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A complete classification of four-dimensional paraKähler Lie algebras

Abstract: We consider paraKähler Lie algebras, that is, even-dimensional Lie algebras $\mathfrak{g}$ equipped with a pair (J, g), where J is a paracomplex structure and g a pseudo-Riemannian metric, such that the fundamental 2-form $\Omega(X, Y) = g(X, JY)$ is symplectic. A complete classification is obtained in dimension four.

Keywords: Lie algebras, paraKähler structures, pseudo-Riemannian homogeneous spaces.

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Dedicated to the Memory of Sergio Console

1 Introduction

Complex manifolds are among the most active research fields in differential geometry. In recent years, their paracomplex analogue attracted a growing number of researchers. While in some aspects the results on paracomplex structures parallel their complex analogues, remarkable differences do occur. Some recent well-written surveys on paracomplex manifolds may be found in [12] and [1].

If we consider an even-dimensional homogeneous manifold, it is a natural problem to determine its invariant complex and paracomplex structures. In particular, several results are known concerning four-dimensional invariant complex and paracomplex structures. Four-dimensional complex, symplectic and pseudo-Kähler Lie algebras were classified in [17]-[19]. Invariant complex and Kähler structures, together with their paracomplex analogues, were classified in [8] on all four-dimensional pseudo-Riemannian manifolds with non-trivial isotropy. Complex and paracomplex structures of four-dimensional generalized symmetric spaces were studied in [6]. Four-dimensional parahypercomplex invariant structures are also known [2, 4, 8].

The above cited results leave us to consider left-invariant paraKähler structures on four-dimensional Lie algebras. A complete classification is provided in [7] and is reported here, together with some results concerning curvature properties of left-invariant paraKähler metrics.

The paper is organized in the following way. In Section 2 we shall provide the classification of four-dimensional paracomplex Lie algebras. Curvature properties of left-invariant paraKähler metrics will be investigated in Section 3.

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2 Four-dimensional paraKähler Lie algebras

2.1 4*D* symplectic Lie algebras

We first report some basic definitions and properties on paracomplex, paraHermitian and paraKähler structures. We may refer to [12] and [1] for further information.

Let M a real smooth manifold of dimension $2n$. A *symplectic structure* on M is a two-form Ω, such that

(s1) $d\Omega = 0$ and
(s2) Ω^n is a volume form on M.

A (1,1)-tensor J on M is called an *almost paracomplex structure* if $J^2 = Id$ and its eigenvalues -1 and $+1$ have the same rank. The integrability of J is expressed by condition

$$[J, J](X, Y) := [JX, JY] + [X, Y] - J[JX, Y] - J[X, JY], \tag{2.1}$$

for all tangent vector fields X, Y. When (2.1) holds, J is called a *paracomplex structure*.

A pseudo-Riemannian metric g on M is said to be *compatible* with J if

$$g(JX, Y) + g(X, JY) = 0, \tag{2.2}$$

for all tangent vector fields X, Y. In such a case, the pair (g, J) is called an *almost paraHermitian structure*. It is well known that if (2.2) holds, then g is necessarily of neutral signature (n, n).

When (2.2) holds, one can consider the fundamental two-form $\Omega(X, Y) := g(X, JY)$ of the almost paraHermitian structure. If Ω is symplectic, then (M, g, J, Ω) is said to be an *almost paraKähler manifold*. In particular, a paraKähler structure is formed by a pseudo-Riemannian metric g, a compatible paracomplex structure J and the associated symplectic form. We explicitly observe that because of the compatibility conditions

$$\Omega(X, Y) = g(X, JY), \quad g(X, Y) = \Omega(JX, Y), \quad \Omega(JX, JY) + \Omega(X, Y) = 0, \tag{2.3}$$

any two among g, J and Ω uniquely determine the third one. For this reason, we simply denote a paraKähler structure by (g, J).

Let now G denote a $2n$-dimensional (simply connected) Lie group and $\mathfrak{g}$ its Lie algebra. If (g, J) is a left-invariant paraKähler structure on M, then tensors g and J are uniquely determined at the algebraic level by corresponding tensors (denoted again by the same symbols) defined on the Lie algebra $\mathfrak{g}$. More precisely, J and g respectively correspond to an endomorphism and an inner product on $\mathfrak{g}$. This fact justifies the following.

Definition 2.1. Let $\mathfrak{g}$ denote a $2n$-dimensional Lie algebra.

(i) A *symplectic structure* on $\mathfrak{g}$ is a closed 2-form ω on $\mathfrak{g}$ of maximal rank, that is, such that $\omega^n \neq 0$.
(ii) A *paraHermitian structure* on $\mathfrak{g}$ is a pair (g, J), where J is a paracomplex structure and g a compatible inner product on $\mathfrak{g}$.
(iii) A *paraKähler structure* on $\mathfrak{g}$ is a paraHermitian structure (g, J), such that its fundamental 2-form ω, defined by $\omega(X, Y) = g(X, JY)$, is symplectic.

A *symplectic* (respectively, *paraHermitian, paraKähler*) *Lie algebra* is a Lie algebra $\mathfrak{g}$ admitting a symplectic (respectively, paraHermitian, paraKähler) structure.

A four-dimensional symplectic Lie algebra is necessarily solvable [13]. The classification of four-dimensional not abelian solvable Lie algebras $\mathfrak{g}$ is the following (see for example Proposition 2.1 of [18]), where in the different cases, $\{e_1, e_2, e_3, e_4\}$ denotes a basis of $\mathfrak{g}$, and only the nonvanishging Lie bracktes $[e_i, e_j]$ are listed.

Table 1: Four-dimensional Symplectic Lie algebras.

