Research Article

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Guaranteed Lower and Upper Bounds for Eigenvalues of Second Order Elliptic Operators in any Dimension

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Abstract: A new method is proposed to provide guaranteed lower bounds for eigenvalues of general second order elliptic operators in any dimension. This method employs a novel generalized Crouzeix–Raviart element which is proved to yield asymptotic lower bounds for eigenvalues of general second order elliptic operators, and a simple post-processing method. As a byproduct, a simple and cheap method is also proposed to obtain guaranteed upper bounds for eigenvalues, which is based on generalized Crouzeix–Raviart element approximate eigenfunctions, an averaging interpolation from the generalized Crouzeix–Raviart element space to the conforming linear element space, and an usual Rayleigh–Ritz procedure. The ingredients for the analysis consist of a crucial projection property of the canonical interpolation operator of the generalized Crouzeix–Raviart element, explicitly computable constants for two interpolation operators. Numerical experiments demonstrate that the guaranteed lower bounds for eigenvalues in this paper are superior to those obtained by the Crouzeix–Raviart element.

Keywords: Generalized Crouzeix–Raviart Element, Eigenvalue Problem, Lower Bound, Upper Bound

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1 Introduction

Finding eigenvalues of partial differential operators is important in the mathematical science. Since exact eigenvalues are almost impossible, many papers and books investigate their bounds from above and below. It is well known that upper bounds for the eigenvalues can always be found by the Rayleigh–Ritz method and conforming subspaces. While the problem of obtaining lower bounds is generally considering more difficult. The study of lower bounds for eigenvalues can date back to several remarkable works, including the intermediate method, the Kato and Lehmann–Goerisch methods, and the homotopy method, see [25] for a review.

The finite element method can effectively approximate eigenvalues with a comprehensive analysis on error estimation, see [3, 30]. Conforming finite element methods can provide upper bounds for eigenvalues. While, some nonconforming finite element methods can give lower bounds of eigenvalues directly when the meshsize is sufficiently small, see [15, 33]. In [15], Hu, Huang and Lin gave a comprehensive survey of the lower bound property of eigenvalues by nonconforming finite element methods and proposed a systematic method that can produce lower bounds for eigenvalues by using nonconforming finite element methods. The theories [15] were limited to asymptotic analysis and it is not easy to check when the meshsize is small enough in practice. Following the theory of [21, 30], Liu and Oishi [26] proposed guaranteed lower bounds for eigenvalues of the Laplace operator in the two dimensions. The main tool therein is an explicit a priori error estimation for the conforming linear element projection. However, for singular eigenfunctions, it needs to compute the explicit a priori error

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estimation by solving an auxiliary problem. Moreover, it is difficult to generalize the idea therein to general second order elliptic operators. Similar guaranteed lower bounds for eigenvalues of both Laplace and biharmonic operators in two dimensions were given by Carstensen et al., see [5, 6], through using the nonconforming Crouzeix–Raviart and Morley elements, respectively. Liu [24] proposed an idea to give guaranteed lower bounds for self-adjoint differential operators and dropped the mesh size condition used in [5, 6]. The generalization to any dimensions can be found in [16]. Recently, some direct lower bounds are obtained by hybrid high-order methods, stabilized nonconforming finite elements and weak Galerkin methods, see [4, 7–11].

The aim of this paper is to propose new methods which are able to obtain both guaranteed lower and upper bounds for eigenvalues of general second order elliptic operators in any dimension. The method for guaranteed lower bounds is derived from asymptotic lower bounds for eigenvalues produced by a generalized Crouzeix–Raviart (GCR hereafter) element proposed herein, and a simple post-processing method. Unlike most methods in the literature, this new method only needs to solve one discrete eigenvalue problem but not involves any base or intermediate eigenvalue problems, and does not need any a priori information concerning exact eigenvalues either. The method can be regarded as an extension to the general second order elliptic operators in any dimension of those due to [26] and [5, 6]. The new method has higher accuracy than those from [26] and [6, 16], see comparisons in Section 7.1. Moreover, this paper drops the mesh-size conditions in [16, Theorem 3.1] for variable coefficients. The approach for guaranteed upper bounds is based on asymptotic upper bounds which are obtained by a postprocessing method firstly proposed in [18, 29], see also [32], and a Rayleigh–Ritz procedure. Compared with [27], this new method does not need to solve an eigenvalue or source problem by a conforming finite element method. The ingredients for the analysis consist of a crucial projection property of the canonical interpolation operator of the GCR element, explicitly computable constants for two interpolation operators. Numerical experiments demonstrate that the guaranteed lower bounds for eigenvalues in this paper are superior to those obtained by the Crouzeix–Raviart element [6].

The remaining paper is organized as follows. Section 2 proposes the GCR element. Section 3 proves asymptotic lower bounds for eigenvalues. Section 4 presents the guaranteed lower bounds for eigenvalues of general elliptic operators. Section 5 provides asymptotic upper bounds for eigenvalues. Section 6 designs guaranteed upper bounds for eigenvalues. Section 7 will give some numerical tests.

2 Preliminaries

In this section, we present second order elliptic boundary value and eigenvalue problems and propose a generalized Crouzeix–Raviart element for them. Throughout this paper, let $\Omega \subset \mathbb{R}^n$ denote a bounded polyhedral Lipschitz domain.

2.1 Second Order Elliptic Boundary Value and Eigenvalue Problems

Given $f \in L^2(\Omega)$, second order elliptic boundary value problems find $u \in H_0^1(\Omega)$ such that

$$(A\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for any } v \in H_0^1(\Omega). \tag{2.1}$$

Here, A is a matrix-valued function on Ω and satisfies

$$(q, q)_{L^2(\Omega)} \lesssim (Aq, q)_{L^2(\Omega)} \quad \text{for any } q \in (L^2(\Omega))^n,$$

where $p \lesssim q$ abbreviates $p \leq Cq$ for some multiplicative mesh-size independent constant $C > 0$ which may be different at different places. Define

$$\|\nabla v\|_A := (A\nabla v, \nabla v)_{L^2(\Omega)}^{\frac{1}{2}}.$$

Hence $\|\nabla \cdot\|_A$ is a norm of $H_0^1(\Omega)$. The matrix $A(x)$ is supposed to be symmetric for all $x \in \Omega$ and each component of A is piecewise Lipschitz continuous on each subdomain of domain Ω.

Second order elliptic eigenvalue problems find $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$(A\nabla u, \nabla v)_{L^2(\Omega)} = \lambda(u, v)_{L^2(\Omega)} \quad \text{for any } v \in H_0^1(\Omega) \text{ and } \|u\| := \|u\|_{L^2(\Omega)} = 1. \tag{2.2}$$

Problem (2.2) has a sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \nearrow +\infty,$$

and corresponding eigenfunctions

$$u_1, u_2, u_3, \ldots,$$

which can be chosen to satisfy

$$(u_i, u_j)_{L^2(\Omega)} = \delta_{ij}, i, j = 1, 2, \ldots .$$

Define

$$E_\ell = \operatorname{span}\{u_1, u_2, \ldots, u_\ell\}. \tag{2.3}$$

Eigenvalues and eigenfunctions satisfy the following well-known Rayleigh–Ritz principle:

$$\lambda_k = \min_{\dim V_k = k, V_k \subset H_0^1(\Omega)} \max_{v \in V_k} \frac{(A\nabla v, \nabla v)_{L^2(\Omega)}}{(v, v)_{L^2(\Omega)}} = \max_{u \in E_k} \frac{(A\nabla u, \nabla u)_{L^2(\Omega)}}{(u, u)_{L^2(\Omega)}}. \tag{2.4}$$

2.2 The Generalized Crouzeix–Raviart Element

Suppose that $\overline{\Omega}$ is covered exactly by shape-regular partitions $\mathcal{T}$ consisting of n-simplices in n dimensions. Let $\mathcal{E}$ denote the set of all $(n-1)$-dimensional subsimplices, and $\mathcal{E}(\Omega)$ denote the set of all the $(n-1)$-dimensional interior subsimplices, and $\mathcal{E}(\partial\Omega)$ denote the set of all the $(n-1)$-dimensional boundary subsimplices. Given $K \in \mathcal{T}$, h_K denotes the diameter of K and $h := \max_{K \in \mathcal{T}} h_K$. Let $|K|$ denote the measure of element K and $|E|$ the measure of $(n-1)$-dimensional subsimplex E. Given $E \in \mathcal{E}$, let ν_E be its unit normal vector and let $[\,\cdot\,]$ be jumps of piecewise functions over E, namely

$$[v] := v|_{K^+} - v|_{K^-}$$

for piecewise functions v and any two elements K^+ and K^- which share the common $(n-1)$-dimensional subsimplex E. Note that $[\,\cdot\,]$ becomes traces of functions on E for boundary subsimplex E.

Given $K \in \mathcal{T}$ and an integer $m \ge 0$, let $P_m(K)$ denote the space of polynomials of degree $\le m$ over K. The simplest nonconforming finite element for problem (2.1) is the Crouzeix–Raviart (CR hereafter) element proposed in [14]. The corresponding element space V_{CR} over $\mathcal{T}$ is defined by

$$V_{\mathrm{CR}} := \Big\{ v \in L^2(\Omega) : v|_K \in P_1(K) \text{ for each } K \in \mathcal{T}, \int_E [v]\, ds = 0 \text{ for all } E \in \mathcal{E}(\Omega) \\ \text{and } \int_E v\, dE = 0 \text{ for all } E \in \mathcal{E}(\partial\Omega) \Big\}.$$

Since the CR element cannot be proved to produce lower bounds for eigenvalues of the Laplace operator on general meshes when eigenfunctions are smooth, see [1, 17]. Hu, Huang and Lin [15] proposed the enriched Crouzeix–Raviart (ECR hereafter) element which was proved to produce lower bounds for eigenvalues of the Laplace operator in the asymptotic sense. The asymptotic expansions of eigenvalues for the ECR element were established in [19]. The ECR element space V_{ECR} is defined by

$$V_{\mathrm{ECR}} := \Big\{ v \in L^2(\Omega) : v|_K \in P_1(K) + \operatorname{span}\Big\{ \sum_{i=1}^n x_i^2 \Big\} \text{ for each } K \in \mathcal{T}, \int_E [v]\, ds = 0 \text{ for all } E \in \mathcal{E}(\Omega) \\ \text{and } \int_E v\, ds = 0 \text{ for all } E \in \mathcal{E}(\partial\Omega) \Big\}.$$

