DE GRUYTER Computational and Mathematical Biophysics 2024; 12: 20230125

Research Article

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An analysis of hybrid impulsive prey-predator-mutualist system on nonuniform time domains

https://doi.org/10.1515/cmb-2023-0125
received September 23, 2023; accepted February 09, 2024

Abstract: In this work, we propose a hybrid impulsive prey-predator-mutualist model on nonuniform time domains. We have investigated the permanence/persistence results for the proposed model using the comparison theorems of impulsive differential equations and some dynamic inequality on the nonuniform time domains. In addition, we have established certain necessary requirements for the uniform asymptotic stability of the almost periodic solution and global attractivity of the proposed model. Furthermore, we provide several numerical examples on nonuniform time domains with computer simulation to demonstrate the viability of the results of the acquired analytical work.

Keywords: timescales, hybrid predator-prey-mutualist model, global attractivity, impulsive effect, Lyapunov functional

MSC 2020: 34N05, 92D25, 34A37, 34A38

1 Introduction

Ecological systems are characterized by the interaction between species and their natural environment [29]. An important type of interaction that affects the population dynamics of all species is predation. Thus, predator-prey models have been the focus of ecological science since the early days of this discipline. In literature, several articles are published related to predator-prey interaction [10,24,27,28,38,46]. But few results are available of prey-predator-mutualist interaction [9,21,34,35,39,45]. Furthermore, it is well known that mutualist species can reduce the capture rate of the predator species to the prey species. Therefore, our main motive for this work is to study the prey-predator-mutualist interaction. Mutualism [42] is one of the most important relationships between the species. Mutualism is a symbiotic association between any two species, and the interaction between the two species is beneficial to both species. Mutualism occurs in many important processes or systems, such as mycorrhizal associations, nitrogen, and even cell organelles [3]. For example, ants prevent herbivores from feeding on plants [7], and ants prevent predators from feeding on aphids [1,41]. In the study by Addicott [2], a general as well as a specific model, in which a mutualist modifies the predation to the benefit of the prey, was considered and analyzed. The mutualist can help the prey in two ways: first, by providing the resources so as to enhance the specific growth rate of the prey and, second, by

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deterring predators from preying upon the prey. Yang et al. [44] studied the permanence and the periodic solution of the periodic predator-prey-mutualist system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= x\left(\Theta_1(s) - b_1(s)x - \frac{c_1(s)z}{d_1(s) + d_2(s)y}\right),\\ \frac{\mathrm{d}y}{\mathrm{d}s} &= y\left(\Theta_2(s) - \frac{y}{d_3(s) + d_4(s)x}\right),\\ \frac{\mathrm{d}z}{\mathrm{d}s} &= z\left(-\Theta_3(s) + \frac{k_1(s)c_1(s)x}{d_1(s) + d_2(s)y}\right),\end{aligned} \tag{1.1}$$

where x, y, and z represent the population density of prey, mutualist, and predator, respectively. The functions $\Theta_i(s)$ $(i = 1, 2, 3)$, $b_i(s)$ $(i = 1, 2)$, $c_1(s)$, $k_1(s)$, and $d_j(s)$ $(j = 1, 2, 3, 4)$ are nonnegative, rd-continuous, and almost periodic functions. For more about prey-predator-mutualist system, one can refer [19,36] and reference therein.

In literature, we have observed that when the size of the population is rarely small or the population has nonoverlapping generations (in the case of semelparous species), then we use a discrete time population dynamical system. One of the famous prey-predator systems at a discrete time is the classical Nicholson-Bailey (NB) system given as follows:

$$\begin{aligned} y_{m+1} &= r y_m e^{-\alpha z_m},\\ z_{m+1} &= c y_m (1 - e^{-\alpha z_m}),\end{aligned} \tag{1.2}$$

where prey grows exponentially in the absence of predator [31]. For more about the discrete time population model, one can see [15,25,43] and references therein. Similarly, the prey-predator-mutualist system (1.1) can be carried over to their discrete time system:

$$\begin{aligned} x(m+1) &= x(m)\exp\left(\Theta_1(m) - b_1(m)x(m) - \frac{c_1(m)z(m)}{d_1(m) + d_2(m)y(m)}\right),\\ y(m+1) &= y(m)\exp\left(\Theta_2(m) - \frac{y(m)}{d_3(m) + d_4(m)x(m)}\right),\\ z(m+1) &= z(m)\exp\left(-\Theta_3(m) + \frac{k_1(m)c_1(m)x(m)}{d_1(m) + d_2(m)y(m)}\right),\end{aligned}$$

where $x(m), y(m)$, and $z(m)$ denote the mth generation population size.

From equations (1.1) and (1.2), we can observe that some species need a continuous time domain for their mathematical modeling and some need a discrete time domain. Besides that, there exist some species whose development cycle depends on both discrete and continuous time domains, e.g., many insect population (like Bumble bees, Pharaoh cicada, Magicicada septendecim, and Magicicada cassinii [26]). These are continuous during the season (and may follow different scheme with variable step size) and die out in winter, while their eggs are in incubating or dormant and then hatch in a new season, giving rise to a nonoverlapping population. Hence, to model these types of species, we need the time domain, which includes both continuous and discrete time. In addition, it had been a challenge for mathematicians to solve this type of problem, which involved combining discrete and continuous time domains into a single mathematical theory. To overcome this type of problem, Hilger [13] proposed the idea of timescales calculus, which will unify the discrete and continuous analysis into a single unified theory. A timescale theory is an arbitrary nonempty closed subset of the real numbers, i.e., $\mathbb{R}$, and is denoted by $\mathbb{T}$. In literature, many authors studied the prey-predator system on timescales, see [12,18,23,30,40] and references therein. Li and Zhang [23] proposed the model with feedback control and established the permanence results on timescales. Dhama and Abbas [12] considered the Leslie-Gower prey-predator model and studied permanence and stability using timescale calculus.

Many times, it is essential to take the model with impulsive effects because, at a specific moment, many species experience a sudden change in their states due to harvesting, natural disasters, and other man-made activities (e.g., pesticides). So, taking the predator-prey model with impulsive effects is more realistic. In literature, many physical problems are characterized by sudden changes in their states. These sudden changes are known as impulsive effects in the system [6]. For more about the impulsive system, one can see [4,17,33,37] and references therein. Li et al. [22] investigated an impulsive prey-predator system as follows:

$$\begin{aligned} z_1'(s) &= z_1(s)(r_1(s) - b_1(s)z_1(s) - c(s)z_2^m(s)), \\ z_2'(s) &= z_2(s)(-r_2(s) + \beta c(s)z_1(s)z_2^{m-1}(s) - b_2(s)z_2(s)), \quad s \neq \tau_k, \\ z_1(\tau_k^+) &= (1 + h_k)z_1(\tau_k), \\ z_2(\tau_k^+) &= (1 + g_k)z_2(\tau_k), \quad s = \tau_k, k = 1, 2, \ldots, \end{aligned} \tag{1.3}$$

where $z_1(s)$ denotes the density of the prey species and $z_2(s)$ denotes the density of predator species. By constructing a suitable Lyapunov function and using the comparison theorem of an impulsive differential equation, the authors studied sufficient conditions that ensure the system's permanence and global attractivity.

Moreover, functional response describes the variation in the number of prey engaged per unit of time per predator as the prey population varies. In 1965, Holling [14] presented three distinct types of functional responses to different kinds of habitats to explain the phenomenon of predation, which built the basic Lotka-Volterra model quite realistic. The Beddington-DeAngelis functional response was proposed by Beddington [5] and DeAngelis et al. [11] in 1975. It is similar to the Holling-type functional response, but it includes an additional term to describe mutual interference by predators.

In literature, not a single publication is available that discusses the analysis of the hybrid impulsive prey-predator-mutualist model on nonuniform time domains. Therefore, motivated by the above observation and discussion, in this work, we analyzed the dynamics of a hybrid impulsive prey-predator-mutualist system on nonuniform time domains given as follows:

$$\Phi_1^{\Delta}(s) = \Theta_1(s) - b_1(s)e^{\Phi_1(s)} - \frac{c_1(s)e^{\Phi_3(s)}}{d_1(s) + d_2(s)e^{\Phi_2(s)}}, \quad s \neq s_k, \quad s \in [s_0, \infty)_{\mathbb{T}}, \quad k \in \mathbb{N}, \tag{1.4}$$

$$\Phi_2^{\Delta}(s) = \Theta_2(s) - \frac{e^{\Phi_2(s)}}{d_3(s) + d_4(s)e^{\Phi_1(s)}}, \quad s \neq s_k, \quad s \in [s_0, \infty)_{\mathbb{T}}, \quad k \in \mathbb{N}, \tag{1.5}$$

$$\Phi_3^{\Delta}(s) = \Theta_4(s) - \Theta_3(s) + \frac{k_1(s)c_1(s)e^{\Phi_1(s)}}{d_1(s) + d_2(s)e^{\Phi_2(s)}} - b_2(s)e^{\Phi_3(s)}, \quad s \neq s_k, \quad s \in [s_0, \infty)_{\mathbb{T}}, \quad k \in \mathbb{N}, \tag{1.6}$$

$$\Phi_1(s_k^+) = \Phi_1(s_k) + \log(1 + h_k), \quad s = s_k, \tag{1.7}$$

$$\Phi_2(s_k^+) = \Phi_2(s_k) + \log(1 + h_k'), \quad s = s_k, \tag{1.8}$$

$$\Phi_3(s_k^+) = \Phi_3(s_k) + \log(1 + h_k''), \quad s = s_k, \tag{1.9}$$

where Φ_1, Φ_2, and Φ_3 represent the population density of prey, mutualist, and predator, respectively. The functions $\Theta_i(s)$ $(i = 1, 2, 3, 4)$, $b_i(s)$ $(i = 1, 2)$, $c_1(s)$, $k_1(s)$, and $d_j(s)$ $(j = 1, 2, 3, 4)$ are nonnegative, rd-continuous, and almost periodic functions. In addition, $\Theta_i(s) (i = 1, 2, 3)$ and $b_i(s) (i = 1, 2)$ are strictly positive functions, and the biological meaning of all the parameters are given below:

(1) $\Theta_1(s)$ is the intrinsic growth rate of prey species Φ_1.
(2) $\Theta_2(s)$ is the intrinsic growth rate of mutualist Φ_2.
(3) $\Theta_4(s)$ is the birth rate of predator species Φ_3.
(4) $\Theta_3(s)$ is the death rate of the predator species Φ_3.
(5) $c_1(s)$ is the coefficient of the functional response.
(6) $k_1(s)$ is called the conversion rate, which denotes the fraction of the prey biomass being converted to predator biomass.
(7) $b_2(s)$ denotes the predator density-dependent rate.
(8) The terms $d_2(s)e^{\Phi_2(s)}$ and $d_4(s)e^{\Phi_1(s)}$ measures the mutual interference between predators and prey species, respectively.

Also, we consider in equation (1.6) that $b_2(s)e^{\Phi_3}$ represents the density restriction term of predator population; this type of assumption is required because the density of any species is restricted by the environment [32].

$\Phi_1(0) > 0$, $\Phi_2(0) > 0$, and $\Phi_3(0) > 0$, s_k is an impulsive point, $0 \leq s_0 < s_1 < s_2 < \cdots < s_k < \cdots$, and $\mathbb{N}$ is the set of positive integers. $\Phi_i(s_k^-) = \lim_{a\to 0^+}\Phi_i(s_k - a)$ and $\Phi_i(s_k^+) = \lim_{a\to 0^+}\Phi_i(s_k + a)$ denote the left and right limit of $\Phi_i(s)$ at $s = s_k$, $\Phi_1(s_k^-) = \Phi_1(s_k)$, $\Phi_2(s_k^-) = \Phi_2(s_k)$, and $\Phi_3(s_k^-) = \Phi_3(s_k)$. Also, we assume that h_k, h_k', and h_k'', $k = 1, 2, \ldots,$ are constants.

Remark 1.1. For $\mathbb{T} = \mathbb{R}$, $x(s) = e^{\Phi_1(s)}$, $y(s) = e^{\Phi_2(s)}$, and $z(s) = e^{\Phi_3(s)}$. Then, Model (1.4)–(1.9) become

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}s} &= x\left(\Theta_1(s) - b_1(s)x - \frac{c_1(s)z}{d_1(s) + d_2(s)y}\right),\\
\frac{\mathrm{d}y}{\mathrm{d}s} &= y\left(\Theta_2(s) - \frac{y}{d_3(s) + d_4(s)x}\right),\\
\frac{\mathrm{d}z}{\mathrm{d}s} &= z\left(-\Theta_3(s) + \frac{k_1(s)c_1(s)x}{d_1(s) + d_2(s)y} - b_2(s)z\right),\\
x(s_k^+) &= (1 + h_k)x(s_k),\\
y(s_k^+) &= (1 + h_k')y(s_k),\\
z(s_k^+) &= (1 + h_k'')z(s_k).
\end{aligned}$$

Remark 1.2. For $\mathbb{T} = \mathbb{Z}$, $x(s) = e^{\Phi_1(s)}$, $y(s) = e^{\Phi_2(s)}$, and $z(s) = e^{\Phi_3(s)}$. Then, Model (1.4)–(1.9) become

$$\begin{aligned}
x(s+1) &= x(s)\exp\left\{\Theta_1(s) - b_1(s)x - \frac{c_1(s)z}{d_1(s) + d_2(s)y}\right\},\\
y(s+1) &= y(s)\exp\left\{\Theta_2(s) - \frac{y}{d_3(s) + d_4(s)x}\right\},\\
z(s+1) &= z(s)\exp\left\{-\Theta_3(s) + \frac{k_1(s)c_1(s)x}{d_1(s) + d_2(s)y} - b_2(s)z\right\},\\
x(s_k^+) &= (1 + h_k)x(s_k),\\
y(s_k^+) &= (1 + h_k')y(s_k),\\
z(s_k^+) &= (1 + h_k'')z(s_k).
\end{aligned}$$

Therefore, by Remarks 1.1 and 1.2, we observe that our model will serve both continuous and discrete time domains. Furthermore, the major findings and motivation of this work are as follows:

- In the existing literature, authors established the dynamics of a prey-predator-mutualist system without impulsive effects, but we are considering the prey-predator-mutualist system with impulsive effects, which shows that our proposed model is more realistic than other existing models.
- In addition, there are no published results for the prey-predator-mutualist system in the nonuniform time domain. Therefore in this article, we establish the hybrid prey-predator-mutualist system on nonuniform time domains, which will unify discrete and continuous time domains into a single unified theory, allowing us to avoid calculating the same results twice, once for discrete time domains and again for continuous time domains.
- Due to the complexity of the model, it is not easy to find the conditions that will ensure the permanence and global attractivity of the proposed model.
- In the end, we have given a numerical example with computer simulation on different-different timescales.

