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Stability analysis of a fractional ordered COVID-19 model

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Abstract: The main purpose of this work is to study transmission dynamics of COVID-19 in Italy 2020, where the first case of Coronavirus disease 2019 (COVID-19) in Italy was reported on 31st January 2020. Taking into account the uncertainty due to the limited information about the Coronavirus (COVID-19), we have taken the modified Susceptible-Asymptomatic-Infectious-Recovered (SAIR) compartmental model under fractional order framework. We have formulated our model by subdividing infectious compartment into two sub compartments (reported and unreported) and introduced hospitalized class. In this work, we have studied the local and global stability of the system at different equilibrium points (disease free and endemic) and calculated sensitivity index for Italy scenario. The validity of the model is justified by comparing real data with the results obtained from simulations.

Keywords: Caputo fractional differential equation; COVID-19; SAIR compartmental model; Stability; Sensitivity index

MSC: 92D30; 37N25

1 Introduction

The respiratory disease caused by novel Coronavirus was first observed in Wuhan City, Hubei Province, China in December 2019. The outbreak of the disease is ongoing worldwide and the World Health Organization named it Coronavirus disease 2019 (COVID-19) on 11 February 2020. On February 21, 2020, the first Italian patient with COVID-19 was diagnosed, a 38-year-old man hospitalized at Codogno Hospital, Lodi, in northern Italy. As of 3 September 2020, Italy has 28,915 active cases; during the elevation of the pandemic, Italy's number of active cases was one of the highest in the World. By 3 September, Italy had tested nearly 5,342,000 people. Due to the limited number of tests performed, the actual number of infected people in Italy, as in other nations, is estimated to be higher than the official count [37]. Due to this reason we have considered two sub classes, namely 'reported symptomatically infected class' and 'unreported symptomatically infected class'. Maria Van Kerkhove, the WHO's technical lead on the COVID-19 pandemic, made it very clear that the actual rates of asymptomatic transmission are not however known [36]. So, we have considered the fact that asymptomatic class does not transmit disease and the main spreading of the Corona virus is happened due to reported infected class and unreported infected class. Some proportion (critical conditions) of reported infected class should be transferred to hospitalized class. Hospitalization capacity is limited for every state, especially ICU facilities and due to this hospitalization of COVID-19 patients have created a capital issue. Introducing hospitalized class, we want to predict the number of COVID-19 patients would be hospitalized which was a major issue in the beginning of pandemic in Italy 2020.

Fractional calculus can be regarded as the generalization of their order where fractional order is replaced with integer order [24, 23]. In systematic study it has been observed that integer order model is a limited case

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of fractional order model where the solution of the fractional order system must converge to the solution of integer order system as the order approaches to one [31]. In that respect there are many fields where fractional order systems are more suitable than integer order systems. Phenomena, which are linked with memory property and affected by hereditary property, cannot be expressed by integer order systems [11]. It is observed that the data garnered from real life phenomena fit better with fractional order systems. We have already done some works in fractional order dynamics [5, 6, 7, 8, 9]. Many researchers have contributed remarkably on various models of COVID-19 [14, 17, 20, 26, 27, 28, 29]. Fractional order modeling is a beneficial approach that has been used to study the nature of diseases because the fractional derivative is a generalization of the integer-order derivative. Also, the integer derivative is local in nature, while the fractional derivative is global. This behavior is very useful for modeling epidemic problems. In addition, the fractional derivative serves to enhance the stability region of the system. The fractional order system provides extra parameter which is useful for better numerical simulations. The previous models are very much productive for studying the transmission of COVID-19 but these models do not contain hospitalized, reported and unreported classes and also the pandemic situation of Italy is not mentioned in the previous research works. These facts along with the advantages of fractional calculus motivate us to construct our model on COVID-19 in Caputo fractional differential equation systems.

In this work, we have constructed a modified fractional order SAIR model for two sub compartment infected classes (reported and unreported) and hospitalized class (section 2). Next, we have calculated basic reproduction number of the model which is very important to know whether a virus is highly contagious or not (section 3). Followed by the uniqueness, non-negativity, boundedness of the solutions have been shown (section 4). We have analyzed local and global stability of our proposed system and also calculated sensitivity indices (section 4). Further we have numerically analyzed the dynamical system with respect to the parameter values connected with Italy scenario (section 5). Section 6 contains some important conclusions.

2 Model formulation

We have constructed the following six compartmental model under fractional order framework using Caputo fractional differential equations:

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha}S(t) = \Lambda^{\alpha} - \delta^{\alpha}S(t) - ({\omega_1}^{\alpha}U(t)S(t) + {\omega_2}^{\alpha}V(t)S(t)),\ S(0) > 0,\\
&{}_{t_0}^{C}D_t^{\alpha}A(t) = ({\omega_1}^{\alpha}U(t)S(t) + {\omega_2}^{\alpha}V(t)S(t)) - (\delta^{\alpha} + \sigma^{\alpha})A(t),\ A(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha}U(t) = f\sigma^{\alpha}A(t) + \eta^{\alpha}V(t) - (\epsilon^{\alpha} + {\delta_u}^{\alpha} + \phi^{\alpha})U(t),\ U(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha}V(t) = (1-f)\sigma^{\alpha}A(t) - (\eta^{\alpha} + \xi^{\alpha} + {\delta_v}^{\alpha})V(t),\ V(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha}H(t) = \epsilon^{\alpha}U(t) - (\gamma^{\alpha} + {\delta_h}^{\alpha})H(t),\ H(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha}R(t) = \phi^{\alpha}U(t) + \xi^{\alpha}V(t) + \gamma^{\alpha}H(t) - \delta^{\alpha}R(t),\ R(0) \geq 0.
\end{aligned}
\tag{I}
$$

where $0 < \alpha < 1$, and ${}_{t_0}^{C}D_t^{\alpha}$ is the notation due to Caputo fractional derivative, $t_0 \geq 0$ is the initial time (it is assumed that $t_0 = 0$). System (I) is dimentionally correct as both sides have same time dimension $time^{-\alpha}$. We have ignored the upper script α of all parameters and the model becomes:

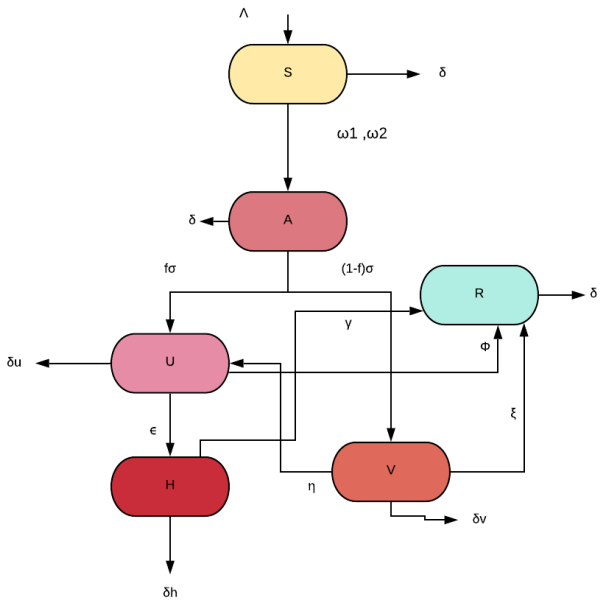

Figure 1: Schematic diagram of system (1)

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\alpha} S(t) = \Lambda - \delta S(t) - (\omega_1 U(t)S(t) + \omega_2 V(t)S(t)), S(0) > 0,\\
&{}_{t_0}^{C}D_t^{\alpha} A(t) = (\omega_1 U(t)S(t) + \omega_2 V(t)S(t)) - (\delta + \sigma)A(t), A(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha} U(t) = f\sigma A(t) + \eta V(t) - (\epsilon + \delta_u + \phi)U(t), U(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha} V(t) = (1-f)\sigma A(t) - (\eta + \xi + \delta_v)V(t), V(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha} H(t) = \epsilon U(t) - (\gamma + \delta_h)H(t), H(0) \geq 0,\\
&{}_{t_0}^{C}D_t^{\alpha} R(t) = \phi U(t) + \xi V(t) + \gamma H(t) - \delta R(t), R(0) \geq 0.
\end{aligned}
\tag{1}
$$

It is assumed that $\delta \leq \delta_u, \delta_v, \delta_h$ i.e., the natural death rate (δ) is lower than the disease induced death rates ($\delta_u, \delta_v, \delta_h$). Here $S(t), A(t), U(t), V(t), H(t)$ and $R(t)$ represent the susceptible, asymptomatically infected, reported symptomatically infected, unreported symptomatically infected, hospitalized population and recovered or removed population at time t respectively. The description of all parameters is given in Table 1.

3 Equilibria and Basic reproduction number

System (1) has two equilibrium points.

1. Disease free equilibrium: $E_0 = (\frac{\Lambda}{\delta}, 0, 0, 0, 0, 0)$
2. Endemic equilibrium: $E_1 = (S^*, A^*, U^*, V^*, H^*, R^*)$.

