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Research Article

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An Analysis of High-Frequency Helmholtz Problems in Domains with Conical Points and Their Finite Element Discretisation

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Abstract: We consider Helmholtz problems in three-dimensional domains featuring conical points. We focus on the high-frequency regime and derive novel sharp upper-bounds for the stress intensity factors of the singularities associated with the conical points. We then employ these new estimates to analyse the stability of finite element discretisations. Our key result is that lowest-order Lagrange finite elements are stable under the assumption that "$\omega^2 h$ is small". This assumption is standard and well known in the case of smooth domains, and we show that it naturally extends to domain with conical points, even when using uniform meshes.

Keywords: Helmholtz Problems, Corner Singularities, Finite Elements, Pollution Effect

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Dedicated to Professor Thomas Apel on the occasion of his 60th birthday.

1 Introduction

High-frequency wave propagation problems play a crucial role in several applications ranging from radar imaging [9] to nanophotonics [12], just to cite a few. When considering complex geometries and/or heterogeneous media, finite element methods are a popular approach to discretise such problems and compute approximate numerical solutions. In this work, we focus on scalar Helmholtz problems which is probably the simplest model problem featuring the difficulties linked with high-frequency waves.

When considering smooth domains, finite element discretisations of Helmholtz problems are now widely covered in the literature; see e.g. [4, 5, 18, 20, 21] and the references therein. In particular, it is now well known that, in the high-frequency regime, the "pollution effect" alters the quality of the finite element solution. More precisely, the quasi-optimality of the finite element solution (which is always guaranteed when considering coercive problems) is lost in the high-frequency regime, unless the mesh is sufficiently refined. For the lowest-degree Lagrange elements in smooth (or convex) non-trapping domains (see Assumption 2.1 below), it is well known that the condition "$\omega^2 h$ is small", where ω is the frequency and h is the mesh size, is necessary and sufficient to ensure the quasi-optimality of the finite element solution.

When dealing with complex geometries and, in particular, with polytopal domains, another concern is the possible presence of singularities in the vicinity of re-entrant corners and edges [6, 16]. These singularities may lower the regularity of the solution and, in turn, diminish the convergence rate of the finite element

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discretisation. For coercive problem, it is now well established that the effect of singularities is essentially local and may be taken care of by local mesh refinements [1, 3, 4]. Specifically, employing geometrically graded meshes close to edges and corners of the boundary permits to restore the optimal convergence rate of the method without adding a significant number of degrees of freedom.

The goal of this work is to analyse the behaviour of corner singularities in the context of high-frequency problems, and their impact on finite element discretisations. Actually, the authors already proposed an analysis of the two-dimensional case in [4], and the purpose of the present work is to extend it to three-dimensional domains with conical points. In fact, we show that the behaviour of corner singularities in three-dimensional domains is fairly similar to the two-dimensional case. In short, we show that the stress intensity factors associated with each singular function decreases as the frequency increases. A direct consequence is that the condition "$\omega^2 h^{1-\varepsilon}$ is small", where $\varepsilon > 0$ can be chosen arbitrarily small (that is almost the same as in the case of a smooth domain), is sufficient to ensure the stability of finite element discretisations in the case of uniform meshes in a domain featuring corners. We also support this last claim by numerical examples.

The remainder of this work is organised as follows. Section 2 makes the model problem we are considering precise and recalls the key assumption that the domain is non-trapping. In Section 3, we consider the case where the domain is actually a cone, thus featuring a single singular point. Section 4 extends this preliminary result to general non-trapping domains via a localisation argument. We build over these abstract results in Section 5, where we establish necessary and sufficient conditions for the stability of finite element discretisations under suitable assumptions on the mesh. In Appendix A, we present technical integration-by-parts results for singular functions.

2 The Setting

In this work, we consider wave propagation problems modelised by the Helmholtz equation in a domain Ω,

$$\begin{cases} -\omega^2 u - \Delta u = f & \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} - i\omega u = 0 & \text{on } \Gamma_{\mathrm{Diss}}, \\ u = 0 & \text{on } \Gamma_{\mathrm{Dir}}, \end{cases} \tag{2.1}$$

where $f\colon \Omega \to \mathbb{C}$ is a given source term, Γ_{Dir} and Γ_{Diss} are two disjoint open subsets of the boundary $\partial\Omega$ of Ω such that $\overline{\Gamma_{\mathrm{Dir}}} \cup \overline{\Gamma_{\mathrm{Diss}}} = \partial\Omega$, and $\omega > 0$ is the angular frequency.

Classically, assuming that $f \in L^2(\Omega)$, we recast (2.1) into the variational problem that consists in looking for the solution $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ to

$$B(u, v) = (f, v) \quad \text{for all } v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega), \tag{2.2}$$

where

$$B(u, v) = -\omega^2 (u, v) - i\omega \langle u, v\rangle_{\Gamma_{\mathrm{Diss}}} + (\nabla u, \nabla v),$$
$$H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) = \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \Gamma_{\mathrm{Dir}}\},$$

γ_0 being the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$, and $(\cdot\,,\cdot)$ means the $L^2(\Omega)$ or $L^2(\Omega)^3$ inner product according to the context.

In the remainder of this work, we assume that problem (2.2) is well-posed and "non-trapping". This is a reasonable assumption often satisfied in applications [2, 11, 17, 22].

Assumption 2.1. We assume that there exists a non-negative constant ω_0 such that, for all $f \in L^2(\Omega)$ and $\omega \geq \omega_0$, problem (2.2) admits a unique solution $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. Furthermore, we assume that

$$\omega \|u\|_{0,\Omega} + |u|_{1,\Omega} \leq C(\Omega, \omega_0)\omega \|f\|_{0,\Omega} \tag{2.3}$$

for some positive constant $C(\Omega, \omega_0)$ that may depend on Ω and ω_0, but neither on ω nor on f.

Typically, Γ_{Dir} corresponds to the boundary of a physical obstacle, whereas Γ_{Diss} is “artificially” introduced to approximate a radiation condition. As a result, we do not cover singularities caused by corners in Γ_{Diss} since it is usually selected as the boundary of a convex domain.

Assumption 2.2. Γ_{Diss} and Γ_{Dir} are connected manifolds. We assume either $\overline{\Gamma_{\mathrm{Dir}}} \cap \overline{\Gamma_{\mathrm{Diss}}} = \emptyset$ or $\overline{\Gamma_{\mathrm{Dir}}} \cap \overline{\Gamma_{\mathrm{Diss}}} = \gamma$ for a one-dimensional curve γ. In the latter case, we assume that the angle formed by Γ_{Diss} and Γ_{Dir} along γ is less than or equal to $\frac{\pi}{2}$. We further assume that Γ_{Diss} is $C^{1,1}$ and that Γ_{Dir} is $C^{1,1}$ except at a finite set $\mathscr{C}$ of “corners” $\boldsymbol{c}$. Finally, we require that there exists $\ell > 0$ such that, for each corner $\boldsymbol{c} \in \mathscr{C}$, the intersection $\Omega \cap B(\boldsymbol{c}, \ell)$ corresponds to $C_{\boldsymbol{c}} \cap B(\boldsymbol{c}, \ell)$, where $C_{\boldsymbol{c}}$ is a cone centred at $\boldsymbol{c}$.

When the domain Ω is convex or smooth, one easily obtains a bound for the $H^2(\Omega)$ norm of the solution from (2.3) by applying a shift theorem for the Laplace operator. However, the class of domains we consider does not have such shifting properties. The analysis of higher-order derivatives of the solution must therefore be carried out carefully by explicitly analysing the corner singularities.

3 The Case of a Cone

Given an open subset G of the unit sphere of $\mathbb{R}^3$ with a $C^{1,1}$ boundary, we consider the truncated cone $\Gamma_{G,R} \subset \mathbb{R}^3$ of radius $R > 0$ defined by

$$\Gamma_{G,R} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in G \text{ and } |\boldsymbol{x}| < R \right\}.$$

Its boundary is split into two parts

$$\Gamma_{\mathrm{Diss}} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in G \text{ and } |\boldsymbol{x}| = R \right\},$$
$$\Gamma_{\mathrm{Dir}} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in \partial G \text{ and } |\boldsymbol{x}| < R \right\},$$

corresponding respectively to the radial and angular portions of $\partial\Gamma_{G,R}$.

Since $\Gamma_{G,R}$ is not necessarily convex at the origin, singularities may occur at the origin for the solution of problem (2.1) in $\Gamma_{G,R}$ with the previous choice of Γ_{Diss} and Γ_{Dir}. Due to [6, Corollary 5.12] and the fact that the angle along Γ_{Dir} and Γ_{Diss} is equal to $\frac{\pi}{2}$, the solution always belongs to $H^s(\Gamma_{G,R})$ for some $s \in (\frac{3}{2}, 2]$ (details are given below). Furthermore, since $\boldsymbol{x} \cdot \boldsymbol{n} = 0$ on Γ_{Dir} and $\boldsymbol{x} \cdot \boldsymbol{n} = R$ on Γ_{Diss}, we can apply [17, Proposition 3.3] to see that Assumption 2.1 is satisfied.

If Ω has a smooth boundary or if it is convex, it easily follows that $u \in H^2(\Omega)$ with its semi-norm $|u|_{2,\Omega}$ explicitly controlled in terms of ω. Since we allow non-convex domains $\Gamma_{G,R}$ here, the solution may present a corner singularity at the origin. Hereafter, we propose a splitting of the solution u into a regular part belonging to $\tilde{u}_r \in H^2(\Gamma_{G,R})$ and a singular part $S \in H^{1+s-\varepsilon}(\Gamma_{G,R})$ for a carefully selected $s > 0$. We show that $|\tilde{u}_r|_{2,\Gamma_{G,R}}$ behaves as $|u|_{2,\Gamma_{G,R}}$ when the domain is regular, and we establish new estimates for $|S|_{1+s-\varepsilon}$.