The rest of the manuscript is arranged as follows: in Section 2, we introduce some preliminary results; in Section 3, we obtain the permanence of the considered system (1.4)–(1.9); in Sections 4 and 5, we analyze the global attractivity and stability of considered system (1.4)–(1.9); in Section 6, we give some numerical examples with computer simulation to show the feasibility of obtained theoretical results.

2 Preliminaries

In this section, we define some basic definitions, lemmas, and useful assumptions.

Definition 2.1. [8] For $s \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and graininess function $\mu : \mathbb{T} \to [0, \infty)$ are defined as follows:
(1) $\sigma(s) \coloneqq \inf\{ß \in \mathbb{T} : ß > s\}$ and $\inf\varnothing = \sup\mathbb{T}$;
(2) $\rho(s) \coloneqq \sup\{ß \in \mathbb{T} : ß < s\}$ and $\sup\varnothing = \inf\mathbb{T}$;
(3) $\mu(s) \coloneqq \sigma(s) - s, \ \forall s \in \mathbb{T}$.

If $\mathbb{T}$ has a left-scattered maximum s_1, then $\mathbb{T}^k = \mathbb{T} - \{s_1\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. [8] If a function $\Phi : \mathbb{T} \to \mathbb{R}$ and let $s \in \mathbb{T}^k$. Then, we define the Δ-derivative of Φ at the point s, to be a number (provided it exists) with property that for each $\varepsilon > 0$, there exists nbd $\mathcal{U}$ of s such that

$$|[\Phi(\sigma(s)) - \Phi(ß)] - \Phi^{\Delta}(s)[\sigma(s) - ß]| \le \varepsilon|\sigma(s) - ß| \quad \forall ß \in \mathcal{U}.$$

If $\mathbb{T} = \mathbb{Z}$, then $\Phi^{\Delta}(s) = \Delta\Phi(s) = \Phi(s+1) - \Phi(s)$.
If $\mathbb{T} = \mathbb{R}$, then $\Phi^{\Delta}(s) = \Phi'(s)$.

Definition 2.3. [8] Suppose that a function $q : \mathbb{T} \to \mathbb{R}$ is called regressive (positive regressive) if $1 + \mu(s)q(s) \ne 0$ $(1 + \mu(s)q(s) > 0)$, $\forall s \in \mathbb{T}$. The set of all regressive (positive regressive) functions is denoted by $\mathcal{R}$ $(\mathcal{R}^+)$.

Definition 2.4. [20] A timescale $\mathbb{T}$ is called an almost periodic timescale if

$$\Pi = \{\omega \in \mathbb{R} : s \pm \omega \in \mathbb{T}, \quad \forall s \in \mathbb{T}\} \ne \{0\}.$$

Lemma 2.1. [20] *Let $\Phi : \mathbb{T} \to \mathbb{R}$ be a continuously increasing function, $\Phi(s) > 0$, $\Phi^{\Delta}(s) \ge 0$ for $s \in \mathbb{T}$. Then,*

$$\frac{\Phi^{\Delta}(s)}{\Phi^{\sigma}(s)} \le [\log(\Phi(s))]^{\Delta} \le \frac{\Phi^{\Delta}(s)}{\Phi(s)}.$$

If $\Phi(s) > 0$ and $\Phi^{\Delta}(s) < 0$ for $s \in \mathbb{T}$, then

$$\frac{\Phi^{\Delta}(s)}{\Phi(s)} \le [\log(\Phi(s))]^{\Delta} \le \frac{\Phi^{\Delta}(s)}{\Phi^{\sigma}(s)}.$$

Lemma 2.2. [20] *Consider $\Phi \in \mathrm{PC}^1[\mathbb{T}, \mathbb{R}]$ and*

$$\begin{aligned} &\Phi^{\Delta}(s) \le (\ge) p(s)\Phi(s) + q(s), \quad s \ne s_k, \quad s \in [s_0, \infty)_{\mathbb{T}}, \\ &\Phi(s_k^+) \le (\ge) d_k\Phi(s_k) + b_k, \quad s = s_k, \quad k \in \mathbb{N}. \end{aligned}$$

Then, for $s \ge s_0 \ge 0$,

$$\Phi(s) \le (\ge)\Phi(s_0) \prod_{s_0<s_k<s} d_k e_p(s, s_0) + \sum_{s_0<s_k<s}\left(\prod_{s_0<s_k<s} d_k e_p(s, s_k)\right) b_k + \int_{s_0}^{s} \prod_{s<s_k<s} d_k e_p(s, \sigma(s))q(s)\Delta s,$$

where $\mathrm{PC}^1 = \{\Phi : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$, *which is rd-continuous* [8] *except at s_k, $k = 1, 2, \ldots$, for which $\Phi(s_k^-)$, $\Phi(s_k^+)$, $\Phi^{\Delta}(s_k^-)$, $\Phi^{\Delta}(s_k^+)$ exists with $\Phi(s_k^-) = \Phi(s_k)$, $\Phi^{\Delta}(s_k^-) = \Phi^{\Delta}(s_k)\}$, and $e_p(\cdot, s_0)$ denotes the exponential function on timescale* [8].

Lemma 2.3. [20] *Suppose that $\Phi \in \mathrm{PC}^1[\mathbb{T}, \mathbb{R}]$, $\nu \le \prod_{s_0<s_k<s} d_k \le \varphi$ for $s \ge s_0$, $-g \in \mathcal{R}^+$.*
(*i*) *If*

$$\begin{aligned} &\Phi^{\Delta}(s) \le m - g\Phi(s), \quad s \ne s_k, \quad s \in [s_0, \infty)_{\mathbb{T}}, \\ &\Phi(s_k^+) \le d_k\Phi(s_k) + b_k, \quad s = s_k, \quad k \in \mathbb{N}, \end{aligned}$$

then for $s \ge s_0$,

$$\Phi(s) \le \Phi(s_0)\varphi e_{(-g)}(s, s_0) + \sum_{s_0<s_k<s} \varphi e_{(-g)}(s, s_k)b_k + \frac{m\varphi}{g}[1 - e_{(-g)}(s, s_0)].$$

In addition, if $m, g > 0$, *we have* $\limsup_{s\to\infty}\Phi(s) \le \frac{m\varphi}{g}$.

(*ii*) *If*

$$\Phi^{\Delta}(s) \ge m - g\Phi(s), \quad s \ne s_k, \quad s \in [s_0, \infty)_{\mathbb{T}},$$
$$\Phi(s_k^+) \ge d_k\Phi(s_k) + b_k, \quad s = s_k, \quad k \in \mathbb{N},$$

then for $s \ge s_0$,

$$\Phi(s) \ge \Phi(s_0)\nu e_{(-g)}(s, s_0) + \sum_{s_0<s_k<s} \nu e_{(-g)}(s, s_k)b_k + \frac{m\nu}{g}[1 - e_{(-g)}(s, s_0)].$$

In addition, if $m, g > 0$, *then* $\liminf_{s\to\infty}\Phi(s) \ge \frac{m\nu}{g}$.

Lemma 2.4. [20] *Suppose that* $-m \in \mathcal{R}^+$, $\Phi \in \mathrm{PC}^1[\mathbb{T}, \mathbb{R}]$, $\Phi(s) > 0$ *for* $s \in \mathbb{T}$, $\nu \le \prod_{s_0<s_k<s} d_k \le \varphi$ *for* $s \ge s_0$.

(*i*) *If*

$$\Phi^{\Delta}(s) \le \Phi^{\sigma}(s)(m - g\Phi(s)), \quad s \ne s_k, \quad s \in [s_0, \infty)_{\mathbb{T}},$$
$$\Phi(s_k^+) \le d_k\Phi(s_k), \quad s = s_k, \quad k \in \mathbb{N},$$

then for $s \ge s_0$,

$$\Phi(s) \le \frac{m\varphi}{g}\left[1 + \left(\frac{m}{g\Phi(s_0)} - 1\right)e_{(-m)}(s, s_0)\right]^{-1}.$$

In addition, if $m, g > 0$, *we have* $\limsup_{s\to\infty}\Phi(s) \le \frac{m\varphi}{g}$.

(*ii*) *If*

$$\Phi^{\Delta}(s) \ge \Phi^{\sigma}(s)(m - g\Phi(s)), \quad s \ne s_k, \quad s \in [s_0, \infty)_{\mathbb{T}},$$
$$\Phi(s_k^+) \ge d_k\Phi(s_k), \quad s = s_k, \quad k \in \mathbb{N},$$

then for $s \ge s_0$,

$$\Phi(s) \ge \frac{m\nu}{g}\left[1 + \left(\frac{m}{g\Phi(s_0)} - 1\right)e_{(-m)}(s, s_0)\right]^{-1}.$$

In addition, if $m, g > 0$, *then* $\liminf_{s\to\infty}\Phi(s) \ge \frac{m\nu}{g}$.

Lemma 2.5. [20] *Consider* $-m \in \mathcal{R}^+$, $\Phi \in \mathrm{PC}^1[\mathbb{T}, \mathbb{R}]$, $\Phi(s) > 0$ *for* $s \in \mathbb{T}$ *and* $\nu \le \prod_{s_0<s_k<s} d_k \le \varphi$ *for* $s \ge s_0$, $\bar{\mu} = \sup_{s\in\mathbb{T}}\mu(s)$. *If*

$$\Phi^{\Delta}(s) \ge \Phi(s)(m - g\Phi(s)), \quad s \ne s_k, \quad s \in [s_0, \infty)_{\mathbb{T}},$$
$$\Phi(s_k^+) \ge d_k\Phi(s_k), \quad s = s_k, \quad k \in \mathbb{N},$$

then for $s \ge s_0$,

$$\Phi(s) \ge \frac{m\nu}{g}\left[1 + \left(\frac{m}{g\Phi(s_0)} - 1\right)e_{(-\frac{m}{1+\bar{\mu}m})}(s, s_0)\right]^{-1}.$$

Furthermore, if $m, g > 0$, *then* $\liminf_{s\to\infty}\Phi(s) \ge \frac{m\nu}{g}$.

We introduce the following notations:

$$\Phi^+ = \sup_{s\in\mathbb{T}} \Phi(s), \quad \Phi^- = \inf_{s\in\mathbb{T}} \Phi(s).$$

3 Permanence

Definition 3.1. [16] Our considered system (1.4)–(1.9) is permanent, if there are the positive constants m_1, m_2, m_3, M_1, M_2, and M_3 such that

$$\liminf_{s\to\infty}\Phi_1(s) \geq m_1, \qquad \limsup_{s\to\infty}\Phi_1(s) \leq M_1,$$
$$\liminf_{s\to\infty}\Phi_2(s) \geq m_2, \qquad \limsup_{s\to\infty}\Phi_2(s) \leq M_2,$$
$$\liminf_{s\to\infty}\Phi_3(s) \geq m_3, \qquad \limsup_{s\to\infty}\Phi_3(s) \leq M_3.$$

We take the following assumptions throughout the manuscript:

(H1) We assume $\{h_k\}, \{h_k'\}$, and $\{h_k''\}$ are the almost periodic sequence. Furthermore, $1 + h_k > 0$, $1 + h_k' > 0$, and $1 + h_k'' > 0$ for $k \in \mathbb{N}$.

(H2) The functions $\Theta_i(s)(i = 1, 2, 3, 4)$, $b_k(s)(k = 1, 2)$, $c_1(s)$, $k_1(s)$, and $d_j(s)$ $(j = 1, 2, 3, 4)$ all are bounded non-negative almost periodic functions on $\mathbb{T}$ such that $\Theta_i^- > 0$, $b_k^- > 0$, $c_1^- > 0$, $k_1^- > 0$, and $d_j^- > 0$.

