Research Article

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An SEIR model with modified saturated incidence rate and Holling type II treatment function

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Abstract: In this article, the behavior of an susceptible exposed infected recovered (SEIR) epidemic model with nonlinear incidence rate and Holling type II treatment function is presented and analyzed. Reproduction number of the model is calculated. Equilibrium points are determined. Disease-free equilibrium exists when $R0$ is below 1. Behavior of disease-free equilibrium is examined at $R0 = 1$. Endemic equilibrium exists when $R0$ crosses 1. Stability of both equilibrium points is investigated locally and globally. Simulation is provided to support the result.

Keywords: SEIR, basic reproduction number, Holling type II function, Lyapunov function

MSC 2020: 34D23, 93A30, 93D20

1 Introduction

Infectious diseases have been a part of human life for a long time already. Evidence tells us that epidemics often end up causing mass deaths. It was after the increase in healthcare, for a certain period of time, the health burden diminished of infectious diseases. However, in recent years, it has emerged that the challenge still exists, especially, in our rapidly changing world since every nation has limited resources to treat the infected. Emerging diseases pose a continuing threat, for example, human immunodeficiency virus in the twentieth century, acute encephalitis syndrome, malaria, cholera, and more recently COVID-19-coronavirus, causing mortality that has proven the necessity of having optimal resources to control an epidemic. Various mathematical models for infectious diseases proposed by many authors (see [1–9,16,18]). To figure out this problem, many treatment functions have been proposed by various researchers [13,14].

Zhang and Xianning [17] introduced the saturated treatment function for the better analysis of real system through the epidemic model. This function is widely known as Holling type II treatment function,

$$h(I) = \frac{aI}{1 + bI},$$

where $a > 0$ is a cure rate and $b \geq 0$ measures the magnitude of the consequence of the infected person being held for treatment. The specialty of this function is that it is defined as continually differentiable and characterizes the situation of limited medical resources.

Whenever we talk about epidemic or pandemic, the incidence rate of the disease in a population is first to be discussed. Previously, bilinear incidence rate βSI was being used more often to assess the new

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infections per unit of time in a population of susceptible. But while pondering about incidence rate, some significant factors must always be considered, like the inhibitory effects caused by the sudden spread of disease in a population. Pathak et al. [11] proposed such saturated incidence rate. This modified incidence rate is defined using sociological and psychological parameters α_1 and α_2, respectively, given by $\frac{\beta I}{1+\alpha_1 S+\alpha_2 I}$. The introduction of these parameters by authors is a modified saturation effect.

In this article, we propose an susceptible exposed infected recovered (SEIR) model with modified saturated incidence rate and Holling type II treatment function. Then, for the model, we calculate $R0$, find out the equilibrium point, and discuss the local and global stability of disease free equilibrium (DFE) and endemic equilibrium (EE). We check the stability of equilibrium at $R0 = 1$. In the next section, we provide simulation to assist theoretical results. In the last section, we discuss the aspects of the proposed model.

2 Proposed mathematical model

The epidemic model we propose and study in this work is an SEIR model with saturated incidence rate and Holling type II treatment function using nonlinear ordinary differential equations (Figure 1).

$$\begin{aligned}
\frac{dS}{dt} &= A - \frac{\beta SI}{1+\alpha_1 s+\alpha_2 I} - \mu S,\\
\frac{dE}{dt} &= \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\mu)E,\\
\frac{dI}{dt} &= \gamma E - (\delta+\mu)I - \frac{aI}{1+bI},\\
\frac{dR}{dt} &= \delta I - \mu R + \frac{aI}{1+bI},
\end{aligned} \tag{2.1}$$

where $S(t) + E(t) + I(t) + R(t) = N(t)$.

In equation (2.1), $S(t)$, $E(t)$, $I(t)$, and $R(t)$ denote the susceptible, exposed, infected, and recovered number of individuals at time t. Other parameters used in the system are infection rate β, recruitment rate A, natural death rate μ, natural recovery rate δ, progression rate γ, sociological parameter $\alpha_1 > 0$, psychological parameter $\alpha_2 > 0$, cure rate $a > O$, and magnitude of the consequence of delaying treatment for an infected individual by $b \geq 0$.

Since the first three equations are free from R, we can reduce system (2.1) into:

$$\begin{aligned}
\frac{dS}{dt} &= A - \frac{\beta SI}{1+\alpha_1 s+\alpha_2 I} - \mu S,\\
\frac{dE}{dt} &= \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\mu)E,
\end{aligned} \tag{2.2}$$

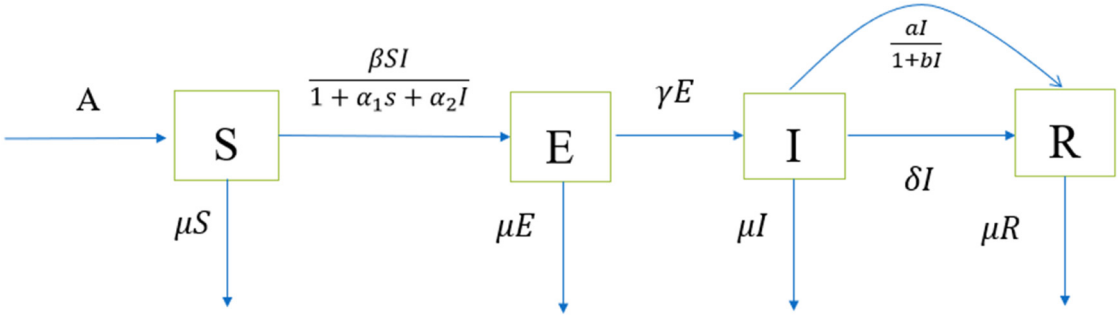

Figure 1: Transfer diagram of the presented model.

$$\frac{dI}{dt} = \gamma E - (\delta + \mu)I - \frac{aI}{1 + bI}.$$

Lemma 2.1. *The set* $\Omega = \left\{(S, E, I, R) : S + E + I + R \le \frac{A}{\mu}, S \ge 0, E \ge 0, I \ge 0, R \ge 0\right\}$ *is a positively invariant system of system* (2.1).