However, the ECR element cannot produce lower bounds for eigenvalues of general second order elliptic operators, which motivates us to generalize the ECR element to more general cases. To this end, let $\bar{A}$ be a piecewise positive-definite constant matrix with respect to $\mathcal{T}$, which is an approximation of A. For example, we can choose $\bar{A}|_K$ to be equal to the value of A at the centroid of K or the integral mean on K. Suppose

$$\bar{A}|_K = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{2.5}$$

Let $\bar{B}$ denote the inverse of $\bar{A}$ as follows:

$$\bar{B}|_K = \bar{A}^{-1}|_K = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}. \tag{2.6}$$

The centroid of K is denoted by mid(K). The coordinate of mid(K) is denoted by $(M_1, M_2, \dots, M_n)$. The vertices of K are denoted by $a_p = (x_{1p}, x_{2p}, \dots, x_{np})$, $1 \le p \le n+1$. Define

$$H = \sum_{i=1}^{n} b_{ii} \sum_{p<q} (x_{ip} - x_{iq})^2 + 2 \sum_{i<j} b_{ij} \sum_{p<q} (x_{ip} - x_{iq})(x_{jp} - x_{jq})$$

and

$$\phi_K = \frac{n+2}{2} - \frac{n(n+1)^2(n+2)}{2H} (x - \mathrm{mid}(K))^T \, \bar{B}|_K \, (x - \mathrm{mid}(K)) \,. \tag{2.7}$$

For two dimensions, the constant H and function ϕ_K are presented as follows, respectively:

$$H = b_{11} \sum_{p<q} (x_{1p} - x_{1q})^2 + b_{22} \sum_{p<q} (x_{2p} - x_{2q})^2 + 2b_{12} \sum_{p<q} (x_{1p} - x_{1q})(x_{2p} - x_{2q}) \tag{2.8}$$

and

$$\phi_K = 2 - \frac{36}{H}(b_{11}(x_1 - M_1)^2 + b_{22}(x_2 - M_2)^2 + 2b_{12}(x_1 - M_1)(x_2 - M_2)). \tag{2.9}$$

Lemma 2.1. *Given $K \in \mathcal{T}$, there holds that*

$$\frac{1}{|K|} \int_K \phi_K dx = 1.$$

Moreover, for any $(n-1)$-dimensional subsimplex $E \subset \partial K$, there holds that

$$\int_E \phi_K \, ds = 0.$$

Proof. Let $\theta_j = \theta_j(x)$, $1 \le j \le n+1$, denote the barycentric coordinates of K associated to vertex a_j. For any integers $\alpha_j \ge 0$, $1 \le j \le n+1$, one has

$$\int_K \theta_1^{\alpha_1} \theta_2^{\alpha_2} \cdots \theta_{n+1}^{\alpha_{n+1}} dx = \frac{\alpha_1! \alpha_2! \cdots \alpha_{n+1}! n!}{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} + n)!} |K|.$$

This leads to

$$\begin{aligned} \int_K (x_i - M_i)(x_j - M_j) \, dx &= \int_K \sum_{p=1}^{n+1} \Big(\theta_p - \frac{1}{n+1}\Big) x_{ip} \sum_{q=1}^{n+1} \Big(\theta_q - \frac{1}{n+1}\Big) x_{jq} \, dx \\ &= \frac{|K|}{(n+1)^2(n+2)} \Big(\sum_{p=1}^{n+1} n x_{ip} x_{jp} - \sum_{p \ne q} x_{ip} x_{jq} \Big) \\ &= \frac{|K|}{(n+1)^2(n+2)} \sum_{p<q} (x_{ip} - x_{iq})(x_{jp} - x_{jq}). \end{aligned}$$

By the definition of ϕ_K in (2.7), this yields

$$\begin{aligned}\frac{1}{|K|}\int_K \phi_K\,dx &= \frac{n+2}{2} - \frac{1}{|K|}\frac{n(n+1)^2(n+2)}{2H}\frac{|K|}{(n+1)^2(n+2)}\sum_{i,j=1}^{n}\sum_{p<q} b_{ij}(x_{ip}-x_{iq})(x_{jp}-x_{jq})\\ &= \frac{n+2}{2} - \frac{n}{2H}H\\ &= 1.\end{aligned}$$

Given an $(n-1)$-dimensional subsimplex $E \subset \partial K$ such that $\theta_1|_E \equiv 0$. A similar equality holds

$$\int_E \theta_2^{\alpha_2}\cdots\theta_{n+1}^{\alpha_{n+1}}\,ds = \frac{\alpha_2!\cdots\alpha_{n+1}!(n-1)!}{(\alpha_2+\cdots+\alpha_{n+1}+n-1)!}|E|.$$

A direct calculation yields

$$\begin{aligned}\int_E (x_i-M_i)(x_j-M_j)\,ds &= \int_E \Big(-\frac{x_{i1}}{n+1} + \sum_{p=2}^{n+1}\Big(\theta_p - \frac{1}{n+1}\Big)x_{ip}\Big)\Big(-\frac{x_{j1}}{n+1} + \sum_{q=2}^{n+1}\Big(\theta_q - \frac{1}{n+1}\Big)x_{jq}\Big)\,ds\\ &= \frac{|E|}{n(n+1)^2}\Big(\sum_{p=1}^{n+1} n x_{ip}x_{jp} - \sum_{p\neq q} x_{ip}x_{jq}\Big)\\ &= \frac{|E|}{(n+1)^2(n+2)}\sum_{p<q}(x_{ip}-x_{iq})(x_{jp}-x_{jq}).\end{aligned}$$

This shows that

$$\int_E \phi_K\,ds = \frac{n+2}{2}|E| - \frac{n(n+1)^2(n+2)}{2H}\frac{|E|}{n(n+1)^2}H = 0,$$

which completes the proof. □

Lemma 2.1 allows for the definition of the following bubble function space:

$$V_{\mathrm{B}} := \{v \in L^2(\Omega) : v|_K \in \operatorname{span}\{\phi_K\} \text{ for all } K \in \mathcal{T}\}.$$

The GCR element space V_{GCR} is then defined by

$$V_{\mathrm{GCR}} := V_{\mathrm{CR}} + V_{\mathrm{B}}. \tag{2.10}$$

If $A(x) \equiv 1$, then $b_{ij} = \delta_{ij}$, $H = \sum_{p<q}|a_p - a_q|^2$ and

$$\phi_K = \frac{n+2}{2} - \frac{n(n+1)^2(n+2)}{2H}\sum_{i=1}^{n}(x_i - M_i)^2 \in \mathrm{ECR}(K).$$

Hence, in this case, $V_{\mathrm{GCR}} = V_{\mathrm{ECR}}$. The GCR element has the following important property.

Lemma 2.2. *Given $v \in V_{\mathrm{GCR}}$, $\bar{A}\nabla v\cdot\nu_E$ is a constant on E for all $E \in \mathcal{E}$.*

Proof. Given $E \in \mathcal{E}$, $x\cdot\nu_E$ is a constant on E. The fact that $\bar{B}$ is the inverse of $\bar{A}$, (2.7) and (2.10) imply that $\bar{A}\nabla v\cdot\nu_E$ is a constant on E. □

2.3 The GCR Element for Second Order Elliptic Boundary Value Problems

The generalized Crouzeix–Raviart element method of problem (2.1) finds $u_{\mathrm{GCR}} \in V_{\mathrm{GCR}}$ such that

$$(A\nabla_{\mathrm{NC}}u_{\mathrm{GCR}}, \nabla_{\mathrm{NC}}v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for any } v \in V_{\mathrm{GCR}}. \tag{2.11}$$

Since $\int_E [v]\,ds = 0$ for all $E \in \mathcal{E}(\Omega)$ and $\int_E v\,ds = 0$ for all $E \in \mathcal{E}(\partial\Omega)$. From the theory of [20], there holds that

$$\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\| \lesssim \|\nabla u - \Pi_0\nabla u\| + \mathrm{osc}(f),$$

where Π_0 denotes the piecewise constant projection, and the oscillation of data reads

$$\operatorname{osc}(f) = \Big(\sum_{K\in\mathcal{T}} h_K^2\Big(\inf_{\bar f\in P_r(K)} \|f-\bar f\|^2_{L^2(K)}\Big)\Big)^{\frac{1}{2}}$$

with arbitrary $r \geq 0$. The optimal convergence of the GCR element follows immediately.

Remark 2.3. Thanks to the definition of (2.10), u_{GCR} can be written as $u_{\mathrm{GCR}} = u_{\mathrm{CR}} + u_{\mathrm{B}}$, where $u_{\mathrm{CR}} \in V_{\mathrm{CR}}$ and $u_{\mathrm{B}} \in V_{\mathrm{B}}$. When A is a piecewise constant matrix-valued function, an integration by parts yields the following orthogonality:

$$(A\nabla u_{\mathrm{CR}}, \nabla\phi_K)_{L^2(K)} = (-\operatorname{div}(A\nabla u_{\mathrm{CR}}), \phi_K)_{L^2(K)} + \sum_{E\subset\partial K}\int_E A\nabla u_{\mathrm{CR}}\cdot\nu_E\phi_K\,ds = 0. \tag{2.12}$$

This leads to

$$(A\nabla u_{\mathrm{B}}, \nabla\phi_K)_{L^2(K)} = (f, \phi_K)_{L^2(K)} \quad \text{for any } K\in\mathcal{T} \tag{2.13}$$

and

$$(A\nabla_{\mathrm{NC}} u_{\mathrm{CR}}, \nabla_{\mathrm{NC}} v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for any } v\in V_{\mathrm{CR}}. \tag{2.14}$$

Consequently, u_{CR} is the discrete solution of problem (2.1) by the CR element. Hence we can solve the GCR element equation (2.11) by solving (2.13) on each K and (2.14) for the CR element, respectively. For general cases, the orthogonality (2.12) does not hold. However, u_{B} can be eliminated a prior by a static condensation procedure.