In this section, we require the following key properties linked to Bessel functions. Before their statements, we recall that, for an arbitrary real number $\nu \in (\frac{1}{2}, 1)$, Bessel functions of first and second kind are defined by

$$J_{\pm\nu}(\rho) = \left(\frac{\rho}{2}\right)^{\pm\nu} \sum_{l=0}^{+\infty} \frac{1}{l!\,\Gamma(\pm\nu + l + 1)} \left(-\frac{\rho^2}{4}\right)^l \quad \text{for all } \rho \geq 0,$$
$$Y_\nu(\rho) = \frac{J_\nu(\rho)\cos(\nu\pi) - J_{-\nu}(\rho)}{\sin(\nu\pi)} \quad \text{for all } \rho \geq 0,$$

while the corresponding Hankel function of second kind is defined by

$$H_\nu^{(2)}(\rho) = J_\nu(\rho) - iY_\nu(\rho) \quad \text{for all } \rho \geq 0.$$

Proposition 3.1. *For all $\alpha \in (\frac{1}{2}, 1)$, there exists $\omega_0 > 0$ large enough such that the following properties hold:*

$$\int_0^R |J_\alpha(\omega r)|^2 r\,dr \le C(\alpha, R, \omega_0)\omega^{-1}, \tag{3.1}$$

$$\int_0^R |H_\alpha^{(2)}(\omega r)|^2 r\,dr = C(\alpha, R, \omega_0)\omega^{-1} + \mathcal{O}(\omega^{-3}), \tag{3.2}$$

$$\frac{Y'_\alpha(\omega R) + iY_\alpha(\omega R) - \frac{Y_\nu(\omega R)}{2\omega R}}{J'_\alpha(\omega R) + iJ_\alpha(\omega R) - \frac{J_\nu(\omega R)}{2\omega R}} = -i + \mathcal{O}(\omega^{-1}), \tag{3.3}$$

$$J_\alpha(\varepsilon\omega)Y'_\alpha(\varepsilon\omega) - J'_\alpha(\varepsilon\omega)Y_\alpha(\varepsilon\omega) = \frac{2}{\varepsilon\pi\omega} \tag{3.4}$$

for all $\omega \ge \omega_0$ and $\varepsilon > 0$.

Proof. Except for (3.3), these results are shown in [4, Proposition 3.1]. For (3.3), consider $\rho \ge 1$. Then it is established in [4, Appendix A] that

$$J'_\nu(\rho) + iJ_\nu(\rho) = i\sqrt{\frac{2}{\pi\rho}}e^{i\kappa} + \mathcal{O}(\rho^{-\frac{3}{2}}) \quad \text{and} \quad Y'_\nu(\rho) + iY_\nu(\rho) = i\sqrt{\frac{2}{\pi\rho}}e^{i\kappa} + \mathcal{O}(\rho^{-\frac{3}{2}})$$

with

$$\kappa = \rho - \frac{\nu\pi}{2} - \frac{\pi}{4}.$$

It is also shown in [4, Lemma A.1] that

$$|J_\nu(\rho)| \le C(\nu)\rho^{-\frac{1}{2}}, \quad |Y_\nu(\rho)| \le C(\nu)\rho^{-\frac{1}{2}}.$$

These properties imply that

$$J'_\nu(\rho) + iJ_\nu(\rho) - \frac{J_\nu(\rho)}{2\rho} = i\sqrt{\frac{2}{\pi\rho}}e^{i\kappa} + \mathcal{O}(\rho^{-\frac{3}{2}}) \tag{3.5}$$

as well as

$$Y'_\nu(\rho) + iY_\nu(\rho) - \frac{Y_\nu(\rho)}{2\rho} = \sqrt{\frac{2}{\pi\rho}}e^{i\kappa} + \mathcal{O}(\rho^{-\frac{3}{2}}).$$

Hence we easily conclude that

$$\frac{Y'_\nu(\rho) + iY_\nu(\rho) - \frac{Y_\nu(\rho)}{2\rho}}{J'_\nu(\rho) + iJ_\nu(\rho) - \frac{J_\nu(\rho)}{2\rho}} = \frac{1}{i} + \mathcal{O}(\rho^{-1}),$$

from which (3.3) follows. □

3.1 Splitting of the Solution

We propose a splitting of the solution u into a regular part $\tilde{u}_r \in H^2(\Gamma_{G,R})$ and a singular part. The singular properties of the Helmholtz operator are strongly linked to the ones of the Laplacian so that our analysis strongly hinges on the work of Grisvard [15] and Dauge [6]. Before stating our result, we first recall that the corner singularities are related to the eigenvalues and eigenvectors of the Laplace–Beltrami operator Δ_{LB} on G with Dirichlet boundary conditions. Namely, denote by $(\nu_j)_{j=1}^\infty$ the sequence of positive eigenvalues of $-\Delta_{\mathrm{LB}}$ with $0 < \nu_1 < \nu_2 \le \cdots$ (counted according to their multiplicities), and let $w_j \in H_0^1(G)$ be its associated eigenvector, namely

$$-\Delta_{\mathrm{LB}} w_j = \nu_j w_j \text{ on } G,$$

that we may suppose to form an orthonormal basis of $L^2(G)$.

Then, for all $j \in \mathbb{N}^* = \{1, 2, \dots\}$, we set

$$\lambda_j = -\frac{1}{2} + \sqrt{\nu_j + \frac{1}{4}}$$

and set

$$\tilde{s}_j = \chi(\rho)\rho^{\lambda_j} w_j(\vartheta),$$

where (ρ, ϑ) are the spherical coordinates and χ is a C^∞ cutoff function that equals 1 in a neighbourhood of the origin and 0 close to Γ_{Diss}. Note that λ_j is always positive and that, for $\nu_j < \frac{3}{4}$ (or equivalently $\lambda_j \in (0, \frac{1}{2})$), $\tilde{s}_j \notin H^2(\Gamma_{G,R})$, while $\Delta \tilde{s}_j \in C^\infty(\overline{\Gamma_{G,R}})$ since $\Delta(\rho^{\lambda_j} w_j(\vartheta)) = 0$.

Now, if $\nu_k \neq \frac{3}{4}$ for all $k \in \mathbb{N}^*$, using [6, Lemma 17.13] or [15, Theorem 8.2.2.8], we can state that the solution u of (2.1) in $\Gamma_{G,R}$ (with the previous choice of Γ_{Dir} and Γ_{Diss}) can be decomposed as

$$u = \tilde{u}_0 + \sum_{0<\lambda_j<\frac{1}{2}} \tilde{c}_{\omega,j}(f)\tilde{s}_j \tag{3.6}$$

where $\tilde{u}_0 \in H^2(\Gamma_{G,R})$ is the regular part, while the second term represents the corner singularity of the solution, $\tilde{c}_{\omega,j}(f) \in \mathbb{C}$ is a constant depending on the data of the problem.

Decomposition (3.6) is especially useful when analysing Laplace problems, as $\rho^{\lambda_j} w_j(\vartheta)$ is a harmonic function. Also, this decomposition will be useful when analysing the approximation properties of finite element spaces. However, it is tricky to directly estimate the constant $\tilde{c}_{\omega,j}(f)$. As a result, we will use the decomposition

$$u = u_0 + \sum_{0<\lambda_j<\frac{1}{2}} c_{\omega,j}(f)s_j \tag{3.7}$$

with $u_0 \in H^2(\Gamma_{G,R})$, and s_j is in the form

$$s_j = \chi(\rho)\rho^{-\frac{1}{2}} J_{\alpha_j}(\omega\rho) w_j(\vartheta),$$

where for shortness we have set $\alpha_j = \lambda_j + \frac{1}{2}$. Indeed, it is easy to check that this function satisfies (see [8, § 6])

$$(\Delta + \omega^2)(\rho^{-\frac{1}{2}} J_{\alpha_j}(\omega\rho) w_j(\vartheta)) = 0, \tag{3.8}$$

and since $J_{\alpha_j}(\omega\rho) \sim (\omega\rho)^{\alpha_j}$ near zero, s_j has the same behaviour as $\tilde{s}_j$ near the origin (see the details below). The advantage of decomposition (3.7) over (3.6) is that the representation of the singularity satisfies (3.8). As a result, decomposition (3.7) is easier to handle, and we shall use it to estimate $c_{\omega,j}(f)$. We easily recover an estimate for $\tilde{c}_{\omega,j}(f)$ in a "post-processing" fashion.

In Theorem 3.2, we show that the solution u can be decomposed according to (3.6) or (3.7). Furthermore, we give a relation between the constants $c_{\omega,j}(f)$ and $\tilde{c}_{\omega,j}(f)$.

Theorem 3.2. *Assume that $\nu_k \neq \frac{3}{4}$ for all $k \in \mathbb{N}^*$. Then, for all $\omega \geq \omega_0$ and $f \in L^2(\Gamma_{G,R})$, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R})$ is the solution to* (2.2)*, then* (3.6) *holds with some function $\tilde{u}_0 \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R}) \cap H^2(\Gamma_{G,R})$ and constants $\tilde{c}_{\omega,j}(f) \in \mathbb{C}$.*

Furthermore, (3.7) *also holds for some function $u_0 \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R}) \cap H^2(\Gamma_{G,R})$, where*

$$c_{\omega,j}(f) = 2^{\alpha_j}\Gamma(\alpha_j + 1)\omega^{-\alpha_j}\tilde{c}_{\omega,j}(f). \tag{3.9}$$

Proof. The existence and uniqueness of $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R})$ being guaranteed, we can look at u as a solution to

$$\begin{cases} -\Delta u = \tilde{f} & \text{in } \Gamma_{G,R}, \\ u = 0 & \text{on } \Gamma_{\mathrm{Dir}}, \\ \nabla u \cdot \boldsymbol{n} = g & \text{on } \Gamma_{\mathrm{Diss}}, \end{cases}$$

where $\tilde{f} = f + \omega^2 u \in L^2(\Gamma_{G,R})$, $g = i\omega u \in \tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})$ (as usual, for $s > 0$, a function g belongs to $\tilde{H}^s(\Gamma_{\mathrm{Diss}})$ if $\tilde{g}$, its extension by zero outside Γ_{Diss}, belongs to $H^s(\partial\Gamma_{G,R})$).