From (2.2), we obtain:

$$\frac{d}{dt}(S + E + I + R) = A - \mu(S + E + I + R) - \delta I - \frac{aI}{1 + bI} \le A - \mu(S + E + I + R) \le A - \mu N.$$

Solving the aforementioned equation and applying $\lim_{t\to\infty} N(t)$, we obtain $N(t) < \frac{A}{\mu}$.

Thus, feasible region invariant with system (2.2) is given by Lemma 2.1.

3 Main results

Reproduction number $R0$ is calculated using the method by van den Driessche and Watmough [12]. The DFE of system (2.2) is at $D_0 = \left(\frac{A}{\mu}, 0, 0\right)$. For calculating $R0$, we focus on infected compartments E and I only; thus, the system furthermore reduces to:

$$\frac{dE}{dt} = \frac{\beta SI}{1 + \alpha_1 S + \alpha_2 I} - (\gamma + \mu)E,$$
$$\frac{dI}{dt} = \gamma E - (\delta + \mu)I - \frac{aI}{1 + bI}. \tag{3.1}$$

Let $x = (E, I)^T$,

$F = \begin{bmatrix} \frac{\beta SI}{1 + \alpha_1 S + \alpha_2 I} \\ 0 \end{bmatrix}$ and $V = \begin{bmatrix} (\gamma + \mu)E \\ -\gamma E + (\delta + \mu)I + \frac{aI}{1 + bI} \end{bmatrix}$.

Jacobian matrix of F and V at $D_0 = \left(\frac{A}{\mu}, 0, 0\right)$ is given by:

$F_1 = \begin{bmatrix} 0 & \frac{\beta A}{\mu + \alpha_1 A} \\ 0 & 0 \end{bmatrix}$, $V_1 = \begin{bmatrix} (\gamma + \mu) & 0 \\ -\gamma & (\delta + \mu) + a \end{bmatrix}$.

The next-generation matrix is given by:

$$F_1 V_1^{-1} = \begin{bmatrix} \frac{\gamma\beta A}{(\mu + \alpha_1 A)(\gamma + \mu)(\delta + \mu + a)} & \frac{\beta A}{(\mu + \alpha_1 A)(\delta + \mu + a)} \\ 0 & 0 \end{bmatrix}.$$

Reproduction number $R0$ is the dominant eigen value of $F_1 V_1^{-1}$

$$R0 = \frac{\gamma\beta A}{(\mu + \alpha_1 A)(\gamma + \mu)(\delta + \mu + a)}. \tag{3.2}$$

3.1 Equilibriums of the system

Setting all rates to zero in system (2.2), that is, setting $\frac{dS}{dt} = 0$, $\frac{dI}{dt} = 0$, $\frac{dE}{dt} = 0$, we obtain:

$$A - \frac{\beta SI}{1 + \alpha_1 s + \alpha_2 I} - \mu S = 0,$$

$$\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}-(\gamma+\mu)E=0, \tag{3.3}$$

$$\gamma E-(\delta+\mu)I-\frac{aI}{1+bI}=0.$$

From (3.3), we obtain:
$I=0$ or $\frac{\gamma\beta S}{(1+\alpha_1 S+\alpha_2 I)(\gamma+\mu)}-\left((\delta+\mu)+\frac{a}{1+bI}\right)=0.$

3.2 Disease-free equilibrium (DFE)

Taking $I=0$ in the remaining equations of system (3.3), we obtain the values of S and E as follows:

$$S=\frac{A}{\mu},$$

$$E=0.$$

Thus, we determined the DFE:

$$D_0=\left(\frac{A}{\mu},0,0\right). \tag{3.4}$$

3.3 Endemic equilibrium (EE)

$$S^*=\frac{(\gamma+\mu)(1+\alpha_2 I^*)\{(1+bI^*)(\delta+\mu)+a\}}{\beta\gamma(1+bI^*)-(\gamma+\mu)\alpha_1\{(1+bI^*)(\delta+\mu)+a\}},$$

$$E^*=\frac{A-\mu S^*}{(\gamma+\mu)}.$$

Substituting the value of S^* in the first equation of system (3.3) yields cubic equation of I^*:

$$\psi_1 I^{*3}+\psi_2 I^{*2}+\psi_3 I^*+\psi_4=0, \tag{3.5}$$

where

$$\psi_1=\beta\gamma b^2(\gamma+\mu)(\delta+\mu)-\alpha_1 b^2(\gamma+\mu)^2(\delta+\mu)^2,$$

$$\psi_2=2b\beta\gamma(\gamma+\mu)(\delta+\mu)+ab\beta\gamma(\gamma+\mu)-2\alpha_1 b(\gamma+\mu)^2(\delta+\mu)^2-2ab\alpha_1(\gamma+\mu)^2(\delta+\mu)+\alpha_2 b\mu(\gamma+\mu)(\delta+\mu)$$
$$-A\beta b^2\gamma^2+A\alpha_1\gamma(\gamma+\mu)(\delta+\mu)b^2,$$

$$\psi_3=\beta\gamma(\gamma+\mu)(\delta+\mu)+a\beta\gamma(\gamma+\mu)-\alpha_1(\gamma+\mu)^2(\delta+\mu)^2-2a\alpha_1(\gamma+\mu)^2(\delta+\mu)-a^2\alpha_1(\gamma+\mu)^2(\delta+\mu)$$
$$+\alpha_2\mu(\gamma+\mu)(\delta+\mu)+\alpha_2\mu(\gamma+\mu)-2Ab\beta+2Ab\alpha_1\gamma(\gamma+\mu)(\delta+\mu)+Aab\alpha_1\gamma(\gamma+\mu),$$

$$\psi_4=a+\mu(\gamma+\mu)-A\beta\gamma^2+A\alpha_1\gamma(\gamma+u)(\delta+\mu)+Aa\alpha_1\gamma(\gamma+\mu).$$