2.4 The GCR Element for Second Order Elliptic Eigenvalue Problems

We consider the discrete eigenvalue problem: Find $(\lambda_{\mathrm{GCR}}, u_{\mathrm{GCR}}) \in \mathbb{R}\times V_{\mathrm{GCR}}$ such that

$$(A\nabla_{\mathrm{NC}} u_{\mathrm{GCR}}, \nabla_{\mathrm{NC}} v)_{L^2(\Omega)} = \lambda_{\mathrm{GCR}}(u_{\mathrm{GCR}}, v)_{L^2(\Omega)} \quad \text{for any } v\in V_{\mathrm{GCR}} \text{ and } \|u_{\mathrm{GCR}}\| = 1. \tag{2.15}$$

Let $Z = \dim V_{\mathrm{GCR}}$. The discrete problem (2.15) admits a sequence of discrete eigenvalues

$$0 < \lambda_{1,\mathrm{GCR}} \leq \lambda_{2,\mathrm{GCR}} \leq \cdots \leq \lambda_{Z,\mathrm{GCR}}$$

and the corresponding eigenfunctions

$$u_{1,\mathrm{GCR}}, u_{2,\mathrm{GCR}}, \dots, u_{Z,\mathrm{GCR}}.$$

Define the discrete counterpart of E_ℓ by

$$E_{\ell,\mathrm{GCR}} = \operatorname{span}\{u_{1,\mathrm{GCR}}, u_{2,\mathrm{GCR}}, \dots, u_{\ell,\mathrm{GCR}}\}. \tag{2.16}$$

Then we have the following discrete Rayleigh–Ritz principle:

$$\lambda_{k,\mathrm{GCR}} = \min_{\dim V_k=k, V_k\subset V_{\mathrm{GCR}}} \max_{v\in V_k} \frac{(A\nabla_{\mathrm{NC}} v, \nabla_{\mathrm{NC}} v)_{L^2(\Omega)}}{(v,v)_{L^2(\Omega)}} = \max_{u\in E_{k,\mathrm{GCR}}} \frac{(A\nabla_{\mathrm{NC}} u, \nabla_{\mathrm{NC}} u)_{L^2(\Omega)}}{(u,u)_{L^2(\Omega)}}. \tag{2.17}$$

According to the theory of nonconforming eigenvalue approximations [2, 15], the following a priori estimate holds true.

Lemma 2.4. *Let u be eigenfunctions of problem* (2.2) *and let u_{GCR} be discrete eigenfunctions of problem* (2.4). *Suppose $u \in H_0^1(\Omega)\cap H^{1+s}(\Omega)$ with $0 < s \leq 1$. Then*

$$\|u - u_{\mathrm{GCR}}\| + h^s\|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A \lesssim h^{2s}|u|_{1+s}. \tag{2.18}$$

We introduce the interpolation operator $\Pi_{\mathrm{GCR}} : H_0^1(\Omega) \to V_{\mathrm{GCR}}$ by

$$\begin{aligned} \int_E \Pi_{\mathrm{GCR}} v\,ds &= \int_E v\,ds \quad \text{for any } E\in\mathcal{E},\\ \int_K \Pi_{\mathrm{GCR}} v\,dx &= \int_K v\,dx \quad \text{for any } K\in\mathcal{T}. \end{aligned} \tag{2.19}$$

Given $w \in V_{\mathrm{GCR}}$, an integration by parts yields

$$(\bar{A}\nabla_{\mathrm{NC}}(v - \Pi_{\mathrm{GCR}}v), \nabla_{\mathrm{NC}}w)_{L^2(\Omega)} = -(v - \Pi_{\mathrm{GCR}}v, \mathrm{div}_{\mathrm{NC}}(\bar{A}\nabla_{\mathrm{NC}}w))_{L^2(\Omega)} + \sum_{K\in\mathcal{T}}\sum_{E\subset\partial K}\int_E (v - \Pi_{\mathrm{GCR}}v)\bar{A}\nabla w \cdot \nu_E \, ds.$$

Since $\mathrm{div}_{\mathrm{NC}}(\bar{A}\nabla_{\mathrm{NC}}w)$ is a piecewise constant on Ω and Lemma 2.2 proves that $\bar{A}\nabla w \cdot \nu_E$ is a constant on the $(n-1)$-dimensional subsimplex E, for any $v \in H_0^1(\Omega)$, the following orthogonality holds true:

$$(\bar{A}\nabla_{\mathrm{NC}}(v - \Pi_{\mathrm{GCR}}v), \nabla_{\mathrm{NC}}w)_{L^2(\Omega)} = 0 \quad \text{for any } w \in V_{\mathrm{GCR}}. \tag{2.20}$$

This orthogonality is important in providing lower bounds for eigenvalues, see more details in the following two sections. Moreover, this yields

$$\|\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v\|_{\bar{A}}^2 + \|\nabla_{\mathrm{NC}}(v - \Pi_{\mathrm{GCR}}v)\|_{\bar{A}}^2 = \|\nabla v\|_{\bar{A}}^2. \tag{2.21}$$

3 Asymptotic Lower Bounds for Eigenvalues

We assume A is a piecewise constant matrix-valued function in this section. Following the theory of [15], we prove that the eigenvalues produced by the GCR element are lower bounds when the meshsize is small enough.

Let (λ, u) and $(\lambda_{\mathrm{GCR}}, u_{\mathrm{GCR}})$ be solutions of (2.2) and (2.15), respectively. First, note that $u - \Pi_{\mathrm{GCR}}u$ has vanishing mean on each $K \in \mathcal{T}$. It follows from the Poincaré inequality that

$$\|u - \Pi_{\mathrm{GCR}}u\| \lesssim h\|\nabla_{\mathrm{NC}}(u - \Pi_{\mathrm{GCR}}u)\|.$$

Suppose $u \in H^{1+s}(\Omega)$, $0 < s \leq 1$. Following from the usual interpolation theory, there holds

$$\|u - \Pi_{\mathrm{GCR}}u\| \lesssim h^{1+s}|u|_{1+s}. \tag{3.1}$$

Theorem 3.1. *Suppose that A is a piecewise constant matrix-valued function. Assume that $u \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$ with $0 < s \leq 1$ and that $h^{2s} \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A^2$. Then*

$$\lambda_{\mathrm{GCR}} \leq \lambda,$$

provided that h is small enough.

Proof. Since A is a piecewise constant matrix-valued function, $A = \bar{A}$, and $\bar{A}$ in (2.20) can be replaced by A. Due to (2.20), an elementary argument as in [1, Lemma 2.2] and [15, 34] proves

$$\lambda - \lambda_{\mathrm{GCR}} = \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A^2 - \lambda_{\mathrm{GCR}}\|\Pi_{\mathrm{GCR}}u - u_{\mathrm{GCR}}\|^2 + \lambda_{\mathrm{GCR}}(\|\Pi_{\mathrm{GCR}}u\|^2 - \|u\|^2). \tag{3.2}$$

The triangle inequality, (2.18) and (3.1) yield

$$\lambda_{\mathrm{GCR}}\|\Pi_{\mathrm{GCR}}u - u_{\mathrm{GCR}}\|^2 \lesssim h^{4s} + h^{2+2s} \lesssim h^{4s}.$$

An algebraic identity and the definition of the interpolation operator Π_{GCR} from (2.19) show

$$\begin{aligned}
\lambda_{\mathrm{GCR}}\left(\|\Pi_{\mathrm{GCR}}u\|^2 - \|u\|^2\right) &= \lambda_{\mathrm{GCR}}(\Pi_{\mathrm{GCR}}u - u, \Pi_{\mathrm{GCR}}u + u)_{L^2(\Omega)} \\
&= \lambda_{\mathrm{GCR}}(\Pi_{\mathrm{GCR}}u - u, \Pi_{\mathrm{GCR}}u + u - \Pi_0(\Pi_{\mathrm{GCR}}u + u))_{L^2(\Omega)} \\
&\lesssim h\|\Pi_{\mathrm{GCR}}u - u\|\|\nabla_{\mathrm{NC}}(\Pi_{\mathrm{GCR}}u + u)\| \\
&\lesssim h^{2+s}.
\end{aligned}$$

The above two estimates and the saturation condition $h^{2s} \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A^2$ imply that the second and third terms on the right-hand of (3.2) are of higher order than the first term. This completes the proof. □

Remark 3.2. Hu, Huang and Lin analyzed the saturation condition in [15]. If the eigenfunctions $u \in H^{1+s}(\Omega)$ with $0 < s < 1$, it was proved that there exist meshes such that the saturation condition $h^s \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A$ holds. In the following lemmas, we will prove the saturation condition $h \lesssim \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A$ provided that $u \in H^2(\Omega)$. For simplicity, we prove it in two dimensions for the GCR element.

Lemma 3.3. *Given $0 \neq u \in H_0^1(\Omega) \cap H^2(\Omega)$, for any triangulation $\mathcal{T}$, there holds that*

$$\sum_{K\in\mathcal{T}}\Big(\Big\|\frac{\partial^2 u}{\partial x_1^2}-\frac{b_{11}}{b_{22}}\frac{\partial^2 u}{\partial x_2^2}\Big\|_{L^2(K)}^2+\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}-\frac{b_{12}}{b_{11}}\frac{\partial^2 u}{\partial x_1^2}\Big\|_{L^2(K)}^2\Big)>0. \tag{3.3}$$

Proof. If (3.3) would not hold, then, for any $K \in \mathcal{T}$, $\|\frac{\partial^2 u}{\partial x_1^2}-\frac{b_{11}}{b_{22}}\frac{\partial^2 u}{\partial x_2^2}\|_{L^2(K)} = 0$. Since $\bar{B}|_K$ is positive-definite, we have $b_{ii} > 0$, $i = 1, 2$. Hence u should be of the form

$$u|_K(x_1,x_2)=\phi\Big(x_1-\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)+\psi\Big(x_1+\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big),$$

where $\phi(\cdot)$ and $\psi(\cdot)$ are two univariate functions. Since $\|\frac{\partial^2 u}{\partial x_1\partial x_2}-\frac{b_{12}}{b_{11}}\frac{\partial^2 u}{\partial x_1^2}\|_{L^2(K)} = 0$, we have

$$(\sqrt{b_{11}b_{22}}+b_{12})\phi''\Big(x_1-\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)=(\sqrt{b_{11}b_{22}}-b_{12})\psi''\Big(x_1+\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big).$$

This yields $\phi'' = \frac{\sqrt{b_{11}b_{22}}-b_{12}}{\sqrt{b_{11}b_{22}}+b_{12}}\psi'' \equiv C$ for some constant C. It is straightforward to derive that

$$\begin{aligned}u|_K&=c_0+c_1\Big(x_1-\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)+c_2\Big(x_1-\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)^2+c_3\Big(x_1+\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)+\frac{\sqrt{b_{11}b_{22}}+b_{12}}{\sqrt{b_{11}b_{22}}-b_{12}}c_2\Big(x_1+\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)^2\\&=c_0+c_1\Big(x_1-\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)+c_3\Big(x_1+\sqrt{\frac{b_{22}}{b_{11}}}x_2\Big)+\frac{2c_2\sqrt{b_{22}}}{\sqrt{b_{11}}(\sqrt{b_{11}b_{22}}-b_{12})}(b_{11}x_1^2+b_{22}x_2^2+2b_{12}x_1x_2)\end{aligned}$$

for some interpolation parameters c_0, c_1, c_2, c_3. Furthermore, since $b_{11}b_{22} - b_{12}^2 > 0$, $b_{11}x_1^2 + b_{22}x_2^2 + 2b_{12}x_1x_2$ cannot be a linear function on any one-dimensional subsimplex of K. The homogenous boundary condition and the continuity indicate that $u \in V_{\mathrm{CR}} \cap H_0^1(\Omega) \cap H^2(\Omega)$. This implies $u \equiv 0$, which contradicts with $u \neq 0$. ☐

Remark 3.4. When the domain is a rectangle, the saturation condition was analyzed in [15]. The theory of [23] does not cover both the ECR and GCR elements, see Corollary 3.3 therein.