(H3) $\Theta_1^+ - b_1^- > 0, \Theta_2^+ > 0, \left(\Theta_4^+ + \frac{k_1^+c_1^+e^{M_1}}{d_1^- + d_2^-} - b_2^-\right) > 0, \left(\Theta_1^- - \frac{c_1^+e^{M_3+\varepsilon}}{d_1^- + d_2^-}\right) > 0, \left(\Theta_4^- - \Theta_3^+ + \frac{k_1^-c_1^-e^{m_1-\varepsilon}}{d_1^+ + d_2^+e^{M_2+\varepsilon}}\right) > 0.$

(H4) We assume that $P = (P_1 + P_1' + P_1'')$, $Q = (P_2 + P_2' + P_2'')$, and $R = (P_3 + P_3' + P_3'') > 0$, where

$$P_1 = \Bigg\{2b_1^-e^{m_1} + d_1^-c_1^-e^{m_3} + \frac{d_2^-c_1^-e^{m_2}e^{m_3}}{(d_1^+ + d_2^+e^{M_2})^2} - \frac{d_2^+c_1^+e^{M_3}e^{M_2}}{(d_1^- + d_2^-e^{m_2})^2} - \mu^2(s)\Big(b_1^+e^{M_1}\Big)^2$$
$$- \frac{\mu(s)b_1^+c_1^+d_1^+e^{M_1+M_3}}{(d_1^- + d_2^-e^{m_2})^2} - \frac{\mu(s)b_1^+d_2^+c_1^+e^{\{M_1+M_2+M_3\}}}{(d_1^- + d_2^-e^{m_2})^2} + \frac{\mu(s)b_1^-d_2^-c_1^-e^{\{m_1+m_2+m_3\}}}{(d_1^+ + d_2^+e_2^M)^2}\Bigg\};$$

$$P_2 = \Bigg\{- \frac{d_2^+c_1^+e^{(M_2+M_3)}}{(d_1^- + d_2^-e^{m_2})^2} + \mu(s)\frac{b_1^-d_2^-c_1^-e^{(m_1+m_2+m_3)}}{(d_1^+ + d_2^+e^{M_2})^2} + \mu(s)\frac{d_1^-d_2^-(c_1^-)^2e^{(m_2+2m_3)}}{(d_1^+ + d_2^+e^{M_2})^4}$$
$$+ \mu(s)\frac{(d_2^-)^2(c_1^-)^2e^{(2m_2+2m_3)}}{(d_1^+ + d_2^+e^{M_2})^4} - \frac{d_2^+c_1^+e^{(M_2+M_3)}}{(d_1^- + d_2^-e^{m_2})^2}\Bigg\};$$

$$P_3 = \Bigg\{d_1^-c_1^-e^{m_3} + \frac{d_2^-c_1^-e^{\{m_2+m_3\}}}{(d_1^+ + d_2^+e^{M_2})^2} - \mu(s)\frac{d_1^+b_1^+c_1^+e^{\{M_1+M_3\}}}{(d_1^- + d_2^-e^{m_2})^2} - \mu(s)\frac{b_1^+d_2^+c_1^+e^{\{M_1+M_2+M_3\}}}{(d_1^- + d_2^-e^{M_2})^2}$$
$$- \left(\frac{mu(s)d_1^+c_1^+e^{M_3}}{(d_1^- + d_2^-e^{m_2})^2}\right)^2 - 2\mu(s)\frac{d_1^+d_2^+(c_1^+)^2e^{\{M_2+2M_3\}}}{(d_1^- + d_2^-e^{m_2})^4} + \mu(s)\frac{d_1^-d_2^-(c_1^-)^2e^{(m_2+2m_3)}}{(d_1^+ + d_2^+e^{M_2})^4}$$
$$- \mu(s)\frac{d_2^+c_1^+e^{(M_2+M_3)}}{(d_1^- + d_2^-e_2^m)^2} + \mu(s)\frac{(d_2^-)^2(c_1^-)^2e^{(2m_2+2m_3)}}{(d_1^+ + d_2^+e^{M_2})^4}\Bigg\};$$

$$P_1' = \Bigg\{- \frac{d_4^+e^{(M_1+M_2)}}{(d_3^- + d_4^-e^{m_1})^2} + \mu(s)\frac{d_3^-d_4^-e^{(m_1+2m_2)}}{(d_3^+ + d_4^+e^{M_1})^4} + \mu(s)\frac{(d_4^-)^2e^{(2m_1+2m_2)}}{(d_3^+ + d_4^+e^{M_1})^4} - \left(\mu(s)\frac{d_4^+e^{(M_1+M_2)}}{d_3^- + d_4^-e^{m_1}}\right)^2\Bigg\};$$

$$P_2' = \Bigg\{\frac{2d_3^-e^{m_2}}{(d_3^+ + d_4^+e^{M_1})^2} + \frac{2d_4^-e^{(m_1+m_2)}}{(d_3^+ + d_4^+e^{M_1})^2} - \frac{d_4^+e^{(M_1+M_2)}}{(d_3^- + d_4^-e^{m_1})^2} - \mu(s)\left(\frac{d_3^+e^{M_2}}{(d_3^- + d_4^-e^{m_1})^2}\right)^2$$
$$- 2\mu(s)\frac{d_3^+d_4^+e^{(M_1+2M_2)}}{(d_3^- + d_4^-e^{m_1})^4} + \mu(s)\frac{d_3^-d_4^-e^{m_1+2m_2}}{(d_3^+ + d_4^+e^{M_1})^4} - \mu(s)\left(\frac{d_4^+e^{(M_1+M_2)}}{(d_3^- + d_4^-e^{m_1})^2}\right)^2 + \frac{(d_4^-)^2e^{(2m_1+2m_2)}}{(d_3^+ + d_4^+e^{M_1})^4}\Bigg\};$$

$$P_3' = 0;$$

$$
\begin{aligned}
P_1'' = \Bigg\{ & - \frac{d_1^+ k_1^+ c_1^+ e^{M_1}}{(d_1^- + d_2^- e^{m_2})^2} - \frac{d_2^+ k_1^+ c_1^+ e^{(M_1+M_2)}}{(d_1^- + d_2^- e^{m_2})^2} + \mu(s) \frac{b_2^- d_1^- k_1^- c_1^- e^{(m_1+m_3)}}{(d_1^+ + d_2^+ e^{M_2})^2} + \mu(s) \frac{b_2^- d_2^- k_1^- c_1^- e^{(2m_2+m_1)}}{(d_1^+ + d_2^+ e^{M_2})^2} \\
& - \mu(s) \left(\frac{d_2^+ k_1^+ c_1^+ e^{(M_1+M_2)}}{(d_1^- + d_2^- e^{M_2})^2} \right)^2 - \mu(s) \frac{d_1^+ k_1^+ c_1^+ e^{M_1}}{(d_1^- + d_2^- e^{m_2})^2} - 2\mu(s) \frac{d_1^+ d_2^+ (k_1^+)^2 (c_1^+)^2 e^{(2M_1+M_2)}}{(d_1^+ + d_2^- e^{m_2})^4} \\
& + \mu(s) \frac{(d_2^-)^2 (k_1^-)^2 (c_1^-)^2 e^{(2m_1+2m_2)}}{(d_1^+ + d_2^+ e^{M_2})^4} + \mu(s) \frac{d_1^- d_2^- (k_1^-)^2 (c_1^-)^2 e^{(2m_1+m_2)}}{(d_1^+ + d_2^+ e^{M_2})^4} \Bigg\}; \\
P_2'' = \Bigg\{ & \frac{d_2^- k_1^- c_1^- e^{(m_1+m_2)}}{(d_1^+ + d_2^+ e^{M_2})^2} - \mu(s) \frac{b_2^+ d_2^+ k_1^+ c_1^+ e^{(M_1+M_2+M_2)}}{(d_1^- + d_2^- e^{m_2})^2} + \mu(s) \frac{d_1^- d_2^- (k_1^-)^2 (c_1^-)^2 e^{(2m_1+2m_2)}}{(d_1^+ + d_2^+ e^{M_2})^4} \\
& + \mu(s) \frac{(d_2^-)^2 (k_1^-)^2 (c_1^-)^2 e^{(2m_1+2m_2)}}{(d_1^+ + d_2^+ e^{M_2})^4} - \mu(s) \left(\frac{d_2^+ k_1^+ c_1^+ e^{(M_1+M_2)}}{(d_1^- + d_2^- e^{m_2})^2} \right)^2 \Bigg\}; \\
P_3'' = \Bigg\{ & 2b_2^- e^{m_3} - \frac{d_1^+ k_1^+ c_1^+ e^{M_1}}{(d_1^- + d_2^- e^{m_2})^2} - \frac{d_2^+ k_1^+ c_1^+ e^{(M_1+M_2)}}{(d_1^- + d_2^- e^{m_2})^2} + \frac{d_2^- k_1^- c_1^- e^{(m_1+m_2)}}{(d_1^+ + d_2^+ e^{M_2})^2} - \mu(s) \left(b_2^+ e^{M_3} \right)^2 \\
& + \mu(s) \frac{b_2^- d_1^- k_1^- c_1^- e^{(m_1+m_3)}}{(d_1^+ + d_2^+ e^{M_2})^2} + \mu(s) \frac{b_2^- d_2^- k_1^- c_1^- e^{(m_1+2m_2)}}{(d_1^+ + d_2^+ e^{M_2})^2} - \mu(s) \frac{b_2^+ d_2^+ k_1^+ c_1^+ e^{(m_1+m_2+m_3)}}{(d_1^- + d_2^- e^{m_2})^2} \Bigg\}.
\end{aligned}
$$

Theorem 3.2. *Let us suppose* (*H*1)–(*H*3) *hold true, then our model* (1.4)–(1.9) *are permanent.*

Proof. From equation (1.4), we have

$$
\begin{aligned}
\Phi_1^{\Delta}(s) &\le \Theta_1^+ - b_1^- e^{\Phi_1(s)} \\
&\le \Theta_1^+ - b_1^- [1 + \Phi_1(s)] \\
&\le [\Theta_1^+ - b_1^-] - b_1^- \Phi_1(s).
\end{aligned}
$$

Then, the above equation with impulsive condition can be written as follows:

$$
\begin{aligned}
\Phi_1^{\Delta}(s) &\le [\Theta_1^+ - b_1^-] - b_1^- \Phi_1(s), \\
\Phi_1(s_k^+) &\le \Phi_1(s_k) + \log(1 + h_k),
\end{aligned}
\tag{3.1}
$$

where $\Theta_1^+ - b_1^- > 0$ and $-b_1^- \in \mathcal{R}^+$. Now, by applying Lemma 2.3 in system (3.1), we obtain

$$
\limsup_{s \to \infty} \Phi_1(s) \le \frac{\Theta_1^+ - b_1^-}{b_1^-} = M_1.
$$

Hence, for arbitrary $\varepsilon > 0$, $\exists\ k_0 > 0$, s.t.

$$
\Phi_1(s) \le M_1 + \varepsilon \quad \forall s > k_0.
$$

From equation (1.5), we have

$$
\begin{aligned}
\Phi_2^{\Delta}(s) &\le \Theta_2^+ - \frac{e^{\Phi_2(s)}}{d_3^+ + d_4^+ e^{M_1+\varepsilon}} \\
&\le \Theta_2^+ - \frac{1 + \Phi_2(s)}{d_3^+ + d_4^+ e^{M_1+\varepsilon}} \\
&\le \Theta_2^+ - \frac{\Phi_2(s)}{d_3^+ + d_4^+ e^{M_1+\varepsilon}}.
\end{aligned}
$$

Then, the above equation with impulsive condition can be written as follows:

$$
\begin{aligned}
\Phi_2^{\Delta}(s) &\le \Theta_2^+ - \frac{\Phi_2(s)}{d_3^+ + d_4^+ e^{M_1+\varepsilon}}, \\
\Phi_2(s_k^+) &\le \Phi_2(s_k) + \log(1 + h_k'),
\end{aligned}
\tag{3.2}
$$

where $-\frac{1}{d_3^+ + d_4^+ e^{M_1+\varepsilon}} \in \mathcal{R}^+$ and $\Theta_2^+ > 0$. Now, by applying Lemma 2.3 in system (3.2), we obtain

$$\limsup_{s\to\infty} \Phi_2(s) \le [d_3^+ + d_4^+ e^{M_1+\varepsilon}]\Theta_2^+.$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_{s\to\infty} \Phi_2(s) \le [d_3^+ + d_4^+ e^{M_1}]\Theta_2^+ = M_2.$$

Hence, for arbitrary small $\varepsilon > 0$, $\exists k_1 > 0$, s.t.

$$\Phi_2(s) \le M_2 + \varepsilon \quad \forall s > k_1.$$

From equation (1.6), we have

$$\begin{aligned}\Phi_3^\Delta(s) &\le \Theta_4^+ + \frac{k_1^+ c_1^+ e^{M_1+\varepsilon}}{d_1^- + d_2^-} - b_2^- e^{\Phi_3(s)}\\ &\le \Theta_4^+ + \frac{k_1^+ c_1^+ e^{M_1+\varepsilon}}{d_1^- + d_2^-} - b_2^- - b_2^- \Phi_3(s).\end{aligned}$$

Then, the above equation with impulsive condition can be written as follows:

$$\begin{aligned}\Phi_3^\Delta(s) &\le \Theta_4^+ + \frac{k_1^+ c_1^+ e^{M_1}}{d_1^- + d_2^-} - b_2^- - b_2^- \Phi_3(s),\\ \Phi_3(s_k^+) &\le \Phi_3(s_k) + \log(1 + h_k''),\end{aligned} \tag{3.3}$$

where $\left(\Theta_4^+ + \frac{k_1^+ c_1^+ e^{M_1}}{d_1^- + d_2^-} - b_2^-\right) > 0$ and $-b_2^- \in \mathcal{R}^+$. Now, by applying Lemma 2.3 in system (3.3), we obtain

$$\limsup_{s\to\infty} \Phi_3(s) \le \frac{[(d_1^- + d_2^-)\Theta_4^+ + k_1^+ c_1^+ e^{M_1+\varepsilon} - b_2^-(d_1^- + d_2^-)]}{[d_1^- + d_2^-]b_2^-}.$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_{s\to\infty} \Phi_3(s) \le \frac{[(d_1^- + d_2^-)\Theta_4^+ + k_1^+ c_1^+ e^{M_1} - b_2^-(d_1^- + d_2^-)]}{[d_1^- + d_2^-]b_2^-} = M_3.$$

Hence, for arbitrary small $\varepsilon > 0$, $\exists k_2 > 0$, s.t.

$$\Phi_3(s) \le M_3 + \varepsilon \quad \forall t > k_2.$$

From equation (1.4), we obtain

$$\Phi_1^\Delta(s) \ge \Theta_1^- - b_1^+ e^{\Phi_1(s)} - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-}.$$

Put $K_1(s) = e^{\Phi_1(s)}$, obviously $K_1(s) > 0$, then the above inequality becomes

$$[\log(K_1(s))]^\Delta \ge \Theta_1^- - b_1^+ K_1(s) - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-}.$$

If $K_1^\Delta(s) \ge 0$, then from Lemma 2.1, we have

$$\begin{aligned}\frac{K_1^\Delta(s)}{K_1(s)} &\ge \Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-} - b_1^+ K_1(s),\\ K_1^\Delta(s) &\ge K_1(s)\left[\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-} - b_1^+ K_1(s)\right].\end{aligned}$$