As g belongs to $\tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})$, by [14], there exists an element $\eta \in H^2(\Gamma_{G,R})$ such that

$$\gamma_0\eta = 0, \quad \gamma_0(\nabla\eta) \cdot \boldsymbol{n} = \tilde{g} = i\omega u \text{ on } \Gamma, \tag{3.10}$$

with the estimate

$$\|\eta\|_{2,\Gamma_{G,R}} \leq C(G,R)\|\tilde{g}\|_{H^{\frac{1}{2}}(\Gamma_{\text{Diss}})} = C(G,R)\omega\|u\|_{H^{\frac{1}{2}}(\Gamma_{\text{Diss}})}$$

for some positive constant $C(G,R)$ that depends only on G and R. Hence, by a trace theorem and estimate (2.3), we deduce that

$$\|\eta\|_{2,\Gamma_{G,R}} \leq C(G,R,\omega_0)\omega\|f\|_{0,\Gamma_{G,R}}. \tag{3.11}$$

Since $\gamma_0\eta = 0$, it is also clear that we have $\eta \in H^1_0(\Gamma_{G,R}) \subset H^1_{\Gamma_{\text{Dir}}}(\Gamma_{G,R})$.

As a result, we see that $v = u - \eta \in H^1_{\Gamma_{\text{Dir}}}(\Gamma_{G,R})$ is the solution to

$$\begin{cases} -\Delta v = h & \text{in } \Gamma_{G,R}, \\ v = 0 & \text{on } \Gamma_{\text{Dir}}, \\ \nabla v \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_{\text{Diss}}, \end{cases} \tag{3.12}$$

where $h = f + \omega^2 u + \Delta\eta \in L^2(\Gamma_{G,R})$. This allows us to apply [6, Lemma 17.4] or [15, Theorem 8.2.2.8], stating that v admits the splitting

$$v = \tilde{v}_0 + \sum_{0<\lambda_j<\frac{1}{2}} \tilde{c}_{\omega,j}\tilde{s}_j, \tag{3.13}$$

where $\tilde{v}_0$ belongs to $H^2(\Gamma_{G,R})$ and $\tilde{c}_{\omega,j} \in \mathbb{C}$ depends continuously on the L^2 norm of h. This yields (3.6) by setting $\tilde{u}_0 = \tilde{v}_0 + \eta$.

Once (3.6) is established, (3.7) and (3.9) directly follow from a careful inspection of the definition of J_α. Indeed, if we isolate the first term in the development of J_{α_j}, we see that

$$\rho^{-\frac{1}{2}}J_{\alpha_j}(\omega\rho)w_j(\vartheta) = \frac{\omega^{\alpha_j}}{2^{\alpha_j}\Gamma(\alpha_j+1)}\rho^{\lambda_j}w_j(\vartheta) + \phi_j,$$

with $\phi_j \in H^1_{\Gamma_{\text{Dir}}}(\Gamma_{G,R}) \cap H^2(\Gamma_{G,R})$. □

3.2 Estimation of $c_{\omega,j}(f)$

For each $\omega \geq \omega_0$, and j such that $\lambda_j < \frac{1}{2}$, it is clear that the mapping $c_{\omega,j}\colon L^2(\Gamma_{G,R}) \to \mathbb{C}$ is a bounded linear form. Then the Riesz representation theorem implies the existence of a unique $w_{\omega,j} \in L^2(\Gamma_{G,R})$ such that

$$c_{\omega,j}(f) = (f, w_{\omega,j}) \quad \text{for all } f \in L^2(\Gamma_{G,R}). \tag{3.14}$$

Lemma 3.3. *For all $\omega \geq \omega_0$, $w_{\omega,j}$ can be characterised as the unique element of $L^2(\Gamma_{G,R})$ satisfying the following conditions (see Appendix A for the definitions used below):*

$$-\omega^2 w_{\omega,j} - \Delta w_{\omega,j} = 0 \quad \text{in } \mathcal{D}'(\Gamma_{G,R}), \tag{3.15}$$

$$\nabla w_{\omega,j}\cdot\boldsymbol{n} + i\omega w_{\omega,j} = 0 \quad \text{in } (\tilde{H}^{\frac{3}{2}}(\Gamma_{\text{Diss}}))', \tag{3.16}$$

$$w_{\omega,j} = 0 \quad \text{in } (\tilde{H}^{\frac{1}{2}}(\Gamma_{\text{Dir}}) \cap V_0^{\frac{1}{2}}(\Gamma_{\text{Dir}}))', \tag{3.17}$$

$$(-\omega^2 s_j - \Delta s_j, w_{\omega,j}) = 1, \tag{3.18}$$

$$(-\omega^2 s_k - \Delta s_k, w_{\omega,j}) = 0 \quad \text{for all } k \neq j \text{ such that } \lambda_k < \frac{1}{2}. \tag{3.19}$$

Proof. First, let us prove that the function $w_{\omega,j} \in L^2(\Gamma_{G,R})$ defined in (3.14) satisfies conditions (3.15)–(3.19).

To prove (3.15), consider $\phi \in \mathcal{D}(\Gamma_{G,R})$, and define $f = -\omega^2\phi - \Delta\phi$. By definition of f, it is clear that $\phi = \phi_R + \sum_{0<\lambda_k<\frac{1}{2}} c_{\omega,k}(f)s_k$ with $\phi_R \in H^2(\Gamma_{G,R})$ and $c_{\omega,k}(f) = 0$ since $\phi \in \mathcal{D}(\Gamma_{G,R})$. As a result,

$$c_{\omega,j}(-\omega^2\phi - \Delta\phi) = (-\omega^2\phi - \Delta\phi, w_{\omega,j}) = 0$$

for all $\phi \in \mathcal{D}(\Gamma_{G,R})$, which is precisely (3.15).

To analyse the boundary conditions satisfied by $w_{\omega,j}$, we use two times Corollary A.5 noticing that $w_{\omega,j}$ belongs to $D(\Delta, L^2(\Gamma_{G,R}))$. Namely, by first taking an arbitrary $\beta \in \tilde{H}^{\frac{1}{2}}(\Gamma_{\text{Dir}}) \cap V_0^{\frac{1}{2}}(\Gamma_{\text{Dir}})$, we consider

$u = \mathcal{R}(0, \beta, 0, 0)$, where $\mathcal{R}$ is the right inverse of T introduced (and fixed) in the proof of Theorem A.4. As $u \in V^2_{00}(\Gamma_{G,R}) \subset H^2(\Gamma_{G,R})$, we have $(\Delta u + \omega^2 u, w_{\omega,j}) = 0$. Hence, using (A.3) on the pair $(u, w_{\omega,j})$, we get

$$0 = (-\omega^2\phi - \Delta\phi, w_{\omega,j}) = \langle \gamma_{\mathrm{Dir}}\nabla u \cdot \boldsymbol{n}, \gamma_{\mathrm{Dir}} w_{\omega,j}\rangle = \langle \beta, \gamma_{\mathrm{Dir}} w_{\omega,j}\rangle.$$

Since β is arbitrary in $\tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}}) \cap V_0^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}})$, we find (3.17).

In the same manner, we take an arbitrary $\gamma \in \tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Diss}})$ and set $\delta = i\omega\gamma$ that belongs to $\tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})$ because $\tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Diss}})$ is continuously embedded into $\tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})$. Then we set $u = \mathcal{R}(0, 0, \gamma, \delta)$ that satisfies $u \in V^2_{00}(\Gamma_{G,R})$ so that $(\Delta u + \omega^2 u, w_{\omega,j}) = 0$ and

$$\gamma_{\mathrm{Diss}}(\nabla u) \cdot \boldsymbol{n} - i\omega\gamma_{\mathrm{Diss}}(u) = 0 \text{ on } \Gamma_{\mathrm{Diss}}.$$

As before, using (A.3) to the pair $(u, w_{\omega,j})$, we get

$$\begin{aligned} 0 &= (-\omega^2\phi - \Delta\phi, w_{\omega,j}) \\ &= \langle \gamma_{\mathrm{Diss}}(\nabla u) \cdot \boldsymbol{n}, \gamma_{\mathrm{Diss}}(w_{\omega,j})\rangle - \langle \gamma_{\mathrm{Diss}}(u), \gamma_{\mathrm{Diss}}(\nabla w_{\omega,j}) \cdot \boldsymbol{n}\rangle \\ &= -\langle \gamma, i\omega\gamma_{\mathrm{Diss}}(w_{\omega,j}) + \gamma_{\mathrm{Diss}}(\nabla w_{\omega,h}) \cdot \boldsymbol{n}\rangle. \end{aligned}$$

Since γ is arbitrary in $\tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Diss}})$, we find (3.16).

Because $\eta = 1$ near the origin, we have $s_j = u_r + s_j$, with $u_r \in H^2(\Gamma_{G,R})$. Also, $\Delta s_j \in L^2(\Gamma_{G,R})$ and s_j satisfies the boundary conditions of problem (2.1). As a result, it is clear that $c_{\omega,j}(-\omega^2 s_j - \Delta s_j) = 1$, and (3.18) follows by the definition (3.14) of $w_{\omega,j}$. Similarly, for k fixed different from j such that $\lambda_k < \frac{1}{2}$, we have $c_{\omega,j}(-\omega^2 s_k - \Delta s_k) = 0$, and (3.19) follows by the definition (3.14) of $w_{\omega,j}$.

We now prove the opposite statement that a function $v_{\omega,j}$ satisfying (3.15)–(3.19) is the function $w_{\omega,j}$ defined by (3.14). Indeed, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R})$ is the unique solution to (2.2) with an arbitrary $f \in L^2(\Gamma_{G,R})$, then the splitting (3.7) and conditions (3.18) and (3.19) satisfied by $v_{\omega,j}$ imply that

$$-(f, v_{\omega,j}) = ((\Delta + \omega^2)u, v_{\omega,j}) = ((\Delta + \omega^2)u_0, v_{\omega,j}) - c_{\omega,j}(f).$$

Then the conclusion follows if we can show that

$$((\Delta + \omega^2)u_0, v_{\omega,j}) = 0. \tag{3.20}$$

Indeed, in such a case, we would have

$$(f, v_{\omega,j} - w_{\omega,j}) = 0 \quad \text{for all } f \in L^2(\Gamma_{G,R}),$$

implying that $v_{\omega,j} = w_{\omega,j}$.