Lie algebra $\mathfrak{g}$	Symplectic structures
$\mathfrak{r}_2\mathfrak{r}_2$	$ae^{12} + be^{13} + ce^{34}, \quad ac \neq 0$
$\mathfrak{r}\mathfrak{h}_3$	$ae^{14} + be^{12} + ce^{23} + pe^{13} + qe^{24}, \quad ac - pq \neq 0$
$\mathfrak{r}\mathfrak{r}_{3,0}$	$ae^{12} + be^{13} + de^{14} + ce^{34}, \quad ac \neq 0$
$\mathfrak{r}\mathfrak{r}_{3,-1}$	$de^{12} + be^{13} + ae^{14} + ce^{23}, \quad ac \neq 0$
$\mathfrak{r}\mathfrak{r}'_{3,0}$	$ae^{12} + be^{13} + ce^{14} + de^{23}, \quad cd \neq 0$
$\mathfrak{r}'_2$	$be^{12} + a(e^{13} - e^{24}) + c(e^{14} + e^{23}), \quad a^2 + c^2 \neq 0$
$\mathfrak{n}_4$	$ae^{12} + de^{14} + be^{24} + ce^{34}, \quad ac \neq 0$
$\mathfrak{r}_{4,0}$	$ae^{14} + be^{23} + de^{24} + ce^{34}, \quad ab \neq 0$
$\mathfrak{r}_{4,-1}$	$ae^{13} + de^{14} + be^{24} + ce^{34}, \quad ab \neq 0$
$\mathfrak{r}_{4,-1,\alpha}, \ \alpha \in]-1,0[$	$ae^{12} + be^{13} + ce^{24} + de^{34}, \quad bc \neq 0$
$\mathfrak{r}_{4,-1,-1}$	$ae^{12} + be^{13} + pe^{14} + ce^{24} + de^{34}, \quad ad - bc \neq 0$
$\mathfrak{r}_{4,\alpha,-\alpha}, \ \alpha \neq -1, 0$	$ae^{14} + be^{23} + ce^{24} + de^{34}, \quad ab \neq 0$
$\mathfrak{r}'_{4,\alpha,0}, \ \alpha > 0$	$ae^{14} + be^{23} + ce^{24} + de^{34}, \quad ab \neq 0$
$\mathfrak{d}_{4,1}$	$a(e^{12} - e^{34}) + be^{14} + ce^{24}, \quad a \neq 0$
$\mathfrak{d}_{4,2}$	$a(e^{12} - e^{34}) + ce^{14} + de^{23} + be^{24}, \quad cd - a^2 \neq 0$
$\mathfrak{d}_{4,\alpha}, \ \alpha \geq \frac{1}{2} (\neq 1, 2)$	$a(e^{12} - e^{34}) + ce^{14} + be^{24}, \quad a \neq 0$
$\mathfrak{d}'_{4,\alpha}, \ \alpha > 0$	$a(e^{12} - \alpha e^{34}) + be^{14} + ce^{24}, \quad a \neq 0$
$\mathfrak{h}_4$	$a(e^{12} - e^{34}) + ce^{14} + be^{24}, \quad a \neq 0$

$$\begin{array}{ll}
\mathfrak{r}_2\mathfrak{r}_2: & [e_1, e_2] = e_2, \ [e_3, e_4] = e_4 \\
\mathfrak{r}\mathfrak{h}_3: & [e_1, e_2] = e_3 \\
\mathfrak{r}\mathfrak{r}_3: & [e_1, e_2] = e_2, \ [e_1, e_3] = e_2 + e_3 \\
\mathfrak{r}\mathfrak{r}_{3,\alpha}: & [e_1, e_2] = e_2, \ [e_1, e_3] = \alpha e_3, \qquad \alpha \in [-1, 1] \\
\mathfrak{r}\mathfrak{r}'_{3,\alpha}: & [e_1, e_2] = \alpha e_2 - e_3, \ [e_1, e_3] = e_2 + \alpha e_3, \qquad \alpha \geq 0 \\
\mathfrak{r}'_2 & [e_1, e_3] = e_3, \ [e_1, e_4] = e_4, \ [e_2, e_3] = e_4, \ [e_2, e_4] = -e_3 \\
\mathfrak{n}_4 & [e_4, e_1] = e_2, \ [e_4, e_2] = e_3 \\
\mathfrak{r}_4 & [e_4, e_1] = e_1, \ [e_4, e_2] = e_1 + e_2, \ [e_4, e_3] = e_2 + e_3 \\
\mathfrak{r}_{4,\alpha} & [e_4, e_1] = e_1, \ [e_4, e_2] = \alpha e_2, \ [e_4, e_3] = e_2 + \alpha e_3 \\
\mathfrak{r}_{4,\mu,\alpha}: & [e_4, e_1] = e_1, \ [e_4, e_2] = \mu e_2, \ [e_4, e_3] = \alpha e_3, \ -1 = \mu \leq \alpha \leq 0 \text{ or } \mu\alpha \neq 0, -1 < \mu \leq \alpha \leq 1 \\
\mathfrak{r}'_{4,\mu,\alpha}: & [e_4, e_1] = \mu e_1, \ [e_4, e_2] = \alpha e_2 - e_3, \ [e_4, e_3] = e_2 + \alpha e_3, \qquad \mu > 0 \\
\mathfrak{d}_4: & [e_1, e_2] = e_3, \ [e_4, e_1] = e_1, \ [e_4, e_2] = -e_2 \\
\mathfrak{d}_{4,\alpha}: & [e_1, e_2] = e_3, \ [e_4, e_3] = e_3, [e_4, e_1] = \alpha e_1, \ [e_4, e_2] = (1-\alpha) e_2, \qquad \alpha \geq \frac{1}{2} \\
\mathfrak{d}'_{4,\alpha}: & [e_1, e_2] = e_3, \ [e_4, e_1] = \frac{\alpha}{2} e_1 - e_2, \ [e_4, e_3] = \alpha e_3, \ [e_4, e_2] = e_1 + \frac{\alpha}{2} e_2, \qquad \alpha \geq 0 \\
\mathfrak{h}_4 & [e_1, e_2] = e_3, \ [e_4, e_3] = e_3, \ [e_4, e_1] = \frac{1}{2} e_1, \ [e_4, e_2] = e_1 + \frac{1}{2} e_2.
\end{array}$$