In order to achieve the desired result, we shall use the operator defined in [15]. Given any $K \in \mathcal{T}$, define $J_{2,K}v \in P_2(K)$ by

$$\int_K \nabla^p J_{2,K}v\,dx=\int_K \nabla^p v\,dx,\qquad p=0,1,2,$$

for any $v \in H^2(K)$. Note that the operator $J_{2,K}$ is well defined. Since $\int_K \nabla^p(v - J_{2,K}v)\,dx = 0$ with $p = 0, 1, 2$, there holds that

$$\|\nabla^{p_1}(v-J_{2,K})v\|_{L^2(K)}\lesssim h_K^{p_2-p_1}\|\nabla^{p_2}(v-J_{2,K})v\|_{L^2(K)}\quad\text{for any } 0\le p_1\le p_2\le 2. \tag{3.4}$$

Finally, define the global operator J_2 by

$$J_2|_K=J_{2,K}\quad\text{for any } K\in\mathcal{T}. \tag{3.5}$$

It follows from the definition of $J_{2,K}$ in (3.5) that

$$\nabla^2 J_{2,K}v=\Pi_0\nabla^2 v.$$

Since piecewise constant functions are dense in the space $L^2(\Omega)$, it follows that

$$\|\nabla_{\mathrm{NC}}^2(v-J_2v)\|\to 0\quad\text{when } h\to 0. \tag{3.6}$$

Lemma 3.5. *Suppose that A is a piecewise constant matrix-valued function. Suppose that $u \in H_0^1(\Omega) \cap H^2(\Omega)$. There holds the following saturation condition:*

$$h\lesssim\|\nabla_{\mathrm{NC}}(u-u_{\mathrm{GCR}})\|_A.$$

Proof. Since A is piecewise constant, when h is small enough, for any $K \in \mathcal{T}$, $A|_K$ is constant. According to Lemma 3.3, there exists constant $\alpha > 0$ such that

$$\alpha<\sum_{K\in\mathcal{T}}\Big(\Big\|\frac{\partial^2 u}{\partial x_1^2}-\frac{b_{11}}{b_{22}}\frac{\partial^2 u}{\partial x_2^2}\Big\|_{L^2(K)}^2+\Big\|\frac{\partial^2 u}{\partial x_1\partial x_2}-\frac{b_{12}}{b_{11}}\frac{\partial^2 u}{\partial x_1^2}\Big\|_{L^2(K)}^2\Big).$$

The fact that $u_{\mathrm{GCR}} \in V_{\mathrm{GCR}}$ plus (2.9) and (2.10) yield that

$$\sum_{K\in\mathcal{T}}\Big(\Big\|\frac{\partial^2 u_{\mathrm{GCR}}}{\partial x_1^2} - \frac{b_{11}}{b_{22}}\frac{\partial^2 u_{\mathrm{GCR}}}{\partial x_2^2}\Big\|^2_{L^2(K)} + \Big\|\frac{\partial^2 u_{\mathrm{GCR}}}{\partial x_1 \partial x_2} - \frac{b_{12}}{b_{11}}\frac{\partial^2 u_{\mathrm{GCR}}}{\partial x_1^2}\Big\|^2_{L^2(K)}\Big) = 0.$$

Let J_2 be defined as in (3.5). It follows from the triangle inequality and the piecewise inverse estimate that

$$\begin{aligned}\alpha &< \sum_{K\in\mathcal{T}}\Big(\Big\|\frac{\partial^2 (u-u_{\mathrm{GCR}})}{\partial x_1^2} - \frac{b_{11}}{b_{22}}\frac{\partial^2 (u-u_{\mathrm{GCR}})}{\partial x_2^2}\Big\|^2_{L^2(K)} + \Big\|\frac{\partial^2 (u-u_{\mathrm{GCR}})}{\partial x_1 \partial x_2} - \frac{b_{12}}{b_{11}}\frac{\partial^2 (u-u_{\mathrm{GCR}})}{\partial x_1^2}\Big\|^2_{L^2(K)}\Big)\\
&\le 2\sum_{K\in\mathcal{T}}\Big(\Big\|\frac{\partial^2 (u-J_2u)}{\partial x_1^2} - \frac{b_{11}}{b_{22}}\frac{\partial^2 (u-J_2u)}{\partial x_2^2}\Big\|^2_{L^2(K)} + \Big\|\frac{\partial^2 (u-J_2u)}{\partial x_1 \partial x_2} - \frac{b_{12}}{b_{11}}\frac{\partial^2 (u-J_2u)}{\partial x_1^2}\Big\|^2_{L^2(K)}\\
&\qquad + \Big\|\frac{\partial^2 (J_2u-u_{\mathrm{GCR}})}{\partial x_1^2} - \frac{b_{11}}{b_{22}}\frac{\partial^2 (J_2u-u_{\mathrm{GCR}})}{\partial x_2^2}\Big\|^2_{L^2(K)} + \Big\|\frac{\partial^2 (J_2u-u_{\mathrm{GCR}})}{\partial x_1 \partial x_2} - \frac{b_{12}}{b_{11}}\frac{\partial^2 (J_2u-u_{\mathrm{GCR}})}{\partial x_1^2}\Big\|^2_{L^2(K)}\Big)\\
&\lesssim \|\nabla^2_{\mathrm{NC}}(u-J_2u)\|^2 + h^{-2}\|\nabla_{\mathrm{NC}}(J_2u-u_{\mathrm{GCR}})\|^2.\end{aligned}$$

The estimate of (3.4) and the triangle inequality lead to

$$1 \lesssim \|\nabla^2_{\mathrm{NC}}(u-J_2u)\|^2 + h^{-2}\|\nabla_{\mathrm{NC}}(u-u_{\mathrm{GCR}})\|^2.$$

Finally, it follows from (3.6) that

$$h^2 \lesssim \|\nabla_{\mathrm{NC}}(u-u_{\mathrm{GCR}})\|^2$$

when the meshsize is small enough, which completes the proof. □

4 Guaranteed Lower Bounds for Eigenvalues

In practice, it is not easy to check whether the meshsize h is small enough in Theorem 3.1. In this section, we propose a new method to provide guaranteed lower bounds for eigenvalues. We follow the idea of [26] and [5, 6] and generalize it to general second order elliptic operators. The mesh-size conditions in [16, Theorem 3.1] for variable coefficients are dropped in this paper. We first present some constants about the matrix-valued function A, which might be depend on h. Define $\mathcal{V}_h := H_0^1(\Omega) + V_{\mathrm{GCR}}$. For any $v \in \mathcal{V}_h$, there exist C_A, $C_{\bar{A}}$, $C_{\bar{A},A}$ and C_∞ such that

$$\|\nabla_{\mathrm{NC}}v\| \le C_A\|\nabla_{\mathrm{NC}}v\|_A, \tag{4.1}$$

$$\|\nabla_{\mathrm{NC}}v\| \le C_{\bar{A}}\|\nabla_{\mathrm{NC}}v\|_{\bar{A}}, \tag{4.2}$$

$$\|\nabla_{\mathrm{NC}}v\|_{\bar{A}} \le C_{\bar{A},A}\|\nabla_{\mathrm{NC}}v\|_A, \tag{4.3}$$

$$\|(A-\bar{A})\nabla_{\mathrm{NC}}v\| \le C_\infty h\|\nabla_{\mathrm{NC}}v\|. \tag{4.4}$$

Define $\eta := C_\infty C_{\bar{A}} C_A C_{\bar{A},A}$.

The following Poincaré inequality can be found in [12].

Lemma 4.1. *Given $K \in \mathcal{T}$, let $w \in H^1(K)$ be a function with vanishing mean. Then*

$$\|w\|_{L^2(K)} \le \frac{h_K}{\pi}\|\nabla w\|_{L^2(K)}.$$

Remark 4.2. Let $j_{1,1} = 3.8317059702$ be the first positive root of the Bessel function of the first kind. In two dimensions, the following improved Poincaré inequality holds from [22]:

$$\|w\|_{L^2(K)} \le \frac{h_K}{j_{1,1}}\|\nabla w\|_{L^2(K)}.$$

Lemma 4.1, Remark 4.2 and the second equation of (2.19) show that, for any $v \in H^1(K)$, there holds that

$$\|v-\Pi_{\mathrm{GCR}}v\|_{L^2(K)} \le C_P h_K\|\nabla(v-\Pi_{\mathrm{GCR}}v)\|_{L^2(K)} \tag{4.5}$$

with $C_P = j_{1,1}^{-1}$ for $n = 2$ and $C_P = \pi^{-1}$ for $n > 2$. The following theorem provides the guaranteed lower bounds for eigenvalues. The proof adopts the techniques in [24, Theroem 2.1] to avoid mesh-size conditions.