Using Lemma A.1, we can apply Corollary A.5 to the pair $(u_0, v_{\omega,j})$ as $u_0 \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R})$ and it satisfies

$$\nabla u_0 \cdot \boldsymbol{n} = i\omega u_0 \text{ on } \Gamma_{\mathrm{Diss}}.$$

We then find that

$$\begin{aligned} ((\Delta + \omega^2)u_0, v_{\omega,j}) = (u_0, (\Delta + \omega^2)v_{\omega,j}) &+ \langle \gamma_{\mathrm{Dir}}(\nabla u_0) \cdot \boldsymbol{n}, \gamma_{\mathrm{Dir}}(v_{\omega,j})\rangle - \langle \gamma_{\mathrm{Dir}}(u_0), \gamma_{\mathrm{Dir}}(\nabla v_{\omega,j}) \cdot \boldsymbol{n}\rangle \\ &+ \langle \gamma_{\mathrm{Diss}}(\nabla u_0) \cdot \boldsymbol{n}, \gamma_{\mathrm{Diss}}(v_{\omega,j})\rangle - \langle \gamma_{\mathrm{Diss}}(u_0), \gamma_{\mathrm{Diss}}(\nabla v_{\omega,j}) \cdot \boldsymbol{n}\rangle. \end{aligned}$$

By the above properties of u_0 and using the properties (3.15)–(3.17) satisfied by $v_{\omega,j}$, we find that (3.20) holds. □

As we consider a simple geometry, the analytical expression of $w_{\omega,j}$ is available, as shown in the following theorem.

Theorem 3.4. *Recalling that w_j is an eigenvector of the Laplace–Beltrami operator, we have*

$$w_{\omega,j}(\boldsymbol{x}) = -\frac{\pi}{2}\rho^{-\frac{1}{2}}\{Y_{\alpha_j}(\omega\rho) - \ell_{\omega,j}J_{\alpha_j}(\omega\rho)\}w_j(\vartheta),$$

with

$$\ell_{\omega,j} = \frac{Y'_{\alpha_j}(\omega R) + iY_{\alpha_j}(\omega R) - \frac{Y_{\alpha_j}(\omega R)}{2\omega R}}{J'_{\alpha_j}(\omega R) + iJ_{\alpha_j}(\omega R) - \frac{J_{\alpha_j}(\omega R)}{2\omega R}}. \tag{3.21}$$

Proof. We are going to show that the function $w_{\omega,j}$ defined above satisfies conditions (3.15)–(3.19). For the sake of simplicity, let us write

$$\sigma(\boldsymbol{x}) = \rho^{-\frac{1}{2}} J_{\alpha_j}(\omega\rho) w_j(\vartheta), \quad \sigma^\star(\boldsymbol{x}) = \rho^{-\frac{1}{2}} Y_{\alpha_j}(\omega\rho) w_j(\vartheta)$$

so that

$$w_{\omega,j} = -\frac{\pi}{2}(\sigma^\star - \ell_{\omega,j}\sigma).$$

We observe that, by construction, both σ and $\sigma^\star$ satisfy the Helmholtz PDE; as a result,

$$-\omega^2 w_{\omega,j} - \Delta w_{\omega,j} = 0,$$

and (3.15) is satisfied. As $w_{\omega,j}$ clearly belongs to $L^2(\Gamma_{G,R})$, we deduce that $w_{\omega,j}$ belongs to

$$D(\Delta, L^2(\Gamma_{G,R})) = \{v \in L^2(\Gamma_{G,R}) : \Delta v \in L^2(\Gamma_{G,R})\};$$

hence Theorem A.4 gives a meaning of its trace on Γ_{Dir} as element of $(\tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}}))'$. Furthermore, since $w_{\omega,j} \in C^\infty(\Gamma_{G,R} \setminus (0,0))$, it is clear that $w_{\omega,j} = 0$ on Γ_{Dir}, and (3.17) holds.

Boundary condition (3.16) is also satisfied by construction. Indeed, if $\boldsymbol{x} = R\vartheta \in \Gamma_{\mathrm{Diss}}$, we have

$$\begin{aligned}(\nabla\sigma \cdot \boldsymbol{n} + i\omega\sigma)(\boldsymbol{x}) &= \Big[\omega R^{-\frac{1}{2}}(J'_\alpha(\omega R) + iJ_\alpha(\omega R)) - \frac{1}{2}R^{-\frac{3}{2}}J_\alpha(\omega R)\Big] w_j(\vartheta),\\ (\nabla\sigma^\star \cdot \boldsymbol{n} + i\omega\sigma^\star)(\boldsymbol{x}) &= \Big[\omega R^{-\frac{1}{2}}(Y'_\alpha(\omega R) + iY_\alpha(\omega R)) - \frac{1}{2}R^{-\frac{3}{2}}Y_\alpha(\omega R)\Big] w_j(\vartheta)\end{aligned}$$

so that $\nabla w_{\omega,j} \cdot \boldsymbol{n} + i\omega w_{\omega,j} = 0$ on Γ_{Diss} if and only if

$$\omega R^{-\frac{1}{2}}(Y'_\alpha(\omega R) + iY_\alpha(\omega R)) - \frac{1}{2}R^{-\frac{3}{2}}Y_\alpha(\omega R) - \ell_{\omega,j}\Big(\omega R^{-\frac{1}{2}}(J'_\alpha(\omega R) + iJ_\alpha(\omega R)) - \frac{1}{2}R^{-\frac{3}{2}}J_\alpha(\omega R)\Big) = 0.$$

Multiplying by $\omega^{-1}R^{\frac{1}{2}}$ and dividing by the factor in front of $\ell_{\omega,j}$, we find (3.21). Note that (3.5) guarantees that this factor is different from zero for ω large enough.

Let us go on with property (3.19). Indeed, as

$$\omega^2 s_k - \Delta s_k = \omega^2 s_k - 2\lambda_j \frac{\partial\chi}{\partial\rho}\rho^{\lambda_j - 1} w_k + \Delta\chi\rho^{\lambda_j} w_k,$$

we have $\omega^2 s_k - \Delta s_k = h(\rho) w_k$ for some radial function h that satisfies

$$\int_0^R |h(\rho)|^2 \rho^2 \, d\rho < \infty.$$

This automatically leads to (3.19) because

$$\int_G w_k(\vartheta) w_j(\vartheta) \, d\sigma(\vartheta) = 0 \quad \text{for all } k \neq j.$$

Hence it remains to show that (3.18) holds. We have

$$-\frac{2}{\pi}(-\omega^2 s_j - \Delta s_j, w_{\omega,j}) = (-\omega^2\chi\sigma - \Delta(\chi\sigma), \sigma^\star) - \ell_{\omega,j}(-\omega^2(\chi\sigma) - \Delta(\chi\sigma), \sigma).$$

We are going to show that

$$(-\omega^2(\chi\sigma) - \Delta(\chi\sigma), \sigma) = 0, \tag{3.22}$$

$$(-\omega^2(\chi\sigma) - \Delta(\chi\sigma), \sigma^\star) = -\frac{2}{\pi}, \tag{3.23}$$

which will conclude the proof. In spirit, the proof relies on simple integration-by-parts techniques. However, because we manipulate functions with low regularity close to the origin, these integrations by parts have to be done carefully.

The technique is then to subtract the ball $B(0,\varepsilon)$ from $\Gamma_{G,R}$ so that all manipulated functions are C^∞ on $D_{G,R,\varepsilon} = \Gamma_{G,R} \setminus B(0,\varepsilon)$. Then integration by parts is allowed on $D_{G,R,\varepsilon}$, and the desired inner products are recovered by letting $\varepsilon \to 0$.

The beginning of the proof of (3.22) and (3.23) is the same. Thus, let us set $\mu = \sigma$ or $\sigma^\star$. Because $\sigma, \mu \in C^\infty(D_{G,R,\varepsilon})$ and $-\omega^2\mu - \Delta\mu = 0$, double integration by parts yields

$$\int\limits_{D_{G,R,\varepsilon}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\mu = -\int\limits_{\partial D_{G,R,\varepsilon}} (\nabla(\chi\sigma)\cdot \boldsymbol{n}\mu - \chi\sigma\nabla\mu\cdot\boldsymbol{n}).$$

Since $\chi\sigma = \nabla(\chi\sigma)\cdot\boldsymbol{n} = 0$ on Γ_{Diss}, $\sigma = \mu = 0$ on $\Gamma_{\mathrm{Dir}} \setminus B_\varepsilon$, and $\chi = 1$ on $B_{\frac{R}{2}}$, we have

$$\begin{aligned}\int\limits_{D_{G,R,\varepsilon}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\mu &= -\int\limits_{\Gamma_{G,R}\cap\partial B_\varepsilon} (\nabla(\chi\sigma)\cdot\boldsymbol{n}\mu - \chi\sigma\nabla\mu\cdot\boldsymbol{n})\rho^2\,d\sigma(\vartheta)\\ &= -\varepsilon^2 \int\limits_{|\boldsymbol{x}|=\varepsilon,\vartheta\in G} (\nabla\sigma(\varepsilon\vartheta)\cdot\boldsymbol{n}\mu(\varepsilon\vartheta) - \sigma(\varepsilon\vartheta)\nabla\mu(\varepsilon\vartheta)\cdot\boldsymbol{n})\,d\sigma(\vartheta).\end{aligned} \tag{3.24}$$

Obviously, when $\mu = \sigma$, the right-hand side of (3.24) vanishes so that

$$\int\limits_{D_{G,R,\varepsilon}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\sigma\,d\sigma(\vartheta) = 0,$$

and (3.22) follows since

$$(-\omega^2(\chi\sigma) - \Delta(\chi\sigma),\sigma) = \int\limits_{\Gamma_{G,R}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\overline{\sigma} = \int\limits_{\Gamma_{G,R}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\sigma = \lim_{\varepsilon\to 0}\int\limits_{D_{G,R,\varepsilon}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\sigma.$$