In the above list, $\mathfrak{r}_2 = \mathit{aff}(\mathbb{R})$ is the Lie algebra of the Lie group of affine motions of $\mathbb{R}$; $\mathfrak{r}'_2$ is the real Lie algebra underlying to the complex Lie algebra $\mathit{aff}(\mathbb{C})$; $\mathfrak{r}\mathfrak{r}_{3,0}$, $\mathfrak{r}\mathfrak{r}_{3,-1}$ and $\mathfrak{r}\mathfrak{h}_3$ are the trivial extensions of the Lie algebra $\mathfrak{e}(2)$ of the group of rigid motions of $\mathbb{R}^2$, the Lie algebra $\mathfrak{e}(1,1)$ of the group of rigid motions of the Minkowski two-space and of the Heisenberg Lie algebra $\mathfrak{h}_3$, respectively.

Four-dimensional symplectic Lie algebras have been completely classified up to isomorphisms in [18]. Denoting by e^{ij} the two-form $e^i \wedge e^j$, the above Table 1 lists all four-dimensional symplectic Lie algebras.

Table 2: Four-dimensional paracomplex Lie algebras.

Lie algebra $\mathfrak{g}$	$\mathbb{R}^2 \bowtie \mathbb{R}^2$	$\mathbb{R}^2 \bowtie aff(\mathbb{R})$	$aff(\mathbb{R}) \bowtie aff(\mathbb{R})$
$\mathbb{R}^4$	$\langle e_1, e_2\rangle \times \langle e_3, e_4\rangle$	no	no
$\mathfrak{r}_2\mathfrak{r}_2$	$\langle e_1, e_3\rangle \ltimes \langle e_2, e_4\rangle$	$\langle e_1 + e_4, e_2\rangle \bowtie \langle e_1, e_3\rangle$	$\langle e_1, e_2\rangle \times \langle e_3, e_4\rangle$
$\mathfrak{r}\mathfrak{h}_3$	$\langle e_2, e_4\rangle \times \langle e_1, e_3\rangle$	no	no
$\mathfrak{r}\mathfrak{r}_3$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_1, e_2\rangle \bowtie \langle e_3, e_4\rangle$	no
$\mathfrak{r}\mathfrak{r}_{3,0}$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_1, e_2\rangle \ltimes \langle e_3, e_4\rangle$	no
$\mathfrak{r}\mathfrak{r}_{3,\alpha}$, $\alpha \neq 0$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_1, e_2\rangle \ltimes \langle e_3, e_4\rangle$	$\langle e_1 + e_4, e_2\rangle \bowtie \langle e_1 - \alpha e_4, e_3\rangle$
$\mathfrak{r}\mathfrak{r}'_{3,\alpha}$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	no	no
$\mathfrak{r}'_2$	$\langle e_1, e_2\rangle \ltimes \langle e_3, e_4\rangle$	$\langle e_1, e_3\rangle \bowtie \langle e_1 - e_4, e_2 + e_3\rangle$	no
$\mathfrak{n}_4$	$\langle e_3, e_4\rangle \bowtie \langle e_1, e_2\rangle$	no	no
$\mathfrak{r}_4$	no	$\langle e_1, e_4\rangle \bowtie \langle e_2, e_3\rangle$	no
$\mathfrak{r}_{4,0}$	$\langle e_2, e_4\rangle \bowtie \langle e_1, e_3\rangle$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	no
$\mathfrak{r}_{4,\alpha}$, $\alpha \neq 0$	no	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_1, e_4\rangle \bowtie \langle e_2, e_4 + \alpha e_3\rangle$
$\mathfrak{r}_{4,\mu,\alpha}$, $\alpha \neq 0, \pm 1$	no	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_4 - e_1, e_2\rangle \bowtie \langle e_1 + e_4, e_3\rangle$
$\mathfrak{r}'_{4,\mu,\alpha}$, $\alpha > 0$	no	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	no
$\mathfrak{d}_4$	no	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_1 - e_3, e_2 + e_4\rangle \bowtie \langle e_4 - e_2, e_1 + e_3\rangle$
$\mathfrak{d}_{4,1}$	$\langle e_2, e_4\rangle \ltimes \langle e_1, e_3\rangle$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_1, e_4\rangle \bowtie \langle e_2 + e_4, e_3\rangle$
$\mathfrak{d}_{4,\alpha}$, $\alpha \geq \frac{1}{2}(\neq 1)$	no	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$\langle e_3, e_4\rangle \bowtie \langle e_4 + \alpha e_2, (1-\alpha)e_1 + \alpha e_3\rangle$
$\mathfrak{h}_4$	no	$\langle e_1, e_4\rangle \bowtie \langle e_2, e_3\rangle$	$\langle e_3, e_4\rangle \ltimes \langle 2e_4 - e_2, e_1 - e_3\rangle$

2.2 Product structures and double Lie algebras

Let $\mathfrak{g}$ denote again an even-dimensional Lie algebra. An *almost product structure* E on $\mathfrak{g}$ is an endomorphism $E : \mathfrak{g} \to \mathfrak{g}$, such that $E^2 = Id$ (and $E \neq \pm Id$) [2]. The integrability of E is expressed by condition

$$E[x, y] = [Ex, y] + [x, Ey] - E[Ex, Ey], \quad \text{for all } x, y \in \mathfrak{g}.$$

Any almost product structure E yields a decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad \text{where} \quad E|\mathfrak{g}_+ = Id, \quad E|\mathfrak{g}_- = -Id.$$

If E is integrable, then $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras. In particular, a paracomplex structure is a product structure $J = E$, with $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$.