Now, unique positive real root of equation (3.5) exists [15] if:
i. $\psi_1>0,\quad \psi_2>0,\quad \psi_3>0$ and $\psi_4<0$,
ii. $\psi_1>0,\quad \psi_2>0,\quad \psi_3<0$ and $\psi_4<0$,
iii. $\psi_1>0,\quad \psi_2<0,\quad \psi_3<0$ and $\psi_4<0$,
where $\psi_1>0$. We can estimate S^* and E^* after we obtain the value of I^*. Thus, there exists a unique EE if the aforementioned inequalities are satisfied under the following conditions.
Let us take $m=(\gamma+\mu),\quad n=(\delta+\mu)$,

$$\psi_2\begin{cases} > 0, \text{when} 2b\beta ymn + ab\beta ym + \alpha_2 b\mu mn + A\alpha_1 ymnb^2 > 2\alpha_1 bm^2n^2 + 2ab\alpha_1 m^2 n + A\beta b^2\gamma^2 \\ < 0, \text{when} 2b\beta ymn + ab\beta ym + \alpha_2 b\mu mn + A\alpha_1 ymnb^2 < 2\alpha_1 bm^2n^2 + 2ab\alpha_1 m^2 n + A\beta b^2\gamma^2, \end{cases}$$

$$\psi_3\begin{cases} > 0, \text{when} \beta ymn + a\beta ym + \alpha_2\mu mn + \alpha_2\mu m + 2Ab\alpha_1 ymn + Aab\alpha_1 ym \\ \quad > \alpha_1 m^2n^2 + 2a\alpha_1 m^2 n + a^2\alpha_1 m^2 n + 2Ab\beta \\ < 0, \text{when} \beta ymn + a\beta ym + \alpha_2\mu mn + \alpha_2\mu m + 2Ab\alpha_1 ymn + Aab\alpha_1 ym \\ \quad < \alpha_1 m^2n^2 + 2a\alpha_1 m^2 n + a^2\alpha_1 m^2 n + 2Ab\beta, \end{cases}$$

$$\psi_4\begin{cases} > 0, \text{when} a + \mu m + A\alpha_1 ym(n + a) > A\beta\gamma^2 \\ < 0, \text{when} a + \mu m + A\alpha_1 ym(n + a) < A\beta\gamma^2. \end{cases}$$

Thus, EE $D^* = (S^*, E^*, I^*)$ exists when the aforementioned conditions are satisfied.

3.4 Stability analysis of equilibria

Theorem 3.1. *For* $R0 < 1$, *DFE* $D_0 = \left(\frac{A}{\mu}, 0, 0\right)$ *is locally asymptotically stable and unstable for* $R0 > 1$.

For this, we construct a Jacobian matrix of system (2.2) at DFE as follows:

$$J(X_0) = \begin{pmatrix} -\mu & 0 & \dfrac{-\beta A}{\mu + \alpha_1 A} \\ 0 & -(\gamma + \mu) & \dfrac{\beta A}{\mu + \alpha_1 A} \\ 0 & \gamma & -(\delta + \mu) - a \end{pmatrix}. \tag{3.6}$$

Thus, $|J(X_0) - \lambda I|$ is given by:

$$\begin{vmatrix} -\mu - \lambda & 0 & \dfrac{-\beta A}{\mu + \alpha_1 A} \\ 0 & -(\gamma + \mu) - \lambda & \dfrac{\beta A}{\mu + \alpha_1 A} \\ 0 & \gamma & -(\mu + \delta) - a - \lambda \end{vmatrix} = 0.$$

That implies

$$(\lambda + \mu)\{\lambda^2 + (2\mu + \delta + \gamma + a)\lambda + (\gamma + \mu)(\mu + \delta + a) - (1 - R0)\} = 0. \tag{3.7}$$

First eigen value of matrix is $\mu < 0$, and other two eigen values are zeros of the equation:

$$\lambda^2 + p\lambda + q = 0,$$

where $p = 2\mu + \gamma + \delta + a > 0$,

$$q = (\mu + r)(\mu + \delta + a)(1 - R0).$$

If $R0 \le 1$, then $q > 0$, so both zeros of equation are negative. So, all the eigen values of the Jacobian matrix at DFE are negative. Therefore, by the Routh–Hurwitz criterion, Theorem 3.1 is proven.

Now, we analyze the global stability of DFE. For this, define:

$$R0^a = \frac{\gamma\beta A}{(\mu + \alpha_1 A)(\gamma + \mu)(\delta + \mu + (a/1 + b(\frac{A}{\mu})))}. \tag{3.8}$$

Theorem 3.2. *DFE, D_0, is globally asymptotically stable if $R0^a < 1$, implying $R0 < 1$.*

Proof. From the first equation of system (2.1), we have $\frac{dS}{dt} \leq A - \mu S$. Solution of $\frac{dx}{dt} = A - \mu x$ is a maximal solution of $S(t)$ and $x \to A/\mu$ as $t \to \infty$. By the comparison theorem, $S(t) \leq A/\mu$ and from set Ω in Lemma 2.1, we obtain $I(t) \leq A/\mu$.