Table 1: Parameters in the system (1)

Λ	Recruitment rate of S
δ	Natural death rate
δ_u	Natural and disease induced death rate of reported class
δ_v	Natural and disease induced death rate of unreported class
δ_h	Natural and disease induced death rate of hospitalized class
ω_1	Transmission coefficient for the reported symptomatically infected compartment U
ω_2	Transmission coefficient for the unreported symptomatically infected compartment V
ϕ	Recovery rate of class U
σ	Rate at which asymptomatic infected becomes symptomatic
f	Fraction amount of population enters to reported symptomatically infected class
ϵ	Rate of hospitalization of symptomatic class
ξ	Rate of recovery of unreported symptomatic class
γ	Recovery rate of hospitalized class
η	Rate of conversion from unreported symptomatic class to reported symptomatic class by PCR testing

Here

$$\begin{aligned} S^\star &= \frac{(\delta+\sigma)A^\star}{\omega_1 U^\star + \omega_2 V^\star} \\ A^\star &= \frac{\Lambda}{\delta+\sigma} - \frac{\delta}{B} \\ \text{where } B &= \frac{\omega_1 f\sigma}{\epsilon+\delta_u+\phi} + \frac{\omega_2(1-f)\sigma}{\eta+\xi+\delta_v} \\ U^\star &= \frac{f\sigma A^\star}{\epsilon+\delta_u+\phi} \\ V^\star &= \frac{(1-f)f\sigma A^\star}{\eta+\xi+\delta_v} \\ H^\star &= \frac{\epsilon U^\star}{\gamma+\delta_h} \\ R^\star &= \frac{\xi V^\star + \phi U^\star + \gamma H^\star}{\delta} \end{aligned} \tag{2}$$

For E_1 to exist in feasible region $\mathbb{R}^6_+$, it is necessary and sufficient that

$$\Lambda\left[\frac{\omega_1 f\sigma}{\epsilon+\delta_u+\phi} + \frac{\omega_2(1-f)\sigma}{\eta+\xi+\delta_v}\right] \geq (\delta+\sigma)\delta$$

The basic reproduction number R_0 provides the average number of secondary (infectious) cases produced by one infective individual. It is calculated as the maximum eigenvalue of the next generation matrix FV^{-1} at disease free equilibrium E_0 [32], where

$$F = \begin{bmatrix} 0 & \omega_1\frac{\Lambda}{\delta} & \omega_2\frac{\Lambda}{\delta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} (\delta + \sigma) & 0 & 0 \\ -f\sigma & \epsilon + \delta_u + \phi & -\eta \\ -(1-f)\sigma & 0 & \eta + \xi + \delta_v \end{bmatrix}$$

Thus, we get

$$R_0 = \frac{\Lambda}{\delta}\left[\frac{\omega_1\sigma(\xi f + \delta_v f + \eta)}{(\eta + \xi + \delta_v)(\epsilon + \delta_u + \phi)} + \frac{\omega_2(1-f)\sigma}{\eta + \xi + \delta_v}\right] \tag{3}$$

Here the first part is due to reported symptomatically infected individuals and the second part is for unreported symptomatically infected individuals. Clearly, each parameter depends on α and so R_0 is a function of α. For analysis purposes, we have fixed the value of α. If we change the value of α, then all other parametric values will be changed and this will change the value of R_0. According to Figure 8, it is clear that the basic reproduction number R_0 will be decreased if the value of α is diminished.

4 Analysis of the model

In this section we have verified the existence, uniqueness, non-negative, boundedness criterion of the solution of system (1). Next we have performed stability analysis.

4.1 Preliminaries

Let us recall some basic theories that are needed for dynamical analysis.

Definition 1 [23] The Caputo fractional derivative with order $\alpha > 0$ for a function $g \in C^n([t_0, \infty+), \mathbb{R})$ is denoted and defined as:

$${}^{C}_{t_0}D^{\alpha}_{t} g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{t_0}^{t} \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \alpha \in (n-1, n), n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} g(t), \alpha = n. \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function, $t \geq t_0$ and n is a natural number. In particular, for $\alpha \in (0, 1)$:

$${}^{C}_{t_0}D^{\alpha}_{t} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{g^{'}(s)}{(t-s)^{\alpha}} ds$$

Lemma 1 (Generalized Mean Value Theorem) [22] Let $0 < \alpha \leq 1$, $\psi(t) \in C[a, b]$ and if ${}^{C}_{0}D^{\alpha}_{t}\psi(t)$ is continuous in $(a, b]$, then

$$\psi(x) = \psi(a) + \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha}.\, {}^{C}_{0}D^{\alpha}_{t}\psi(\zeta)$$

where $0 \leq \zeta \leq x$, $\forall x \in (a, b]$.

Remark: If ${}^{C}_{0}D^{\alpha}_{t}\psi(t) \geq 0 \left({}^{C}_{0}D^{\alpha}_{t}\psi(t) \leq 0\right), t \in (a, b)$, then $\psi(t)$ is a non-decreasing (non-increasing) function for $t \in [a, b]$.

Definition 2 [24] One parametric and two parametric Mittag-Leffler functions are described as follows:

$$E_\alpha(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha k + 1)} \text{ and } E_{\alpha_1,\alpha_2}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha_1 k + \alpha_2)}, \text{ where } \alpha, \alpha_1, \alpha_2 \in \mathbb{R}_+.$$

Theorem 1 [18] *Let $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$. Assume $g(t)$ is continuously differentiable function up to order $(n-1)$ on $[t_0, \infty)$ and n^{th} derivative of $g(t)$ exists with exponential order. If ${}_{t_0}^{C}D_t^\alpha g(t)$ is piecewise continuous on $[t_0, \infty)$, then*

$$\mathscr{L}\left\{{}_{t_0}^{C}D_t^\alpha g(t)\right\} = s^\alpha F(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} g^j(t_0),$$

where $F(s) = \mathscr{L}\left\{g(t)\right\}$ denotes the Laplace transform of $g(t)$.

Theorem 2 [16] *Let C be the complex plane. For any $\alpha_1, \alpha_2 \in \mathbb{R}_+$ and $M \in \mathbb{C}$, then*

$$\mathscr{L}\left\{t^{\alpha_2-1} E_{\alpha_1,\alpha_2}(Mt^{\alpha_1})\right\} = \frac{s^{\alpha_1-\alpha_2}}{(s^{\alpha_1} - M)},$$

for $\mathscr{R}(s) > \|M\|^{\frac{1}{\alpha_1}}$, where $\mathscr{R}(s)$ represents the real part of the complex number s, and E_{α_1,α_2} is the Mittag-Leffler function.

Theorem 3 [23] Consider the following fractional-order system:

$${}_{t_0}^{C}D_t^\alpha X(t) = \Psi(X), X_{t_0} = (x_{t_0}^1, x_{t_0}^2, \ldots, x_{t_0}^n), x_{t_0}^i > 0, i = 1, 2, .., n$$

with $0 < \alpha < 1$, $X(t) = (x^1(t), x^2(t), \ldots, x^n(t))$ and $\Psi(X) : [t_0, \infty) \to \mathbb{R}^{n\times n}$. The equilibrium points of this system are evaluated by solving the following system of equations: $\Psi(X) = 0$. These equilibrium points are locally asymptotically stable iff each eigenvalue λ_i of the Jacobian matrix $J(X) = \frac{\partial(\Psi_1, \Psi_2, \ldots, \Psi_n)}{\partial(x^1, x^2, \ldots, x^n)}$ calculated at the respective equilibrium points satisfy $\left|\arg(\lambda_i)\right| > \frac{\alpha\pi}{2}$.

4.2 Existence and uniqueness

Lemma 2 [19] Consider the system:

$${}_{t_0}^{C}D_t^\alpha x(t) = g(t, x), t_0 > 0 \tag{4}$$

with initial condition $x(t_0) = x_{t_0}$, where $\alpha \in (0, 1]$, $g : [t_0, \infty) \times \Omega \to \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, if local lipschitz condition is satisfied by $g(t, x)$ with respect to x, then there exists a solution of (4) on $[t_0, \infty) \times \Omega$ which is unique.

To study the existence and uniqueness of system (1), let us consider the region $\Omega \times [t_0, \gamma]$,where $\Omega = \{(S, A, U, V, H, R) \in \mathbb{R}^6 : \max(|S|, |A|, |U|, |V|, |H|, |R|) \le M\}$ and $\gamma < +\infty$. Denote $X = (S, A, U, V, H, R)$ and $\overline{X} = (\bar{S}, \bar{A}, \bar{U}, \bar{V}, \bar{H}, \bar{R})$. Consider a mapping $L(X) = (L_1(X), L_2(X), L_3(X), L_4(X), L_5(X), L_6(X))$, where

$$L_1(X) = \Lambda - \delta S - \omega_1 SU - \omega_2 SV$$

$$L_2(X) = \omega_1 SU + \omega_2 SV - (\delta + \sigma)A$$

$$L_3(X) = f\sigma A + \eta V - (\epsilon + \delta_u + \phi)U$$

$$L_4(X) = (1-f)\sigma A - (\eta + \xi + \delta_v)V$$

$$L_5(X) = \epsilon U - (\gamma + \delta_h)H$$

$$L_6(X) = \xi V + \phi U + \gamma H - \delta R$$

For any $X, \overline{X} \in \Omega$:

$$
\begin{aligned}
&\left\|L(X) - L(\overline{X})\right\| \\
&= \left|L_1(X) - L_1(\overline{X})\right| + \left|L_2(X) - L_2(\overline{X})\right| + \left|L_3(X) - L_3(\overline{X})\right| + \left|L_4(X) - L_4(\overline{X})\right| \\
&+ \left|L_5(X) - L_5(\overline{X})\right| + \left|L_6(X) - L_6(\overline{X})\right| \\
&= \left|-\delta S - \omega_1(US - \bar{U}\bar{S}) - \omega_2(VS - \bar{V}\bar{S})\right| \\
&+ \left|\omega_1(US - \bar{U}\bar{S}) + \omega_2(VS - \bar{V}\bar{S}) - (\delta + \sigma)(A - \bar{A})\right| \\
&+ \left|f\sigma(A - \bar{A}) + \eta(V - \bar{V}) - (\epsilon + \delta_u + \phi)(U - \bar{U})\right| + \left|(1 - f)\sigma(A - \bar{A}) - (\eta + \delta_v + \xi)(U - \bar{U})\right| \\
&+ \left|\epsilon(U - \bar{U}) - (\gamma + \delta_h)(H - \bar{H})\right| \\
&+ \left|\xi(V - \bar{V}) + \phi(U - \bar{U}) + \gamma(H - \bar{H}) - \delta(R - \bar{R})\right| \\
&\leq \delta\left|S - \bar{S}\right| + 2\omega_1\left|US - \bar{S}\bar{U}\right| + 2\omega_2\left|VS - \bar{S}\bar{V}\right| + (\delta + 2\sigma)\left|A - \bar{A}\right| \\
&+ \{2\epsilon + \delta_u + 2\phi\}\left|U - \bar{U}\right| + \{2\eta + 2\xi + \delta_v\}\left|V - \bar{V}\right| \\
&+ \{2\gamma + \delta_h\}\left|H - \bar{H}\right| + \delta\left|R - \bar{R}\right| \\
&\leq (\delta + 2\omega_1 M + 2\omega_2 M)\left|S - \bar{S}\right| + (\delta + 2\sigma)\left|A - \bar{A}\right| \\
&+ \{2\epsilon + \delta_u + 2\phi + 2\omega_1 M + 2\omega_2 M\}\left|U - \bar{U}\right| + \{2\eta + 2\xi + \delta_v\}\left|V - \bar{V}\right| \\
&+ \{2\gamma + \delta_h\}\left|H - \bar{H}\right| + \delta\left|R - \bar{R}\right| \\
&\leq F_1\left|S - \bar{S}\right| + F_2\left|A - \bar{A}\right| + F_3\left|U - \bar{U}\right| + F_4\left|V - \bar{V}\right| + F_5\left|H - \bar{H}\right| + F_6\left|R - \bar{R}\right| \\
&\leq F\left\|X - \overline{X}\right\|, \text{ where } F = \max\{F_1, F_2, F_3, F_4, F_5, F_6\},
\end{aligned}
$$

and

$$
\begin{aligned}
F_1 &= (\delta + 2\omega_1 M + 2\omega_2 M) \\
F_2 &= (\delta + 2\sigma) \\
F_3 &= \{2\epsilon + \delta_u + 2\phi + 2\omega_1 M + 2\omega_2 M\} \\
F_4 &= \{2\eta + 2\xi + \delta_v\} \\
F_5 &= \{2\gamma + \delta_h\} \\
F_6 &= \delta
\end{aligned}
$$

Hence $L(X)$ satisfies Lipschitz's condition with respect to X. Therefore, Lemma 2 confirms that there exists a unique solution $X(t)$ of system (1) with initial condition $X(0) = (S(0), A(0), U(0), V(0), H(0), R(0))$. The following theorem is the consequence of this result.

Theorem 4. There exists a unique solution $X(t) \in \Omega$ of system (1) for all $t \geq 0$ with initial condition $X(0) = (S(0), A(0), U(0), V(0), H(0), R(0)) \in \Omega = \{(S, A, U, V, H, R) \in \mathbb{R}^6 : \max(|S|, |A|, |U|, |V|, |H|, |R|) \leq M\}$.

4.3 Non-negativity and boundedness

In this section we have established the criterion for feasibility of the solution of system (1). Suppose $\mathbb{R}_+$ stands for the set of all non-negative real numbers and $\Gamma_+ = \left\{(S, A, U, V, H, R) \in \mathbb{R}_+^6\right\}$.

Theorem 5 The solutions $X(t) = (S, A, U, V, H, R)$ of system (1) remain in Γ_+ if $X(0) = (S(0), A(0), U(0), V(0), H(0), R(0)) \in \Gamma_+$.

Proof:

$$ {}_{t_0}^{C}D_t^{\alpha} S(t)\big|_{S(t)=0} = \Lambda > 0 \tag{i} $$

$$ {}_{0}^{C}D_t^{\alpha} A(t)\big|_{A(t)=0} = \omega_1 US + \omega_2 VS \tag{ii} $$

$$ {}_{0}^{C}D_t^{\alpha} U(t)\big|_{U(t)=0} = f\sigma A + \eta V \tag{iii} $$

$$ {}_{0}^{C}D_t^{\alpha} V(t)\big|_{V(t)=0} = (1-f)\sigma A \tag{iv} $$

$$ {}_{0}^{C}D_t^{\alpha} H(t)\big|_{H(t)=0} = \epsilon U \tag{v} $$

$$ {}_{0}^{C}D_t^{\alpha} R(t)\big|_{R(t)=0} = \xi V + \phi U + \gamma H \tag{vi} $$

From (i), we have

$$ {}_{t_0}^{C}D_t^{\alpha} S(t)|_{S(t)=0} = \Lambda > 0. $$

From Lemma 1, we can say $S(t)$ is increasing in a neighborhood of time t where $S(t) = 0$ and $S(t)$ cannot cross the axis $S(t) = 0$. Hence, $S(t) > 0$ for all $t \geq 0$. Now, we claim that the solution $A(t)$ starts from Γ_+ and remains non-negative. If not then there exists τ_1 such that $A(t)$ crosses $A(t) = 0$ axis at $t = \tau_1$ for the first time and the following conditions hold

$$ \begin{cases} A(t) > 0, \text{ for } 0 \leq t < \tau_1, \\ A(\tau_1) = 0, \\ A(\tau_1^+) < 0. \end{cases} $$

From (ii), we have ${}_{0}^{C}D_t^{\alpha} A(t)\big|_{P_1(\tau_1)=0} = \omega_1 U(\tau_1)S(\tau_1) + \omega_2 V(\tau_1)S(\tau_1)$.

Now, two cases arise.

Case1: If $\omega_1 U(\tau_1)S(\tau_1) + \omega_2 V(\tau_1)S(\tau_1) > 0$ then by the remark of Lemma 1 we can say that $A(t)$ is non-decreasing in a neighborhood of $t = \tau_1$ and which concludes $A(\tau_1^+) = 0$. Hence, we have arrived at a contradiction.

Case2: In this case we have $\omega_1 U(\tau_1)S(\tau_1) + \omega_2 V(\tau_1)S(\tau_1) < 0$, which implies any one or both of $U(\tau_1)$, $V(\tau_1)$ must be negative. Now we have two sub-cases.

Sub-case1: If $V(\tau_1) < 0$, then there exists a τ_2 such that $0 < \tau_2 < \tau_1$ with

$$ \begin{cases} V(t) > 0, \text{ for } 0 \leq t < \tau_2, \\ V(\tau_2) = 0, \\ V(\tau_2^+) < 0. \end{cases} $$

From (iv), we have

$$ {}_{0}^{C}D_t^{\alpha} V(t)\big|_{V(\tau_2)=0} = (1-f)\sigma A(\tau_2) > 0 $$

which contradicts our assumption that $V(\tau_2^+) < 0$. Therefore, we have $V(t) \geq 0, \forall t \in [0, \infty)$.

Sub-case2: If $U(\tau_2) < 0$, then we can find a τ_3 such that $0 < \tau_3 < \tau_2 < \tau_1$ with

$$ \begin{cases} U(t) > 0, \text{ for } 0 \leq t < \tau_3, \\ U(\tau_3) = 0, \\ U(\tau_3^+) < 0. \end{cases} $$

From (iii), we have

$$ {}_0^C D_t^\alpha U(t)\big|_{U(\tau_3)=0} = f\sigma A(\tau_3) + \eta V(\tau_3) > 0 $$

which contradicts our assumption that $U(\tau_3^+) < 0$. Therefore, we have $U(t) \geq 0, \forall t \in [0, \infty)$ and also $A(t) \geq 0, \forall t \in [0, \infty)$.

Again from (v),(vi) we have $H(t), R(t) \geq 0, \forall t \in [0, \infty)$

Thus, all solutions of system (1) starting in Γ_+ are confined in the region Γ_+.

Theorem 6 (Boundedness). Solutions $X(t) = (S, A, U, V, H, R)$ of system (1) are uniformly bounded.

Proof: From first equation of (1), it is noted that

$$ {}_0^C D_t^\alpha x(t) \leq \Lambda - \delta S $$

Taking Laplace transforms on both sides, we have

$$ s^\alpha \mathscr{L}\{S(t)\} - s^{\alpha-1} S(0) + \delta \mathscr{L}\{S(t)\} \leq \frac{\Lambda}{s}, \text{ where } \mathscr{L}\{\cdot\} \text{ is the Laplace transform operator} $$

$$ \Rightarrow \mathscr{L}\{S(t)\} \leq A \frac{s^{\alpha-(1+\alpha)}}{s^\alpha + \delta} + S(0) \frac{s^{\alpha-1}}{s^\alpha + \delta} $$

Taking inverse Laplace transforms (using Theorem 2):

$$ S(t) \leq S(0) E_{\alpha,1}(-\delta t^\alpha) + \Lambda t^\alpha E_{\alpha,\alpha+1}(-\delta t^\alpha) \tag{5} $$

$$ \therefore S(t) \leq M_1 [E_{\alpha,1}(-\delta t^\alpha) + \delta t^\alpha E_{\alpha,\alpha+1}(-\delta t^\alpha)] = \frac{M_1}{\Gamma(1)} = M_1, $$

where $M_1 = \max\left\{\frac{\Lambda}{\delta}, S(0)\right\}$ and since from the properties of Mittag Leffler function [15]:

$$ E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)} $$

In this case

$$ E_{\alpha,1}(-\delta t^\alpha) = (-\delta t^\alpha) E_{\alpha,\alpha+1}(-\delta t^\alpha) + \frac{1}{\Gamma(1)} \tag{6} $$