Product structures are in a one-to-one correspondence with double Lie algebras. Three Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ form a *double Lie algebra* (which will be denoted by $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_-$) if $\mathfrak{g}_+$, $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces.

The above definitions yield that the eigenspaces of a product structure E on $\mathfrak{g}$ give rise to a double Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_-$. Conversely, if $\mathfrak{g} = \mathfrak{g}_+ \bowtie \mathfrak{g}_-$, then such a decomposition uniquely determines a product structure $E : \mathfrak{g} \to \mathfrak{g}$, defined by taking $E|\mathfrak{g}_+ = Id$ and $E|\mathfrak{g}_- = -Id$.

Let now $\mathfrak{g}$ denote a four-dimensional Lie algebra, equipped with a paracomplex structure J. Then, $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_- = 2$. As the only two non-isomorphic two-dimensional Lie algebras are $\mathbb{R}^2$ and $aff(\mathbb{R}) = \mathfrak{r}_2$, the possible decompositions of $\mathfrak{g}$ as double Lie algebra are exactly the folllowing: either $\mathbb{R}^2 \bowtie \mathbb{R}^2$, $\mathbb{R}^2 \bowtie aff(\mathbb{R})$ or $aff(\mathbb{R}) \bowtie aff(\mathbb{R})$.

Using the correspondence between paracomplex structures and double Lie algebras with $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_- = 2$, four-dimensional solvable Lie algebras admitting a paracomplex structure were completely classified in [2]. We report their classification in the above Table 2, rewritten here with respect to the same bases $\{e_1, e_2, e_3, e_4\}$ we used in Table 1. We shall use the standard notations: $\mathfrak{g} = \mathfrak{g}_+ \ltimes \mathfrak{g}_-$ for the semidirect product of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ (when $[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-$) and $\mathfrak{g} = \mathfrak{g}_+ \times \mathfrak{g}_-$ for the direct product of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ (when $[\mathfrak{g}_+, \mathfrak{g}_-] = 0$).

Table 3: Four-dimensional paraKähler Lie algebras.

Lie algebra $\mathfrak{g}$	Paracomplex structures	Symplectic fundamental two-forms
$\mathfrak{r}_2\mathfrak{r}_2$	$\langle e_1, e_3\rangle \ltimes \langle e_2, e_4\rangle$	$ae^{12} + ce^{34},\ ac \neq 0$
$\mathfrak{r}\mathfrak{h}_3$	$\langle e_2, e_4\rangle \ltimes \langle e_1, e_3\rangle$	$ae^{14} + be^{12} + ce^{23},\ ac \neq 0$
$\mathfrak{r}\mathfrak{r}_{3,0}$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$ae^{12} + be^{13} + ce^{34}, ac \neq 0$
$\mathfrak{r}\mathfrak{r}_{3,-1}$	$\langle e_1, e_2\rangle \ltimes \langle e_3, e_4\rangle$	$be^{13} + ae^{14} + ce^{23}, ac \neq 0$
$\mathfrak{r}_2'$	$J_1 : \langle e_1, e_2\rangle \ltimes \langle e_3, e_4\rangle$ $J_2 : \langle e_1, e_3\rangle \bowtie \langle e_1 - e_4, e_2 + e_3\rangle$	$a(e^{13} - e^{24}) + c(e^{14} + e^{23}),\ a^2 + c^2 \neq 0,$ $c(e^{14} + e^{23}),\ c \neq 0$
$\mathfrak{n}_4$	$\langle e_1, e_4\rangle \bowtie \langle e_2, e_3\rangle$	$ae^{12} + be^{24} + ce^{34}, ac \neq 0$
$\mathfrak{r}_{4,0}$	$\langle e_2, e_4\rangle \bowtie \langle e_1, e_3\rangle$	$ae^{14} + be^{23} + ce^{34}, ab \neq 0$
$\mathfrak{r}_{4,-1}$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$ae^{13} + be^{24} + ce^{34}, ab \neq 0$
$\mathfrak{r}_{4,-1,\alpha},\ \alpha \in]-1, 0[$	$J_1 : \langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$ $J_2 : \langle e_4 - e_1, e_2\rangle \bowtie \langle e_1 + e_4, e_3\rangle$	$ae^{12} + be^{13} + ce^{24} + de^{34}, bc \neq 0$ $a(e^{12} - e^{24}) + b(e^{13} + e^{34}), ab \neq 0$
$\mathfrak{r}_{4,-1,-1}$	$J_1 : \langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$ $J_2 : \langle e_4 - e_1, e_2\rangle \bowtie \langle e_1 + e_4, e_3\rangle$	$ae^{12} + be^{13} + ce^{24} + de^{34},\ ad - bc \neq 0$ $a(e^{12} - e^{24}) + pe^{14} + b(e^{13} + e^{34}),\ ab \neq 0$
$\mathfrak{r}_{4,-\mu,\mu},\ \mu \in]0, 1[$	$\langle e_4 - e_1, e_2\rangle \bowtie \langle e_1 + e_4, e_3\rangle$	$ae^{14} + be^{23}, ab \neq 0$
$\mathfrak{d}_{4,1}$	$J_1 : \langle e_2, e_4\rangle \ltimes \langle e_1, e_3\rangle$ $J_2 : \langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$a(e^{12} - e^{34}) + be^{14}, a \neq 0$ $a(e^{12} - e^{34}) + ce^{24}, a \neq 0$
$\mathfrak{d}_{4,2}$	$J_1 : \langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$ $J_2 : \langle e_3, e_4\rangle \bowtie \langle 2e_2 + e_4, e_3 - e_1\rangle$	$a(e^{12} - e^{34}) + be^{24},\ a \neq 0$ $c(2e^{14} + e^{23}) + be^{24},\ c \neq 0$
$\mathfrak{d}_{4,\alpha},\ \alpha \geq \frac{1}{2}(\neq 1, 2)$	$\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$	$a(e^{12} - e^{34}) + be^{24}, a \neq 0$
$\mathfrak{h}_4$	$\langle e_1, e_4\rangle \bowtie \langle e_2, e_3\rangle$	$a(e^{12} - e^{34}) + be^{24}, a \neq 0$