Let us define the Lyapunov function:

$$L = \gamma E + (\gamma + \mu)I.$$

Therefore,

$$\begin{aligned}\frac{dL}{dt} &= \left[\frac{\gamma\beta S}{1 + \alpha_1 S + \alpha_2 I} - (\gamma + \mu)\left(\delta + \mu + \frac{a}{1 + bI}\right)\right]I \\ &\leq \left[\frac{\gamma\beta S}{1 + \alpha_1 S + \alpha_2 I} - (\gamma + \mu)\left(\delta + \mu + \frac{a}{1 + bI}\right)\right]I \\ &\leq \left[\frac{\gamma\beta A}{\mu + \alpha_1 A} - (\gamma + \mu)\left(\delta + \mu + \frac{a}{1 + b(A/\mu)}\right)\right]I \quad \leq 0,\end{aligned}$$

and $\frac{dL}{dt} = 0$ iff $I = 0$.

$\{(S, E, I) \in \Omega, \frac{dL}{dt} = 0\}$ is the singleton set. Thus, by the Lasalle–Lyapunov theorem, DFE is asymptotically stable globally. □

Theorem 3.3. *DFE, D_0, is unstable at $R0 = 1$; thus, a positive equilibrium is found when $R0$ crosses 1.*

To check the stability of DFE, we make use of center manifold theory [12]. To do this, let us suppose: $x_1 = S$, $x_2 = E$, and $x_3 = I$. Define $X = (x_1, x_2, x_3)^T$ in such a way that system (2.2) can be written as $\frac{dX}{dt} = F(x)$, where $F = (f_1, f_2, f_3)$. Therefore,

$$\frac{dx_1}{dt} = A - \frac{\beta x_1 x_3}{1 + \alpha_1 x_1 + \alpha_2 x_3} - \mu x_1 = f_1,$$

$$\frac{dx_2}{dt} = \frac{\beta x_1 x_3}{1 + \alpha_1 x_1 + \alpha_2 x_3} - (\gamma + \mu)x_2 = f_2, \tag{3.9}$$

$$\frac{dx_3}{dt} = -\gamma x_2 - (\delta + \mu)x_3 - \frac{a x_3}{1 + b x_3} = f_3. \tag{3.10}$$

Now, at $R0 = 1$, $\beta = \beta^* = \frac{(\mu + \alpha_1 A)(\gamma + \mu)(\delta + \mu + a)}{\gamma A}$.

Let J^* be the Jacobian matrix at $R0 = 1$ and $\beta = \beta^*$. Then,

$J^* = \begin{pmatrix} -\mu & 0 & \frac{-\beta^* A}{\mu + \alpha_1 A} \\ 0 & -(\gamma + \mu) & \frac{\beta^* A}{\mu + \alpha_1 A} \\ 0 & \gamma & -(\delta + \mu) - a \end{pmatrix}$. Let $w = [w_1, w_2, w_3]$ and $u = [u_1, u_2, u_3]^T$ be the left and right eigen vectors of J^* correlating with zero eigen value. Then,

$[w_1, w_2, w_3] = \left[0, \quad \frac{\gamma}{(\gamma + \mu)}, \quad 1\right]$ and $[u_1, u_2, u_3]^T = \left[\frac{-\beta^* A}{\mu + \alpha_1 A}, \quad \frac{-\beta^* A}{(\gamma + \mu)(\mu + \alpha_1 A)}, \quad 1\right]$.

Now, to obtain the bifurcation coefficients a_1 and a_2, the nonzero partial derivatives of system (3.9) calculated at D_0 are as follows:

$$\frac{\partial^2 f_1}{\partial x_1 \partial x_3} = \frac{\partial^2 f_1}{\partial x_3 \partial x_1} = \frac{\partial^2 f_1}{\partial x_1^2} = -\frac{\mu^2 \beta^*}{(\mu + \alpha_1 A)^2}, \frac{\partial^2 f_1}{\partial x_2^2} = \frac{2\alpha_2 \mu \beta^* A}{(u + \alpha_1 A)^2}, \frac{\partial^2 f_2}{\partial x_3^2} = -\frac{2\alpha_2 \mu \beta^* A}{(u + \alpha_1 A)^2}, \frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_1} = \frac{\mu^2 \beta^*}{(\mu + \alpha_1 A)^2},$$

$$\frac{\partial^2 f_3}{\partial x_3^2} = \frac{b}{2},$$

$$a_1 = \sum_{k,\ i,\ j=1}^{3} w_k u_i u_j \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} \right)_{D_0},$$

$$a_1 = -\frac{2\gamma\mu\beta^* A}{(\gamma+\mu)(\mu+\alpha_1 A)^2}\left\{\frac{\beta^*}{(\mu+\alpha_1 A)} + \alpha_2\right\} + \frac{b}{2} < 0.$$

$$\text{If } a_1 \text{ satisfies } \frac{b}{2} < \frac{2\gamma\mu\beta^* A}{(\gamma+\mu)(\mu+\alpha_1 A)^2}\left\{\frac{\beta^*}{(\mu+\alpha_1 A)} + \alpha_2\right\}, \tag{3.11}$$

$\frac{\partial^2 f_1}{\partial x_3 \partial \beta^*} = -\frac{A}{(\mu+\alpha_1 A)}$, $\frac{\partial^2 f_2}{\partial x_3 \partial \beta^*} = \frac{A}{(\mu+\alpha_1 A)}$,

$$a_2 = \sum_{k,i=1}^{3} w_k u_i \left(\frac{\partial^2 f_k}{\partial x_i \partial \beta^*} \right)_{D_0},$$

$$a_2 = \frac{\gamma A}{(\gamma+\mu)(\mu+\alpha_1 A)} = \frac{(\delta+\mu+a)}{\beta^*} > 0.$$

Since $a_1 < 0$ when (3.11) is satisfied and $a_2 > 0$, D_0 becomes unstable and a positive equilibrium is found when $R0$ crosses 1.