Let, $N(t) = S(t) + A(t) + U(t) + V(t) + H(t) + R(t)$ represents the total population, then

$$ \begin{aligned} {}_0^C D_t^\alpha N(t) &= {}_0^C D_t^\alpha S(t) + {}_0^C D_t^\alpha A(t) + {}_0^C D_t^\alpha U(t) + {}_0^C D_t^\alpha V(t) + {}_0^C D_t^\alpha H(t) + {}_0^C D_t^\alpha R(t) \\ &= \Lambda - \{\delta S(t) + \delta A(t) + \delta_u U(t) + \delta_v V(t) + \delta_h H(t) + \delta R(t)\} \\ &\leq \Lambda - \delta_m N(t), \text{ where } \delta_m = \min\{\delta_u, \delta_v, \delta_h, \delta\} \end{aligned} $$

Therefore,

$$ {}_0^C D_t^\alpha N(t) + \delta_m N(t) \leq \Lambda $$

Applying Laplace transformation, we have (using Theorem 1):

$$ s^\alpha F(s) - s^{\alpha-1} N(0) + \delta_m F(s) \leq \frac{\Lambda}{s}, \text{ where } F(s) = \mathscr{L}\{N(t)\} $$

$$ \Rightarrow F(s) \leq \Lambda \frac{s^{-1}}{s^\alpha + \delta_m} + \frac{N(0) s^{\alpha-1}}{s^\alpha + \delta_m} = \frac{s^{\alpha-1} N(0)}{s^\alpha + \delta_m} + \frac{\Lambda s^{\alpha-(1+\alpha)}}{s^\alpha + \delta_m} $$

Taking inverse Laplace transforms (using Theorem 2):

$$N(t) = N(0)E_{\alpha,1}(-\delta_m t^\alpha) + \Lambda t^\alpha E_{\alpha,\alpha+1}(-\delta_m t^\alpha) \tag{7}$$

From (6) and (7), we get

$$N(t) \le M_2[E_{\alpha,1}(-\delta_m t^\alpha) + \delta_m t^\alpha E_{\alpha,\alpha+1}(-\delta_m t^\alpha)] = \frac{M_2}{\Gamma(1)} = M_2,$$

where $M_2 = \max\left\{\frac{\Lambda}{\delta_m}, N(0)\right\}$

Thus $S(t)$, $N(t)$ are bounded and hence the solutions $X(t) = (S, A, U, V, H, R)$ are bounded uniformly in $\{(S, A, U, V, H, R) | S + A + U + V + H + R \le M_2; S \le M_1\}$ for $t \in [0, \infty)$ □

4.4 Local stability

For stability analysis, let us reduce system (1) by discarding last two equations as H, R do not appear in first four equations of system (1). If we study the dynamics of S, A, U, V, then the dynamics of H, R also be obtained from them. The reduced system is as follows:

$$\begin{aligned}
&{}_{t_0}^{C}D_t^\alpha S(t) = \Lambda - \delta S(t) - (\omega_1 U(t)S(t) + \omega_2 V(t)S(t)), S(0) > 0,\\
&{}_{t_0}^{C}D_t^\alpha A(t) = (\omega_1 U(t)S(t) + \omega_2 V(t)S(t)) - (\delta + \sigma)A(t), A(0) \ge 0,\\
&{}_{t_0}^{C}D_t^\alpha U(t) = f\sigma A(t) + \eta V(t) - (\epsilon + \delta_u + \phi)U(t), U(0) \ge 0,\\
&{}_{t_0}^{C}D_t^\alpha V(t) = (1-f)\sigma A(t) - (\eta + \xi + \delta_v)V(t), V(0) \ge 0.
\end{aligned} \tag{8}$$

The equilibrium points of (8) are as follows:

1. Disease free equilibrium: $E_0^\star = (\frac{\Lambda}{\delta}, 0, 0, 0)$
2. Endemic equilibrium: $E_1^\star = (S^\star, A^\star, U^\star, V^\star)$.

Here

$$\begin{aligned}
S^\star &= \frac{(\delta+\sigma)A^\star}{\omega_1 U^\star + \omega_2 V^\star}\\
A^\star &= \frac{\Lambda}{\delta+\sigma} - \frac{\delta}{B}\\
&\text{where } B = \frac{\omega_1 f\sigma}{\epsilon+\delta_u+\phi} + \frac{\omega_2(1-f)\sigma}{\eta+\xi+\delta_v}\\
U^\star &= \frac{f\sigma A^\star}{\epsilon+\delta_u+\phi}\\
V^\star &= \frac{(1-f)f\sigma A^\star}{\eta+\xi+\delta_v}
\end{aligned} \tag{9}$$

For $E_1^\star$ to exist in feasible region $\mathbb{R}_+^4$, it is necessary and sufficient that

$$\Lambda\left[\frac{\omega_1 f\sigma}{\epsilon+\delta_u+\phi} + \frac{\omega_2(1-f)\sigma}{\eta+\xi+\delta_v}\right] \ge (\delta+\sigma)\delta$$

To analyze the local stability of endemic equilibrium $E_1^\star$, we need the followings:

Definition 3 [13]: The discriminant $\nabla(f)$ of a polynomial $f(x) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \ldots + \alpha_n$ is defined by

$$\nabla(f) = (-1)^{\frac{n(n-1)}{2}} |S_n(f, f')|$$

$S_n(f, g)$ is the Sylvester matrix of $f(x)$ and $g(x)$ of order $(n+l)\times(n+l)$ where $g(x) = x^l + \beta_1 x^{l-1} + \beta_2 x^{l-2} + \ldots + \beta_l$.

For $n = 3$, we have $f(x) = x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3$ and $f'(x) = 3x^2 + 2\alpha_1 x + \alpha_2$.

$$|S_3(f, f')| = \begin{vmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 3 & 2\alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 3 & 2\alpha_1 & \alpha_2 & 0 \\ 0 & 0 & 3 & 2\alpha_1 & \alpha_2 \end{vmatrix} = -18\alpha_1\alpha_2\alpha_3 - (\alpha_1\alpha_2)^2 + 4\alpha_1^2\alpha_3 + 4\alpha_2^2 + 27\alpha_3^2$$

Hence $\nabla(f) = -|S_3(f, f')| = 18\alpha_1\alpha_2\alpha_3 + (\alpha_1\alpha_2)^2 - 4\alpha_1^2\alpha_3 - 4\alpha_2^2 - 27\alpha_3^2$

Lemma 3 [2]: If $\nabla(P)$ is the discriminant of the characteristic equation $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ of Jacobian matrix of system (1) evaluated at equilibrium point, then for $n = 3$ the system is asymptotically stable if any of the following conditions hold:

1. $\nabla(P) > 0$, $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$
2. $\nabla(P) < 0$, $a_1 \geq 0$, $\alpha_2 \geq 0$, $a_3 > 0$ and $\alpha < \frac{2}{3}$
3. $\nabla(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1 a_2 = a_3$ and $\alpha \in (0, 1)$.

For $n = 4$, $P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$

$$\nabla(P) = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 1 & a_1 & a_2 & a_3 & a_4 \\ 4 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4 & 3a_1 & 2a_2 & a_3 \end{vmatrix}$$

Lemma 4 [21]:

1. If $\nabla(P) > 0$, $a_1 > 0$, $a_2 < 0$ and $\alpha > \frac{2}{3}$, then the equilibrium point is unstable.
2. If $\nabla(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ and $\alpha < \frac{1}{3}$, then the equilibrium point is asymptotically stable.
3. If $\nabla(P) < 0$, $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, $a_4 > 0$ and $\alpha < \frac{1}{3}$, then the equilibrium point is unstable.
4. If $\nabla(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_2 = \frac{a_1 a_4}{a_3} + \frac{a_3}{a_1}$, then the equilibrium point is asymptotically stable for $\alpha \in (0, 1)$.
5. $a_4 > 0$ is the necessary condition for the equilibrium point E^* is to be locally asymptotically stable.

At the disease-free equilibrium point E_0^*, the Jacobian matrix is given by

$$J\left\{\left(\frac{\Lambda}{\delta},0,0,0\right)\right\}=\begin{bmatrix}-\delta & 0 & -\frac{\Lambda}{\delta}\omega_1 & -\frac{\Lambda}{\delta}\omega_2\\ 0 & -(\delta+\sigma) & \frac{\Lambda}{\delta}\omega_1 & \frac{\Lambda}{\delta}\omega_2\\ 0 & f\sigma & -(\epsilon+\delta_u+\phi) & \eta\\ 0 & (1-f)\sigma & 0 & -(\eta+\xi+\delta_v)\end{bmatrix}$$

Characteristic equation of this matrix is: $(\lambda + d)P(\lambda) = 0$, where $P(\lambda) \equiv \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$,

$$c_1 = -(K_1 + K_5 + K_3)$$

$$c_2 = K_1K_5 + K_1K_9 + K_5K_9 - K_2K_4 - K_3K_7 - K_6K_8$$

$$c_3 = -K_1K_5K_9 + K_1K_6K_8 + K_2K_4K_9 - K_2K_6K_7 - K_3K_4K_8 + K_3K_7K_5$$

and

$$\begin{aligned}K_1 &= -(\delta+\sigma)\\ K_2 &= \frac{\Lambda}{\delta}\omega_1\\ K_3 &= \frac{\Lambda}{\delta}\omega_2\\ K_4 &= f\sigma\\ K_5 &= -(\epsilon+\delta_u+\phi)\\ K_6 &= \eta\\ K_7 &= (1-f)\sigma\\ K_8 &= 0\\ K_9 &= -(\xi+\eta+\delta_v)\end{aligned} \tag{10}$$

So, λ_i, $i = 1, 2, 3$, can be found from the equation: $P(\lambda) \equiv \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$. Suppose $\nabla(P) = 18c_1c_2c_3 + (c_1c_2)^2 - 4c_1^2c_3 - 4c_2^2 - 27c_3^2$, then by Lemma 3 the disease-free equilibrium point $E_0^\star$ is locally asymptotically stable if any of the following conditions holds good :

1. $\nabla(P) > 0$, $c_1 > 0$, $c_3 > 0$ and $c_1c_2 > c_3$
2. $\nabla(P) < 0$, $c_1 \geq 0$, $c_2 \geq 0$, $c_3 > 0$ and $\alpha < \frac{2}{3}$
3. $\nabla(P) < 0$, $c_1 > 0$, $c_2 > 0$, $c_1c_2 = c_3$ and $\alpha \in (0, 1)$

The following theorem is the consequence of these discussions.