2.3 The classification of 4*D* paraKähler Lie algebras

Let now $\mathfrak{g}$ denote a four-dimensional paraKähler Lie algebra. Then, $\mathfrak{g}$ is equipped both with a left-invariant symplectic structure Ω and a left-invariant paracomplex structure J, satisfying the compatibility condition (2.3). Since we aim to classify paraKähler structures on $\mathfrak{g}$ up to paracomplex isomorphisms, by the same argument used in [19] for the Kähler case, it suffices to investigate the compatibility between *an arbitrary* symplectic structure on $\mathfrak{g}$ and *a representative* of the class of its paracomplex structures. Thus, we start from four-dimensional symplectic Lie algebras, as listed in Table 1, and check the compatibility condition (2.3) for the corresponding paracomplex structures (when they exist) listed in Table 2. The classification result we obtain is the following.

Theorem 2.2. *A non-abelian four-dimensional paraKähler Lie algebra is isomorphic to one of the Lie algebras, endowed with paracomplex and compatible symplectic structures as listed in the Table 3.*

Proof. The proof of above classification listed in Table 3 follows from a case-by-case argument. For any Lie algebra listed both in Tables 1 and 2, we checked the compatibility condition (2.3) between an arbitrary symplectic structure, and a representative in each class of paracomplex structures.

We report below the calculations for the case $\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2$. In this case, any symplectic structure is of the form $\Omega = ae^{12} + be^{13} + ce^{34}$, with $ac \neq 0$. On the other hand, this Lie algebra admits three paracomplex structures corresponding to non-isomorphic double Lie algebras, which we denote here by $J_1 : \langle e_1, e_3\rangle \ltimes \langle e_2, e_4\rangle$, $J_2 : \langle e_1 + e_4, e_2\rangle \bowtie \langle e_1, e_3\rangle$ and $J_3 : \langle e_1, e_2\rangle \times \langle e_3, e_4\rangle$.

It is easily seen that compatibility condition (2.3) holds between J_1 and Ω if and only if $b = 0$, while when we apply (2.3) to J_2 (respectively, J_3) and Ω, we find $a = 0$ (respectively, $c = 0$), which contradicts the fact that Ω is symplectic. Therefore, the Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ admits a two-parameter family of paraKähler structures, namely, all and the ones determined by the paracomplex structure J_1, together with any of the compatible symplectic structures $\Omega = ae^{12} + ce^{34}$, with $ac \neq 0$. This leads to the case we listed in Table 3.

We observe that the Lie algebra $\mathfrak{rr}'_{3,0}$ admits both symplectic and paracomplex structures. However, these structures never satisfy compatibility condition (2.3). Consequently, $\mathfrak{rr}'_{3,0}$ does not admit any paraKähler structure. □

3 Curvature properties of left-invariant paraKähler metrics

Let (J, Ω) denote a paraKähler structure on a four-dimensional Lie algebra $\mathfrak{g}$. Then, the corresponding compatible left-invariant pseudo-Riemannian metric g is uniquely determined by $g(X, Y) = \Omega(JX, Y)$. Because of the above Table 3, this gives us 20 families of invariant pseudo-Riemannian metrics on four-dimensional solvable Lie algebras, each family depending on a number of parameters varying from one to four.

In order to investigate some curvature properties of these metrics, we shall now describe the general approach to the description of their Levi-Civita connection and curvature. Let $\mathfrak{g}$ denote a Lie algebra, $\{e_1, \dots, e_n\}$ a basis of $\mathfrak{g}$ and g any left-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. Then, g uniquely defines its invariant linear Levi-Civita connection, which is described in terms of the map $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, such that $\Lambda(x)(y) = \nabla_x y$ for all $x, y \in \mathfrak{g}$. Explicitly, one has

$$\Lambda(x)(y) = \tfrac{1}{2}[x, y] + v(x, y), \qquad \text{for all } x, y \in \mathfrak{g}, \tag{3.1}$$

where $v : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is symmetric and uniquely determined by condition

$$2g(v(x, y), z) = g(x, [z, y]) + g(y, [z, x]), \qquad \text{for all } x, y, z \in \mathfrak{g}.$$

The curvature tensor is then described in terms of the map

$$\begin{aligned} R : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ (x, y) &\mapsto [\Lambda(x), \Lambda(y)] - \Lambda([x, y]) \end{aligned} \tag{3.2}$$

Finally, the Ricci tensor ϱ of g, described in terms of its components with respect to $\{e_i\}$, is given by

$$\varrho(e_i, e_j) = \sum_{r=1}^{4} R_{ri}(e_r, e_j), \qquad \text{for all } i, j. \tag{3.3}$$

By means of equations (3.1)-(3.3) (and the covariant derivatives of tensors R and ϱ, when needed) one can study curvature properties of any left-invariant metric g on a given Lie algebra $\mathfrak{g}$. Applied to left-invariant paraKähler metrics on four-dimensional Lie algebras, they show a wide range of different behaviours. In some cases, all left-invariant paraKähler metrics are flat. In some other cases (when the paracomplex Lie algebra corresponds to some "complicated" double Lie algebra), the curvature does not seem to have any special feature at all. In between these two extreme cases, one can find several interesting examples, also including the existence of non-trivial left-invariant Ricci solitons on Lie algebras $\mathfrak{r}_{4,-1,\alpha}$, for any $\alpha \in]-1, 0[$.