3.5 Endemic equilibrium (EE)

Theorem 3.4. *EE, $D^* = (S^*, E^*, I^*)$, is locally asymptotically stable when $R0 > 1$*

$$\text{iff} \left(\mu + \delta + \frac{a}{(1+bI^*)^2}\right) < \frac{\beta S^*}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}. \tag{3.12}$$

Jacobian matrix at EE:

$$J(D^*) = \begin{pmatrix} -\frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} - \mu & 0 & -\frac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} \\ \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} & -(\gamma+\mu) & \frac{\beta S^*(1+\alpha_1 S^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} \\ 0 & \gamma & -(\delta+\mu) - \frac{a}{(1+bI^*)^2} \end{pmatrix}.$$

Characteristic equation of $J(D^*)$ is given by:

$$\kappa^3 + s_1\kappa^2 + s_2\kappa + s_3 = 0,$$

where

$$s_1 = 3\mu + \delta + \gamma + \frac{a}{(1+bI^*)^2} + \frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2},$$

$$s_2 = \left(2\mu + \delta + \gamma + \frac{a}{(1+bI^*)^2}\right)\left(\mu + \left(\frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right)\right) + \mu\left(\mu + \delta + \frac{a}{(1+bI^*)^2}\right) + \gamma\left[\left(\mu + \delta + \frac{a}{(1+bI^*)^2}\right) - \frac{\beta S^*}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right],$$

$$s_3 = (\gamma+\mu)\left(\mu + \delta + \frac{a}{(1+bI^*)^2}\right)\left(\frac{\beta I^*(1+\alpha_2 I^*)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right) + \mu\left[\mu\left(\mu + \delta + \frac{a}{(1+bI^*)^2}\right) + \gamma\left\{\left(\mu + \delta + \frac{a}{(1+bI^*)^2}\right) - \frac{\beta S^*}{(1+\alpha_1 S^* + \alpha_2 I^*)^2}\right\}\right].$$

Using the Routh–Hurwitz criterion, clearly, $s_1 > 0$, also $s_2 > 0$, $s_3 > 0$, and $\Delta = s_1 s_2 - s_3 > 0$ when inequality (3.12) holds. Thus, $D^* = (S^*, E^*, I^*)$ is locally asymptotically stable when $R0 > 1$.

Theorem 3.5. *EE, $D^* = (S^*, E^*, I^*)$, is globally asymptotically stable when $R0 > 1$.*

For system (2.2), Jacobian matrix is as follows:

$$J = \begin{pmatrix} -\dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - \mu & 0 & -\dfrac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2} \\ \dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} & -(\gamma+\mu) & \dfrac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2} \\ 0 & \gamma & -(\delta+\mu) - \dfrac{a}{(1+bI)^2} \end{pmatrix}.$$

Second compound matrix is as follows:

$$J^{\sigma} = \begin{pmatrix} -\dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \gamma & \dfrac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2} & \dfrac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2} \\ \gamma & -\dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \delta - \dfrac{a}{(1+bI)^2} & 0 \\ 0 & \dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} & -(\delta+\gamma+2\mu) - \dfrac{a}{(1+bI)^2} \end{pmatrix}.$$

Choosing

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & E/I & 0 \\ 0 & 0 & E/I \end{bmatrix} \text{ implies } K^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I/E & 0 \\ 0 & 0 & I/E \end{bmatrix}.$$

$$\text{Thus, } K_l = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E'I - EI'}{I^2} & 0 \\ 0 & 0 & \frac{E'I - EI'}{I^2} \end{bmatrix}.$$

$$\text{Therefore, } K_l K^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E'}{E} - \frac{I'}{I} & 0 \\ 0 & 0 & \frac{E'}{E} - \frac{I'}{I} \end{bmatrix}.$$

$$\text{And } KJ^{\sigma}K^{-1} = \begin{pmatrix} -\frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \gamma & \frac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E} & \frac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E} \\ \gamma\frac{E}{I} & -\frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \delta - \frac{a}{(1+bI)^2} & 0 \\ 0 & \frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} & -(\delta+\gamma+2\mu) - \frac{a}{(1+bI)^2} \end{pmatrix}.$$

Now,

$B = K_l K^{-1} + KJ^{\sigma}K^{-1}$ implies

$$B = \begin{pmatrix} -\dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \gamma & \dfrac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\dfrac{I}{E} & \dfrac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\dfrac{I}{E} \\ \gamma\dfrac{E}{I} & \dfrac{E'}{E} - \dfrac{I'}{I} - \dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \delta - \dfrac{a}{(1+bI)^2} & 0 \\ 0 & \dfrac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} & \dfrac{E'}{E} - \dfrac{I'}{I} - (\delta+\gamma+2\mu) - \dfrac{a}{(1+bI)^2} \end{pmatrix}.$$

In block form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$B_{11} = \left[-\frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \gamma \right], B_{12} = \left[\frac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E} \quad \frac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E} \right], B_{21} = \begin{bmatrix} \gamma\frac{E}{I} \\ 0 \end{bmatrix},$$

$$B_{22} = \begin{bmatrix} \frac{E'}{E} - \frac{I'}{I} - \frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \delta - \frac{a}{(1+bI)^2} & 0 \\ \frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} & \frac{E'}{E} - \frac{I'}{I} - (\delta+\gamma+2\mu) - \frac{a}{(1+bI)^2} \end{bmatrix}.$$