Theorem 7

The disease free equilibrium $E_0^\star$ of system (1) is asymptotically stable if any of the following conditions hold:

1. $\nabla(P) > 0$, $c_1 > 0$, $c_3 > 0$ and $c_1c_2 > c_3$
2. $\nabla(P) < 0$, $c_1 \geq 0$, $c_2 \geq 0$, $c_3 > 0$ and $\alpha < \frac{2}{3}$
3. $\nabla(P) < 0$, $c_1 > 0$, $c_2 > 0$, $c_1c_2 = c_3$ and $\alpha \in (0, 1)$

where $\nabla(P) = 18c_1c_2c_3 + (c_1c_2)^2 - 4c_1^2c_3 - 4c_2^2 - 27c_3^2$,

$$c_1 = -(K_1 + K_5 + K_3)$$

$$c_2 = K_1K_5 + K_1K_9 + K_5K_9 - K_2K_4 - K_3K_7 - K_6K_8$$

$$c_3 = -K_1K_5K_9 + K_1K_6K_8 + K_2K_4K_9 - K_2K_6K_7 - K_3K_4K_8 + K_3K_7K_5$$

where K_i, $i = 1, 2\ldots, 9$ are given in (10).

Now, Jacobian matrix of system (1) at endemic equilibrium E_1^* is given by

$$J(S^*, A^*, U^*, V^*) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}, \text{ where}$$

$$\begin{aligned} b_{11} &= -\delta - (\omega_1 U^* + \omega_2 V^*) \\ b_{12} &= 0 \\ b_{13} &= -S^* \omega_1 \\ b_{14} &= S^* \omega_2 \\ b_{22} &= -(\delta + \sigma) \\ b_{23} &= S^* \omega_1 \\ b_{24} &= S^* \omega_2 \\ b_{31} &= 0 \\ b_{32} &= f\sigma \\ b_{33} &= -(\epsilon + \delta_u + \phi) \\ b_{11} &= \eta \\ b_{41} &= 0 \\ b_{42} &= (1-f)\sigma \\ b_{43} &= 0 \\ b_{44} &= -(\xi + \eta + \delta_v) \end{aligned} \tag{11}$$

Characteristic equation of this matrix is: $Q(\lambda) \equiv \lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0$, where

$$\begin{aligned} d_1 &= -(b_{11} + b_{12} + b_{13} + b_{14}) \\ d_2 &= -b_{12}b_{21} - b_{13}b_{31} - b_{23} \\ &+ b_{32} + b_{22}b_{33} - b_{14}b_{41} - b_{24}b_{42} - b_{34}b_{43} + (b_{22} + b_{33})b_{44} + b_{11}(b_{22} + b_{33} + b_{44}) \\ d_3 &= b_{11}(b_{23}b_{32} - b_{22}b_{33} + b_{24}b_{42} + b_{34}b_{43}) \\ &+ b_{14}(b_{22}b_{41} + b_{33}b_{41} - b_{21}b_{42} - b_{31}b_{43}) + b_{24}(b_{33}b_{42} - b_{32}b_{43}) \\ &+ b_{34}(b_{22}b_{43} - b_{23}b_{42}) - b_{44}(b_{11}b_{22} - b_{23}b_{32} + (b_{11} + b_{22})b_{33}) \\ &+ b_{13}(-b_{21}b_{32} - b_{34}b_{41} + b_{31}(b_{22} + b_{44})) + b_{12}(-b_{23}b_{31} - b_{24}b_{41} + b_{21}(b_{33} + b_{44})) \\ d_4 &= b_{12}b_{41}(b_{24}b_{33} - b_{23}b_{34}) + b_{11}b_{42}(b_{23}b_{34} - b_{24}b_{33}) \\ &+ b_{24}b_{43}(b_{32}b_{11} - b_{12}b_{31}) + b_{34}b_{43}(b_{12}b_{21} - b_{11}b_{22}) + b_{14}\{b_{23}(b_{32}b_{41} - b_{31}b_{42}) \\ &+ b_{21}(b_{33}b_{42} - b_{32}b_{43}) + b_{22}(b_{31}b_{43} - b_{33}b_{41})\} + \{b_{12}(b_{23}b_{31} - b_{21}b_{33}) + b_{11}(b_{22}b_{33} - b_{32}b_{23})\} \\ &+ b_{13}\{b_{22}(b_{34}b_{41} - b_{31}b_{44}) + b_{24}(b_{31}b_{42} - b_{32}b_{41}) + b_{21}(b_{32}b_{44} - b_{34}b_{42})\} \end{aligned} \tag{12}$$

Therefore, we have the following theorem:

Theorem 8 The endemic equilibrium E_1^* of system (1) is asymptotically stable if any of the following conditions hold:

1. If $\nabla(Q) < 0$, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $d_4 > 0$ and $\alpha < \frac{1}{3}$
2. If $\nabla(Q) < 0$, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $d_4 > 0$ and $d_2 = \frac{d_1 d_4}{d_3} + \frac{d_3}{d_1}$, for $\alpha \in (0, 1)$,

where $Q(\lambda) = \lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4$ is the Jacobian matrix at endemic equilibrium E_1^* and d_i's (i=1,2,3,4) can be found from (11) and (12).

4.5 Global Stability

First, let us state the following important lemmas.

Lemma 5 [19] Suppose $u(t) \in \mathbb{R}_+$be a continuous and differentiable function, then for any time $t \geq t_0$, ${}_{t_0}^{C}D_t^{\alpha}\left[u(t) - u^{\star} - u^{\star}\ln\frac{u(t)}{u^{\star}}\right] \leq \left(1 - \frac{u^{\star}}{u(t)}\right){}_{t_0}^{C}D_t^{\alpha}u(t), u^{\star} \in \mathbb{R}_+, \forall \alpha \in (0,1)$.

Lemma 6 [10](*Uniform Asymptotic Stability Theorem*) Consider the non-autonomous system

$$ {}_{0}^{C}D_t^{\alpha}x(t) = f(t,x),\ x \in \Omega \subseteq \mathbb{R}^n \tag{13}$$

Let $x^{\star}$ be an equilibrium point of the system ($x^{\star} \in \Omega \subseteq \mathbb{R}^n$) and $\Phi(t,x(t)) : [0,\infty)\times\Omega \to \mathbb{R}$ be a continuously differentiable function such that

$${}_{0}^{C}D_t^{\alpha}\Phi(t,x(t)) \leq -\Theta_3(x),$$

$$\Theta_1(x) \leq \Phi(t,x(t)) \leq \Theta_2(x), \forall \alpha \in (0,1), \forall x(t) \in \Omega$$

where $\Theta_i,\ i = 1,2,3$, are continuous positive definite functions in Ω. Then the equilibrium point $x^{\star}$ of system (13) is globally stable.

Theorem 9. The disease free equilibrium $E_0^{\star}$ of system (8) is globally asymptotically stable if $\omega_1 M_1 \leq (\epsilon + \delta_u + \phi)$ and $\omega_2 M_1 \leq (\eta + \xi + \delta_v)$, where $M_1 = \max\left\{\frac{\Lambda}{\delta}, S(0)\right\}$.

Proof: We have considered a positive definite function:

$$L = A + U + V$$

Clearly $L \geq 0$ and $L = 0$ only at $(0,0,0)$.

Taking α order Caputo derivative ${}_{0}^{C}D_t^{\alpha}$ of L along the solution of system (8), we have

$${}_{0}^{C}D_t^{\alpha}L = {}_{0}^{C}D_t^{\alpha}A + {}_{0}^{C}D_t^{\alpha}U + {}_{0}^{C}D_t^{\alpha}V$$

$$\leq (\omega_1 M_1 U + \omega_2 M_1 V) - (\delta + \sigma)A + f\sigma A - (\epsilon + \delta_u + \phi)U + (1-f)\sigma A - (\eta + \xi + \delta_v)V \text{ (since } S \leq M_1)$$
$$= -\delta A + [\omega_1 M_1 - (\epsilon + \delta_u + \phi)]U + [\omega_2 M_1 - (\eta + \xi + \delta_v)]V$$

Hence, ${}_{0}^{C}D_t^{\alpha}L \leq 0$ if $\omega_1 M_1 \leq (\epsilon + \delta_u + \phi)$ and $\omega_2 M_1 \leq (\eta + \xi + \delta_v)$.
Therefore, using Lemma 6:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} U(t) = \lim_{t\to\infty} V(t) = 0.$$

Hence in the limit $S(t)$ is given by the solutions of ${}_{0}^{C}D_t^{\alpha}S(t) = \Lambda - \delta S$. Since $S(0) > 0$, the theorem follows. □

Theorem 10. The endemic equilibrium $E_1^{\star}(S^{\star}, A^{\star}, U^{\star}, V^{\star})$ of system (8) is globally asymptotically stable.