On the other hand, to prove (3.23), since

$$\begin{aligned}-\nabla\sigma\cdot\boldsymbol{n} &= \frac{\partial\sigma}{\partial\rho} = \Big[-\frac{1}{2}\rho^{-\frac{3}{2}}J_{\alpha_j}(\varepsilon\omega) + \omega\rho^{-\frac{1}{2}}J'_{\alpha_j}(\varepsilon\omega)\Big]w_j(\vartheta) \quad \text{on } \partial B_\varepsilon,\\ -\nabla\sigma^\star\cdot\boldsymbol{n} &= \Big[-\frac{1}{2}\rho^{-\frac{3}{2}}Y_{\alpha_j}(\varepsilon\omega) + \omega\rho^{-\frac{1}{2}}Y'_{\alpha_j}(\varepsilon\omega)\Big]w_j(\vartheta) \quad \text{on } \partial B_\varepsilon,\end{aligned}$$

by (3.24) with $\mu = \sigma^\star$, we have

$$\int\limits_{D_{G,R,\varepsilon}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\sigma^\star = \varepsilon\omega(J'_\alpha(\varepsilon\omega)Y_\alpha(\varepsilon\omega) - J_\alpha(\varepsilon\omega)Y'_\alpha(\varepsilon\omega))\int\limits_G |w_j(\vartheta)|^2\,d\sigma(\vartheta).$$

As we have that $\int_G |w_j(\vartheta)|^2\,d\sigma(\vartheta) = 1$ and recalling (3.4) from Proposition 3.1, we have

$$J'_\alpha(\varepsilon\omega)Y_\alpha(\varepsilon\omega) - J_\alpha(\varepsilon\omega)Y'_\alpha(\varepsilon\omega) = -\frac{2}{\varepsilon\pi\omega},$$

and we obtain

$$\int\limits_{D_{G,R,\varepsilon}} (-\omega^2\chi\sigma - \Delta(\chi\sigma))\sigma^\star = -\frac{2}{\pi}.$$

Then (3.23) holds by letting $\varepsilon \to 0$. □

Corollary 3.5. *We have*

$$\Big|w_{\omega,j}(\boldsymbol{x}) + i\frac{\pi}{2}\rho^{-\frac{1}{2}}H^{(2)}_{\alpha_j}(\omega\rho)w_j(\vartheta)\Big| \le C(G,R,\omega_0)\omega^{-1}\rho^{-\frac{1}{2}}|J_{\alpha_j}(\omega\rho)||w_j(\vartheta)|, \tag{3.25}$$

$$\|w_{\omega,j}\|_{0,\Gamma_{G,R}} = C(G,R,\omega_0)(\omega^{-\frac{1}{2}} + \mathcal{O}(\omega^{-\frac{3}{2}})). \tag{3.26}$$

Proof. Estimate (3.25) directly follows from property (3.3) from Proposition 3.1. Then, because of (3.25), we have

$$\|w_{\omega,j}\|^2_{0,\Gamma_{G,R}} = C(G,R,\omega_0)\bigg(\int\limits_0^R |H^{(2)}_{\alpha_j}(\omega\rho)|^2\rho\,d\rho + \mathcal{O}(\omega^{-2})\int\limits_0^R |J_{\alpha_j}(\omega\rho)|^2\rho\,d\rho\bigg).$$

In addition, from (3.1) and (3.2), we have

$$\|w_{\omega,j}\|^2_{0,\Gamma_{G,R}} = C(G,R,\omega_0)(\omega^{-1} + \mathcal{O}(\omega^{-3})),$$

and (3.26) follows. □

We are now ready to establish the main result of this section.

Theorem 3.6. *The estimate*

$$|c_{\omega,j}(f)| \leq C(G,R,\omega_0)\omega^{-\frac{1}{2}}\|f\|_{0,\Gamma_{G,R}} \tag{3.27}$$

holds for all $\omega \geq \omega_0$ and $f \in L^2(\Gamma_{G,R})$. Furthermore, estimate (3.27) *is optimal in the sense that, for all $\omega \geq \omega_0$, there exists an $f \in L^2(\Gamma_{G,R})$ such that*

$$|c_{\omega,j}(f)| \geq C(G,R,\omega_0)\omega^{-\frac{1}{2}}\|f\|_{0,\Gamma_{G,R}}. \tag{3.28}$$

Proof. By definition, for all $\omega \geq \omega_0$ and $f \in L^2(\Gamma_{G,R})$, we have

$$|c_{\omega,j}(f)| = |(f, w_{\omega,j})| \leq \|f\|_{0,\Gamma_{G,R}}\|w_{\omega,j}\|_{0,\Gamma_{G,R}},$$
$$|c_{\omega,j}(w_{\omega,j})| = |(w_{\omega,j}, w_{\omega,j})| = \|w_{\omega,j}\|^2_{0,\Gamma_{G,R}}.$$

But Corollary 3.5 shows that

$$C_1(G,R,\omega_0)\omega^{-\frac{1}{2}} \leq \|w_{\omega,j}\|_{0,\Gamma_{G,R}} \leq C_2(G,R,\omega_0)\omega^{-\frac{1}{2}} \quad \text{for all } \omega \geq \omega_0,$$

assuming that ω_0 is large enough. As a result, we have (3.27), and (3.28) follows by taking $f = w_{\omega,j}$. □

Corollary 3.7. *The estimate*

$$|\tilde{c}_{\omega,j}(f)| \leq C(G,R,\omega_0)\omega^{\lambda_j}\|f\|_{0,\Gamma_{G,R}} \tag{3.29}$$

holds for all $\omega \geq \omega_0$ and $f \in L^2(\Gamma_{G,R})$. Furthermore, estimate (3.29) *is optimal in the sense that, for all $\omega \geq \omega_0$, there exists an $f \in L^2(\Gamma_{G,R})$ such that*

$$|\tilde{c}_{\omega,j}(f)| \geq C(G,R,\omega_0)\omega^{\lambda_j}\|f\|_{0,\Gamma_{G,R}}.$$

Proof. Direct consequence of the previous theorem, identity (3.9) and recalling that $\lambda_j = \alpha_j - \frac{1}{2}$. □

3.3 Behaviour of the Regular Part

To complete our analysis, we now need to study the regular part $\tilde{u}_r \in H^2(\Gamma_{G,R})$.

Theorem 3.8. *Assume that $\nu_k \neq \frac{3}{4}$ for all $k \in \mathbb{N}^*$. Then, for all $\omega \geq \omega_0$ and $f \in L^2(\Gamma_{G,R})$, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R})$ is the solution to* (2.2), *then* (3.6) *holds with some constants $\tilde{c}_{\omega,j}(f) \in \mathbb{C}$ satisfying* (3.29) *and a function*

$$\tilde{u}_0 \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R}) \cap H^2(\Gamma_{G,R})$$

satisfying

$$\|\tilde{u}_0\|_{2,\Gamma_{G,R}} \leq C(G,R,\omega_0)\omega\|f\|_{0,\Gamma_{G,R}}. \tag{3.30}$$

Proof. We proceed as in Theorem 3.2 and use the lifting $\eta \in H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R}) \cap H^2(\Gamma_{G,R})$ satisfying (3.10) and (3.11). Then we let $\upsilon = u - \eta$ so that υ is the solution of (3.12) with $h = \Delta u - \Delta\eta = f + \omega^2 u - \Delta\eta$ that belongs to $L^2(\Gamma_{G,R})$ and satisfies

$$\|h\|_{L^2(\Gamma_{G,R})} \lesssim \omega\|f\|_{L^2(\Gamma_{G,R})},$$

owing to (2.3) and (3.11). Now, using the splitting (3.13) of υ, we have

$$\Delta\tilde{\upsilon}_0 = \Delta\upsilon - \sum_{0<\lambda_j<\frac{1}{2}} \tilde{c}_{\omega,j}(f)\Delta(\tilde{s}_j) = -\tilde{h},$$

where

$$\tilde{h} = h + \sum_{0<\lambda_j<\frac{1}{2}} \tilde{c}_{\omega,j}(f)\Delta(\tilde{s}_j)$$

Since $\Delta(\tilde{s}_j)$ is in $L^2(\Gamma_{G,R})$, we deduce from (3.11) and (3.29) that

$$\|\tilde{h}\|_{0,\Gamma_{G,R}} \lesssim \Big(\omega + \sum_{0<\lambda_j<\frac{1}{2}} \omega^{\lambda_j}\Big)\|f\|_{0,\Gamma_{G,R}} \lesssim \omega\|f\|_{0,\Gamma_{G,R}}, \tag{3.31}$$

this last estimate following from the fact that $\lambda_j < \frac{1}{2}$.

Then we see that $\tilde{v}_0 \in H^2(\Gamma_{G,R})$ is the solution to a Laplace problem with mixed boundary condition (3.12), and using [15, Theorem 8.2.2.1] near the origin and the fact that the angle along Γ_{Dir} and Γ_{Diss} is equal to $\frac{\pi}{2}$, we have

$$\|\tilde{v}_0\|_{2,\Gamma_{G,R}} \le C(G,R)(\|\tilde{h}\|_{0,\Gamma_{G,R}} + \|\tilde{v}_0\|_{0,\Gamma_{G,R}}).$$

By applying the Poincaré inequality, we further see that

$$\|\tilde{v}_0\|_{0,\Gamma_{G,R}} \le C(G,R)|\tilde{v}_0|_{1,\Gamma_{G,R}} \le C(G,R)\|\tilde{h}\|_{0,\Gamma_{G,R}},$$

Using this estimate, we find $\|\tilde{v}_0\|_{2,\Gamma_{G,R}} \lesssim \|\tilde{h}\|_{0,\Gamma_{G,R}}$, and with the help of (3.31), we arrive at

$$\|\tilde{v}_0\|_{2,\Gamma_{G,R}} \lesssim \omega\|f\|_{0,\Gamma_{G,R}}.$$

This estimate and (3.11) lead to (3.30), recalling that $\tilde{u}_0 = \tilde{v}_0 + \eta$. □

4 The General Case

We now consider the general case of Helmholtz problems that satisfy Assumptions 2.1 and 2.2.