If vectors of $\mathbb{R}^3$ are given by (p, q, r). Then, norm in $\mathbb{R}^3$ can be selected as $|p, q, r| = \max\{|p|, |q|, /r/\}$. Let $\mathscr{F}(B) \le \sup\{j_1, j_2\}$, where $\mathscr{F}$ denotes Lozinski measure [10] and $j_1 = \mathscr{F}(B_{11}) + |B_{12}|$, $j_2 = \mathscr{F}(B_{22}) + |B_{21}|$.

$$|B_{12}| = \frac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E}, |B_{21}| = \gamma\frac{E}{I},$$

$$\mathscr{F}(B_{11}) = -\frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \gamma,$$

$$\mathscr{F}(B_{22}) = \frac{E'}{E} - \frac{I'}{I} - (\delta+2\mu) - \frac{a}{(1+bI)^2},$$

$$j_1 = -\frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - 2\mu - \gamma + \frac{\beta S(1+\alpha_1 S)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E},$$

$$j_2 = \gamma\frac{E}{I} + \frac{E'}{E} - \frac{I'}{I} - (\delta+2\mu) - \frac{a}{(1+bI)^2}.$$

Second and third equations of system (2.2) can be rewritten as:

$$\frac{E'}{E} + \gamma + \mu = \frac{\beta SI}{(1+\alpha_1 s+\alpha_2 I)E}, \tag{3.13}$$

$$\frac{I'}{I} + \delta + \mu + \frac{a}{1+bI} = \gamma\frac{E}{I}. \tag{3.14}$$

Substitution of (3.13) and (3.14), respectively, in j_1 and j_2 results in:

$$j_1 = \frac{E'}{E} - \mu - \frac{\beta I(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2} - \frac{\beta S(1+\alpha_2 I)}{(1+\alpha_1 S+\alpha_2 I)^2}\frac{I}{E} \le \frac{E'}{E} - \mu,$$

$j_2 = \frac{E'}{E} - \mu + \frac{a}{(1+bI)^2} \le \frac{E'}{E} - \mu,$
iff $\frac{a}{(1+bI)^2} < \mu.$

Thus, $\mathscr{F}(B) \le \sup\{j_1, j_2\} \le \frac{E'}{E} - \mu.$
And so,

$$\bar{y}_2 = \frac{1}{t}\int_0^t \mathscr{F}(B)ds \le \ln\frac{E(t)}{E(t')} + \frac{1}{t}\int_0^{t'} \mathscr{F}(B)\mathrm{d}s - \mu,$$

implies, $\bar{y}_2 = \lim_{t\to\infty} \sup\sup \frac{1}{t}\int_0^t \mathscr{F}(B)\mathrm{d}s < -\mu < 0$

implies $\bar{y}_2 < 0$.

Thus, $D^* = (S^*, E^*, I^*)$ is globally asymptotically stable.

4 Example

For understanding the model better, we provide an example through the simulation of data. First, we consider the scenario when $R0 < 1$. For this, we take $A = 7$, $\mu = 0.2$, $\beta = 0.5$, $\delta = 0.12$, $\gamma = 0.03$, $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $a = 1$, and $b = 1$ when $R0 = 0.1504$. The initial values of all the four compartments in population are $S(0) = 10$, $E(0) = 5$, $I(0) = 5$, and $R(0) = 5$. Then, we obtain $D_0 = (S_0, I_0, E_0) = (35, 0, 0)$. The following results can be seen in Figure 2. In this case, it can be clearly seen that DFE D_0 is approached and sustains as per Theorems 3.1 and 3.2. A whole lot of population is susceptible in less time when an infectious disease

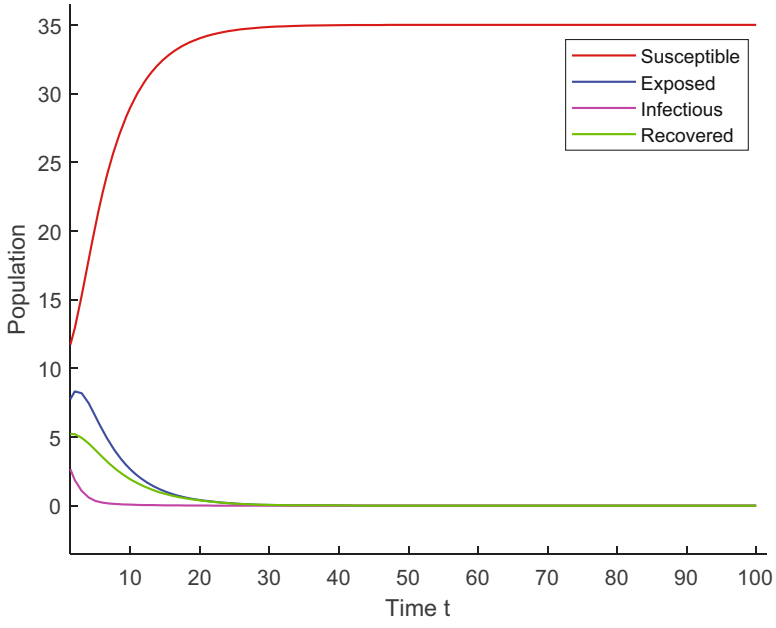

Figure 2: Behavior of S, E, I, and R when $R0 < 1$.