We first classify the cases where all left-invariant paraKähler metrics are flat, obtaining the following.

Theorem 3.1. *Let g denote a left-invariant paraKähler metric on a four-dimensional Lie algebra $\mathfrak{g}$. Then, g is a flat metric in all the cases listed below:*

(1) $\mathfrak{g}$ is either $\mathfrak{r}_2\mathfrak{r}_2$, $\mathfrak{rh}_3$, $\mathfrak{rr}_{3,-1}$, $\mathfrak{rr}_{3,0}$, $\mathfrak{r}_{4,-1}$, $\mathfrak{n}_4$, $\mathfrak{d}_{4,1}$ or $\mathfrak{d}_{4,\alpha}$ ($\alpha \geq 1/2$, $\alpha \neq 1, 2$);

(2) $\mathfrak{g} = \mathfrak{r}'_2$ and g is compatible with $J_1 : \langle e_1, e_2\rangle \ltimes \langle e_3, e_4\rangle$;

(3) $\mathfrak{g} = \mathfrak{r}_{4,-1,-1}$ and g is compatible with $J_1 : \langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$;

(4) $\mathfrak{g} = \mathfrak{d}_{4,2}$ and g is compatible with $J_1 : \langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$.

Proof. We report below the details for the case of $\mathfrak{r}_2\mathfrak{r}_2$. Let $\{e_1, e_2, e_3, e_4\}$ denote the basis we used to describe the Lie brackets and the symplectic and paracomplex structures of $\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2$. By the above Table 3, left-invariant paraKähler metrics on $\mathfrak{r}_2\mathfrak{r}_2$ are all and the ones of the form

$$g = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & c & 0 \end{pmatrix}, \qquad ac \neq 0$$

with respect to $\{e_i\}$. Applying (3.1), we find

$$\Lambda_{e_1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{e_2} = \Lambda_{e_4} = 0, \quad \Lambda_{e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Computing $R(e_i, e_j)$ as in (3.2), it is then easy to conclude that $R(e_i, e_j) = 0$ for all indices i, j and so, g is flat. □

We explicitly observe that flat examples do also occur for the remaining cases. (For example, left-invariant paraKähler metrics on $\mathfrak{r}_{4,-\mu,\mu}$, compatible with the paracomplex structure J_1, are flat if and only if $b = 0$ (see Theorem 3.3, *(a)*).) Theorem 3.1 lists the four-dimensional Lie algebras, for which *all* left-invariant paraKähler metrics compatible with a given paracomplex structure, are flat.

In the Riemannian case, it is well-known (see for example [15]) that any irreducible symmetric space is Einstein . With regard to paraKähler metrics on the Lie algebra $\mathfrak{r}_{4,-\mu,\mu}$, counterexamples occur, as shown by the following result.

Theorem 3.2. *Let g denote a left-invariant paraKähler metric on the four-dimensional Lie algebra* $\mathfrak{r}_{4,-\mu,\mu}$ *(with* $\mu \in]-1,1[, \mu \neq 0$*). Then, g is symmetric but not Einstein.*

Proof. Consider the Lie algebra $\mathfrak{g} = \mathfrak{r}_{4,-\mu,\mu}$ and the corresponding basis $\{e_1, e_2, e_3, e_4\}$ we used in the previous Section. By Table 3, with respect to $\{e_i\}$, left-invariant paraKähler metrics on $\mathfrak{r}_{4,-\mu,\mu}$ are all and the ones of the form

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \qquad ab \neq 0.$$

From (3.1) we get

$$\Lambda_{e_1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{e_2} = \Lambda_{e_3} = 0, \quad \Lambda_{e_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so, by (3.2), the only non-vanishing curvature components with respect to $\{e_i\}$ are $R(e_1, e_4)e_1 = -e_4$, and the ones obtained by this one using the symmetries of the curvature tensor. It is then easy to check that all the components of ∇R with respect to $\{e_i\}$ vanish identically and so, g is symmetric.

On the other hand, computing the Ricci tensor with respect to $\{e_i\}$ by means of (3.3), we find

$$\varrho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Appplying to the above descriptions of g and ϱ the Einstein condition $\varrho = \lambda g$, we conclude that it is never satisfied and so, g is not Einstein. Also the weaker equation (3.4), characterizing Ricci solitons, does not admit any solution. □

We now prove the existence of nontrivial left-invariant paraKähler Ricci soliton metrics on the Lie algebra $\mathfrak{r}_{4,-1,\alpha}$. We briefly recall that a *Ricci soliton* is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field V, such that

$$\mathcal{L}_V g + \varrho = \lambda g, \tag{3.4}$$

where $\mathcal{L}_V$ and ϱ respectively denote the Lie derivative in the direction of V and the Ricci tensor and λ is a real number. Einstein manifolds are regarded as trivial Ricci solitons, since they satisfy (3.4) taking $V = 0$ (or a Killing vector field). A Ricci soliton is said to be *shrinking*, *steady* or *expanding*, according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Ricci solitons are the self-similar solutions of the *Ricci flow*. We may refer to [11] for a recent survey and further references on Ricci solitons.