Recall that ℓ is defined in Assumption 2.2 as the smallest distance between two corners of Γ_{Dir}. As a result, for each corner $\boldsymbol{c} \in \mathscr{C}$, the intersection $\Omega \cap B(\boldsymbol{c},\ell)$ corresponds to the truncated cone

$$\Gamma_{\boldsymbol{c}} = \Big\{\boldsymbol{x} \in \mathbb{R}^3 : \frac{\boldsymbol{x}-\boldsymbol{c}}{|\boldsymbol{x}-\boldsymbol{c}|} \in G_{\boldsymbol{c}} \text{ and } |\boldsymbol{x}-\boldsymbol{c}| < \ell\Big\},$$

where $G_{\boldsymbol{c}}$ is a subset of the unit sphere of $\mathbb{R}^3$. We then denote by $0 < \nu_1^{\boldsymbol{c}} < \nu_2^{\boldsymbol{c}} \le \cdots$ the eigenvalues of the Laplace–Beltrami operator on $G_{\boldsymbol{c}}$, and we set

$$\lambda_j^{\boldsymbol{c}} = -\frac{1}{2} + \sqrt{\nu_j^{\boldsymbol{c}} + \frac{1}{4}}$$

for $j \in \mathbb{N}^\star$. We will also denote by $w_j^{\boldsymbol{c}}$ the eigenfunction associated with $\nu_j^{\boldsymbol{c}}$.

Then, if $\chi \in C^\infty(\mathbb{R})$ is a cutoff function so that $\chi(\rho) = 1$ if $\rho < \frac{\ell}{3}$ and $\chi(\rho) = 0$ if $\rho > \frac{2\ell}{3}$, we can introduce the singular function

$$\tilde{s}_j^{\boldsymbol{c}}(\boldsymbol{x}) = \chi^{\boldsymbol{c}}(\boldsymbol{x}) r_{\boldsymbol{c}}^{\lambda_j^{\boldsymbol{c}}} w_j^{\boldsymbol{c}}(\vartheta),$$

where $\chi^{\boldsymbol{c}}(\boldsymbol{x}) = \chi(r_{\boldsymbol{c}}(\boldsymbol{x}))$ and $r_{\boldsymbol{c}}(\boldsymbol{x})$ is the distance from $\boldsymbol{x}$ to $\boldsymbol{c}$.

With this notation, the general case then easily follows from a localisation argument presented in [4, Theorem 4.1], where the "cone case" is applied to the neighbourhood of each corners of Γ_{Dir}. For the sake of shortness, we do not reproduce the proof here.

Theorem 4.1. *For all $\omega \ge \omega_0$ and $f \in L^2(\Omega)$, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ is the solution to (2.2), there exist a function*

$$\tilde{u}_r \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \cap H^2(\Omega)$$

and constants $c^j_{\omega,j}(f) \in \mathbb{C}$ such that

$$u = \tilde{u}_r + \sum_{\boldsymbol{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\boldsymbol{c}}<\frac{1}{2}} \tilde{c}^{\boldsymbol{c}}_{\omega,j}(f)\tilde{s}_j^{\boldsymbol{c}},$$

and it holds that

$$|\tilde{c}^{\mathbf{c}}_{\omega,j}(f)| \leq C(\Omega, \omega_0)\omega^{\lambda^{\mathbf{c}}_j}\|f\|_{0,\Omega} \tag{4.1}$$

for all $\mathbf{c} \in \mathscr{C}$ *and* $0 < \lambda^{\mathbf{c}}_j < \frac{1}{2}$, *whereas*

$$|\tilde{u}_r|_{2,\Omega} \leq C(\Omega, \omega_0)\omega\|f\|_{0,\Omega}. \tag{4.2}$$

5 Frequency-Explicit Stability of Finite Element Discretisations

5.1 The Finite Element Space

In this section, we assume that we are given a mesh $\mathcal{T}_h$ of Ω. Each element $K \in \mathcal{T}_h$ is obtained from a reference element $\hat{K}$ through a bi-Lipschitz mapping F_K. We then consider a $\mathcal{P}_1$ finite element space

$$V_h = \{v_h \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) : v_h|_K \circ F_K^{-1} \in \mathcal{P}_1(\hat{K}) \text{ for all } K \in \mathcal{T}_h\}.$$

We will further assume that there exists an interpolation operator such that $\mathcal{I}_h : H^{1+s}(\Omega) \cap H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \to V_h$ with

$$|v - \mathcal{I}_h v|_{\ell,\Omega} \leq C(\Omega, s)h^{1+s-\ell}|v|_{1+s,\Omega} \tag{5.1}$$

for all $s \in (\frac{1}{2}, 1]$; see [24, § 3.1].

The solution $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ to problem (2.2) is then approximated by a function $u_h \in V_h$ satisfying

$$B(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \tag{5.2}$$

For the sake of simplicity, we assume in the remainder of this work that

$$\omega h \leq 1. \tag{5.3}$$

This assumption is natural and means that the number of elements per wavelength is bounded from below. We will actually see that more restrictive conditions on h must be met anyway to ensure the well-posedness of the finite element problem.

In order to simplify notation, we introduce the ω-dependent norm

$$|||v|||^2 = \omega^2\|v\|^2_{0,\Omega} + |v|^2_{1,\Omega} \quad \text{for all } v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega),$$

that is equivalent to the standard $H^1(\Omega)$ norm, but is more natural considering the properties of "usual" solutions to (2.2).

We start our analysis with an interpolation result for the singular functions.

Lemma 5.1. *For all* $\mathbf{c} \in \mathscr{C}$ *and* $0 < \lambda^{\mathbf{c}}_j < \frac{1}{2}$, *we have*

$$|||\tilde{s}^{\mathbf{c}}_j - \mathcal{I}_h\tilde{s}^{\mathbf{c}}_j||| \leq C(\Omega, \varepsilon)h^{\lambda^{\mathbf{c}}_j + \frac{1}{2} - \varepsilon} \tag{5.4}$$

for all $\varepsilon > 0$.

Proof. It is clear that $\tilde{s}^{\mathbf{c}}_j \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. Besides, simple computations show that $\tilde{s}^{\mathbf{c}}_j \in H^{\frac{3}{2}+\lambda^{\mathbf{c}}_j-\varepsilon}(\Omega)$. Since $\tilde{s}^{\mathbf{c}}_j$ only depends on Ω, (5.4) easily follows from (5.1) and the assumption (5.3) that $\omega h \leq 1$. □

Then the core of our theoretical findings is the following frequency-explicit interpolation estimate for the solution to the Helmholtz problem.

Lemma 5.2. *For* $\phi \in L^2(\Omega)$, *define* $u_\phi \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ *as the solution to*

$$B(u_\phi, v) = (\phi, v) \quad \textit{for all } v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega).$$

Then we have

$$|||u_\phi - \mathcal{I}_h u_\phi||| \leq C(\Omega, \omega_0, \varepsilon)(\omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon} + \omega h)\|\phi\|_{0,\Omega}, \tag{5.5}$$

where $\lambda = \min_{\mathbf{c}\in\mathscr{C}} \min_j \lambda_j^{\mathbf{c}}$. Furthermore, estimate (5.5) also holds for the function $u_\phi^\star$ defined as the unique element of $H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, solution to

$$B(v, u_\phi^\star) = (v, \phi) \quad \text{for all } v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega).$$

Proof. We recall that we have the decomposition

$$u_\phi = \tilde{u}_r + \sum_{\mathbf{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\mathbf{c}}<\frac{1}{2}} \tilde{c}^{\mathbf{c}}_{\omega,j}(\phi)\tilde{s}_j^{\mathbf{c}},$$

where $\tilde{s}_j^{\mathbf{c}} \in H^{\frac{3}{2}+\lambda_j^{\mathbf{c}}-\varepsilon}(\Omega) \cap H^1_0(\Omega)$ for all $\varepsilon > 0$ and $\tilde{u}_r \in H^2(\Omega) \cap H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. Hence, since

$$u_\phi - \mathcal{I}_h u_\phi = (\tilde{u}_r - \mathcal{I}_h\tilde{u}_r) + \sum_{\mathbf{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\mathbf{c}}<\frac{1}{2}} \tilde{c}^{\mathbf{c}}_{\omega,j}(\phi)(\tilde{s}_j - \mathcal{I}_h\tilde{s}_j),$$

we have

$$|||u_\phi - \mathcal{I}_h u_\phi||| \le |||\tilde{u}_r - \mathcal{I}_h\tilde{u}_r||| + \sum_{\mathbf{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\mathbf{c}}<\frac{1}{2}} |\tilde{c}^{\mathbf{c}}_{\omega,j}(\phi)|\,|||\tilde{s}_j^{\mathbf{c}} - \mathcal{I}_h\tilde{s}_j^{\mathbf{c}}|||$$

Recalling that $\omega h \le 1$, we obtain from (5.1) and (4.2) that

$$|||\tilde{u}_r - \mathcal{I}_h\tilde{u}_r||| \le C(\Omega)h|\tilde{u}_r|_{2,\Omega} \le C(\Omega, \omega_0, \gamma)\omega h\|\phi\|_{0,\Omega}. \tag{5.6}$$

On the other hand, recalling (4.1) and (5.4), we have

$$\begin{aligned}\sum_{\mathbf{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\mathbf{c}}<\frac{1}{2}} |\tilde{c}^{j}_{\omega,j}(\phi)|\,|||\tilde{s}_j - \mathcal{I}_h\tilde{s}_j||| &\le C(\Omega, \omega_0, \varepsilon) \sum_{\mathbf{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\mathbf{c}}<\frac{1}{2}} \omega^{\lambda_j^{\mathbf{c}}} h^{\lambda_j^{\mathbf{c}}+\frac{1}{2}-\varepsilon}\|\phi\|_{0,\Omega}\\ &\le C(\Omega, \omega_0, \varepsilon)\omega^{-\frac{1}{2}+\varepsilon} \sum_{\mathbf{c}\in\mathscr{C}} \sum_{0<\lambda_j^{\mathbf{c}}<\frac{1}{2}} (\omega h)^{\lambda_j^{\mathbf{c}}+\frac{1}{2}-\varepsilon}\|\phi\|_{0,\Omega}.\end{aligned}$$

Since $\omega h \le 1$, we have $(\omega h)^{\lambda_j^{\mathbf{c}}+\frac{1}{2}-\varepsilon} \le (\omega h)^{\lambda+\frac{1}{2}-\varepsilon}$, and it follows that

$$\omega^{1-l}\sum_{j=1}^{N} |\tilde{c}^{j}_{\omega,j}(\phi)||\tilde{s}_j - \mathcal{I}_h\tilde{s}_j|_{l,\Omega} \le C(\Omega, \omega_0, \varepsilon)\omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon}\|\phi\|_{0,\Omega}. \tag{5.7}$$

Then (5.5) follows from (5.6) and (5.7). □

Using Lemma 5.2, we can easily derive sufficient condition for the quasi-optimality of the finite element solution following the so-called Schatz argument [10, 20, 25]. Interestingly, the resulting quasi-optimality, "$\omega^2 h^{1-\varepsilon}$ is small", with $\varepsilon > 0$ arbitrarily small, is almost the same as in the case of a smooth domain in the absence of singularity.