Proof: Consider a positive definite function:

$$\begin{aligned} V = &\left(S - S^{\star} - S^{\star}\ln\frac{S}{S^{\star}}\right) + \left(A - A^{\star} - A^{\star}\ln\frac{A}{A^{\star}}\right) \\ &+c_1\left(U - U^{\star} - U^{\star}\ln\frac{U}{U^{\star}}\right) + c_2\left(V - V^{\star} - V^{\star}\ln\frac{V}{V^{\star}}\right), \end{aligned} \tag{14}$$

where $c_1 = \min\left\{\frac{\omega_1 S^* U^*}{f\sigma A^*}, \frac{\omega_2 S^* V^*}{\eta V^*}\right\}$, $c_2 = \frac{\omega_2 S^* V^*}{(1-f)\sigma A^*}$. It is observed that $V \geq 0$ and $V = 0$ only at E_1^*. Taking α order Caputo derivative ${}_0^C D_t^\alpha$ of V and using Lemma 5, we have

$$\begin{aligned} {}_0^C D_t^\alpha(V) \leq & \left(1 - \frac{S^*}{S}\right) {}_0^C D_t^\alpha S + \left(1 - \frac{A^*}{A}\right) {}_0^C D_t^\alpha A \\ & + c_1 \left(1 - \frac{U^*}{U}\right) {}_0^C D_t^\alpha U + c_2 \left(1 - \frac{V^*}{V}\right) {}_0^C D_t^\alpha V \end{aligned} \tag{15}$$

From steady-state of equilibrium point (9), we have

$$\begin{aligned} \Lambda &= \delta S^* + \omega_1 S^* U^* + \omega_2 S^* V^* \\ \omega_1 S^* U^* + \omega_2 S^* V^* &= (\delta + \sigma) A^* \\ f\sigma A^* + \eta V^* &= (\epsilon + \delta_v + \phi) V^* \\ (1-f)\sigma A^* &= (\eta + \xi + \delta_v) V^* \end{aligned} \tag{16}$$

Let

$$s = \frac{S}{S^*}, a = \frac{A}{A^*}, u = \frac{U}{U^*}, v = \frac{V}{V^*}.$$

From (15) and (16), we have

$$\begin{aligned} {}_0^C D_t^\alpha(V) \leq & \frac{(S-S^*)}{S}\left[-\delta(S-S^*) - \omega_1(US - U^*S^*) - \omega_2(VS - V^*S^*)\right] \\ & \left(1 - \frac{A^*}{A}\right)\left[\omega_1 SU + \omega_2 SV - (\omega_1 S^* U^* + \beta_2 S^* V^*)\frac{A}{A^*}\right] \\ & + c_1\left(1 - \frac{U}{U^*}\right)\left[f\sigma A + \eta V - \frac{(f\sigma A^* + \eta V^*)U}{U^*}\right] + c_2\left(1 - \frac{V}{V^*}\right)\left[(1-f)\sigma A - \frac{((1-f)\sigma A^*)V}{V^*}\right] \\ = & -\frac{\delta}{S}(S-S^*)^2 + \omega_1 S^* U^*\left[(1-\frac{1}{s})(1-su) + (1-\frac{1}{a})su - (1-\frac{1}{a})a\right] \\ & + \omega_1 S^* V^*\left[(1-\frac{1}{s})(1-sv) + (1-\frac{1}{a})sv - (1-\frac{1}{a})a\right] \\ & + c_1 f\sigma A^*[(a-u)(1-\frac{1}{u})] + c_2\eta V^*[(v-u)(1-\frac{1}{u})] + c_2(1-f)\sigma A^*[(a-v)(1-\frac{1}{v})] \\ \leq & -\frac{\delta}{S}(S-S^*)^2 + \omega_1 U^* S^*\left(3 - \frac{1}{s} - \frac{su}{a} - \frac{a}{u}\right) \\ & + \omega_2 S^* V^*\left(4 - u - \frac{1}{s} - \frac{sv}{a} - \frac{v}{u} - \frac{a}{v} - \frac{1}{v}\right) \end{aligned} \tag{17}$$

since $c_1 = \min\left\{\frac{\omega_1 S^* U^*}{f\sigma A^*}, \frac{\omega_2 S^* V^*}{\eta V^*}\right\}$, $c_2 = \frac{\omega_2 S^* V^*}{(1-f)\sigma A^*}$

Using the inequality $A.M. \geq G.M.$, we have: $3 - \frac{1}{s} - \frac{su}{a} - \frac{a}{u} \leq 0$; $6 - u - \frac{1}{s} - \frac{sv}{a} - \frac{v}{u} - \frac{a}{v} - \frac{1}{v} \leq 0$. From relation (17) it is clear that ${}_0^C D_t^\alpha(V) \leq 0$ and thus ${}_0^C D_t^\alpha(V)$ is negative definite with respect to E_1^*. Thus E_1^* is globally asymptotically stable by Lemma 6. □

Table 2: Parametric values of system (1) corresponding to the situation in Italy

Parameters	Values
Λ	1183^{α}
δ	0.0006^{α}
δ_u	0.0007^{α}
δ_v	0.0007^{α}
δ_h	0.00067^{α}
ω_1	$(0.1 \times 10^{-}5)^{\alpha}$
ω_2	$(0.1 \times 10^{-}5)^{\alpha}$
ϕ	0.05^{α}
σ	0.07^{α}
f	0.5
ϵ	0.001^{α}
ξ	0.05^{α}
γ	0.06^{α}
η	0.1^{α}
α	0.85

Table 3: Sensitivity indices of different parameters of system (1) corresponding to Table 2

Parameters	sensitivity index
Λ	+1
σ	+1
ω_1	+0.8290
ω_2	+0.1710
ϕ	-0.8018
ξ	-0.1664
f	-0.0033
ϵ	-0.1064
η	-0.0022

4.6 Sensitivity analysis

The basic reproduction number (R_0) depends on several parameters and value of $R_0 = 2.6784$ according to Table 2. To examine the sensitivity of R_0 to any parameter (say, θ), normalized forward sensitivity index with respect to each parameter has been computed as follows [3, 32]:

$$\Omega_{\theta}^{R_0} = \frac{\partial R_0}{\partial \theta} \frac{\theta}{R_0}$$

The sensitivity index may depend on some system parameters but also can be constant or independent of some parameters. These values are very much important to estimate the sensitivity of parameters which should be done cautiously, since a small perturbation in a parameter causes relevant quantitative changes. Merely in the estimation of a parameter with lower value of sensitivity index does not demand to deal cautiously, because a small perturbation in that parameter causes small changes. Since we cannot control $\delta_u, \delta, \delta_v$, we have concentrated on the parameters $\omega_1, \omega_2, \sigma, \phi, \xi, \epsilon, \eta, f$. Now, $\Omega_{\theta}^{R_0} = +1$ indicates that increasing (decreasing) θ by a given proportion increases (decreases) R_0 by same proportion and $\Omega_{\theta}^{R_0} = -1$ means that increasing (decreasing) θ by a given proportion decreases (increases) R_0 by same proportion. The values of sensitivity indexes for the parameters $\omega_1, \omega_2, \sigma, \phi, \xi, \epsilon, \eta, f$ corresponding to Table 2 is given in Table 3.

Table 4: Day wise Reported and Death cases between 1st March 2020 and 20th April 2020

Day	Reported Cases	Deaths
1/3/20	1528	12
6/3/20	3919	41
11/3/20	8518	169
16/3/20	20607	369
21/3/20	37859	629
26/3/20	57469	714
31/03/20	75744	840
5/04/20	91137	683
10/04/20	98129	572
15/04/20	105252	580
20/04/20	10804	455

5 Numerical simulations

For numerical simulations, we have used MatLab interface along with Predictor-corrector PECE method for fractional differential equations introduced by Roberto Garrappa [12]. We have considered Table 2 for the scenario of Italy due to COVID-19 in the period 1st March 2020 to 20th April 2020. We have performed numerical simulations to compare the results of our model with the real data from various reports published by WHO [33] and worldometer [4]. The total population of Italy is around 6.04×10^7 [35]. We have taken $t = 1$ day as time unit and $t = 50$ as the final time. The recruitment rate has been calculated as $\frac{7.2\times N(=6.04\times 10^7)}{1000\times 365} = 1183$ per day. In Italy, birth rate per 1000 inhabitants is 7.2 [30]. Table 4 recommends the reported symptomatically infected cases and deaths from 1st March 2020 to 20th April 2020, we have taken 5 days gap between two successive reports [4]. We have considered system (I) rather than system (1) for numerical simulations as the parameters of system (I) contains α which gives better results. From Figure 2, it has been observed that in initial stage the time series of reported class is fitted with actual data and for death rate the time series is relevant for first 15 days (Figure 3). Next, we have simulated our model with estimated parametric values (Table 2) from 3rd September (present scenario in Italy) to 31st October (60 days) (Figure 5-6). Figure 4 indicates that for $\alpha = 0.85$ the time series of reported class is closer to the realistic scenario. The death cases have been computed at time t due to COVID-19 as $\delta_u U(t) + \delta_v V(t) + \delta_h H(t)$. The number of death cases according to the time series on 6/03/20, 11/03/20, 16/03/20, 21/03/20 are 29, 168, 327, 435 which are quite close to the real data. Similarly, on 21/03/20, 26/03/20, 31/03/20, 5/04/20 number of reported cases according to time series are 36370, 59150, 78430, 93500. Figure 7 depicts that for slight changes in η the reported cases increases, but the unreported cases decreases rapidly and number of hospitalized class is also increased. The variation of R_0 with order of derivative is depicted in Figure 8.