Theorem 4.3. *Let λ_ℓ and $\lambda_{\ell,\mathrm{GCR}}$ be the ℓ–th eigenvalues of* (2.2) *and* (2.15), *respectively. Then there holds that*

$$\frac{\lambda_{1,\mathrm{GCR}}}{1+\frac{\lambda_{1,\mathrm{GCR}}^2 C_P^4 C_A^4 h^4}{\beta+\lambda_{1,\mathrm{GCR}} C_P^2 C_A^2 h^2}+\frac{\eta^2 h^2}{1-\beta}} \le \lambda_1, \tag{4.6}$$

and for any $0<\beta<1$,

$$\frac{\lambda_{\ell,\mathrm{GCR}}}{1+\frac{\lambda_{\ell,\mathrm{GCR}}^2 C_P^4 C_A^4 h^4}{\beta+\lambda_{\ell,\mathrm{GCR}} C_P^2 C_A^2 h^2}+\frac{\eta^2 h^2}{1-\beta}+\lambda_{\ell,\mathrm{GCR}}\lambda_{1,\mathrm{GCR}}^{-1} C_A^2 C_\infty^2 h^2} \le \lambda_\ell \quad \textit{for any } \ell>1 \tag{4.7}$$

Proof. Since $\|\cdot\|$ is compact in $\mathcal{V}_h$ with respect to $\|\nabla_{\mathrm{NC}}\cdot\|_A$ (see [31]), resulting from the argument of compactness (see e.g. [2]), there exist $0<\bar\lambda_1\le\bar\lambda_2\le\cdots$ such that

$$\begin{aligned}\bar\lambda_\ell &= \min_{\dim V_\ell=\ell, V_\ell\subset\mathcal{V}_h}\ \max_{v\in V_\ell}\frac{(A\nabla_{\mathrm{NC}}v,\nabla_{\mathrm{NC}}v)_{L^2(\Omega)}}{(v,v)_{L^2(\Omega)}}\\ &= \max_{\dim W=\ell-1, W\subset\mathcal{V}_h}\ \min_{v\in W^\perp}\frac{(A\nabla_{\mathrm{NC}}v,\nabla_{\mathrm{NC}}v)_{L^2(\Omega)}}{(v,v)_{L^2(\Omega)}},\end{aligned} \tag{4.8}$$

where $W^\perp$ denotes the orthogonal complement of W in $\mathcal{V}_h$ with respect to $(A\nabla_{\mathrm{NC}}\cdot,\nabla_{\mathrm{NC}}\cdot)$. Since $H_0^1(\Omega)\subset\mathcal{V}_h$, $\lambda_\ell\ge\bar\lambda_\ell$ due to the Rayleigh–Ritz principle. Further, by choosing W in (4.8) as $E_{\ell-1,\mathrm{GCR}}$ (see (2.16)), a lower bound for λ_ℓ is obtained:

$$\lambda_\ell\ge\bar\lambda_\ell\ge\min_{v\in E_{\ell-1,\mathrm{GCR}}^\perp}\frac{(A\nabla_{\mathrm{NC}}v,\nabla_{\mathrm{NC}}v)_{L^2(\Omega)}}{(v,v)_{L^2(\Omega)}}. \tag{4.9}$$

Let $E_{\ell-1,\mathrm{GCR}}^{\perp,h}$ denote the orthogonal complement of $E_{\ell-1,\mathrm{GCR}}$ in V_{GCR} with respect to $(A\nabla_{\mathrm{NC}}\cdot,\nabla_{\mathrm{NC}}\cdot)$, i.e.,

$$V_{\mathrm{GCR}}=E_{\ell-1,\mathrm{GCR}}\oplus E_{\ell-1,\mathrm{GCR}}^{\perp,h}.$$

For any $v\in E_{\ell-1,\mathrm{GCR}}^\perp$ with $\|v\|=1$, the following decomposition holds:

$$v=\Pi_{\mathrm{GCR}}v+(v-\Pi_{\mathrm{GCR}}v)=(w_1+w_2)+(v-\Pi_{\mathrm{GCR}}v) \tag{4.10}$$

with $w_1\in E_{\ell-1,\mathrm{GCR}}$, $w_2\in E_{\ell-1,\mathrm{GCR}}^{\perp,h}$ and satisfying

$$(w_1,w_2)_{L^2(\Omega)}=(A\nabla_{\mathrm{NC}}w_1,\nabla_{\mathrm{NC}}w_2)_{L^2(\Omega)}=0.$$

This and (2.20) lead to

$$\begin{aligned}(A\nabla_{\mathrm{NC}}w_1,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)} &= (A\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)}\\ &= (\bar A\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)}+((A-\bar A)\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)}\\ &= (\bar A\nabla_{\mathrm{NC}}v,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)}+((A-\bar A)\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)}.\end{aligned}$$

Moreover, since $w_1\in E_{\ell-1,\mathrm{GCR}}$, $v\in E_{\ell-1,\mathrm{GCR}}^\perp$, a combination with assumptions of A in (4.1) and (4.4) shows

$$\begin{aligned}(A\nabla_{\mathrm{NC}}w_1,\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)} &= ((\bar A-A)\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v),\nabla_{\mathrm{NC}}w_1)_{L^2(\Omega)}\\ &\le \big\|(A-\bar A)\nabla_{\mathrm{NC}}v\big\|\,\|\nabla_{\mathrm{NC}}w_1\|\\ &\le C_A C_\infty h\|\nabla_{\mathrm{NC}}v\|_A\|\nabla_{\mathrm{NC}}w_1\|_A.\end{aligned}$$

It follows from (2.17) that w_1 satisfies

$$\|w_1\|\le\lambda_{1,\mathrm{GCR}}^{-\frac12}\|\nabla_{\mathrm{NC}}w_1\|_A\le\lambda_{1,\mathrm{GCR}}^{-\frac12}C_A C_\infty h\|\nabla_{\mathrm{NC}}v\|_A. \tag{4.11}$$

As for $w_2\in E_{\ell-1,\mathrm{GCR}}^{\perp,h}$,

$$\|w_2\|\le\lambda_{\ell,\mathrm{GCR}}^{-\frac12}\|\nabla_{\mathrm{NC}}w_2\|_A\le\lambda_{\ell,\mathrm{GCR}}^{-\frac12}\|\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v\|_A. \tag{4.12}$$

An elementary manipulation yields the following decomposition:

$$\|\nabla v\|_A^2=\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|_A^2+\|\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v\|_A^2+2(A(\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v),\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v)_{L^2(\Omega)}. \tag{4.13}$$

For the first term of (4.13), it follows from (4.1) and (4.5) that

$$\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|_A^2 \geq \frac{1}{C_P^2 C_A^2 h^2}\|v-\Pi_{\mathrm{GCR}}v\|^2. \tag{4.14}$$

The second term of (4.13) can be analyzed by (4.12) and (4.11) as

$$\begin{aligned}\|\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v\|_A^2 &\geq \lambda_{\ell,\mathrm{GCR}}\|w_2\|^2\\ &= \lambda_{\ell,\mathrm{GCR}}(\|v-\Pi_{\mathrm{GCR}}v\|^2+\|v\|^2-2(v-\Pi_{\mathrm{GCR}}v,v)_{L^2(\Omega)}-\|w_1\|^2)\\ &\geq \lambda_{\ell,\mathrm{GCR}}(\|v-\Pi_{\mathrm{GCR}}v\|^2+\|v\|^2-2(v-\Pi_{\mathrm{GCR}}v,v)_{L^2(\Omega)}-\lambda_{1,\mathrm{GCR}}^{-1}C_A^2C_\infty^2h^2\|\nabla_{\mathrm{NC}}v\|_A^2).\end{aligned} \tag{4.15}$$

By the second equation of (2.19), we have

$$2(v-\Pi_{\mathrm{GCR}}v,v)_{L^2(\Omega)} = 2(v-\Pi_{\mathrm{GCR}}v,v-\Pi_0 v)_{L^2(\Omega)}.$$

Since $\int_K \Pi_0 v\,dx = \int_K v\,dx$, the same estimate of (4.5) holds true for Π_0. This, (4.1) and the Young inequality reveal for any $\delta_1 > 0$ that

$$\begin{aligned}2(v-\Pi_{\mathrm{GCR}}v,v-\Pi_0v)_{L^2(\Omega)} &\leq 2\|v-\Pi_{\mathrm{GCR}}v\|\|v-\Pi_0v\| \leq 2C_Ph\|v-\Pi_{\mathrm{GCR}}v\|\|\nabla v\|\\ &\leq 2C_PC_Ah\|v-\Pi_{\mathrm{GCR}}v\|\|\nabla_{\mathrm{NC}}v\|_A\\ &\leq C_P^2C_A^2h^2\delta_1\|v-\Pi_{\mathrm{GCR}}v\|^2+\frac{1}{\delta_1}\|\nabla_{\mathrm{NC}}v\|_A^2.\end{aligned}$$

The third term of (4.13) has the following decomposition:

$$\begin{aligned}2(A(\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)),\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v)_{L^2(\Omega)} &= 2(\bar{A}(\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v),\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v)_{L^2(\Omega)}\\ &\quad+2((A-\bar{A})\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v),\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v)_{L^2(\Omega)}.\end{aligned} \tag{4.16}$$

Thanks to (2.20), the first term in the above equation equals zero. It remains to estimate the second term, which can be estimated by (4.1)–(4.4), (2.21) and the Young inequality that

$$\begin{aligned}2((A-\bar{A})\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v),\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v)_{L^2(\Omega)} &\leq 2C_{\bar{A}}\|(A-\bar{A})\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|\|\nabla_{\mathrm{NC}}\Pi_{\mathrm{GCR}}v\|_{\bar{A}}\\ &\leq 2C_{\bar{A}}C_\infty h\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|\|\nabla_{\mathrm{NC}}v\|_{\bar{A}}\\ &\leq 2\eta h\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|_A\|\nabla_{\mathrm{NC}}v\|_A\\ &\leq \delta_2\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|_A^2+\frac{\eta^2h^2}{\delta_2}\|\nabla_{\mathrm{NC}}v\|_A^2,\end{aligned}$$

where $\eta = C_\infty C_{\bar{A}}C_AC_{\bar{A},A}$ and $\delta_2 > 0$ is arbitrary. By substituting (4.14)–(4.16) into (4.13), we obtain, for any $0 < \beta < 1$, that

$$\begin{aligned}\lambda_\ell \geq \|\nabla_{\mathrm{NC}}v\|_A^2 &\geq \Big(\frac{\beta}{C_P^2C_A^2h^2}+\lambda_{\ell,\mathrm{GCR}}-\lambda_{\ell,\mathrm{GCR}}C_P^2C_A^2h^2\delta_1\Big)\|v-\Pi_{\mathrm{GCR}}v\|^2+(1-\beta-\delta_2)\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{GCR}}v)\|_A^2\\ &\quad-\Big(\frac{\lambda_{\ell,\mathrm{GCR}}}{\delta_1}+\frac{\eta^2h^2}{\delta_2}+\lambda_{\ell,\mathrm{GCR}}\lambda_{1,\mathrm{GCR}}^{-1}C_A^2C_\infty^2h^2\Big)\|\nabla_{\mathrm{NC}}v\|_A^2+\lambda_{\ell,\mathrm{GCR}}\|v\|^2.\end{aligned}$$