Theorem 5.3. *For all $\varepsilon > 0$, there exists a constant $C_\star(\Omega, \omega_0, \varepsilon)$ such that, for all $\omega \ge \omega_0$ and h such that $\omega^2 h^{1-\varepsilon} \le C_\star(\Omega, \omega_0, \varepsilon)$, problem (5.2) admits a unique solution $u_h \in V_h$. Furthermore, in this case, the finite element solution u_h is quasi-optimal,*

$$|||u - u_h||| \le C(\Omega, \omega_0)|||u - \mathcal{I}_h u|||, \tag{5.8}$$

and the error estimate

$$|||u - u_h||| \le C(\Omega, \omega_0, \varepsilon)(\omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon} + \omega h)\|f\|_{0,\Omega} \tag{5.9}$$

holds.

Proof. The proof uses the standard Schatz argument. Let $u_h \in V_h$ be any solution to (5.2). We introduce $\xi \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, the solution to $B(v, \xi) = (v, u - u_h)$ for all $v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, so that

$$\|u - u_h\|^2_{0,\Omega} = B(u - u_h, \xi) = B(u - u_h, \xi - \mathcal{I}_h\xi).$$

By definition of ξ, recalling (5.5), we have

$$|||\xi - \mathcal{I}_h\xi||| \leq C(\Omega, \omega_0)(\omega h + \omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon})\|u - u_h\|_{0,\Omega}$$

so that

$$\begin{aligned}\|u - u_h\|_{0,\Omega}^2 &= B(u - u_h, \xi - \mathcal{I}_h\xi)\\ &\leq C(\Omega)|||u - u_h||| \cdot |||\xi - \mathcal{I}_h\xi|||\\ &\leq C(\Omega, \omega_0)(\omega h + \omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon})|||u - u_h||| \cdot \|u - u_h\|_{0,\Omega}.\end{aligned}$$

It follows that

$$\|u - u_h\|_{0,\Omega} \leq C(\Omega, \omega_0)(\omega h + \omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon})|||u - u_h|||.$$

Now, we write

$$\begin{aligned}|||u - u_h|||^2 &\leq \operatorname{Re} B(u - u_h, u - u_h) + 2\omega^2\|u - u_h\|_{0,\Omega}^2\\ &= \operatorname{Re} B(u - u_h, u - \mathcal{I}_h u_h) + 2\omega^2\|u - u_h\|_{0,\Omega}^2\\ &\leq C(\Omega, \omega_0)\{|||u - u_h||| \cdot |||u - \mathcal{I}_h u||| + \omega^2(\omega h + \omega^{-\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon})^2|||u - u_h|||^2\},\end{aligned}$$

and simplifying by $|||u - u_h|||$, we obtain

$$\{1 - C(\Omega, \omega_0)(\omega^2 h + \omega^{\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon})^2\}|||u - u_h||| \leq C(\Omega, \omega_0, \varepsilon)|||u - \mathcal{I}_h u|||. \tag{5.10}$$

Recalling that $\omega h \leq 1$, since $\lambda \geq 0$, we have

$$\omega^{\frac{1}{2}+\varepsilon}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon} \leq \omega^{\frac{1}{2}+\varepsilon}(\omega h)^{\frac{1}{2}-\varepsilon} = \omega^{\frac{1}{2}+\varepsilon}\omega^{\frac{1}{2}+\varepsilon}h^{\frac{1}{2}-\varepsilon} = \omega h^{\frac{1}{2}-\varepsilon} = (\omega^2 h^{1-2\varepsilon})^{\frac{1}{2}}$$

Hence, assuming that $\omega^2 h^{1-2\varepsilon}$ is small enough, we have

$$C(\Omega, \omega_0, \varepsilon)(\omega^2 h + \omega^{-\frac{1}{2}}(\omega h)^{\lambda+\frac{1}{2}-\varepsilon})^2 \leq \frac{1}{2},$$

and (5.8) follows from (5.10). Finally, (5.9) follows from (5.8) and (5.5).

The uniqueness of u_h is a direct consequence of (5.9), and existence follows since u_h is defined as the solution of a finite-dimensional square linear system. □

6 Numerical Example

We consider one numerical example that illustrates our key findings. The domain is the revolution cone

$$\Omega = \left\{ \boldsymbol{x} = (\rho\cos\phi\sin\theta, \rho\sin\phi\sin\theta, \rho\cos\theta) : \begin{array}{l} 0 < \rho < 1,\\ 0 < \theta < \theta_\star,\\ 0 < \phi < 2\pi \end{array} \right\}$$

of opening $\theta_\star = \arccos(z_\star)$, where $z_\star \sim -0.958504193084082$ is a zero of the Legendre function $P_\lambda^0(z)$ with $\lambda = 0.25$. The boundary $\partial\Omega$ is split into $\Gamma_{\mathrm{Diss}} = \{\boldsymbol{x} \in \partial\Omega; |\boldsymbol{x}| = 1\}$ and $\Gamma_{\mathrm{Dir}} = \partial\Omega \setminus \Gamma_{\mathrm{Diss}}$. A cutoff of the domain along the $y = 0$ plane is represented on Figure 1. Then, following [7], the singular function

$$\boldsymbol{x} \to \rho^\lambda P_\lambda^0(\cos(\theta))$$

belongs to the kernel of Laplace operator. Similarly, the function

$$S(\boldsymbol{x}) = \rho^{-\frac{1}{2}} J_{\lambda+\frac{1}{2}}(\omega r)P_\lambda^0(\cos(\theta))$$

is in the kernel of the Helmholtz operator with frequency ω. For our example, the analytic solution will be

$$u = \chi(\boldsymbol{x})S(\boldsymbol{x}),$$

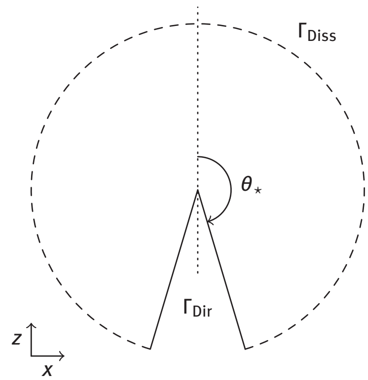

Figure 1: Cut of the domain.

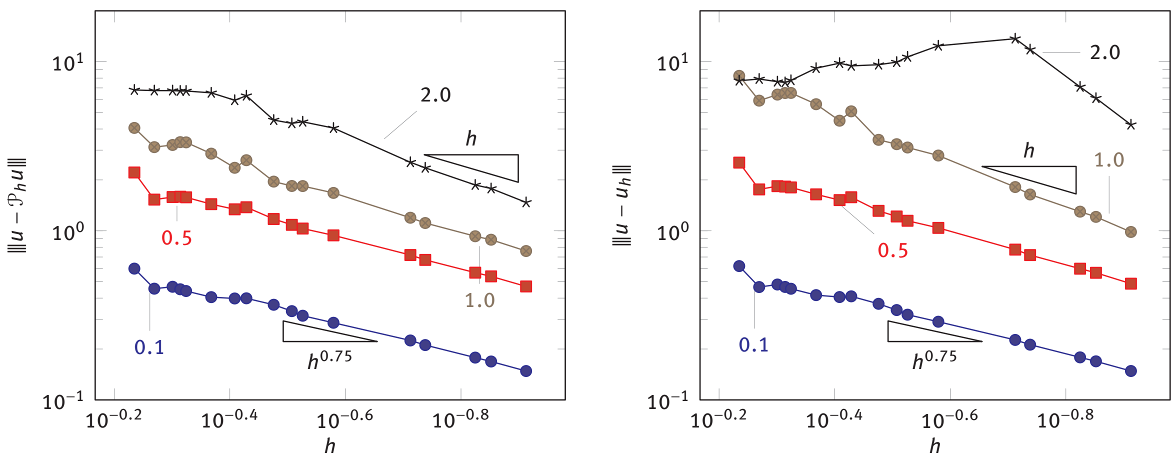

Figure 2: Convergence as the mesh is refined.

where χ is a smooth radial cutoff function that equals one close to 0 and zero close to 1. Since S is the solution to the Helmholtz problem (without the boundary condition on Γ_{Diss}), the right-hand side corresponding to u is given by

$$f = \Delta\chi S + 2\nabla\chi \cdot \nabla S.$$

Notice that, since the supports of $\nabla\chi$ and $\Delta\chi$ do not contain $\mathbf{0}$, the function f is smooth.

We consider a series of meshes generated by the software package `gmsh` [13]. Contrary to our theoretical analysis, these meshes consist of straight elements, with nodes exactly lying on the boundary of Ω. However, the resulting "variational crime" is expected to result in an error smaller than the best-approximation error since we are working with lowest-order Lagrange elements.

Figure 2 reports the convergence histories of the finite element solutions u_h and the best approximations $\mathcal{P}_h u$ of u in the $|||\cdot|||$ norm for a sequence of uniform meshes and the frequencies $\omega = 2\pi\nu$ with $\nu = 0.1, 0.5, 1.0$ and 2.0. We make two important observations. First, for the lower frequencies, we observe the expected convergence rate $\mathcal{O}(h^{\lambda+\frac{1}{2}}) = \mathcal{O}(h^{0.75})$ for both the best-approximation and finite element error. Second, we also see that, for the higher frequencies, the optimal rate $\mathcal{O}(h)$ is achieved on the considered sequence of meshes. As predicted by our analysis (in particular (5.5) and (5.9)), this means that, for larger frequencies, the deteriorated convergence rate due to the singularity is only observed asymptotically for very fine meshes with $h \leq h_\star(\omega)$, with $h_\star(\omega)$ decreasing as ω increases.