6 Conclusion

Fractional calculus plays an important role in real dynamical processes, including the cases of epidemic spreading. Here we have studied on the evolution of a modified SAIR epidemic model, incorporating memory effects. We have taken the hospitalized class and subdivided symptomatically infected class into two subclasses, one is reported class and other is unreported class. Our model is giving a well approximation of the reality of Italy outbreak and predicting daily number of confirmed cases due to COVID-19. The number of hospitalized persons is relevant to provide an approximation of the Intensive Care Units (ICU) required. The model we have studied for Italy can also be employed to study the reality of the other countries. The value of order of derivative α may vary region to region. It is observed that the dynamics of the system (1) depends on

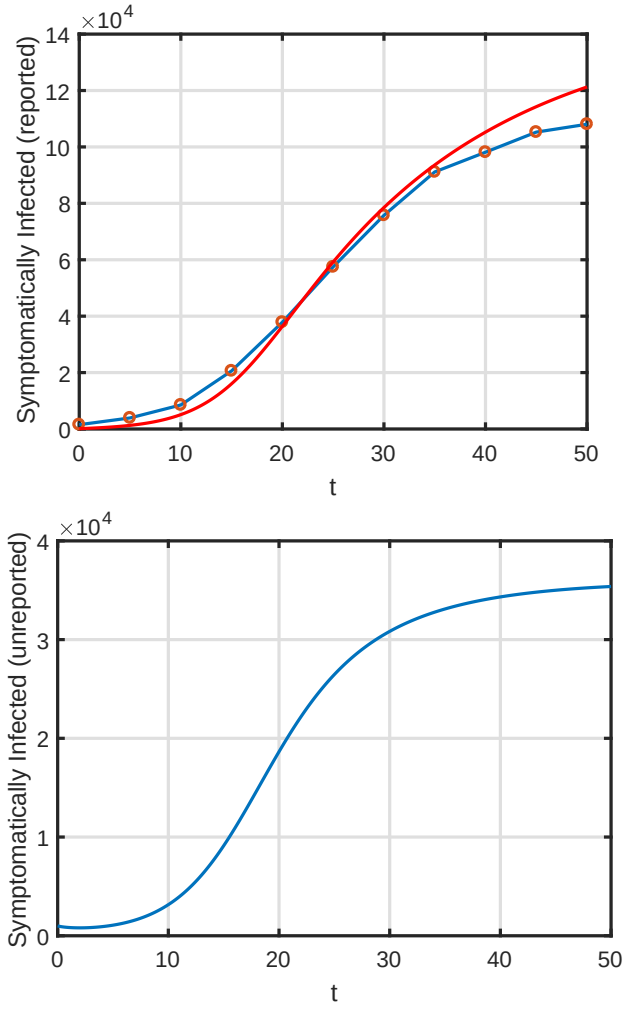

Figure 2: Time series of symptomatically infected class (both reported and unreported) system (1) for Table 2 and real data (from 1st march 2020 to 20th April 2020)

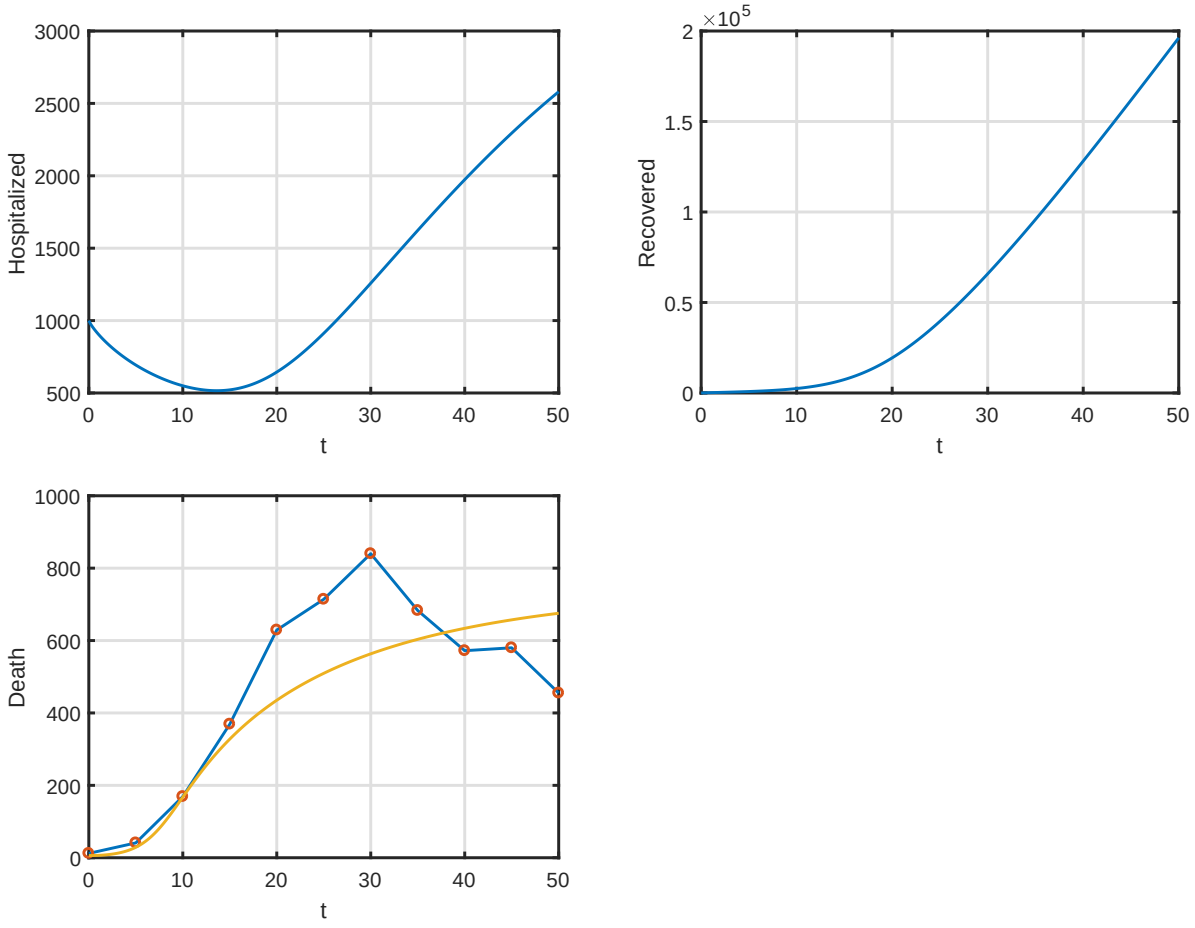

Figure 3: Time series of hospitalized(H), recovered (R) , death classes and real death cases from 1st march 2020 to 20th April 2020 with correspondence to Table 2

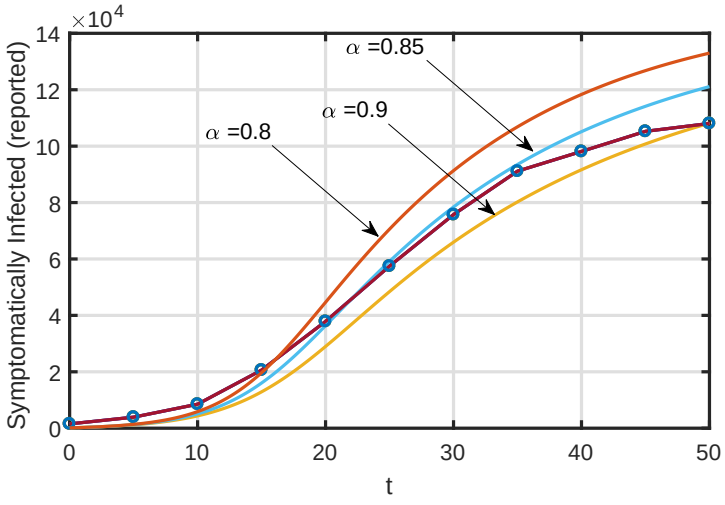

Figure 4: Time series of symptomatically infected variable (reported) system (1) for Table 2 different $\alpha = 0.8, 0.85, 0.9$

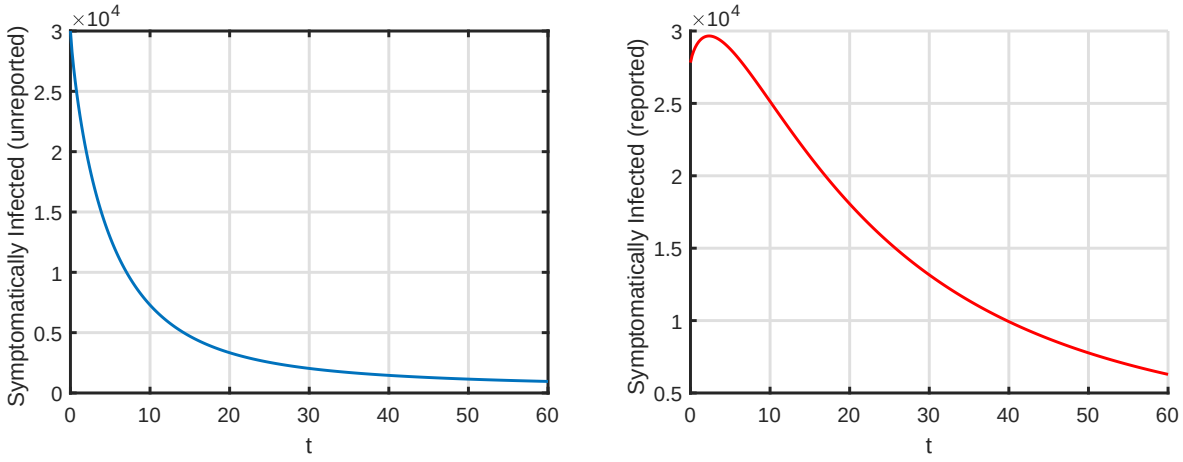

Figure 5: Time series of symptomatically infected (reported and) with correspondence to Table 2 from 3rd September 2020