If $M = G/H$ is a homogeneous space, a *homogeneous Ricci soliton* on M is a G-invariant metric g, for which equation (3.4) holds. In particular, by an *invariant Ricci soliton* we mean a homogeneous one, such that equation (3.4) is satisfied by an invariant vector field. It is a natural question to determine which homogeneous manifolds G/H admit a G-invariant Ricci soliton [16]. Some recent results in this framework, both for Riemannian and pseudo-Riemannian metrics, have been obtained in [3],[5],[9],[10],[16].

It is then a natural problem to study left-invariant Ricci solitons on four-dimensional Lie algebras, looking for possible examples among metrics with a special geometrical meaning, as paraKähler metrics. We recall that in Riemannian settings, most of the known examples of Ricci solitons are Kähler manifolds, where equation (3.4) holds for a holomorphic vector field (see for example [11, 14]). Also this fact makes natural to study paraKähler Ricci solitons. We prove the following result.

Theorem 3.3. *Consider the four-dimensional Lie algebra $\mathfrak{r}_{4,-1,\alpha}$ (with $\alpha \in]-1, 0[$). With respect to the basis $\{e_1, .., e_4\}$, left-invariant paraKähler metrics on $\mathfrak{r}_{4,-1,\alpha}$ compatible with the paracomplex structure J_1 : $\langle e_1, e_4\rangle \ltimes \langle e_2, e_3\rangle$ and corresponding Ricci tensors are respectively given by*

$$g = \begin{pmatrix} 0 & a & b & 0 \\ a & 0 & 0 & -c \\ b & 0 & 0 & -d \\ 0 & -c & -d & 0 \end{pmatrix}, \quad ad - bc \neq 0, \ \ \text{and} \tag{3.5}$$

$$\varrho = \begin{pmatrix} -\frac{(1+\alpha)^2 a^2 b^2}{2(ad-bc)^2} & 0 & 0 & \frac{(1+\alpha)ab\{(1+\alpha)bc - ad + bc\}}{2(ad-bc)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{(1+\alpha)ab\{(1+\alpha)bc - ad + bc\}}{2(ad-bc)^2} & 0 & 0 & -\frac{bc(1+\alpha)(2bc + \alpha bc - ad)}{2(ad-bc)^2} \end{pmatrix}. \tag{3.6}$$

Then:

(a) The following properties are equivalent: (i) g is flat; (ii) g is Einstein; (iii) $b = 0$.

(b) Whenever $a = 0 \neq bc$, g is a (steady) left-invariant Ricci soliton, for which equation (3.4) *holds with*

$$V = -\frac{(\alpha+1)(\alpha+2)}{4c} e_2 \quad \text{and} \quad \lambda = 0.$$

Proof. The description (3.5) of paraKähler metrics on $\mathfrak{r}_{4,-\mu,\mu}$, compatible with the paracomplex structure J_1, follows from Table 3. We the apply (3.1) and we get

$$\Lambda_{e_1} = \begin{pmatrix} 0 & 0 & 0 & \frac{bc(1+\alpha)}{2(ad-bc)} \\ 0 & 0 & \frac{b^2(1+\alpha)}{2(ad-bc)} & 0 \\ 0 & 0 & -\frac{ab(1+\alpha)}{2(ad-bc)} & 0 \\ 0 & 0 & 0 & \frac{ab(1+\alpha)}{2(ad-bc)} \end{pmatrix}, \quad \Lambda_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{b^2(1+\alpha)}{2(ad-bc)} & 0 & 0 & -\frac{bd(1+\alpha)}{2(ad-bc)} \\ -\frac{ab(1+\alpha)}{2(ad-bc)} & 0 & 0 & \frac{bc(1+\alpha)}{2(ad-bc)} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{e_4} = \begin{pmatrix} \frac{2\,ad-bc+\alpha\,bc}{2(ad-bc)} & 0 & 0 & -\frac{cd(1+\alpha)}{ad-bc} \\ 0 & -1 & -\frac{bd(1+\alpha)}{2(ad-bc)} & 0 \\ 0 & 0 & \frac{2\,\alpha\,da-\alpha\,bc+bc}{2(ad-bc)} & 0 \\ \frac{ab(1+\alpha)}{2(ad-bc)} & 0 & 0 & -\frac{bc+\alpha\,da}{ad-bc} \end{pmatrix}.$$

We then use (3.2) and determine the curvature of g. Explicitly, we find that with respect to $\{e_i\}$, the curvature tensor is completely determined by

$$R(e_1, e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{b^3(1+\alpha)^2 a}{4(ad-bc)^2} & 0 & 0 & \frac{ab^2 d(1+\alpha)^2}{4(ad-bc)^2} \\ \frac{a^2 b^2(1+\alpha)^2}{4(ad-bc)^2} & 0 & 0 & -\frac{ab^2 c(1+\alpha)^2}{4(ad-bc)^2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R(e_1, e_4) = \begin{pmatrix} \frac{ab^2 c(1+\alpha)^2}{4(ad-bc)^2} & 0 & 0 & \frac{bc(1+\alpha)(2ad-bc(3+\alpha))}{4(ad-bc)^2} \\ 0 & 0 & \frac{b^2(1+\alpha)(3+\alpha)}{4(ad-bc)} & 0 \\ 0 & 0 & -\frac{ab(1+\alpha)}{2(ad-bc)} & 0 \\ \frac{a^2 b^2(1+\alpha)^2}{4(ad-bc)^2} & 0 & 0 & \frac{ab(1+\alpha)(2ad-bc(3+\alpha))}{4(ad-bc)^2} \end{pmatrix},$$

$$R(e_3, e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{b^3 c(1+\alpha)^2}{4(ad-bc)^2} & 0 & 0 & \frac{b^2 cd(1+\alpha)^2}{4(ad-bc)^2} \\ \frac{ab^2 c(1+\alpha)^2}{4(ad-bc)^2} & 0 & 0 & -\frac{b^2 c^2(1+\alpha)^2}{4(ad-bc)^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The above equations for $R(e_i, e_j)$ yield that g is flat if and only if $b = 0$. Next, we use (3.3) to calculate the Ricci tensor and find the above equation (3.6), which implies that g is Einstein (more precisely, Ricci-flat) if and only if $b = 0$.

