := e^{c(t)-c(t+t_\gamma)} \tag{3.25}$$

(note that, in particular, this expression does not depend on t).

Assume, for a contradiction, that c is not Γ-invariant. By (3.25), there exists $\gamma_0 \in \Gamma$ such that $\rho_{\gamma_0} < 1$. The map ψ_{γ_0} is a contraction of the complete metric space (N, d_N), where d_N is the distance induced by g_N. By the Banach fixed point theorem, ψ_{γ_0} has a unique fixed point $x_0 \in N$ and

$$\lim_{k\to\infty} \psi_{\gamma_0}^k(x) = x_0, \qquad \forall\, x \in N. \tag{3.26}$$

Let γ be any element of Γ. For every integer $k \in \mathbb{N}$ we have

$$y_k := (\gamma_0^k \circ \gamma \circ \gamma_0^{-k})(0, 0, x_0) = (s_\gamma, t_\gamma, \psi_{\gamma_0}^k(\psi_\gamma(x_0))),$$

so by (3.26), the sequence $\{y_k\}$ converges to $(s_\gamma, t_\gamma, x_0) =: y_0$. Since the action of Γ is totally discontinuous, this implies that $y_k = y_0$ for k sufficiently large, whence $\psi_\gamma(x_0) = x_0$ for every $\gamma \in \Gamma$.

Consider now the continuous map $f : \mathbb{R}^2 \times N \to \mathbb{R}_+$ defined by $f(s, t, x) := e^{c(t)}d_N(x, x_0)$. Using (3.25) an immediate induction shows that

$$c(nt_{\gamma_0}) = c(0) - n\ln(\rho_{\gamma_0}), \qquad \forall\, n \in \mathbb{Z},$$

thus showing that c is onto on $\mathbb{R}$. In particular, f is onto on $\mathbb{R}_+$.

For every $\gamma \in \Gamma$ we have using (3.25):

$$
\begin{aligned}
(\gamma^\star f)(s, t, x) &= f(s + s_\gamma, t + t_\gamma, \psi_\gamma(x)) = e^{c(t+t_\gamma)} d_N(\psi_\gamma(x), x_0) \\
&= e^{c(t+t_\gamma)} d_N(\psi_\gamma(x), \psi_\gamma(x_0)) = \rho_\gamma e^{c(t+t_\gamma)} d_N(x, x_0) = e^{c(t)} d_N(x, x_0) \\
&= f(s, t, x).
\end{aligned}
$$

Thus f is Γ-invariant and induces a continuous map $\tilde{f} : \Gamma\backslash(\mathbb{R}^2 \times N) \to \mathbb{R}$. Since f is onto, $\tilde{f}$ is also onto, contradicting the fact that the action of Γ on $\mathbb{R}^2 \times N$ is co-compact. $\qquad\square$

Summarizing, we have proved:

Theorem 3.5. *Let (M, g, J, θ) be a compact lcK manifold of complex dimension $n > 2$ admitting a non-trivial parallel vector field V. Then the following (exclusive) possibilities occur:*

(i) The Lee form θ is a (non-zero) constant multiple of $V^\flat$, so M is a Vaisman lcK manifold.

(ii) (M, g, Ω, θ) is globally conformally Kähler and there exists a complete simply connected Kähler manifold (N, g_N, Ω_N) of real dimension $2n - 2$, a smooth real function $c : \mathbb{R} \to \mathbb{R}$ and a discrete co-compact group Γ acting freely and totally discontinuously on $\mathbb{R}^2 \times N$, preserving the metric $\mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N$, the Hermitian 2-form $\mathrm{d}s \wedge \mathrm{d}t + e^{2c(t)}\Omega_N$ and the vector fields ∂_s and ∂_t, such that M is diffeomorphic to $\Gamma\backslash(\mathbb{R}^2 \times N)$, and the structure (g, Ω, θ) corresponds to $(\mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N, \mathrm{d}s \wedge \mathrm{d}t + e^{2c(t)}\Omega_N, \mathrm{d}c)$ through this diffeomorphism.

Example 3.6. Typically, one can obtain examples of type (ii) by taking (N, g_N, Ω_N) to be any compact Kähler manifold, c any T-periodic function, and Γ the group of isometries of $(\mathbb{R}^2 \times N, \mathrm{d}s^2 + \mathrm{d}t^2 + e^{2c(t)}g_N)$ generated by the maps $\gamma_1 : (s, t, x) \mapsto (s + 1, t, x)$ and $\gamma_2 : (s, t, x) \mapsto (s, t + T, x)$.

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