Then, the above equation with impulsive condition can be written as follows:

$$\begin{aligned} K_1^{\Delta}(s) &\geq K_1(s)\left[\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-} - b_1^+ K_1(s)\right], \\ K_1(s_k^+) &\geq k_1(s_k)(1 + h_k), \end{aligned} \tag{3.4}$$

where $\left(\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-}\right) > 0$, $b_1^+ > 0$, and $-\left(\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-}\right) \in \mathcal{R}^+$. Then, by applying Lemma 2.5 in system (3.4), we obtain

$$\liminf_{s\to\infty} \Phi_1(s) \geq \log\left[\frac{[d_1^- + d_2^-]\Theta_1^- - c_1^+ e^{M_3+\varepsilon}}{[d_1^- + d_2^-]b_1^+}\right]. \tag{3.5}$$

If $K_1^{\Delta}(s) < 0$, then from Lemma 2.1, we have

$$\begin{aligned} \frac{K_1^{\Delta}(s)}{K_1^{\sigma}(s)} &\geq \Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-} - b_1^+ K_1(s), \\ K_1^{\Delta}(s) &\geq K_1^{\sigma}(s)\left[\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-} - b_1^+ K_1(s)\right]. \end{aligned}$$

Then, the above equation with impulsive condition can be written as follows:

$$\begin{aligned} K_1^{\Delta}(s) &\geq K_1^{\sigma}(s)\left[\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-} - b_1^+ K_1(s)\right], \\ K_1(s_k^+) &\geq k_1(s_k)(1 + h_k), \end{aligned} \tag{3.6}$$

where $\left(\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-}\right) > 0$, $b_1^+ > 0$, and $-\left(\Theta_1^- - \frac{c_1^+ e^{M_3+\varepsilon}}{d_1^- + d_2^-}\right) \in \mathcal{R}^+$. Then, by applying Lemma 2.4 in system (3.6), we obtain

$$\liminf_{s\to\infty} \Phi_1(s) \geq \log\left[\frac{[d_1^- + d_2^-]\Theta_1^- - c_1^+ e^{M_3+\varepsilon}}{[d_1^- + d_2^-]b_1^+}\right]. \tag{3.7}$$

Therefore, from above expression (3.5) and (3.7), $\varepsilon \to 0$, we can write

$$\liminf_{s\to\infty} \Phi_1(s) \geq \log\left[\frac{[d_1^- + d_2^-]\Theta_1^- - c_1^+ e^{M_3}}{[d_1^- + d_2^-]b_1^+}\right] = m_1.$$

Hence, for arbitrary small $\varepsilon > 0$, $\exists\ k_3 > 0$, s.t.

$$\Phi_1(s) \geq m_1 - \varepsilon \quad \forall s > k_3.$$

From equation (1.5), we obtain

$$\Phi_2^{\Delta}(s) \geq \Theta_2^- - \frac{e^{\Phi_2(s)}}{d_3^- + d_4^-}.$$

Put $K_2(s) = e^{\Phi_2(s)}$, obviously $K_2(s) > 0$, then the above inequality

$$[\log K_2(s)]^{\Delta} \geq \Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-}.$$

If $K_2^{\Delta}(s) \geq 0$, then from Lemma 2.1, we obtain

$$\begin{aligned} \frac{K_2^{\Delta}(s)}{K_2(s)} &\geq \Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-}, \\ K_2^{\Delta}(s) &\geq K_2(s)\left[\Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-}\right]. \end{aligned}$$

Then, the above equation with impulsive condition can be written as follows:

$$K_2^{\Delta}(s) \geq K_2(s)\left[\Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-}\right], \qquad (3.8)$$
$$K_2(s_k^+) \geq K_2(s_k)(1 + h_k'),$$

where $\Theta_2^-, \frac{1}{d_3^- + d_4^-} > 0$, and $-\Theta_2^- \in \mathcal{R}^+$. Then, by applying Lemma 2.5 in system (3.8), we obtain

$$\liminf_{s\to\infty} \Phi_2(s) \geq \log[\Theta_2^-(d_3^- + d_4^-)]. \qquad (3.9)$$

If $K_2^{\Delta}(s) < 0$, then from Lemma (2.1), we obtain

$$\frac{K_2^{\Delta}(s)}{K_2^{\sigma}(s)} \geq \Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-},$$
$$K_2^{\Delta}(s) \geq K_2^{\sigma}(s)\left[\Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-}\right].$$

Then, the above equation with impulsive condition can be written as follows:

$$K_2^{\Delta}(s) \geq K_2^{\sigma}(s)\left[\Theta_2^- - \frac{K_2(s)}{d_3^- + d_4^-}\right], \qquad (3.10)$$
$$K_2(s_k^+) \geq K_2(s_k)(1 + h_k'),$$

where $\Theta_2^-, \frac{1}{d_3^- + d_4^-} > 0$, and $-\Theta_2^- \in \mathcal{R}^+$. Then, by applying Lemma 2.4 in system (3.10), we obtain

$$\liminf_{s\to\infty} \Phi_2(s) \geq \log[\Theta_2^-(d_3^- + d_4^-)]. \qquad (3.11)$$

Therefore, from equations (3.9) and (3.11), we obtain

$$\liminf_{s\to\infty} \Phi_2(s) \geq \log[\Theta_2^-(d_3^- + d_4^-)] = m_2.$$

Hence, for arbitrary small $\varepsilon > 0$, $\exists k_4 > 0$, s.t.

$$\Phi_2(s) \geq m_2 - \varepsilon \quad \forall s > k_4.$$

From equation (1.6), we obtain

$$\Phi_3^{\Delta}(s) \geq \Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ e^{\Phi_3(s)}.$$

Put $K_3(s) = e^{\Phi_3(s)}$, obviously $K_3(s) > 0$, then the above inequality becomes

$$[\log(K_3(s))]^{\Delta} \geq \Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s).$$

If $K_3^{\Delta}(t) \geq 0$, then from Lemma 2.1, we obtain

$$\frac{K_3^{\Delta}(s)}{K_3(s)} \geq \Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s),$$
$$K_3^{\Delta}(s) \geq K_3(s)\left[\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s)\right].$$

Then, the above equation with impulsive condition can be written as follows:

$$K_3^{\Delta}(s) \geq K_3(s)\left[\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s)\right], \qquad (3.12)$$
$$K_3(s_k^+) \geq K_3(s_k)(1 + h_k''),$$

where $\left(\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}}\right)$, $b_2^+ > 0$, and $-\left(\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}}\right) \in \mathcal{R}^+$. Then, by applying Lemma 2.5 in system (3.12), we obtain

$$\liminf_{s\to\infty} \Phi_3(s) \geq \log\left\{\frac{\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}}}{b_2^+}\right\}. \tag{3.13}$$

If $K_3^\Delta(s) < 0$, then from Lemma 2.1, we obtain

$$\frac{K_3^\Delta(s)}{K_3^\sigma(s)} \geq \Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s),$$
$$K_3^\Delta(s) \geq K_3^\sigma(s)\left[\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s)\right].$$

Then, the above equation with impulsive condition can be written as follows:

$$\begin{aligned} K_3^\Delta(s) &\geq K_3^\sigma(s)\left[\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}} - b_2^+ K_3(s)\right], \\ K_3(s_k^+) &\geq K_3(s_k)(1 + h_k''), \end{aligned} \tag{3.14}$$

where $\left(\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}}\right)$, $b_2^+ > 0$, and $-\left(\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}}\right) \in \mathcal{R}^+$. Then, by applying Lemma 2.4 in system (3.14), we obtain

$$\liminf_{s\to\infty} \Phi_3(s) \geq \log\left\{\frac{\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1-\varepsilon}}{d_1^+ + d_2^+ e^{M_2+\varepsilon}}}{b_2^+}\right\}. \tag{3.15}$$

Therefore, by equations (3.13) and (3.15), $\varepsilon \to 0$, we can write

$$\liminf_{s\to\infty} \Phi_3(s) \geq \log\left\{\frac{\Theta_4^- - \Theta_3^+ + \frac{k_1^- c_1^- e^{m_1}}{d_1^+ + d_2^+ e^{M_2}}}{b_2^+}\right\} = m_3.$$

Thus, from the above calculations, we obtain

$$\begin{aligned} &\liminf_{s\to\infty} \Phi_1(s) \geq m_1, && \limsup_{s\to\infty} \Phi_1(s) \leq M_1, \\ &\liminf_{s\to\infty} \Phi_2(s) \geq m_2, && \limsup_{s\to\infty} \Phi_2(s) \leq M_2, \quad \text{and} \\ &\liminf_{s\to\infty} \Phi_3(s) \geq m_3, && \limsup_{s\to\infty} \Phi_3(s) \leq M_3. \end{aligned}$$

Hence, by Definition 3.1, our considered system (1.4)–(1.9) will be permanent. □

4 Global attractivity

Here, we will study about the global attractivity of considered model (1.4)–(1.9).

Suppose that the set $\bar{\Gamma} = \{\Phi_1, \Phi_2, \Phi_3 : m_1 \leq \Phi_1 \leq M_1, m_2 \leq \Phi_2 \leq M_2, m_3 \leq \Phi_3 \leq M_3\}$ is the solution set of our proposed system (1.4)–(1.9).

Lemma 4.1. [20] *Suppose that* $(H1)$–$(H4)$ *hold. Then,* $\bar{\Gamma} \neq \emptyset$.

Definition 4.1. [22] System (1.4)–(1.9) is said to be globally attractive if any two positive solutions: $X(s) = (\Phi_1(s), \Phi_2(s), \Phi_3(s))$ with $\Phi_1(0) > 0, \Phi_2(0) > 0$, and $\Phi_3(0) > 0$ and $Y(s) = (y_1(s), y_2(s), y_3(s))$ with $y_1(0) > 0, y_2(0) > 0, y_3(0) > 0$, of system (1.4)–(1.9) satisfy

$$\lim_{s\to\infty}|\Phi_i(s) - y_i(s)| = 0, \quad i = 1, 2, 3.$$

Theorem 4.2. *Assume that the assumptions* ($H1$)–($H4$) *hold true. Assume further that there exists a positive constant* $\kappa > 0$ *and* $-\kappa \in \mathcal{R}^+$, *where* $\kappa = \min\{(P_1 + P_1' + P_1''), (P_2 + P_2' + P_2''), (P_3 + P_3' + P_3'')\}$. *Then, the species* Φ_1, Φ_2, *and* Φ_3 *are globally attractive.*

Proof. From Theorem 3.2, we observe that considered model (1.4)–(1.9) has a bounded solution satisfying

$$\begin{aligned} m_1 &\le \Phi_1(s) \le M_1,\\ m_2 &\le \Phi_2(s) \le M_2,\\ m_3 &\le \Phi_3(s) \le M_3. \end{aligned}$$

Hence, $|\Phi_1(s)| \le A_1$, $|\Phi_2(s)| \le A_2$, and $|\Phi_3(s)| \le A_3$, where $A_1 = \max\{m_1, M_1\}$, $A_2 = \max\{m_2, M_2\}$, and $A_3 = \max\{m_3, M_3\}$ for all $(\Phi_1, \Phi_2, \Phi_3) \in \mathbb{R}^3$, and we define a norm on $\mathbb{R}^3$:

$$||(\Phi_1(s), \Phi_2(s), \Phi_3(s))|| = \sup_{s\in\mathbb{T}}|\Phi_1(s)| + \sup_{s\in\mathbb{T}}|\Phi_2(s)| + \sup_{s\in\mathbb{T}}|\Phi_3(s)|.$$

Suppose that $X(s) = (\Phi_1(s), \Phi_2(s), \Phi_3(s))$ and $Y(s) = (y_1(s), y_2(s), y_3(s))$ be any two solutions of system (1.4)–(1.9), then $||X|| \le C$ and $||Y|| \le C$, where $C = \sum_{i=1}^{4} A_i$. Now, we have the following:

$$\begin{aligned}
\Phi_1^\Delta(s) &= \Theta_1(s) - b_1(s)e^{\Phi_1(s)} - \frac{c_1(s)e^{\Phi_3(s)}}{d_1(s) + d_2(s)e^{\Phi_2(s)}},\\
\Phi_2^\Delta(s) &= \Theta_2(s) - \frac{e^{\Phi_2(s)}}{d_3(s) + d_4(s)e^{\Phi_1(s)}},\\
\Phi_3^\Delta(s) &= \Theta_4(s) - \Theta_3(s) + \frac{k_1(s)c_1(s)e^{\Phi_1(s)}}{d_1(s) + d_2(s)e^{\Phi_2(s)}} - b_2(s)e^{\Phi_3(s)},\\
\Phi_1(s_k^+) &= \Phi_1(s_k) + \log(1 + h_k),\\
\Phi_2(s_k^+) &= \Phi_2(s_k) + \log(1 + h_k'),\\
\Phi_3(s_k^+) &= \Phi_3(s_k) + \log(1 + h_k'')
\end{aligned}$$

and

$$\begin{aligned}
y_1^\Delta(s) &= \Theta_1(s) - b_1(s)e^{y_1(s)} - \frac{c_1(s)e^{y_3(s)}}{d_1(s) + d_2(s)e^{y_2(s)}},\\
y_2^\Delta(s) &= \Theta_2(s) - \frac{e^{y_2(s)}}{d_3(s) + d_4(s)e^{y_1(s)}},\\
y_3^\Delta(s) &= \Theta_4(s) - \Theta_3(s) + \frac{k_1(s)c_1(s)e^{y_1(s)}}{d_1(s) + d_2(s)e^{y_2(s)}} - b_2(s)e^{y_3(s)},\\
y_1(s_k^+) &= y_1(s_k) + \log(1 + h_k),\\
y_2(s_k^+) &= y_2(s_k) + \log(1 + h_k'),\\
y_3(s_k^+) &= y_3(s_k) + \log(1 + h_k'').
\end{aligned}$$