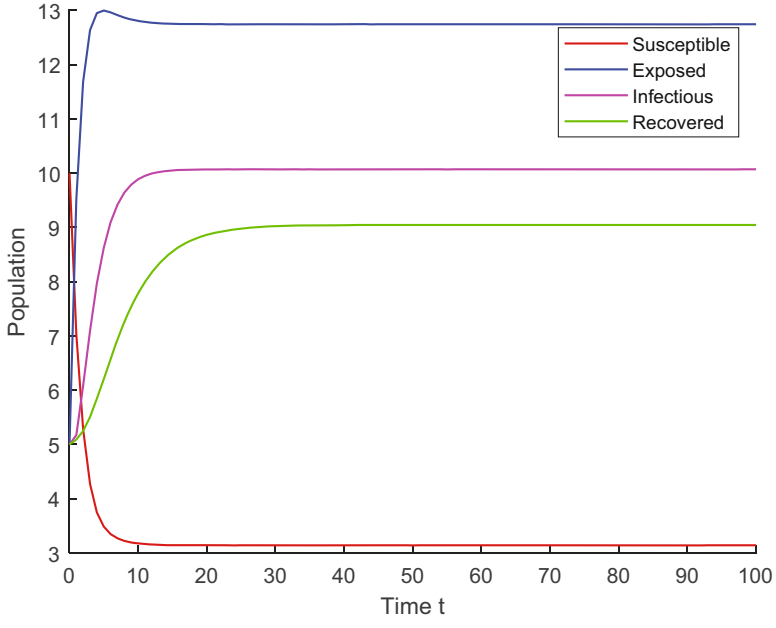

Figure 3: Behavior of S, E, I, and R when $R0 < 1$.

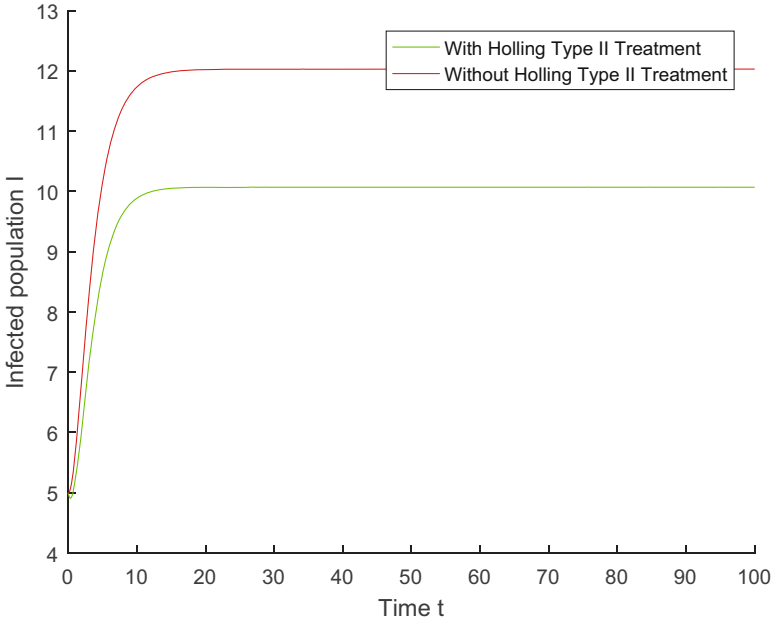

Figure 4: Effect of the introduced treatment on infected.

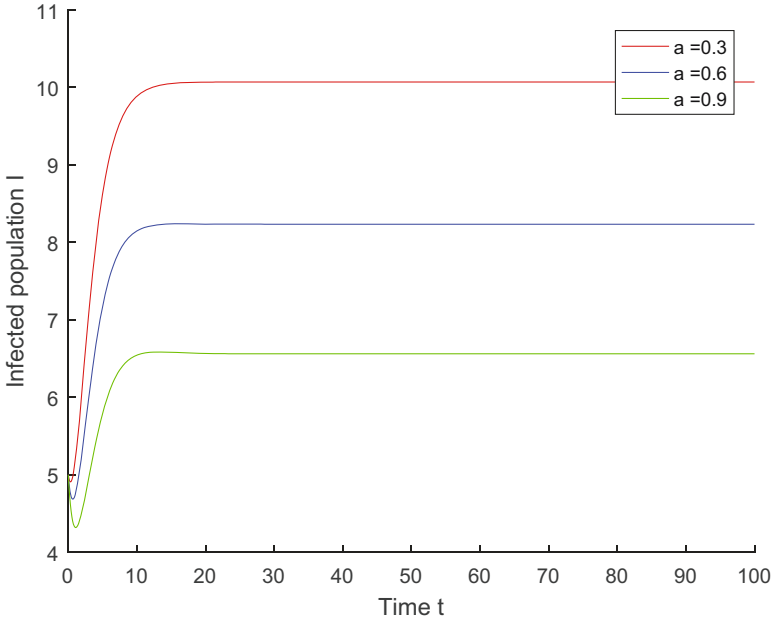

Figure 5: Effect of cure rate.

emerges but due to lower reproduction number, the average number of exposed and infected population remains low and approaches DFE.

For the case, when $R0 > 1$, $A = 7$, $\mu = 0.2$, $\beta = 1$, $\delta = 0.12$, $\gamma = 0.3$, $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $a = 0.3$, $b = 0.4$ results in $R0$ =2.9453 with initial values $S(0) = 10$, $E(0) = 5$, $I(0) = 5$, $R(0) = 5$, which results in EE $D^* = (S^*, E^*, I^*, R^*) = (3.14, 12.74, 10.07, 9.05)$ when inequality (3.12) is satisfied. Susceptible population decreases over time as a result of increased exposed and infected population, which also achieves a constant rate as they approach EE, which can be seen in Figure 3.