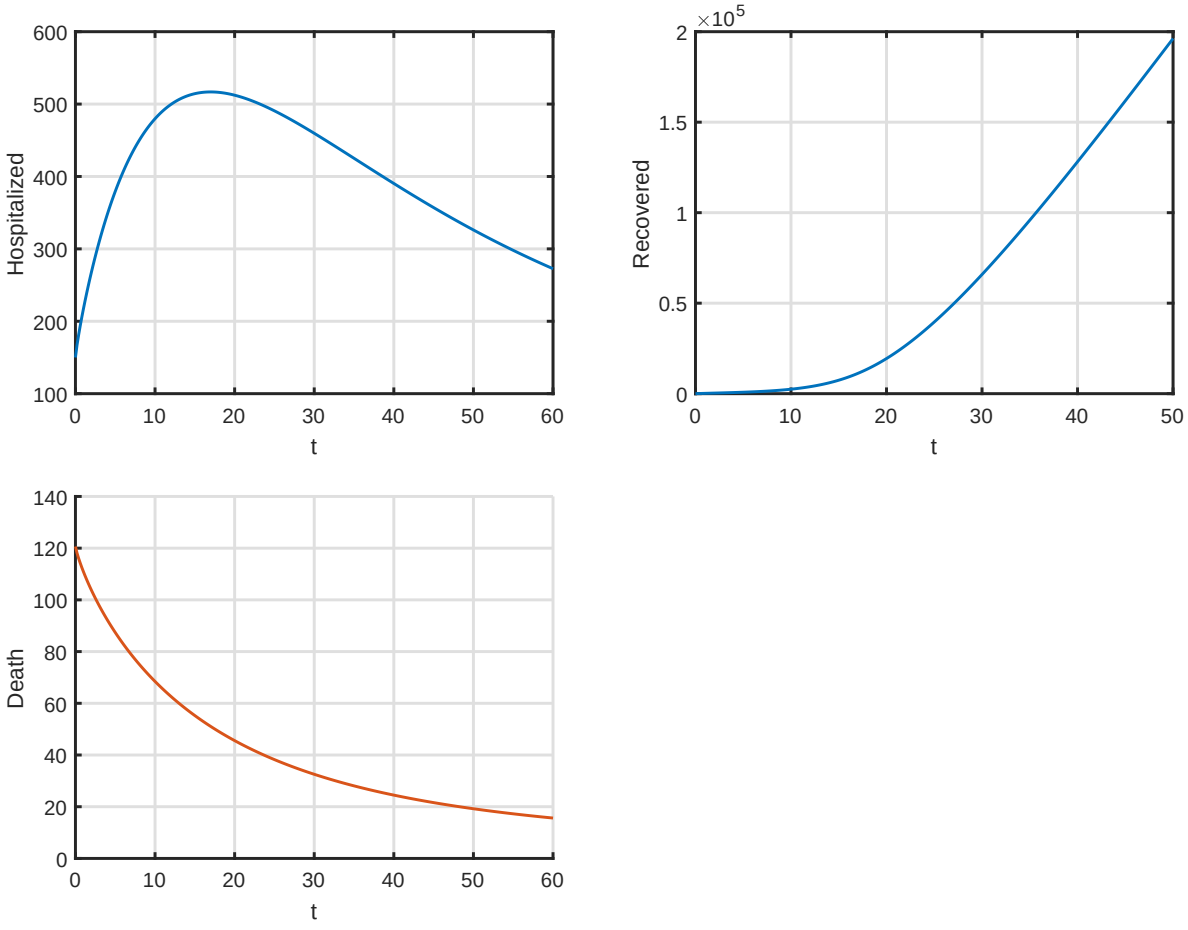

Figure 6: Time series of hospitalized, recovered and death cases with correspondence to Table 2 from 3rd September 2020

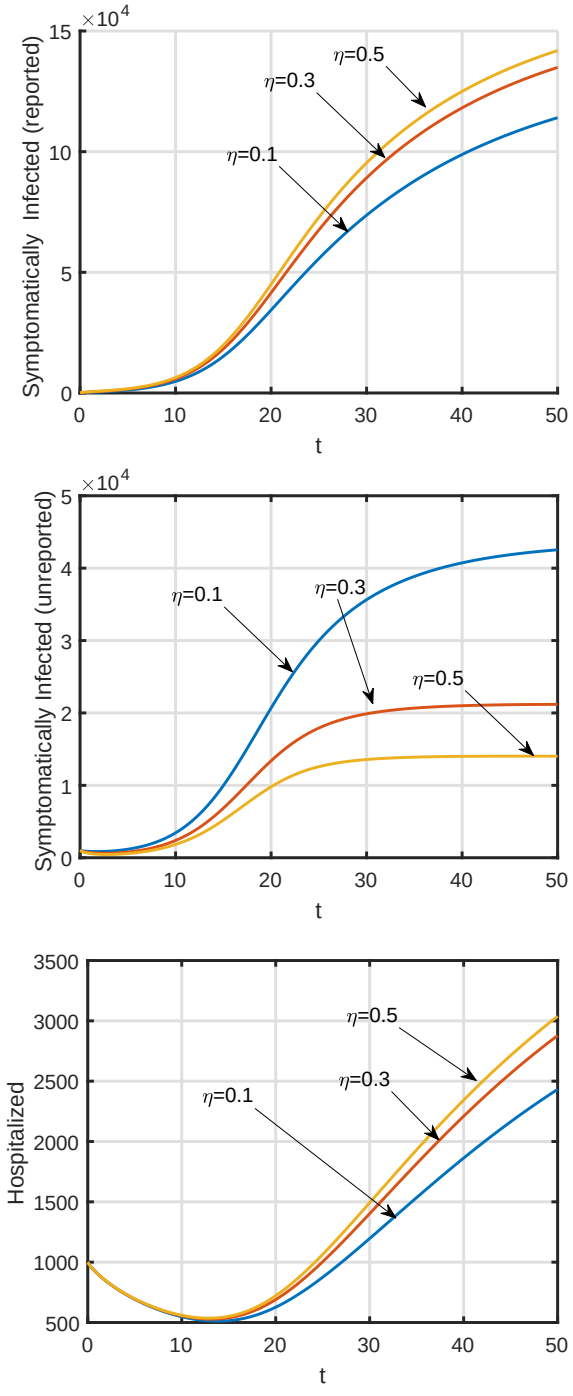

Figure 7: Time series of hospitalized, infected cases (reported and unreported) with correspondence to Table 2 from 1st March 2020 to 20th April 2020 for different values of $\eta = 0.1, 0.3, 0.5$

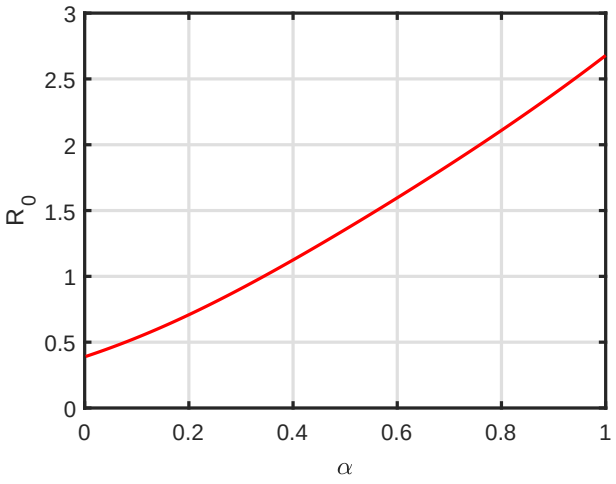

Figure 8: Variation of R_0 with α according to Table 2

the robustness of memory effects, which is controlled by the order of fractional derivative α. The results will be different if we change the order of derivative for same set of parametric values (Figure 4). This shows that order of derivative plays an important role in simulations of system (1). From Table 2, it is clear that the values of parameter depends on order of derivative α. We will get different sets of parametric values if we change the value of α. In our context, we have fixed the value of differentiation at 0.85 which is more suitable for the real scenario. From Figures 5-6, we can conclude that the number of reported case will be decreasing at the end of October 2020. The morbidity rate will also be under control in the last week of October 2020. Rapid PCR (Polymerase Chain Reaction) will increase the reported cases which may convey the situation under control.

No model is perfect for COVID-19 scenario. There are many factors required to be considered for modeling COVID-19. The situation in different countries in this pandemic is unlike. From analytical study of our model it seems to us that if we are able to decrease disease transmission rates (ω_1, ω_2), then we will achieve disease-free state quickly. These are also consistent with the conditions of stability (see Theorem 9). The social distancing, mobility of asymptomatic individual, herd immunity, vaccination, improvement of health conditions are the key factors and essential in complex and accurate modeling. Though our model has these limitations, but even then this model can ray on the research on Mathematical study of COVID-19 spreading.

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Abbreviations:
COVID-19, Coronavirus disease 2019; SAIR, Susceptible-Asymptomatic-Infectious-Recovered; PCR, Polymerase Chain Reaction; FDE, Fractional Differential Equation; matlab, matrix laboratory; WHO, World Health Organization.

Availability of data and materials: The data used to support the findings of this work are available at

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Authors' contributions: The main idea of this paper has been proposed by Guruprasad Samanta. All authors have jointly carried out the research and worked together on the manuscript. All authors have read and approved the final manuscript.

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