Now, we consider the Lyapunov function define by $\mathbb{T} \times \bar{\Gamma} \times \bar{\Gamma}$ as follows:

$$V(s, X, Y) = \sum_{i=1}^{3}(\Phi_i - y_i)^2.$$

The Dini derivative of $V(s, X, Y)$ is given as follows:

$$D^+V(s,X,Y)=\sum_{i=1}^{3}[2(\Phi_i(s)-y_i(s))+\mu(s)(\Phi_i(s)-y_i(s))^{\Delta}][\Phi_i(s)-y_i(s)]^{\Delta}=\sum_{i=1}^{3}(2v_i+\mu(s)v_i^{\Delta}(s))v_i^{\Delta}(s),$$

where $v_i=\Phi_i(s)-y_i(s)$, $i=1,2,3$. Therefore, we can write

$$D^+V(s,X,Y)=V_1(s)+V_2(s)+V_3(s),$$

where V_1, V_2, and V_3 are given as follows:

$$V_1(s)=(2v_1+\mu(s)v_1^{\Delta}(s))v_1^{\Delta}(s),\quad V_2(s)=(2v_2+\mu(s)v_2^{\Delta}(s))v_2^{\Delta}(s),\quad V_3(s)=(2v_3+\mu(s)v_3^{\Delta}(s))v_3^{\Delta}(s).$$

Now,

$$\begin{aligned}v_1^{\Delta}(s)&=\Phi_1^{\Delta}(s)-y_1^{\Delta}(s)\\&=\Theta_1(s)-b_1(s)e^{\Phi_1(s)}-\frac{c_1(s)e^{\Phi_3(s)}}{d_1(s)+d_2(s)e^{\Phi_2(s)}}-\Theta_1(s)+b_1(s)e^{y_1(s)}+\frac{c_1(s)e^{y_3(s)}}{d_1(s)-d_2(s)e^{y_2(s)}}\\&=-b_1(s)[e^{\Phi_1(s)}-e^{y_1(s)}]-\frac{c_1(s)e^{\Phi_3(s)}}{d_1(s)+d_2(s)e^{\Phi_2(s)}}+\frac{c_1(s)e^{y_3(s)}}{d_1(s)+d_2(s)e^{y_2(s)}}\\&=-b_1(s)e^{\Phi_1i_1(s)}v_1(s)-\frac{d_1(s)c_1(s)e^{\Phi_1i_3(s)}v_3(s)+d_2(s)c_1(s)[e^{y_2(s)}e^{\Phi_1i_3(s)}v_3(s)-e^{y_3(s)}e^{\Phi_1i_2(s)}v_2(s)]}{[d_1(s)+d_2(s)e^{\Phi_2(s)}][d_1(s)+d_2(s)e^{y_2(s)}]}.\end{aligned}$$

The Δ-derivative of $v_2(s)$ is given as follows:

$$\begin{aligned}v_2^{\Delta}(s)&=\Phi_2^{\Delta}(s)-y_2^{\Delta}(s)\\&=\Theta_2(s)-\frac{e^{\Phi_2(s)}}{d_3(s)+d_4(s)e^{\Phi_1(s)}}-\Theta_2(s)+\frac{e^{y_2(s)}}{d_3(s)-d_4(s)e^{y_1(s)}}\\&=-\frac{e^{\Phi_2(s)}}{d_3(s)+d_4(s)e^{\Phi_1(s)}}+\frac{e^{y_2(s)}}{d_3(s)-d_4(s)e^{y_1(s)}}\\&=-\frac{d_3(s)e^{\Phi_1i_2(s)}v_2(s)+d_4(s)[e^{y_1(s)}e^{\Phi_1i_2(s)}v_2(s)-e^{y_2(s)}e^{\Phi_1i_1(s)}v_1(s)]}{[d_3(s)+d_4(s)e^{\Phi_1(s)}][d_3(s)+d_4(s)e^{y_1(s)}]}.\end{aligned}$$

The Δ-derivative of $v_3(s)$ is given as follows:

$$\begin{aligned}v_3^{\Delta}(s)&=\Phi_3^{\Delta}(s)-y_3^{\Delta}(s)\\&=-\Theta_3(s)+\frac{k_1(s)c_1(s)e^{\Phi_1(s)}}{d_1(s)+d_2(s)e^{\Phi_2(s)}}-b_2(s)e^{\Phi_3(s)}+\Theta_3(s)-\frac{k_1(s)c_1(s)e^{y_1(s)}}{d_1(s)-d_2(s)e^{y_2(s)}}+b_2(s)e^{y_3(s)}\\&=-b_2(s)e^{\Phi_3(s)}+b_2(s)e^{y_3(s)}+\frac{k_1(s)c_1(s)e^{\Phi_1(s)}}{d_1(s)+d_2(s)e^{\Phi_2(s)}}-\frac{k_1(s)c_1(s)e^{y_1(s)}}{d_1(s)+d_2(s)e^{y_2(s)}}\\&=-b_2(s)e^{\Phi_1i_3(s)}v_3(s)+\frac{d_1(s)k_1(s)c_1(s)e^{\Phi_1i_1(s)}v_1(s)+d_2(s)k_1(s)c_1(s)[e^{y_2(s)}e^{\Phi_1i_1(s)}v_1(s)-e^{y_1(s)}e^{\Phi_1i_2(s)}v_2(s)]}{[d_1(s)+d_2(s)e^{\Phi_2(s)}][d_1(s)+d_2(s)e^{y_2(s)}]}.\end{aligned}$$

From the above equations, we obtain

$$\begin{aligned}v_1^{\Delta}(s)&=-b_1(s)e^{\Phi_1i_1(s)}v_1(s)-\frac{d_1(s)c_1(s)e^{\Phi_1i_3(s)}v_3(s)+d_2(s)c_1(s)[e^{y_2(s)}e^{\Phi_1i_3(s)}v_3(s)-e^{y_3(s)}e^{\Phi_1i_2(s)}v_2(s)]}{[d_1(s)+d_2(s)e^{\Phi_2(s)}][d_1(s)+d_2(s)e^{y_2(s)}]},\\v_2^{\Delta}(s)&=-\frac{d_3(s)e^{\Phi_1i_2(s)}v_2(s)+d_4(s)[e^{y_1(s)}e^{\Phi_1i_2(s)}v_2(s)-e^{y_2(s)}e^{\Phi_1i_1(s)}v_1(s)]}{[d_3(s)+d_4(s)e^{\Phi_1(s)}][d_3(s)+d_4(s)e^{y_1(s)}]},\\v_3^{\Delta}(s)&=-b_2(s)e^{\Phi_1i_3(s)}v_3(s)+\frac{d_1(s)k_1(s)c_1(s)e^{\Phi_1i_1(s)}v_1(s)+d_2(s)k_1(s)c_1(s)[e^{y_2(s)}e^{\Phi_1i_1(s)}v_1(s)-e^{y_1(s)}e^{\Phi_1i_2(s)}v_2(s)]}{[d_1(s)+d_2(s)e^{\Phi_2(s)}][d_1(s)+d_2(s)e^{y_2(s)}]},\\v_1(s_k^+)&=v_1(s_k),\\v_2(s_k^+)&=v_2(s_k),\\v_3(s_k^+)&=v_3(s_k).\end{aligned}$$

Hence,

$$
\begin{aligned}
V_1(s) &= (2v_1(s) + \mu(s)v_1^{\Delta}(s))v_1^{\Delta}(s) \\
&\leq \Bigg\{- \ 2b_1^- e^{m_1} - \ d_1^- c_1^- e^{m_3} - \ \frac{d_2^- c_1^- e^{m_2} e^{m_3}}{(d_1^+ + d_2^+ e^{M_2})^2} + \frac{d_2^+ c_1^+ e^{M_3} e^{M_2}}{(d_1^- + d_2^- e^{m_2})^2} + \mu^2(s)(b_1^+ e^{M_1})^2 \\
&\quad + \frac{\mu(s) b_1^+ c_1^+ d_1^+ e^{M_1+M_3}}{(d_1^- + d_2^- e^{m_2})^2} + \frac{\mu(s) b_1^+ d_2^+ c_1^+ e^{\{M_1+M_2+M_3\}}}{(d_1^- + d_2^- e^{m_2})^2} - \frac{\mu(s) b_1^- d_2^- c_1^- e^{\{m_1+m_2+m_3\}}}{(d_1^+ + d_2^+ e_2^{M})^2}\Bigg\} \times v_1^2 \\
&\quad + \Bigg\{\frac{d_2^+ c_1^+ e^{(M_2+M_3)}}{(d_1^- + d_2^- e^{m_2})^2} - \ \mu(s) \frac{b_1^- d_2^- c_1^- e^{(m_1+m_2+m_3)}}{(d_1^+ + d_2^+ e^{M_2})^2} - \ \mu(s) \frac{d_1^- d_2^- (c_1^-)^2 e^{(m_2+2m_3)}}{(d_1^+ + d_2^+ e^{M_2})^4} \\
&\quad - \ \mu(s) \frac{(d_2^-)^2 (c_1^-)^2 e^{(2m_2+2m_3)}}{(d_1^+ + d_2^+ e^{M_2})^4} + \frac{d_2^+ c_1^+ e^{(M_2+M_3)}}{(d_1^- + d_2^- e^{m_2})^2}\Bigg\} \times v_2^2 \\
&\quad + \Bigg\{-d_1^- c_1^- e^{m_3} - \ \frac{d_2^- c_1^- e^{\{m_2+m_3\}}}{(d_1^+ + d_2^+ e^{M_2})^2} + \mu(s) \frac{d_1^+ b_1^+ c_1^+ e^{\{M_1+M_3\}}}{(d_1^- + d_2^- e^{m_2})^2} + \mu(s) \frac{b_1^+ d_2^+ c_1^+ e^{\{M_1+M_2+M_3\}}}{(d_1^- + d_2^- e^{M_2})^2} \\
&\quad + \left(\frac{\mu(s) d_1^+ c_1^+ e^{M_3}}{(d_1^- + d_2^- e^{m_2})^2}\right)^2 + 2\mu(s) \frac{d_1^+ d_2^+ (c_1^+)^2 e^{\{M_2+2M_3\}}}{(d_1^- + d_2^- e^{m_2})^4} - \ \mu(s) \frac{d_1^- d_2^- (c_1^-)^2 e^{(m_2+2m_3)}}{(d_1^+ + d_2^+ e^{M_2})^4} \\
&\quad + \mu(s) \frac{d_2^+ c_1^+ e^{(M_2+M_3)}}{(d_1^- + d_2^- e_2^{m})^2} - \ \mu(s) \frac{(d_2^-)^2 (c_1^-)^2 e^{(2m_2+2m_3)}}{(d_1^+ + d_2^+ e^{M_2})^4}\Bigg\} \times v_3^2 \\
&\leq -P_1 v_1^2 - \ P_2 v_2^2 - \ P_3 v_3^2.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
V_2(s) &= (2v_2 + \mu(s)v_2^{\Delta}(s))v_2^{\Delta}(s) \\
&= \left\{2v_2(s) + \mu(s)\left(\frac{-d_3 e^{\Phi_1 i_2} v_2}{(d_3 + d_4 e^{\Phi_1 i_1})^2} - \ \frac{d_4 e^{y_1} e^{\Phi_1 i_2} v_2}{(d_3 + d_4 e^{\Phi_1 i_1})^2} + \frac{d_4 e^{y_2} e^{\Phi_1 i_1} v_1}{(d_3 + d_4 e^{\Phi_1 i_1})^2}\right)\right\} \\
&\quad \times \left(\frac{-d_3 e^{\Phi_1 i_2} v_2}{(d_3 + d_4 e^{\Phi_1 i_1})^2} - \ \frac{d_4 e^{y_1} e^{\Phi_1 i_2} v_2}{(d_3 + d_4 e^{\Phi_1 i_1})^2} + \frac{d_4 e^{y_2} e^{\Phi_1 i_1} v_1}{(d_3 + d_4 e^{\Phi_1 i_1})^2}\right) \\
&\leq \Bigg\{\frac{d_4^+ e^{(M_1+M_2)}}{(d_3^- + d_4^- e^{m_1})^2} - \ \mu(s) \frac{d_3^- d_4^- e^{(m_1+2m_2)}}{(d_3^+ + d_4^+ e^{M_1})^4} - \ \mu(s) \frac{(d_4^-)^2 e^{(2m_1+2m_2)}}{(d_3^+ + d_4^+ e^{M_1})^4} \\
&\quad + \left(\mu(s) \frac{d_4^+ e^{(M_1+M_2)}}{(d_3^- + d_4^- e^{m_1})^2}\right)^2\Bigg\} \times v_1^2 + \Bigg\{\frac{-2d_3^- e^{m_2}}{(d_3^+ + d_4^+ e^{M_1})^2} - \ \frac{2d_4^- e^{(m_1+m_2)}}{(d_3^+ + d_4^+ e^{M_1})^2} \\
&\quad + \frac{d_4^+ e^{(M_1+M_2)}}{(d_3^- + d_4^- e^{m_1})^2} + \mu(s)\left(\frac{d_3^+ e^{M_2}}{(d_3^- + d_4^- e^{m_1})^2}\right)^2 + 2\mu(s) \frac{d_3^+ d_4^+ e^{(M_1+2M_2)}}{(d_3^- + d_4^- e^{m_1})^4} \\
&\quad - \ \mu(s) \frac{d_3^- d_4^- e^{m_1+2m_2}}{(d_3^+ + d_4^+ e^{M_1})^4} + \mu(s)\left(\frac{d_4^+ e^{(M_1+M_2)}}{(d_3^- + d_4^- e^{m_1})^2}\right)^2 - \ \mu(s) \frac{(d_4^-)^2 e^{(2m_1+2m_2)}}{(d_3^+ + d_4^+ e^{M_1})^4}\Bigg\} \times v_2^2.
\end{aligned}
$$