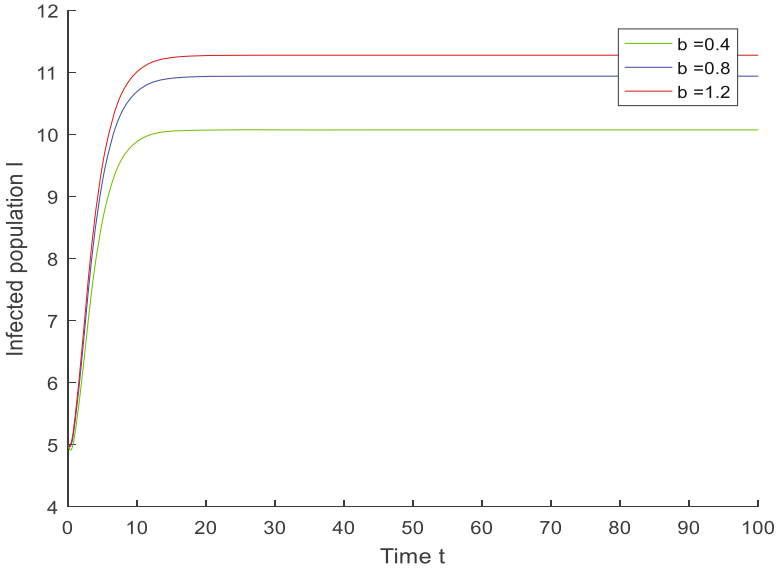

Figure 6: Effect of delay in treatment.

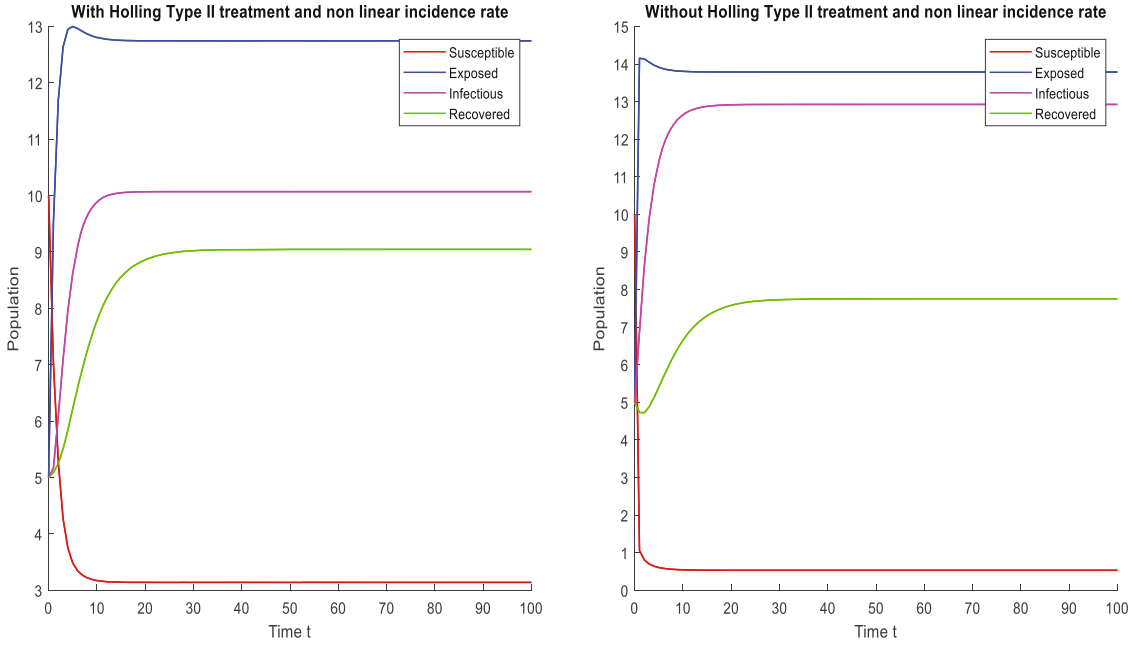

Figure 7: Comparison of the introduced model with classic SEIR model.

In this article, we have called out Holling type II function as a better strategy to read out treatment of infected population since there is a significant difference. In the same time frame, while using the function, the infected number of individuals is less as compared to not using the function (Figure 4). As cure rate increases, we see a significant drop in infected population. Thus, in order to eradicate a disease, higher cure rate is absolutely essential (Figure 5). The longer it takes to treat the infected, the higher the number of infected; as a result of slow recovery, infectious disease spreads more widely (Figure 6).

We also discuss the difference in results with Holling type II function and nonlinear incidence rate, which is used in this article for all the compartments with the classic SEIR model in Figure 7. We can observe that with the introduction of the aforementioned functions when $R0 > 1$, there is a significant

increase in susceptible and recovered population, whereas there is a noticeable drop in exposed and infected. Thus, taking psychological and sociological effect into consideration helps us to curtail the infection by reducing the exposed population. Also, timely treatment reduces the infected individuals and joins the healthy recovered population.

5 Discussion

In this work, we investigated the SEIR model with saturated incidence rate and Holling type II treatment rate. With the aid of the expression, we derived in (3.8), we measured the $R0$ value to determine the probability of epidemic and checked the behavior of population in the same time frame using graphs with the help of MATLAB. Note that the reproduction number $R0$ relies only on social behavior parameter α_1 related to susceptible individual, not on psychological impact α_2 of infected in population. This indicates that the reduction of social awareness in infected and rise in protective measure in susceptible are likely to have an effect on transmission of disease in the population. In order to grasp the psychological effect of the model, we make some further estimates by checking the existence and stability of EE. $R0$ also relies on the cure rate a of infected individuals, implying higher the value of a, lesser the $R0$ and lesser the amount of infected. So, cure rate should be higher to control epidemic. Sooner the infected people receive appropriate medical help, the quicker they heal and lot quicker we can control the epidemic. In this model, $R0$ does not depend on the magnitude of the consequence of the infected individual being delayed for treatment b; to understand this impact, we did further assessment through EE and found that more delay in treatment of the infected will lead to slow recovery and they remain infected for longer time period causing the breakout of disease. Thus, to control an epidemic, authorities should be well equipped with proper resources to avoid the delay.

The model can be used to describe the behavior of infectious diseases with latency period, such as COVID-19, influenza, and smallpox.

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