Let $\delta_1 = \frac{\beta+\lambda_{\ell,\mathrm{GCR}}C_P^2C_A^2h^2}{\lambda_{\ell,\mathrm{GCR}}C_P^4C_A^4h^4}$, $\delta_2 = 1-\beta$. This yields

$$\begin{aligned}0 &\leq \|\nabla_{\mathrm{NC}}v\|_A^2\Big(1+\frac{\lambda_{\ell,\mathrm{GCR}}}{\delta_1}+\frac{\eta^2h^2}{\delta_2}+\lambda_{\ell,\mathrm{GCR}}\lambda_{1,\mathrm{GCR}}^{-1}C_A^2C_\infty^2h^2\Big)-\lambda_{\ell,\mathrm{GCR}}\|v\|^2\\ &\leq \lambda_\ell\Big(1+\frac{\lambda_{\ell,\mathrm{GCR}}}{\delta_1}+\frac{\eta^2h^2}{\delta_2}+\lambda_{\ell,\mathrm{GCR}}\lambda_{1,\mathrm{GCR}}^{-1}C_A^2C_\infty^2h^2\Big)-\lambda_{\ell,\mathrm{GCR}}.\end{aligned}$$

This concludes (4.7). As for $\ell = 1$, it follows from the fact that $w_1 = 0$, one can easily prove (4.6). □

Remark 4.4. When A is a piecewise constant matrix-valued function, (4.7) yields

$$\frac{\lambda_{\ell,\mathrm{GCR}}}{1+\frac{\lambda_{\ell,\mathrm{GCR}}^2C_P^4C_A^4h^4}{1+\lambda_{\ell,\mathrm{GCR}}C_P^2C_A^2h^2}} \leq \lambda_\ell. \tag{4.17}$$

For the Laplace operator in two dimensions considered in [6], as we shall find in Section 7, the guaranteed lower bounds of this paper are more accurate than those from [6] by the CR element numerically, see (7.1) below. On uniform triangulations, this can be proven asymptotically for sufficiently smooth eigenfunctions by using the asymptotic expansions of the eigenvalues by the CR and ECR elements from [19]. Recall the expansions [19, Theorems 3.14 and 4.4] as follows:

$$\lambda - \lambda_{\mathrm{CR}} = J - \frac{\lambda^2}{144} H^2 + O(h^4 |\ln h| |u|^2_{H^4(\Omega)}), \tag{4.18}$$

$$\lambda - \lambda_{\mathrm{ECR}} = J + O(h^4 |\ln h| |u|^2_{H^4(\Omega)}) \tag{4.19}$$

with $H = \sum_{p<q} (x_{1p} - x_{1q})^2 + \sum_{p<q} (x_{2p} - x_{2q})^2$ as in (2.8) and

$$J = h^2 \Big(\frac{\gamma_{RT}^{11}}{4} \|\partial_{x_1 x_1} u - \partial_{x_2 x_2} u\|^2 + \gamma_{RT}^{22} \|\partial_{x_1 x_2} u\|^2 + \gamma_{RT}^{12} \int_\Omega (\partial_{x_1 x_1} u - \partial_{x_2 x_2} u) \partial_{x_1 x_2} u \, dx \Big),$$

and see other notation $\gamma_{RT}^{11}, \gamma_{RT}^{12}, \gamma_{RT}^{22}$ in [19]. The guaranteed lower bounds $\frac{\lambda_{\ell,\mathrm{CR}}}{1+(j_{1,1}^{-2}+48^{-1})\lambda_{\ell,\mathrm{CR}} h^2}$ by the CR element from [6], (4.18) and $H \le 3h^2$ show

$$\begin{aligned} \lambda_\ell - \frac{\lambda_{\ell,\mathrm{CR}}}{1 + (j_{1,1}^{-2} + 48^{-1}) \lambda_{\ell,\mathrm{CR}} h^2} &= J - \frac{\lambda_\ell^2}{144} H^2 + \lambda_{\ell,\mathrm{CR}}^2 (j_{1,1}^{-2} + 48^{-1}) h^2 + O(h^4) \\ &\ge J - \frac{\lambda_\ell^2}{48} h^2 + \lambda_\ell^2 (j_{1,1}^{-2} + 48^{-1}) h^2 + O(h^4) \\ &\ge J + \lambda_\ell^2 j_{1,1}^{-2} h^2 + O(h^4). \end{aligned} \tag{4.20}$$

For the Laplace operator, the GCR element is the ECR element as mentioned in Section 2.2. The guaranteed lower bounds (4.17) by the GCR element from (4.17) with $C_P = j_{1,1}^{-1}$, $C_A = 1$ and (4.19) show

$$\lambda_\ell - \frac{\lambda_{\ell,\mathrm{GCR}}}{1 + \frac{\lambda_{\ell,\mathrm{GCR}}^2 j_{1,1}^{-4} h^4}{1+\lambda_{\ell,\mathrm{GCR}} j_{1,1}^{-2} h^2}} = J + O(h^4). \tag{4.21}$$

The combination of (4.20) and (4.21) leads that for sufficiently small mesh size

$$\frac{\lambda_{\ell,\mathrm{GCR}}}{1 + \frac{\lambda_{\ell,\mathrm{GCR}}^2 j_{1,1}^{-4} h^4}{1+\lambda_{\ell,\mathrm{GCR}} j_{1,1}^{-2} h^2}} > \frac{\lambda_{\ell,\mathrm{CR}}}{1 + (j_{1,1}^{-2} + 48^{-1}) \lambda_{\ell,\mathrm{CR}} h^2}.$$

5 Asymptotic Upper Bounds for Eigenvalues

It is well known that conforming finite element methods provide upper bounds for eigenvalues, but it needs to compute an extra eigenvalue problem. Here we present a simple postprocessing method to provide uppers bound for eigenvalues by the GCR element, see more details in [18, 29].

For any $v \in V_{\mathrm{GCR}}$, define the interpolation $\Pi_{\mathrm{CR}} : V_{\mathrm{GCR}} \to V_{\mathrm{CR}}$ by

$$\int_E \Pi_{\mathrm{CR}} v \, ds = \int_E v \, ds \text{ for any } E \in \mathcal{E}.$$

It is straightforward to see that $v - \Pi_{\mathrm{CR}} v \in V_{\mathrm{B}}$. Furthermore, the standard interpolation theory of [13] gives

$$\|v - \Pi_{\mathrm{CR}} v\| \lesssim h \|\nabla_{\mathrm{NC}} (v - \Pi_{\mathrm{CR}} v)\| \lesssim h^2 \|\nabla_{\mathrm{NC}}^2 v\|, \tag{5.1}$$

An integration by parts leads to the following orthogonality:

$$(\nabla_{\mathrm{NC}} (v - \Pi_{\mathrm{CR}} v), \nabla_{\mathrm{NC}} \Pi_{\mathrm{CR}} v)_{L^2(\Omega)} = 0. \tag{5.2}$$

For any $v \in V_{\mathrm{CR}}$, define the interpolation $\Pi_{\mathrm{c}} : V_{\mathrm{CR}} \to V_{\mathrm{c}} := V_{\mathrm{CR}} \cap H_0^1(\Omega)$ by

$$(\Pi_{\mathrm{c}} v)(z) = \begin{cases} 0, & z \in \partial\Omega, \\ \dfrac{1}{|\omega_z|} \sum\limits_{K \in \omega_z} v|_K(z), & z \notin \partial\Omega, \end{cases} \tag{5.3}$$

where ω_z is the union of elements containing vertex z, $|\omega_z|$ is the number of elements containing vertex z. The following lemma was proved in [18, 29, 32].

Lemma 5.1. *Let $v \in V_{\mathrm{CR}}$. For any $w \in H_0^1(\Omega)$, there holds that*

$$\begin{aligned} \|v - \Pi_{\mathrm{c}} v\| &\lesssim h \|\nabla_{\mathrm{NC}}(v - w)\|, \\ \|\nabla_{\mathrm{NC}}(v - \Pi_{\mathrm{c}} v)\| &\lesssim \|\nabla_{\mathrm{NC}}(v - w)\|. \end{aligned}$$

Then (5.1) and Lemma 5.1 yield the following result.

Corollary 5.2. *Let u and u_{GCR} be eigenfunctions of* (2.2) *and* (2.15), *respectively. Suppose that $u \in H^{1+s}(\Omega)$, $0 < s \le 1$. There holds that*

$$\begin{aligned} \|u_{\mathrm{GCR}} - \Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}})\| &\lesssim h^{1+s} |u|_{1+s}, \\ \|\nabla_{\mathrm{NC}}(u_{\mathrm{GCR}} - \Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}}))\|_A &\lesssim h^s |u|_{1+s}. \end{aligned}$$

Define the Rayleigh quotient

$$\lambda_c = \frac{(A \nabla \Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}}), \Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}}))_{L^2(\Omega)}}{(\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}}), \Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}}))_{L^2(\Omega)}}.$$

Theorem 5.3. *Suppose (λ, u) is an eigenpair of* (2.2) *and $u \in H^{1+s}(\Omega)$, $0 < s \le 1$. Then*

$$|\lambda - \lambda_{\mathrm{c}}| \lesssim h^{2s} |u|_{1+s}.$$

Moreover, $\lambda_{\mathrm{c}} \ge \lambda$ provided that h is small enough.