Therefore,

$$
V_2(s) \leq -P_1' v_1^2 - \ P_2' v_2^2 - \ P_3' v_3^2.
$$

Now,

$$
\begin{aligned}
V_3(s) &= (2v_3(s) + \mu(s)v_3^{\Delta}(s))v_3^{\Delta}(s) \\
&= \Bigg\{2v_3(s) + \mu(s)\Bigg(-b_2(s)e^{\Phi_1 i_3(s)}v_3(s) + \frac{d_1(s)k_1(s)c_1(s)e^{\Phi_1 i_1}v_1(s)}{(d_1(s) + d_2(s)e^{\Phi_1 i_2})^2} \\
&\quad + \frac{d_2(s)k_1(s)c_1(s)e^{y_2(s)}e^{\Phi_1 i_1(s)}v_1(s)}{(d_1(s) + d_2(s)e^{\Phi_1 i_2(s)})^2} - \frac{d_2(s)k_1(s)c_1(s)e^{y_1(s)}e^{\Phi_1 i_2(s)}v_2(s)}{(d_1(s) + d_2(s)e^{\Phi_1 i_2(s)})^2}\Bigg)\Bigg\} \\
&\quad \times \Bigg(-b_2(s)e^{\Phi_1 i_3(s)}v_3(s) + \frac{d_1(s)k_1(s)c_1(s)e^{\Phi_1 i_1}v_1(s)}{(d_1(s) + d_2(s)e^{\Phi_1 i_2})^2} + \frac{d_2(s)k_1(s)c_1(s)e^{y_2(s)}e^{\Phi_1 i_1(s)}v_1(s)}{(d_1(s) + d_2(s)e^{\Phi_1 i_2(s)})^2} \\
&\quad - \frac{d_2(s)k_1(s)c_1(s)e^{y_1(s)}e^{\Phi_1 i_2(s)}v_2(s)}{(d_1(s) + d_2(s)e^{\Phi_1 i_2(s)})^2}\Bigg) \\
&\le \Bigg\{\frac{d_1^+k_1^+c_1^+e^{M_1}}{(d_1^- + d_2^-e^{m_2})^2} + \frac{d_2^+k_1^+c_1^+e^{(M_1+M_2)}}{(d_1^- + d_2^-e^{m_2})^2} - \mu(s)\frac{b_2^-d_1^-k_1^-c_1^-e^{(m_1+m_3)}}{(d_1^+ + d_2^+e^{M_2})^2} \\
&\quad - \mu(s)\frac{b_2^-d_2^-k_1^-c_1^-e^{(2m_2+m_1)}}{(d_1^+ + d_2^+e^{M_2})^2} + \mu(s)\left(\frac{d_2^+k_1^+c_1^+e^{(M_1+M_2)}}{(d_1^- + d_2^-e^{M_2})^2}\right)^2 + \mu(s)\left(\frac{d_1^+k_1^+c_1^+e^{M_1}}{(d_1^- + d_2^-e^{m_2})^2}\right)^2 \\
&\quad + 2\mu(s)\frac{d_1^+d_2^+(k_1^+)^2(c_1^+)^2e^{(2M_1+M_2)}}{(d_1^- + d_2^-e^{m_2})^4} - \mu(s)\frac{(d_2^-)^2(k_1^-)^2(c_1^-)^2e^{(2m_1+2m_2)}}{(d_1^+ + d_2^+e^{M_2})^4} \\
&\quad - \mu(s)\frac{d_1^-d_2^-(k_1^-)^2(c_1^-)^2e^{(2m_1+m_2)}}{(d_1^+ + d_2^+e^{M_2})^4}\Bigg\} \times v_1^2 \\
&\quad + \Bigg\{\frac{-d_2^-k_1^-c_1^-e^{(m_1+m_2)}}{(d_1^+ + d_2^+e^{M_2})^2} + \mu(s)\frac{b_2^+d_2^+k_1^+c_1^+e^{(M_1+M_2+M_3)}}{(d_1^- + d_2^-e^{m_2})^2} + \mu(s)\left(\frac{d_2^+k_1^+c_1^+e^{(M_1+M_2)}}{(d_1^- + d_2^-e^{m_2})^2}\right)^2 \\
&\quad - \mu(s)\frac{d_1^-d_2^-(k_1^-)^2(c_1^-)^2e^{(2m_1+2m_2)}}{(d_1^+ + d_2^+e^{M_2})^4} - \mu(s)\frac{(d_2^-)^2(k_1^-)^2(c_1^-)^2e^{(2m_1+2m_2)}}{(d_1^+ + d_2^+e^{M_2})^4}\Bigg\} \times v_2^2 \\
&\quad + \Bigg\{-2b_2^-e^{m_3} + \frac{d_1^+k_1^+c_1^+e^{M_1}}{(d_1^- + d_2^-e^{m_2})^2} + \frac{d_2^+k_1^+c_1^+e^{(M_1+M_2)}}{(d_1^- + d_2^-e^{m_2})^2} - \frac{d_2^-k_1^-c_1^-e^{(m_1+m_2)}}{(d_1^+ + d_2^+e^{M_2})^2} \\
&\quad + \mu(s)\big(b_2^+e^{M_3}\big)^2 - \mu(s)\frac{b_2^-d_1^-k_1^-c_1^-e^{(m_1+m_3)}}{(d_1^+ + d_2^+e^{M_2})^2} - \mu(s)\frac{b_2^-d_2^-k_1^-c_1^-e^{(m_1+2m_2)}}{(d_1^+ + d_2^+e^{M_2})^2} \\
&\quad + \mu(s)\frac{b_2^+d_2^+k_1^+c_1^+e^{(m_1+m_2+m_3)}}{(d_1^- + d_2^-e^{m_2})^2}\Bigg\} \times v_3^2 \\
&\le -P_1''v_1^2 - P_2''v_2^2 - P_3''v_3^2.
\end{aligned}
$$

Therefore,

$$D^+V(s, X, Y) \le -(P_1 + P_1' + P_1'')v_1^2 - (P_2 + P_2' + P_2'')v_2^2 - (P_3 + P_3' + P_3'')v_3^2 \le -\kappa V(s, X, Y).$$

For $s = s_k$, $k \in \mathbb{N}$, we have

$$
\begin{aligned}
V(s_k^+, X(s_k^+), Y(s_k^+)) &= (\Phi_1(s_k^+) - y_1(s_k^+))^2 + (\Phi_2(s_k^+) - y_2(s_k^+))^2 + (\Phi_3(s_k^+) - y_3(s_k^+))^2 \\
&= [\Phi_1(s_k) + \log(1 + h_k) - y_1(s_k) - \log(1 + h_k)]^2 \\
&\quad + [\Phi_2(s_k) + \log(1 + h_k') - y_2(s_k) - \log(1 + h_k')]^2 \\
&\quad + [\Phi_3(s_k) + \log(1 + h_k'') - y_3(s_k) - \log(1 + h_k'')]^2 \\
&\le V(s_k, X(s_k), y(s_k)).
\end{aligned}
$$

Thus, form the above calculation, $D^+V(s, X, Y)$ can be written as follows:

$$D^+V(s, X, Y) \le -\kappa V(s, X, Y).$$

Now, the solution of above inequality is

$$V(s, X, Y) \le V(s_0, X, Y)e^{-\kappa(s-s_0)}.$$

Therefore, $V(s, X, Y) \to 0$ as $s \to \infty$, i.e.,

$$\sum_{i=1}^{3}(\Phi_i - y_i)^2 \to 0 \quad \text{as } s \to \infty.$$

Furthermore, we can say that

$$\lim_{s\to\infty}|\Phi_i - y_i| = 0, \quad i = 1, 2, 3.$$

Thus, the proof is completed. □

5 Almost periodic solutions

Theorem 5.1. *Suppose that* (*H*1)–(*H*4) *hold with* $\kappa > 0$ *and* $-\kappa \in \mathcal{R}^+$. *Then, system* (1.4)–(1.9) *has a solution that is unique and uniformly asymptotically stable.*

Proof. From permanence results, we observe that (1.4)–(1.9) has a bounded solution satisfying

$$m_1 \le \Phi_1(s) \le M_1,$$
$$m_2 \le \Phi_2(s) \le M_2,$$
$$m_3 \le \Phi_3(s) \le M_3.$$

Hence, $|\Phi_1(s)| \le A_1, |\Phi_2(s)| \le A_2$, and $|\Phi_3(s)| \le A_3$, where $A_1 = \max\{m_1, M_1\}$, $A_2 = \max\{m_2, M_2\}$, $A_3 = \max\{m_3, M_3\}$ for all $(\Phi_1, \Phi_2, \Phi_3) \in \mathbb{R}^n$, we define a norm on $\mathbb{R}^3$

$$\|(\Phi_1(s), \Phi_2(s), \Phi_3(s))\| = \sup_{s\in\mathbb{T}}|\Phi_1(s)| + \sup_{s\in\mathbb{T}}|\Phi_2(s)| + \sup_{s\in\mathbb{T}}|\Phi_3(s)|.$$

Suppose that $X(s) = (\Phi_1(s), \Phi_2(s), \Phi_3(s))$, and $Y(s) = (y_1(s), y_2(s), y_3(s))$ be any two solutions of system (1.4)–(1.9), then $\|X\| \le C$ and $\|Y\| \le C$, where $C = \sum_{i=1}^{4} A_i$. Now, we consider the Lyapunov function defined by $\mathbb{T} \times \bar{\Gamma} \times \bar{\Gamma}$ is

$$V^*(s, X, Y) = \sum_{i=1}^{3}(\Phi_i - y_i)^2,$$

which satisfy the properties (1), (2) of ([20], Lemma(4.1)). Again, for property (3) of ([20], Lemma (4.1)), we obtain

$$\begin{aligned}
V^*(s_k^+, X(s_k^+), Y(s_k^+)) &= (\Phi_1(s_k^+) - y_1(s_k^+))^2 + (\Phi_2(s_k^+) - y_2(s_k^+))^2 + (\Phi_3(s_k^+) - y_3(s_k^+))^2 \\
&= [\Phi_1(s_k) + \log(1 - P_1) - y_1(s_k) - \log(1 - P_1)]^2 \\
&\quad + [\Phi_2(s_k) + \log(1 - P_1) - y_2(s_k) - \log(1 - P_1)]^2 \\
&\quad + [\Phi_3(s_k) + \log(1 - P_1) - y_3(s_k) - \log(1 - P_1)]^2 \\
&\le V^*(s_k, X(s_k), y(s_k)).
\end{aligned}$$

Hence, property (3) also satisfied. In addition, from Theorem 4.2, we have

$$D^+V^*(s, X, Y) \le -\kappa V^*(s, X, Y),$$

where $\kappa = \min\{(P_1 + P_1' + P_1''), (P_2 + P_2' + P_2''), (P_3 + P_3' + P_3'')\}$. For, $\kappa > 0$, we obtain property (4) of ([20], Lemma 4.1) also holds. Hence, all the properties of ([20], Lemma 4.1) are satisfied. Thus, by ([20], Lemma 4.1), there exists a unique almost periodic solution of considered system (1.4)–(1.9), which is uniformly asymptotically stable. This completes the proof. □

6 Example

Here, we will discuss some numerical examples with computer simulation.

Example 6.1. We are considering $\mathbb{T} = \mathbb{R}$, i.e., $\mu = 0$. Also, the impulsive points $s_1 = 15$ and $s_2 = 30$ and choose the following parameters:

$$\Theta_1 = 2.3 + 0.001\cos(5\pi s); \quad \Theta_2 = 1.9 + 0.1\sin(5\pi s); \quad \Theta_3 = 0.003; \quad \Theta_4 = 2.3 + 0.01\cos(5\pi s);$$
$$b_1 = 1.5 + 0.001\cos(5\pi s); \quad b_2 = 2.3 + 0.01\cos(5\pi s); \quad c_1 = 0.5 + 0.01\sin(5\pi s);$$
$$d_1 = 1.3 + 0.001\sin(5\pi s); \quad d_2 = 1.1 + 0.01\sin(5\pi s); \quad d_3 = 0.9; \quad d_4 = 0.005;$$
$$k_1 = 2.82 + 0.1\sin(5\pi s); \quad h_1 = 1/\exp(6), \quad h_1' = 1/\exp(3.5), \quad h_1'' = 1/\exp(5.4);$$
$$h_2 = -0.003, \quad h_2' = -0.04, h_2'' = -0.007.$$

We can easily see that

$$\Theta_1^+ = 2.3010,\ \Theta_1^- = 2.2990,\ \Theta_2^+ = 2,\ \Theta_2^- = 1.8000,\ \Theta_3^+ = 0.0030,\ \Theta_3^- = 0.0030,\ \Theta_4^+ = 2.3100,$$
$$\Theta_4^- = 2.2900,\ b_1^+ = 1.5010,\ b_1^- = 1.4990,\ b_2^+ = 2.3100,\ b_2^- = 2.2900,\ c_1^+ = 0.5100,\ c_1^- = 0.4900,$$
$$d_1^+ = 1.3010,\ d_1^- = 1.2990, d_2^+ = 1.1100,\ d_2^- = 1.090,\ d_3^+ = 0.9000,\ d_3^- = 0.9000,\ k_1^+ = 2.9200,$$
$$d_4^+ = 0.0050,\ d_4^- = 0.0050,\ k_1^- = 2.7200.$$