Finally, we can use the previous description of the Levi-Civita connection of g to compute the Lie derivative $\mathcal{L}_V g$ of the metric g with respect to an arbitrary vector field $V = \sum V_i e_i \in \mathfrak{g}$. We find

$$\mathcal{L}_V g = \begin{pmatrix} 0 & 0 & -b\,(1+\alpha)\,V_4 & -aV_2 + b\alpha V_3 \\ 0 & 0 & 0 & aV_1 - cV_4 \\ -b\,(1+\alpha)\,V_4 & 0 & 0 & bV_1 + \alpha d V_4 \\ -aV_2 a + b\alpha V_3 & aV_1 - cV_4 & bV_1 + \alpha d V_4 & 2cV_2 c - 2d\alpha V_3 \end{pmatrix}.$$

Using the above equations for g, ϱ and $\mathcal{L}_V g$, we find that, excluding the trivial Einstein case $b = 0$, equation (3.4) holds if and only if $a = 0 \neq bc$, $\lambda = 0$ and $V = -\frac{(\alpha+1)(\alpha+2)}{4c} e_2$. Hence, for $a = 0$, left-invariant metrics (3.5) give a three-parameter family of non-trivial steady Ricci solitons, for any admissible value of α. Note that $(\alpha + 1)(\alpha + 2) \neq 0$, since $\alpha \in]-1, 0[$.

We observe that, by equations (3.5) and (3.6), the *Ricci operator* Q of g (defined by condition $\varrho(X,Y) = g(QX,Y)$) is given by

$$Q = \begin{pmatrix} 0 & \frac{ab^2(\alpha+1)(\alpha+2)}{2(ad-bc)^2} & \frac{a^2b(\alpha+1)}{2(ad-bc)^3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{b(\alpha+1)(a^2d^2+(\alpha+2)b^2c^2-(\alpha+3)adbc)}{2(ad-bc)^3} & 0 & 0 \end{pmatrix}.$$

Thus, Q is two-step nilpotent (and not diagonalizable, except in the trivial case when $b = 0$). □

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References

[1] D.V. Alekseevsky, C. Medori, A. Tomassini, *Homogeneous para-Kähler Einstein manifolds*, Russian Math. Surveys, **64** (2009), 1–43.

[2] A. Andrada, M.L. Barberis, I.G. Dotti, G. Ovando, *Product structures on four-dimensional solvable Lie algebras*, Homology, Homotopy and Applications, **7** (2005), 9–37.

[3] P. Baird and L. Danielo, *Three-dimensional Ricci solitons which project to surfaces*, J. Reine Angew. Math., **608** (2007), 65–91.

[4] N. Blazić, S. Vukmirović, *Four-dimensional Lie algebras with a para-hypercomplex structure*, Rocky Mountain J. Math., **40** (2010), 1391–1439.

[5] M. Brozos-Vazquez, G. Calvaruso, E. Garcia-Rio and S. Gavino-Fernandez, *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math., **188** (2012), 385–403.

[6] G. Calvaruso, *Symplectic, complex and Kähler structures on four-dimensional generalized symmetric spaces*, Diff. Geom. Appl., **29** (2011), 758–769.

[7] G. Calvaruso, *Four-dimensional paraKähler Lie algebras: classification and geometry*, Houston J. Math., to appear.

[8] G. Calvaruso and A. Fino, *Complex and paracomplex structures on homogeneous pseudo-Riemannian four-manifolds*, Int. J. Math., **24** (2013), 1250130, 28 pp.

[9] G. Calvaruso and A. Fino, *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces*, Canad. J. Math., **64** (2012), 778–804.

[10] G. Calvaruso and A. Fino, *Four-dimensional pseudo-Riemannian homogeneous Ricci soliton*, Arxiv: 1111.6384. To appear in Int. J. Geom. Methods Mod. Phys.

[11] H.-D. Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, 1–38, Adv. Lect. Math. (ALM), **11**, Int. Press, Somerville, MA, 2010.

[12] V. Cruceanu, P. Fortuny and P.M. Gadea, *A survey on paracomplex geometry*, Rocky Mount. J. Math., **26** (1996), 83–115.

[13] B.Y. Chu, *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc., **197** (1974), 145–159.

[14] A.S. Dancer and M.Y. Wang, *Some new examples on non-Kähler Ricci solitons*, Math. Res. Lett., **16** (2009), no. 2, 349–363.

[15] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata, **7** (1978), 259–280.

[16] J. Lauret, *Ricci solitons solvmanifolds*, J. Reine Angew. Math., **650** (2011), 1–21.

[17] G. Ovando, *Invariant complex structures on solvable real Lie groups*, Manuscripta Math., **103**, (2000), 19–30.

[18] G. Ovando, *Four-dimensional symplectic Lie algebras*, Beiträge Algebra Geom., **47**(2006), no. 2, 419–434.

[19] G. Ovando, *Invariant pseudo-Kähler metrics in dimension four*, J. Lie Theory, **16** (2006), 371–391.