Proof. The proof is similar to that of [29, Theorem 3.4] and [32, Theorem 4.1]. Let $w = \Pi_c(\Pi_{\mathrm{CR}} u_{\mathrm{GCR}})$. An elementary manipulation leads

$$\begin{aligned} \|\nabla(u - w)\|_A^2 &= (A\nabla(u - w), \nabla(u - w))_{L^2(\Omega)} = \lambda + \|w\|^2 \lambda_c - 2(A\nabla u, \nabla w)_{L^2(\Omega)} \\ &= \lambda + \|w\|^2 \lambda_c - 2\lambda(u, w)_{L^2(\Omega)} \\ &= \|w\|^2(\lambda_c - \lambda) + \lambda \|u - w\|^2. \end{aligned} \tag{5.4}$$

Thanks to (2.18) and Corollary 5.2, it holds that

$$\|\nabla(u - w)\|_A \le \|\nabla_{\mathrm{NC}}(u - u_{\mathrm{GCR}})\|_A + \|\nabla_{\mathrm{NC}}(u_{\mathrm{GCR}} - w)\|_A \lesssim h^s |u|_{1+s} \tag{5.5}$$

and

$$\|u - w\| \le \|u - u_{\mathrm{GCR}}\| + \|u_{\mathrm{GCR}} - w\| \lesssim (h^{2s} + h^{1+s}) |u|_{1+s} \lesssim h^{2s} |u|_{1+s}. \tag{5.6}$$

On the other hand $|\|w\| - \|u\|| \le \|u - w\| \lesssim h^{2s} |u|_{1+s}$. Hence $\|w\|$ is bounded. Substituting (5.5) and (5.6) into (5.4) yields

$$|\lambda - \lambda_{\mathrm{c}}| \lesssim h^{2s} |u|_{1+s}.$$

The following saturation condition holds, see [15]:

$$h^s \lesssim \|\nabla(u - w)\|_A.$$

Hence, when h is small enough, $\|u - w\|$ is of higher order than $\|\nabla(u - w)\|_A$. This and (5.4) yield that

$$0 \le \|w\|^2(\lambda_c - \lambda),$$

which completes the proof. □

6 Guaranteed Upper Bounds for Eigenvalues

Since λ_c is the upper bound of λ in the asymptotic sense, we propose a method to guarantee upper bounds for eigenvalues. Suppose (λ_ℓ, u_ℓ) be the ℓ-th eigenpair of (2.2) and $E_{\ell,\mathrm{GCR}}$ be defined in (2.16). Define

$$\lambda_{\ell,c}^m := \sup_{v\in\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}E_{\ell,\mathrm{GCR}})} \frac{(A\nabla v, \nabla v)_{L^2(\Omega)}}{(v,v)_{L^2(\Omega)}}. \tag{6.1}$$

Lemma 6.1. *Suppose that $u_\ell \in H^{1+s}(\Omega)$ with $0 < s \le 1$. Then*

$$|\lambda_{\ell,\mathrm{c}}^m - \lambda_\ell| \lesssim h^{1+s}|u|_{1+s}.$$

Proof. Following the theory of [2], there holds that

$$|\lambda_{\ell,\mathrm{c}}^m - \lambda_\ell| \lesssim \Big(\inf_{v\in\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}E_{\ell,\mathrm{GCR}})} \|\nabla(v-u_\ell)\|_A\Big)^2 \lesssim \|\nabla(\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}u_{\ell,\mathrm{GCR}}) - u_\ell)\|_A^2.$$

Hence, the above result and (5.5) yield that

$$|\lambda_{\ell,\mathrm{c}}^m - \lambda_\ell| \lesssim h^{2s}|u|_{1+s}.$$

This completes the proof. □

Assume $\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}E_{\ell,\mathrm{GCR}})$ is ℓ-dimensional. The Rayleigh–Ritz principle (2.4) implies that $\lambda_{\ell,c}^m$ is the upper bound of λ_ℓ. We propose some conditions in the following lemma to guarantee that $\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}E_{\ell,\mathrm{GCR}})$ is ℓ-dimensional.

Lemma 6.2. *Suppose there exist computable constants β_1 and β_2 such that*

$$\begin{aligned} \|v-\Pi_{\mathrm{CR}}v\| &\le \beta_1 h\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{CR}}v)\| \quad && \text{for any } v\in V_{\mathrm{GCR}},\\ \|w-\Pi_{\mathrm{c}}w\| &\le \beta_2 h\|\nabla_{\mathrm{NC}}w\| && \text{for any } w\in V_{\mathrm{CR}}.\end{aligned}$$

Then $\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}E_{\ell,\mathrm{GCR}})$ is ℓ-dimensional provided that

$$h < \frac{1}{(\beta_1+\beta_2)C_A\sqrt{\lambda_{\ell,\mathrm{GCR}}}}. \tag{6.2}$$

Proof. For any $v = \sum_{k=1}^{\ell}\xi_i u_{i,\mathrm{GCR}}$ and $\|v\| = 1$, the triangle inequality yields

$$\begin{aligned}\|v-\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}v)\| &\le \|v-\Pi_{\mathrm{CR}}v\| + \|\Pi_{\mathrm{CR}}v - \Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}v)\|\\ &\le \beta_1 h\|\nabla_{\mathrm{NC}}(v-\Pi_{\mathrm{CR}}v)\| + \beta_2 h\|\nabla_{\mathrm{NC}}\Pi_{\mathrm{CR}}v\|.\end{aligned}$$

Due to (5.2) and the constant in (4.1), there holds the following estimate:

$$\begin{aligned}\|v-\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}v)\| &\le (\beta_1+\beta_2)h\|\nabla_{\mathrm{NC}}v\| \le (\beta_1+\beta_2)C_A h\|\nabla_{\mathrm{NC}}v\|_A\\ &\le (\beta_1+\beta_2)C_A h\sqrt{\lambda_{\ell,\mathrm{GCR}}}.\end{aligned}$$

Then the condition for h in (6.2) yields

$$\|\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}v)\| \ge 1 - \|v-\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}v)\| \ge 1-(\beta_1+\beta_2)C_A h\sqrt{\lambda_{\ell,\mathrm{GCR}}} > 0.$$

Hence, $\Pi_{\mathrm{c}}(\Pi_{\mathrm{CR}}E_{\ell,\mathrm{GCR}})$ is ℓ-dimensional. □

Remark 6.3. Note that (6.2) is not a strict condition. Indeed, to obtain good approximation of the ℓ-the eigenvalue λ_ℓ by finite element methods, $\lambda_\ell h^2 \lesssim 1$ is always required.

We show that β_1 is computable. Note that $(v-\Pi_{CR}v)|_K \in \mathrm{span}\{\phi_K\}$, where ϕ_K is defined as in (2.7). For each $K\in\mathcal{T}$, we can find a positive constant β_K such that

$$\|\phi_K\|_{L^2(K)} \le \beta_K\|\nabla\phi_K\|_{L^2(K)}.$$

Then we take

$$\beta_1 = \frac{\max_{K\in\mathcal{T}}\{\beta_K\}}{h}.$$

There are several results concerning the constant for the interpolation operator Π_{CR} in two dimensions, see for instance [5, 28]. Recall C_P from (4.5). We present the result in [5, 16] as follows:

$$\|v - \Pi_{\mathrm{CR}} v\|_{L^2(K)} \le \sqrt{C_P^2 + \frac{1}{2n(n+1)(n+2)}}\, h_K \|\nabla(v - \Pi_{\mathrm{CR}} v)\|_{L^2(K)} \quad \text{for any } v \in H^1(K).$$

Hence we can choose

$$\beta_1 = \sqrt{C_P^2 + \frac{1}{2n(n+1)(n+2)}}. \tag{6.3}$$

Next, we analyze the computable constant β_2. To this end, we define

$$\xi = \max_{K\in\mathcal{T}} \max_{K'\cap K\neq\varnothing} \frac{|K'|}{|K|} \tag{6.4}$$

and

$$N = \max_{z\in\mathcal{V}} |\omega_z|, \tag{6.5}$$

where $\mathcal{V}$ denotes the set of all the vertices of $\mathcal{T}$ and $|\omega_z|$ denotes the number of elements containing vertex z.

Lemma 6.4. *For any $w \in V_{\mathrm{CR}}$, it holds that*

$$\|w - \Pi_{\mathrm{c}} w\| \le \frac{(n-1)N\sqrt{\xi}}{n} h \|\nabla_{\mathrm{NC}} w\|.$$

Proof. Given an element $K \in \mathcal{T}$, let a_p, $1 \le p \le n+1$, be its vertices and let θ_p be the corresponding barycentric coordinates. Then

$$w|_K = \sum_{p=1}^{n+1} w|_K(a_p)\theta_p \quad \text{and} \quad (\Pi_{\mathrm{c}} w)|_K = \sum_{p=1}^{n+1} \bar{w}_p \theta_p,$$

where

$$\bar{w}_p = \frac{1}{|\omega_{a_p}|} \sum_{K'\in\omega_{a_p}} w|_{K'}(a_p),$$

as defined in (5.3). This gives

$$\begin{aligned}
\|w - \Pi_{\mathrm{c}} w\|^2 &= \sum_{K\in\mathcal{T}} \|w - \Pi_{\mathrm{c}} w\|^2_{L^2(K)} \\
&= \sum_{K\in\mathcal{T}} \Big\| \sum_{p=1}^{n+1} w|_K(a_p)\theta_p - \sum_{p=1}^{n+1} \bar{w}_p\theta_p \Big\|^2_{L^2(K)} \\
&\le \sum_{K\in\mathcal{T}} \sum_{p,q=1}^{n+1} |(w|_K(a_p) - \bar{w}_p)(w|_K(a_q) - \bar{w}_q)| (\theta_p, \theta_q)_{L^2(K)}.
\end{aligned}$$

An explicit calculation that $(\theta_p, \theta_q)_{L^2(K)} = \frac{|K|}{(n+1)(n+2)}(1 + \delta_{pq})$ leads to

$$\|w - \Pi_{\mathrm{c}} w\|^2 \le \sum_{K\in\mathcal{T}} \frac{|K|}{n+1} \sum_{p=1}^{n+1} |w|_K(a_p) - \bar{w}_p|^2.$$

It follows from the definitions of the interpolation operator Π_{c} in (5.3) and N in (6.5) that

$$\begin{aligned}
\|w - \Pi_{\mathrm{c}} w\|^2 &\le \sum_{K} \frac{|K|}{n+1} \sum_{p=1}^{n+1} \sup_{K'\cap a_p\neq\varnothing} |w|_K(a_p) - w|_{K'}(a_p)|^2 \\
&\le \sum_{K\in\mathcal{T}} \frac{|K|}{n+1} \sum_{p=1}^{n+1} \frac{N}{4} \sum_{E'\in\mathcal{E}, E'\cap a_p\neq\varnothing} |[w]|^2_{L^\infty(E')} \\
&= \sum_{K\in\mathcal{T}} \frac{N|K|}{4(n+1)} \sum_{p=1}^{n+1} \sum_{E'\in\mathcal{E}, E'\cap a_p\neq\varnothing} |[w]|^2_{L^\infty(E')}.
\end{aligned} \tag{6.6}$$

Given $E' \in \mathcal{E}$, suppose that $|[w]|$ achieves the maximum at point z' and the centroid of E' is M'. Let $\tau_{E'}$ denote the tangent vector of E' from M' to z'. Since $\int_{E'}[w]\,ds = 0$ and $[w] \in P_1(E')$, this yields

$$\begin{aligned}|[w](z')| &= \left|\int_{M'}^{z'} \left[\frac{\partial w}{\partial \tau_{E'}}\right] ds\right| \le |z' - M'| \|[\nabla w]\|_{L^\infty(E')} \\ &\le \frac{n-1}{n} h_{E'} \|[\nabla w]\|_{L^\infty(E')} = \frac{(n-1)h_{E'}}{n|E'|^{\frac{1}{2}}} \|[\nabla w]\|_{L^2(E')}.\end{aligned} \tag{6.7}$$