Thus, the system (1.4)–(1.9) becomes

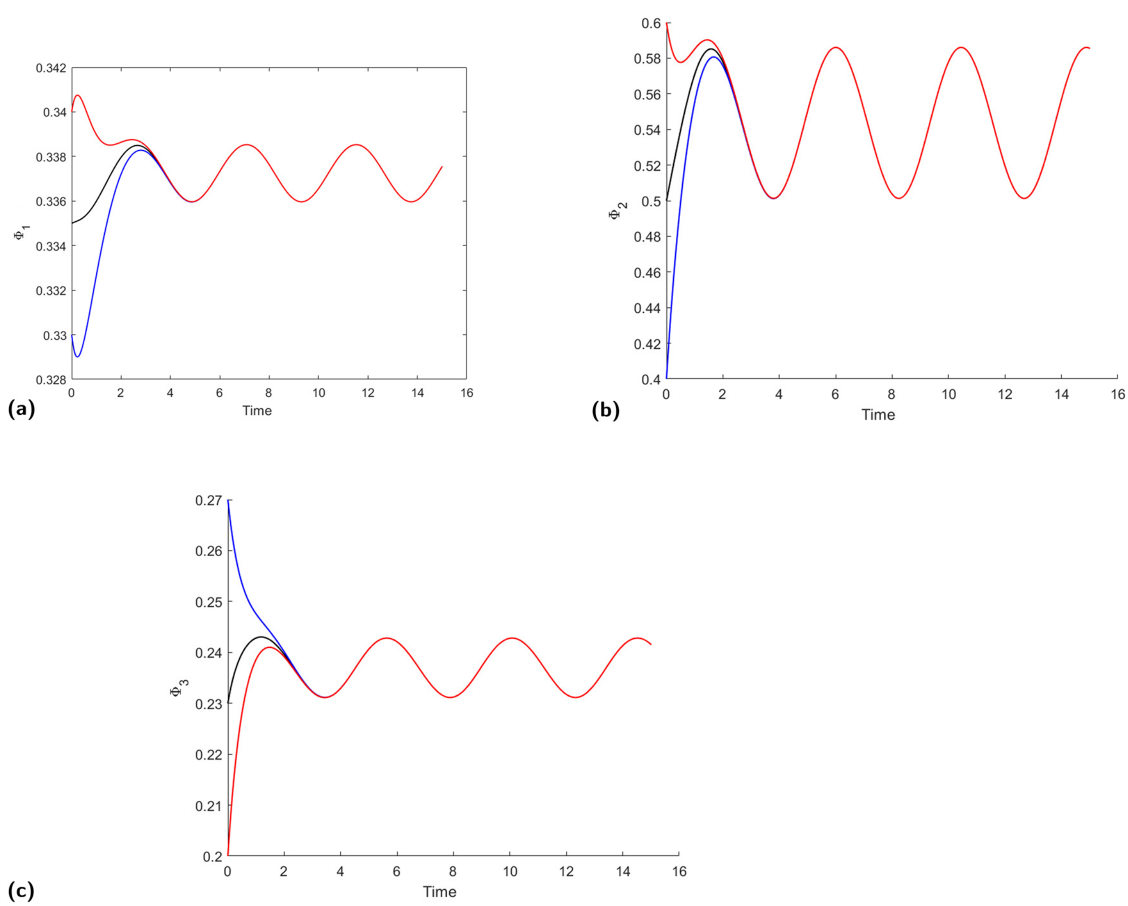

Figure 1: Global attractivity of species Φ_1, Φ_2, Φ_3 with time domain $\mathbb{T} = \mathbb{R}$. (a) Prey (Φ_1) with initial conditions (0.335, 0.33, 0.34), (b) mutualist (Φ_2) with initial conditions (0.5, 0.4, 0.6), and (c) predator (Φ_3) with initial conditions (0.23, 0.27, 0.2).

$$
\begin{aligned}
\Phi_1^{\Delta}(s) &= 2.3 + 0.001\cos(5\pi s) - (1.5 + 0.001\cos(5\pi s))e^{\Phi_1(s)} \\
&\quad - \frac{(0.5 + 0.01\sin(5\pi s))e^{\Phi_3(s)}}{(1.3 + 0.001\sin(5\pi s)) + (1.1 + 0.01\sin(5\pi s))(s)e^{\Phi_2(s)}}, \quad s \neq s_k,\ s \in [0,\infty)_{\mathbb{R}}, \\
\Phi_2^{\Delta}(s) &= 1.9 + 0.1\sin(5\pi s) - \frac{e^{\Phi_2(s)}}{0.9 + 0.005e^{\Phi_1(s)}}, \quad s \neq s_k,\ s \in [0,\infty)_{\mathbb{R}}, \\
\Phi_3^{\Delta}(s) &= 2.3 + 0.01\cos(5\pi s) - 0.003 + \frac{(2.82 + 0.1\sin(5\pi s))(0.5 + 0.01\sin(5\pi s))e^{\Phi_1(s)}}{(1.3 + 0.001\sin(5\pi s)) + (1.1 + 0.01\sin(5\pi s))e^{\Phi_2(s)}} \\
&\quad - (2.3 + 0.01\cos(5\pi s))e^{\Phi_3(s)}, \quad s \neq s_k,\ s \in [0,\infty)_{\mathbb{R}}, \\
\Phi_1(s_k^+) &= \Phi_1(s_k) + \log(1 + h_k), \quad s = s_k, \quad k = 1, 2, \\
\Phi_2(s_k^+) &= \Phi_2(s_k) + \log(1 + h_k'), \quad s = s_k, \quad k = 1, 2, \\
\Phi_3(s_k^+) &= \Phi_3(s_k) + \log(1 + h_k''), \quad s = s_k, \quad k = 1, 2.
\end{aligned} \tag{6.1}
$$

After doing some simple calculation, we obtain

$m_1 = 0.2649$, $m_2 = 0.4749$, $m_3 = 0.0793$, $M_1 = 0.5350$, $M_2 = 1.8171$, $M_3 = 0.4735$, $P = 1.7316$, $Q = 2.3220$, and $R = 3.4916$.

Therefore, $\kappa = \min\{P, Q, R\} = 1.7316 > 0$. Hence, all the assumptions of Theorems 4.2 and 5.1 hold true. Thus, our considered model is globally attractive. Furthermore,

$$
\begin{aligned}
&\liminf_{s\to\infty}\Phi_1(s) \geq 0.2649, && \limsup_{s\to\infty}\Phi_1(s) \leq 0.5350, \\
&\liminf_{s\to\infty}\Phi_2(s) \geq 0.4749, && \limsup_{s\to\infty}\Phi_2(s) \leq 1.8171, \\
&\liminf_{s\to\infty}\Phi_3(s) \geq 0.0793, && \limsup_{s\to\infty}\Phi_3(s) \leq 0.4735.
\end{aligned}
$$

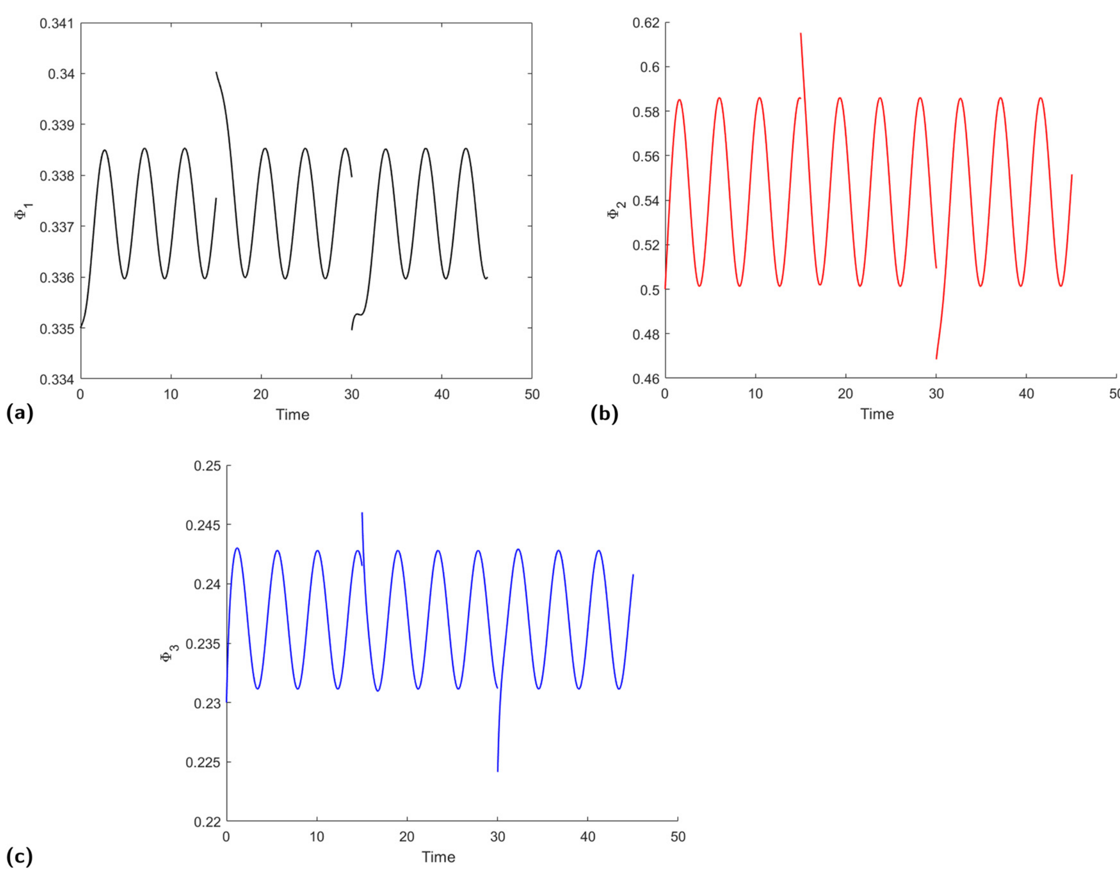

Figure 2: Impulsive effects at $s_1 = 15$ and $s_2 = 30$ with $\mathbb{T} = \mathbb{R}$ and initial conditions ($\Phi_1(0) = 0.335$, $\Phi_2(0) = 0.5$, and $\Phi_3(0) = 0.23$). (a) $h_1(15) = 1/(\exp(6))$ and $h_2(30) = -0.003$; (b) $h_1'(15) = 1/(\exp)(3.5)$ and $h_2'(30) = -0.04$; and (c) $h_1''(15) = 1/(\exp(5.4))$ and $h_2''(30) = -0.007$.

Therefore, our system is permanent.

Example 6.2. We are considering $\mathbb{T} = \mathbb{Z}$, i.e., $\mu = 1$ and impulsive point $s_1 = 23$. Also, consider the following:

$$\Theta_1 = 0.5 + 0.001\cos(1/5s);\ \Theta_2 = 0.23 + 0.001\sin(1/5s);\ \Theta_3 = 0.2 + 0.001\cos(1/5s);$$
$$b_1 = 0.2 + 0.001\cos(1/5s);\ b_2 = 1.0 + 0.001\cos(1/5s);\ c_1 = 0.006 + 0.001\sin(1/5s);$$
$$d_1 = 7.9 + 0.1\sin(1/5s);\ d_2 = 0.01 + 0.01\sin(1/5s);\ d_3 = 5.5 + 0.001\cos(1/5s);$$
$$\Theta_4 = 1.4 + 0.01\cos(1/5s);\ d_4 = 0.01 + 0.001\cos(1/5s);$$
$$k_1 = 6.1 + 0.001\sin(1/5s);\ h_1 = 1/\exp(7),\ \ h_1' = -0.583,\ \ h_1'' = -0.516.$$

We can easily see that

$$\Theta_1^+ = 0.5010,\ \Theta_1^- = 0.4990,\ \Theta_2^+ = 0.2300,\ \Theta_2^- = 0.2290,\ \Theta_3^+ = 0.2010,\ \Theta_3^- = 0.1990,\ \Theta_4^+ = 1.4100,$$
$$\Theta_4^- = 1.3900, b_1^+ = 0.2010,\ b_1^- = 0.1990,\ b_2^+ = 1.0010,\ b_2^- = 0.9990,\ c_1^+ = 0.0060,\ c_1^- = 0.0050,$$
$$d_1^+ = 7.9998,\ d_1^- = 7.8000,\ d_2^+ = 0.0190,\ d_2^- = 0.000002,\ d_3^+ = 5.5010,\ d_3^- = 5.4990,\ k_1^+ = 6.1010,$$
$$d_4^+ = 0.0110,\ d_4^- = 0.0090,\ k_1^- = 6.0990.$$

Thus, the system (1.4)–(1.9) becomes

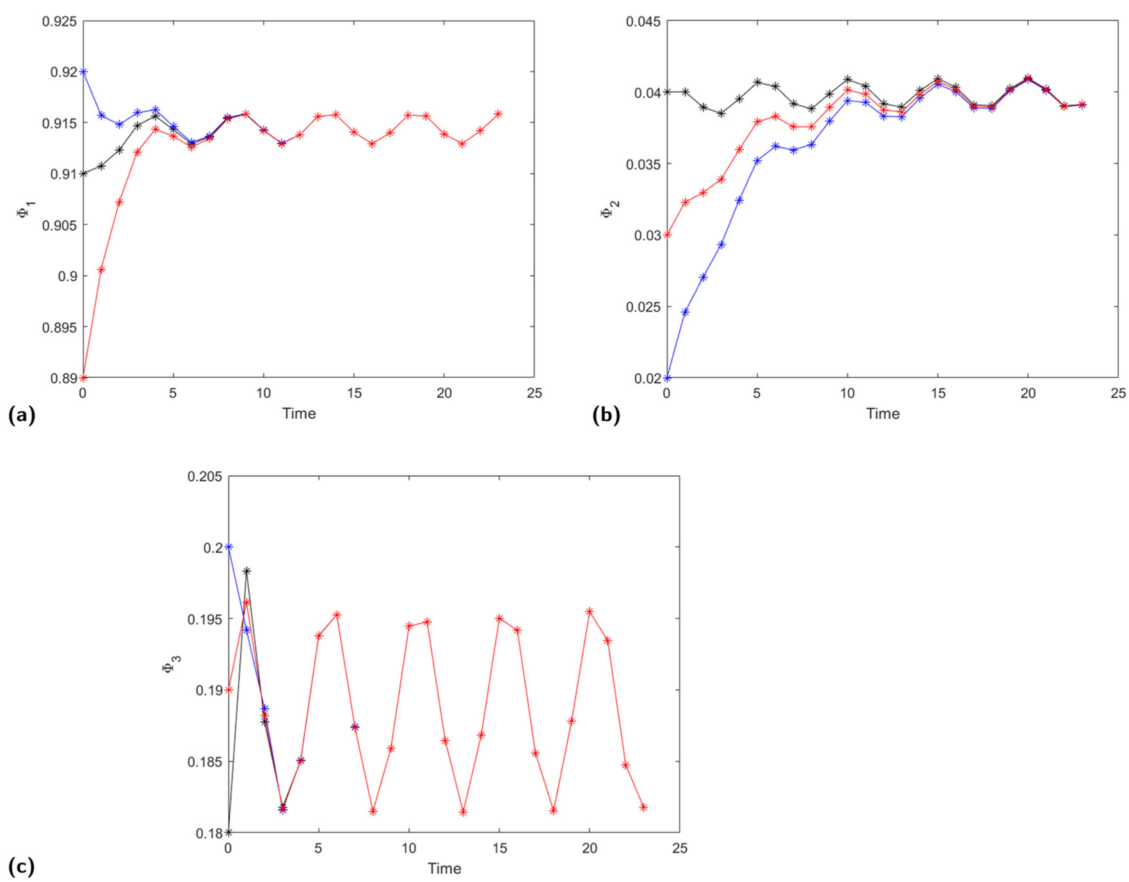

Figure 3: Global attractivity of species Φ_1, Φ_2, and Φ_3 with $\mathbb{T} = \mathbb{Z}$. (a) Prey (Φ_1) with initial data (0.91, 0.92, 0.89), (b) mutualist (Φ_2) with initial conditions (0.04, 0.02, 0.03), and (c) predator (Φ_3) with initial conditions (0.18, 0.20, 0.19).