Substituting (6.7) into (6.6) gives that

$$\|w - \Pi_c w\|^2 \le \sum_K \frac{(n-1)^2 N|K|}{4n^2(n+1)} \sum_{p=1}^{n+1} \sum_{E' \in \mathcal{E}, E' \cap a_p \neq \varnothing} h_{E'}^2 \|[\nabla w]\|_{L^2(E')}^2.$$

Since $\nabla_{\mathrm{NC}} w$ is a piecewise constant, the following trace inequality holds:

$$\|[\nabla w]\|_{L^2(E')}^2 \le \frac{2|E'|}{|K_1|} \|\nabla w\|_{L^2(K_1)}^2 + \frac{2|E'|}{|K_2|} \|\nabla w\|_{L^2(K_2)}^2.$$

Hence

$$\|w - \Pi_c w\|^2 \le \sum_{K \in \mathcal{T}} \frac{N(n-1)^2|K|}{n^2(n+1)} \sum_{p=1}^{n+1} \sum_{K' \cap a_p \neq \varnothing} \frac{h_{E'}^2}{|K'|} \|\nabla w\|_{L^2(K')}^2.$$

By the definition of ξ in (6.4), there holds that

$$\|w - \Pi_c w\|^2 \le \frac{(n-1)^2 N^2 \xi}{n^2} h^2 \sum_{K \in \mathcal{T}} \|\nabla w\|_{L^2(K)}^2.$$

This completes the proof. □

7 Numerical Results

7.1 The Laplace Operator

In this example, the L-shape domain $\Omega = (0,1)^2/[0.5,1]^2$ and $A(x) \equiv 1$. We compare the lower bounds provided by the CR and GCR elements. Let $\lambda_{\ell,\mathrm{CR}}$ be the ℓ-th eigenvalues by the CR element. Carstensen and Gedickel [6] give the guaranteed lower bounds

$$\mathrm{GLB}_{\ell,\mathrm{CR}} = \frac{\lambda_{\ell,\mathrm{CR}}}{1 + (j_{1,1}^{-2} + 48^{-1})\lambda_{\ell,\mathrm{CR}} h^2}. \tag{7.1}$$

By the GCR element, Theorem 4.3 and $C_P = j_{1,1}^{-1}$ give the guaranteed lower bounds

$$\mathrm{GLB}_{\ell,\mathrm{GCR}} = \frac{\lambda_{\ell,\mathrm{GCR}}}{1 + \frac{\lambda_{\ell,\mathrm{GCR}}^2 j_{1,1}^{-4} h^4}{1 + \lambda_{\ell,\mathrm{GCR}} j_{1,1}^{-2} h^2}}. \tag{7.2}$$

Note that the modifications $\lambda_{\ell,\mathrm{GCR}} - \mathrm{GLB}_{\ell,\mathrm{GCR}} = O(h^4)$ in (7.2) are of higher order than $\lambda_{\ell,\mathrm{CR}} - \mathrm{GLB}_{\ell,\mathrm{CR}} = O(h^2)$ in (7.1). Table 1 and Table 2 show the results of first and 20th eigenvalues, respectively. For comparison, the discrete eigenvalues $\lambda_{\ell,\mathrm{P1}}$ by the conforming P1 element are computed as upper bounds. Due to the fact that $V_{\mathrm{CR}} \subset V_{\mathrm{GCR}}$, $\lambda_{\ell,\mathrm{GCR}}$ is smaller than $\lambda_{\ell,\mathrm{CR}}$. However, the guaranteed lower bounds produced by the GCR element are larger than those by the CR element.

7.2 General Second Elliptic Operators

In this example, let $\Omega = (0,1)^2$, and

$$A(x) = \begin{pmatrix} x_1^2 + 1 & x_1 x_2 \\ x_1 x_2 & x_2^2 + 1 \end{pmatrix}.$$

h	$\lambda_{1,\mathrm{CR}}$	$\mathrm{GLB}_{1,\mathrm{CR}}$	$\lambda_{1,\mathrm{GCR}}$	$\mathrm{GLB}_{1,\mathrm{GCR}}$	$\lambda_{1,\mathrm{P1}}$
0.707107	24	11.6092	21.4979	16.4175	
0.353553	32.7371	24.0013	31.1326	29.4946	56.3170
0.176777	36.5336	33.1658	35.9771	35.7822	43.0976
0.088388	37.8448	36.8751	37.6910	37.6761	39.8639
0.044194	38.2993	38.0462	38.2596	38.2586	38.9633
0.022097	38.4619	38.3978	38.4519	38.4518	38.6918
0.011049	38.5219	38.5058	38.5194	38.5194	38.6048
0.005524	38.5446	38.5406	38.5440	38.5440	38.5754

Table 1: The first eigenvalue of L-shape domain.

h	$\lambda_{20,\mathrm{CR}}$	$\mathrm{GLB}_{20,\mathrm{CR}}$	$\lambda_{20,\mathrm{GCR}}$	$\mathrm{GLB}_{20,\mathrm{GCR}}$	$\lambda_{20,\mathrm{P1}}$
0.353553	454.2769	75.0788	298.6560	105.7197	
0.176777	307.4914	165.7926	280.6304	229.3926	722.3323
0.088388	387.1673	305.0883	372.4979	360.6719	500.4567
0.044194	401.4816	375.3058	397.2255	396.1748	429.3377
0.022097	405.0899	398.0864	403.9846	403.9127	412.1292
0.011049	406.0462	404.2640	405.7671	405.7625	407.8798
0.005524	406.3103	405.8627	406.2404	406.2401	406.8021

Table 2: The 20th eigenvalue of L-shape domain.

By a direct computation, the eigenvalues of $A(x)$ are $x_1^2 + x_2^2 + 1$ and 1, and $|A - \bar{A}|_\infty \le \min\{\frac{4}{3}h, 1\}$. The constants in (4.1)–(4.4) are

$$C_A = 1, C_{\bar{A}} = 1, \quad C_{\bar{A},A} = \min\Big\{\sqrt{1 + \frac{8}{3}h}, \sqrt{3}\Big\}, \quad C_\infty = \min\Big\{\frac{8}{3}, \frac{2}{h}\Big\}$$

and

$$\eta = C_\infty C_{\bar{A}} C_A C_{\bar{A},A} = \min\Big\{\frac{8}{3}, \frac{2}{h}\Big\} \min\Big\{\sqrt{1 + \frac{8}{3}h}, \sqrt{3}\Big\}.$$

To compute the guaranteed lower and upper bounds for the first eigenvalue, it does not need the mesh-size condition in (6.2). As for the 20th eigenvalue, we compute $\lambda_{20,c}^m$ as a upper bound of λ_{20}. Since the computations are on uniform partitions, the constants in (6.4) and (6.5) are

$$\xi = 1, \quad N = 6, \quad \beta_2 = \frac{N\sqrt{\xi}}{2} = 3.$$

We use the estimate of β_1 in (6.3). Let $\beta_1 \approx 0.2984$. The condition in (6.2) reads

$$h < \frac{1}{(\beta_1 + \beta_2) C_A \sqrt{\lambda_{20,\mathrm{GCR}}}} = \frac{1}{(0.2984 + 3)\sqrt{\lambda_{20,\mathrm{GCR}}}} =: h_1.$$

Let $\beta = \frac{1}{2}$ in Theorem 4.3. The GCR element gives the guaranteed lower bounds

$$\mathrm{GLB}_{1,\mathrm{GCR}} = \frac{\lambda_{1,\mathrm{GCR}}}{1 + \frac{\lambda_{1,\mathrm{GCR}}^2 C_A^4 h^4}{0.5 j_{1,1}^4 + \lambda_{1,\mathrm{GCR}} j_{1,1}^2 C_A^2 h^2} + 2\eta^2 h^2},$$

and for any $\ell > 1$,

$$\mathrm{GLB}_{\ell,\mathrm{GCR}} = \frac{\lambda_{\ell,\mathrm{GCR}}}{1 + \frac{\lambda_{\ell,\mathrm{GCR}}^2 C_A^4 h^4}{0.5 j_{1,1}^4 + \lambda_{\ell,\mathrm{GCR}} j_{1,1}^2 C_A^2 h^2} + 2\eta^2 h^2 + \lambda_{\ell,\mathrm{GCR}} \lambda_{1,\mathrm{GCR}}^{-1} C_A^2 C_\infty^2 h^2}.$$

Table 3 and Table 4 show the results of the first and 20th eigenvalues, respectively. From Table 4, we find that when $h \le 0.0110$, the condition $h < h_1$ is guaranteed. Actually, when $h \le 0.1768$, $\Pi_c(\Pi_{\mathrm{CR}} E_{20,\mathrm{GCR}})$ is already 20-dimensional and $\lambda_{20,c}^m$ is thus a guaranteed upper bound of λ_{20}.

h	$\lambda_{1,\mathrm{GCR}}$	$\mathrm{GLB}_{1,\mathrm{GCR}}$	$\lambda_{1,\mathrm{P1}}$	$\lambda_{1,c}$
1.4142	22.93710	0.82825		
0.7071	22.73488	1.00339	39	39
0.3536	25.38568	5.61741	30.22432	30.68603
0.1768	26.29812	15.84612	27.52878	27.63606
0.0884	26.54494	23.33235	26.85419	26.86946
0.0442	26.60805	25.80609	26.68551	26.68745
0.0221	26.62394	26.42955	26.64332	26.64356
0.0110	26.62792	26.58041	26.63277	26.63280
0.0055	26.62892	26.61720	26.63013	26.63013

Table 3: The first eigenvalue of square domain.

h	h_1	$\lambda_{20,\mathrm{GCR}}$	$\mathrm{GLB}_{20,\mathrm{GCR}}$	$\lambda_{20,\mathrm{P1}}$	$\lambda_{20,\mathrm{c}}$	$\lambda_{20,\mathrm{c}}^{m}$
0.3536	0.0197	236.8297	22.0631		348.5134	
0.1768	0.0173	305.4755	87.9449	576.1674	620.3720	720.0317
0.0884	0.0159	362.8685	224.2311	427.1357	424.3606	433.1020
0.0442	0.0156	378.9545	330.4063	394.1451	394.3686	394.7023
0.0221	0.0155	383.2543	370.0130	387.0340	387.0722	387.0910
0.0110	0.0155	384.3485	380.9748	385.2930	385.2979	385.2991
0.0055	0.0155	384.6233	383.7771	384.8595	384.8601	384.8601

Table 4: The 20th eigenvalue of square domain.

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