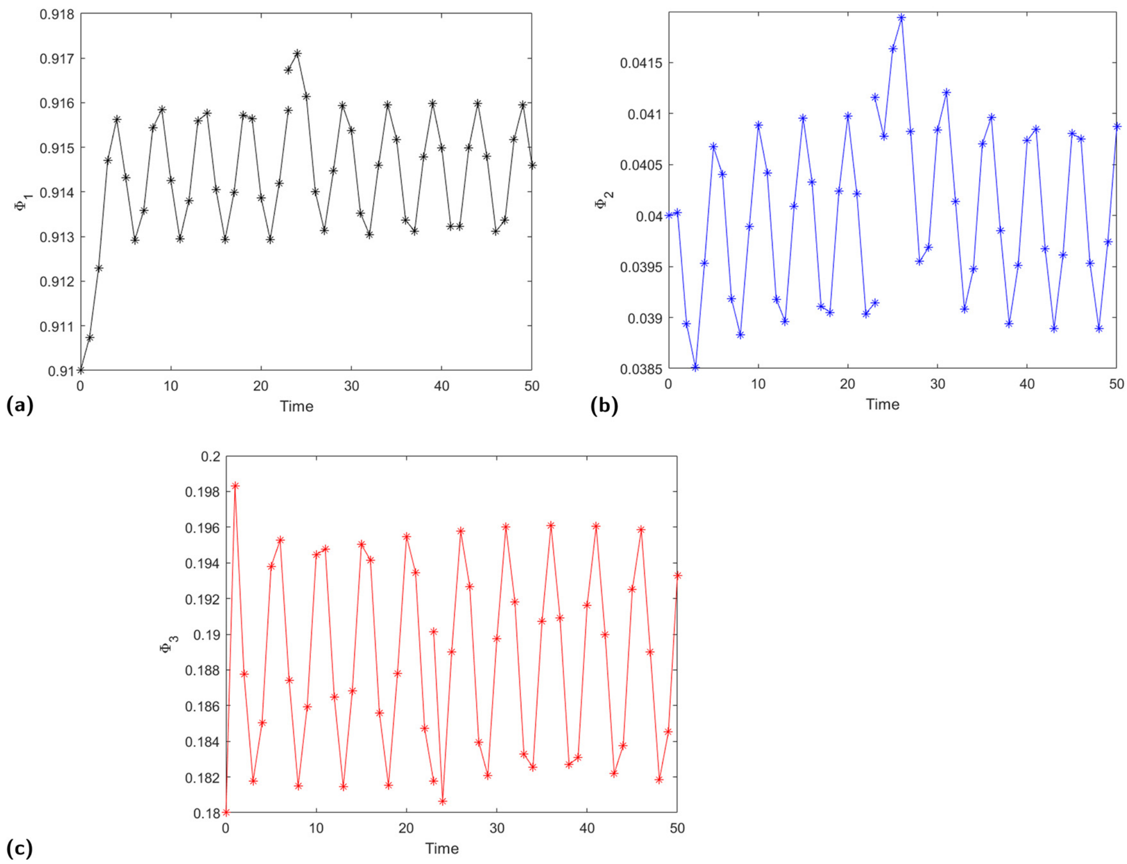

Figure 4: Impulsive effects at $s_1 = 23$ with timescale $\mathbb{T} = \mathbb{Z}$ and initial data ($\Phi_1(0) = 0.91$, $\Phi_2(0) = 0.04$, and $\Phi_3(0) = 0.18$). (a) $h_1(23) = 1/(\exp(7))$, (b) $h_1'(23) = -0.583$, and (c) $h_1''(23) = -0.516$.

$$
\begin{aligned}
\Phi_1^{\Delta}(s) &= 0.5 + 0.001\cos(1/5s) - (0.2 + 0.001\cos(1/5s))e^{\Phi_1(s)} \\
&\quad - \frac{(0.006 + 0.001\sin(1/5s))e^{\Phi_3(s)}}{(7.9 + 0.1\sin(1/5s)) + (0.01 + 0.01\sin(1/5s))e^{\Phi_2(s)}}, \quad s \neq s_1, \quad s \in [0,\infty)_{\mathbb{Z}}, \\
\Phi_2^{\Delta}(s) &= 0.23 + 0.001\sin(1/5s) - \frac{e^{\Phi_2(s)}}{(5.5 + 0.001\cos(1/5s)) + (0.01 + 0.001\cos(1/5s))e^{\Phi_1(s)}}, \quad s \neq s_1, \\
&\quad s \in [0,\infty)_{\mathbb{Z}}, \\
\Phi_3^{\Delta}(s) &= 1.4 + 0.01\cos(1/5s) - (0.2 + 0.001\cos(1/5s)) - (1.0 + 0.001\cos(1/5s))e^{\Phi_3(s)} \\
&\quad + \frac{(6.1 + 0.001\sin(1/5s))(0.006 + 0.001\sin(1/5s))e^{\Phi_1(s)}}{(7.9 + 0.1\sin(1/5s)) + (0.01 + 0.01\sin(1/5s))e^{\Phi_2(s)}}, \quad s \neq s_1, \quad s \in [0,\infty)_{\mathbb{Z}}, \\
\Phi_1(s_k^+) &= \Phi_1(s_k) + \log(1 + 1/\exp(7)), \quad s = s_1, \\
\Phi_2(s_k^+) &= \Phi_2(s_k) + \log(1 - 0.582), \quad s = s_1, \\
\Phi_3(s_k^+) &= \Phi_3(s_k) + \log(1 - 0.51), \quad s = s_1.
\end{aligned}
\tag{6.2}
$$

After doing some simple calculation, we obtain

$m_1 = 0.9065$, $m_2 = 0.2289$, $m_3 = 0.1800$, $M_1 = 1.5176$, $M_2 = 1.2823$, $M_3 = 0.4364$, $P = 0.1687$, $Q = 0.0135$, and $R = 0.0238$.

Thus, $\kappa = \min\{P, Q, R\} = 0.0135 > 0$. Hence, all the conditions of Theorems 4.2 and 5.1 hold true. Thus, our system is globally attractive. In addition,

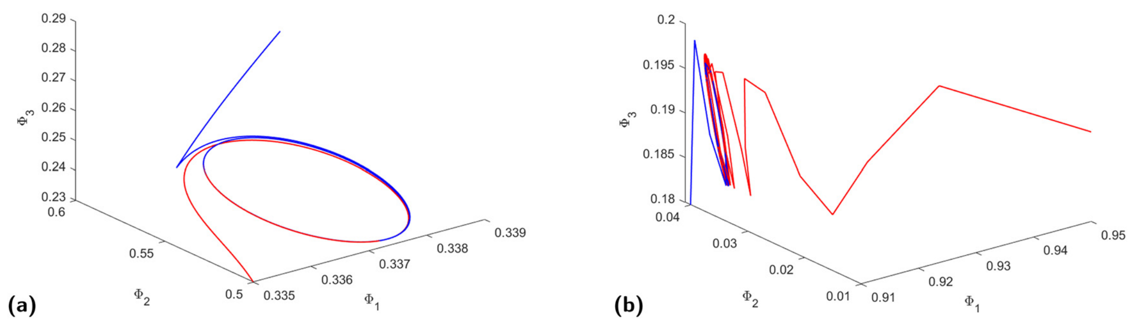

Figure 5: Phase diagram of species Φ_1, Φ_2, and Φ_3 with time domain $\mathbb{T} = \mathbb{R}$ and $\mathbb{Z}$. (a) $\Phi_1(0) = 0.335$, $\Phi_2(0) = 0.5$, and $\Phi_3(0) = 0.23$ and (b) $\Phi_1(0) = 0.95$, $\Phi_2(0) = 0.01$, and $\Phi_3(0) = 0.19$.

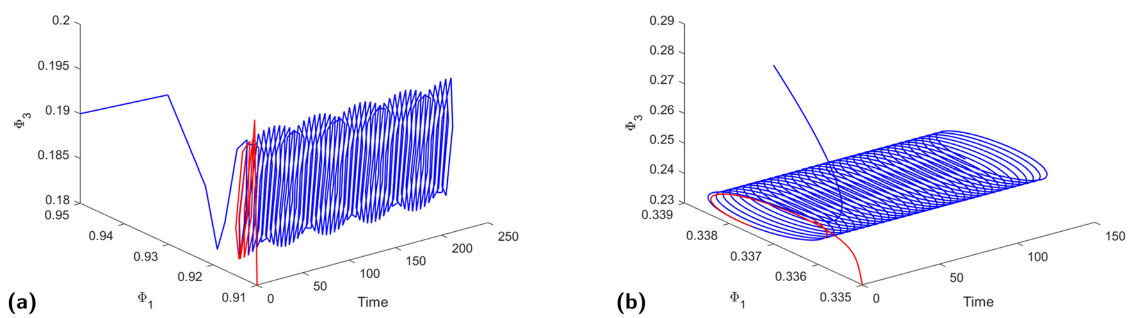

Figure 6: Phase diagram of species Φ_1 and Φ_3 with time domain $\mathbb{T} = \mathbb{Z}$ and $\mathbb{R}$. (a) $\Phi_1(0) = 0.335$ and $\Phi_3(0) = 0.23$ and (b) $\Phi_1(0) = 0.95$ and $\Phi_3(0) = 0.19$.

$$\liminf_{s\to\infty} \Phi_1(s) \geq 0.9065, \qquad \limsup_{s\to\infty} \Phi_1(s) \leq 1.5176,$$
$$\liminf_{s\to\infty} \Phi_2(s) \geq 0.2289, \qquad \limsup_{s\to\infty} \Phi_2(s) \leq 1.2823,$$
$$\liminf_{s\to\infty} \Phi_3(s) \geq 0.1800, \qquad \limsup_{s\to\infty} \Phi_3(s) \leq 0.4364.$$

Therefore, our system is permanent.

Remark 6.1. In Figures 1 and 2, we choose $\mathbb{T} = \mathbb{R}$ and analyze the following results:

(1) From Figure 1, we can observe the feasibility of global attractivity of the considered system (6.1). In Figure 1(a)–(c), we have taken the different–different initial conditions and observed that the trajectory of prey, mutualist, and predator concerning corresponding initial conditions follow the same trend after some saturation points. From these figures, we can conclude that the considered system (6.1) is globally attractive. Furthermore, Figures 5(a) and 6(b) show the phase diagram when $\mathbb{T} = \mathbb{R}$.

(2) Figure 2 shows the feasibility of impulsive effects. In Figure 2(a)–(c), we can see that, when we change the input of impulsive function from some saturation value, the corresponding values of species also change, respectively.

Remark 6.2. In Figures 3 and 4, we choose $\mathbb{T} = \mathbb{Z}$ and analyze the following results:

(1) From Figure 3, we can observe the feasibility of global attractivity of the considered system (6.2). In Figure 3(a)–(c), we have taken the different-different initial conditions and observed that the trajectory of prey, mutualist, and predator concerning corresponding initial conditions follow the same trend after some saturation points. From these figures, we can conclude that the considered system (6.2) is globally attractive. In addition, Figures 5(b) and 6(a) show the phase diagram when $\mathbb{T} = \mathbb{Z}$.

(2) Figure 4 shows the feasibility of impulsive effects. In Figure 4(a)–(c), we can see that when we change the input of impulsive function from some saturation value, the corresponding values of species also increase or decrease, respectively.

7 Conclusion

This manuscript presents a hybrid impulsive prey-predator-mutualist model on nonuniform time domains. The primary goal of this study is to establish the necessary conditions to guarantee the considered system's long-term survival and global attractivity. Furthermore, we have discussed the stability of system (1.4)–(1.9). Moreover, numerical examples and computer simulations further demonstrate the viability of theoretical solutions. In this work, our main focus is to study the dynamics of prey-predator-mutualist model with impulsive effects on timescales. But in future, we can extend these results in prey-predator system with Allee effect on timescales, prey-predator system with feedback control strategy and diffusive effects on timescales, etc.

Acknowledgements: We are very thankful to the associate editor and anonymous reviewers for their constructive comments and suggestions that help us improve the manuscript.

Funding information: This research received no specific grant from any funding agency, commercial or nonprofit sectors.

Author contributions: Anil Kumar: methodology, investigation, conceptualization, editing. Muslim Malik: supervision, methodology, investigation, conceptualization, editing.

Conflict of interest: The authors have no conflicts of interest to disclose.

Ethical approval: This research did not required ethical approval.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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