For more details, we also refer the reader to the numerical experiments in [4]. There, the 2D case is considered so that finer meshes and higher frequencies may be easily considered, leading to an even better illustration of this phenomenon.

7 Conclusion

Following the lines of our analysis of polygonal domains [4], we analyse high-frequency Helmholtz problems in three-dimensional domains with conical points. We derive sharp estimates on the stress intensity factors associated with each corner singularity. Besides, we show that finite element discretisations with lowest-order Lagrange elements are stable under the usual condition that "$\omega^2 h$ is small enough", even in the presence of corner singularities on uniform meshes. We also present numerical experiments highlighting the fact that the corner singularities have a lesser impact on finite element discretisations for larger frequencies.

A An Embedding and Green's Formula

We recall that, for all $n \in \mathbb{N}$, $V_0^n(\Gamma_{G,R})$ is the weighted Sobolev space of order n defined by

$$V_0^n(\Gamma_{G,R}) = \{u \in L^2(\Gamma_{G,R}) : \rho^{|\alpha|-n} D^\alpha u \in L^2(\Gamma_{G,R}) \text{ for all } \alpha \in \mathbb{N}^3 \text{ such that } |\alpha| \leq n\},$$

that is a Hilbert space equipped with its natural inner product; see [6, Appendix AA].

We first recall the following result; see [6, Theorem AA.7].

Lemma A.1. *The embedding $H^1_{\Gamma_{\mathrm{Dir}}}(\Gamma_{G,R}) \cap H^2(\Gamma_{G,R}) \hookrightarrow V_0^2(\Gamma_{G,R})$ holds true.*

Let us now introduce the space

$$X = \tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}) \cap V_0^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}) \times \tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}}) \cap V_0^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}}) \times \tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Diss}}) \times \tilde{H}^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}}),$$

where, as usual, for an open subset Γ_0 of the boundary of $\partial\Gamma_{G,R}$, and any $s > 0$, $\tilde{H}^s(\Gamma_0)$ is the set of functions in $H^s(\Gamma_0)$ such that its extension by zero outside Γ_0 is in $H^s(\partial\Gamma_{G,R})$ in a neighbuorhood of the boundary of Γ_0, while for $s \in \mathbb{N}$, $V_0^{s+\frac{1}{2}}(\Gamma_{\mathrm{Dir}})$ is the trace on Γ_{Dir} of functions in $V_0^{s+1}(\Gamma_{G,R})$, equipped with the quotient norm; see [19, (6.1.2)].

For a smooth function $u \in C^\infty(\overline{\Gamma_{G,R}})$, let us denote by $\gamma_{\mathrm{Dir}} u$ (resp. $\gamma_{\mathrm{Dir}} \frac{\partial u}{\partial n}$) the restriction of u (resp. $\frac{\partial u}{\partial n}$) to Γ_{Dir}, and similarly $\gamma_{\mathrm{Diss}} u$ (resp. $\gamma_{\mathrm{Diss}} \frac{\partial u}{\partial n}$) the restriction of u (resp. $\frac{\partial u}{\partial n}$) to Γ_{Diss}. We recall that the trace operator

$$T: u \to \left(\gamma_{\mathrm{Dir}} u, \gamma_{\mathrm{Dir}} \frac{\partial u}{\partial n}, \gamma_{\mathrm{Diss}} u, \gamma_{\mathrm{Diss}} \frac{\partial u}{\partial n}\right)$$

can be continuously extended from $C^\infty(\overline{\Gamma_{G,R}})$ to a linear and continuous operator from $H^2(\Gamma_{G,R})$ to

$$H^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}) \times H^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}}) \times H^{\frac{3}{2}}(\Gamma_{\mathrm{Diss}}) \times H^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}}).$$

But, according to [14], this mapping is not surjective and its range is not closed.

We then introduce the space

$$V_{00}^2(\Gamma_{G,R}) := \{u \in V_0^2(\Gamma_{G,R}) : Tu \in X\}.$$

As $V_{00}^2(\Gamma_{G,R})$ is continuously embedded into $H^2(\Gamma_{G,R})$, T will be a continuous mapping from $V_{00}^2(\Gamma_{G,R})$ into X. It turns out that this mapping is onto.

Theorem A.2. *The mapping T from $V_{00}^2(\Gamma_{G,R})$ into X is surjective.*

Proof. Let us fix $(\alpha, \beta, \gamma, \delta) \in X$. First, by the definition of $V_0^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}})$ and $V_0^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}})$, [19, equivalence (6.1.6)] and a standard lifting theorem in usual Sobolev spaces, there exists $\psi \in V_0^2(\Gamma_{G,R})$ such that

$$u = \alpha \quad \text{and} \quad \frac{\partial u}{\partial n} = \beta \text{ on } \Gamma_{\mathrm{Dir}}.$$

Now fix a radial and smooth cutoff function η such that $\eta = 1$ in a neighbourhood of 0 and $\eta = 0$ on $\rho \geq \frac{R}{2}$. Then, by [14, Lemme], there exists $v \in H^2(\Gamma_{G,R})$ such that $Tv = ((1-\eta)\alpha, (1-\eta)\beta, \gamma, \delta)$. Now we fix another radial and smooth cutoff function ψ such that $\psi = 1$ on the support of $1 - \eta$ and $\psi = 0$ near 0 so that $\chi(1-\eta) = (1-\eta)$ and therefore $T(\chi v) = ((1-\eta)\alpha, (1-\eta)\beta, \gamma, \delta)$. We conclude by noticing that $\eta u + \chi v$ belongs to $V^2_{00}(\Gamma_{G,R})$ and satisfies $T(\eta u + \chi v) = (\alpha, \beta, \gamma, \delta)$. □

Let us now introduce the space (see [15, 23])

$$D(\Delta, L^2(\Gamma_{G,R})) := \{u \in L^2(\Gamma_{G,R}) : \Delta u \in L^2(\Gamma_{G,R})\},$$

that is a Banach space with the natural norm

$$\|u\|_{D(\Delta, L^2(\Gamma_{G,R}))} := \|u\|_{L^2(\Gamma_{G,R})} + \|\Delta u\|_{L^2(\Gamma_{G,R})} \quad \text{for all } u \in D(\Delta, L^2(\Gamma_{G,R})).$$

Since the boundary of $\Gamma_{G,R}$ is Lipschitz, the proof of [23, Lemma 1.36] can be used to obtain the next result.

Lemma A.3. *The space $\mathcal{D}(\overline{\Gamma_{G,R}})$ is dense in $D(\Delta, L^2(\Gamma_{G,R}))$.*

Theorem A.4. *The trace mapping T from $H^2(\Gamma_{G,R})$ to $H^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}) \times H^{\frac{1}{2}}(\Gamma_{\mathrm{Dir}}) \times H^{\frac{3}{2}}(\Gamma_{\mathrm{Diss}}) \times H^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})$ admits a unique continuous extension from in $D(\Delta, L^2(\Gamma_{G,R}))$ into X'.*

Proof. As Theorem A.2 guarantees that T is continuous and surjective from $V^2_{00}(\Gamma_{G,R})$ into X, it admits right continuous inverses; we therefore fix one of them that we denote by $\mathcal{R}$.

Now, for $u \in H^2(\Gamma_{G,R})$ and $v \in V^2_{00}(\Gamma_{G,R})$, by a standard Green's formula (as $V^2_{00}(\Gamma_{G,R}) \subset H^2(\Gamma_{G,R})$), we have

$$\int_{\partial\Gamma_{G,R}} \left(\frac{\partial \bar{u}}{\partial n} v - \bar{u}\frac{\partial v}{\partial n}\right) d\sigma = \int_{\Omega} (\Delta \bar{u} r v - \bar{u}\Delta v)\, dx. \tag{A.1}$$

Splitting the left integral into Γ_{Dir} and Γ_{Diss}, and using Cauchy–Schwarz's inequality, we find

$$\left| \int_{\Gamma_{\mathrm{Dir}}} \left(\frac{\partial \bar{u}}{\partial n} v - \bar{u}\frac{\partial v}{\partial n}\right) d\sigma + \int_{\Gamma_{\mathrm{Diss}}} \left(\frac{\partial \bar{u}}{\partial n} v - \bar{u}\frac{\partial v}{\partial n}\right) d\sigma \right| \le \|u\|_{D(\Delta, L^2(\Gamma_{G,R}))} \|v\|_{V^2_0(\Gamma_{G,R})}. \tag{A.2}$$

Now fix $u \in H^2(\Gamma_{G,R})$ and define the linear and continuous mapping

$$\ell : X \to \mathbb{C} : (\alpha, \beta, \gamma, \delta) \to \int_{\Gamma_{\mathrm{Dir}}} \left(\frac{\partial \bar{u}}{\partial n}\alpha - \bar{u}\beta\right) + \int_{\Gamma_{\mathrm{Diss}}} \left(\frac{\partial \bar{u}}{\partial n}\gamma - \bar{u}\delta\right).$$

Now, for a fixed $(\alpha, \beta, \gamma, \delta) \in X$, denote $v = \mathcal{R}(\alpha, \beta, \gamma, \delta)$, that consequently satisfies $Tv = (\alpha, \beta, \gamma, \delta)$ and consequently

$$\ell((\alpha, \beta, \gamma, \delta)) = \int_{\Gamma_{\mathrm{Dir}}} \left(\frac{\partial \bar{u}}{\partial n} v - \bar{u}\frac{\partial v}{\partial n}\right) d\sigma + \int_{\Gamma_{\mathrm{Diss}}} \left(\frac{\partial \bar{u}}{\partial n} v - \bar{u}\frac{\partial v}{\partial n}\right) d\sigma.$$

By estimate (A.2), we find that

$$|\ell((\alpha, \beta, \gamma, \delta))| \le \|u\|_{D(\Delta, L^2(\Gamma_{G,R}))} \|\mathcal{R}(\alpha, \beta, \gamma, \delta)\|_{V^2_0(\Gamma_{G,R})} \lesssim \|u\|_{D(\Delta, L^2(\Gamma_{G,R}))} \|(\alpha, \beta, \gamma, \delta)\|_X.$$

This means that $\|\ell\|_{X'} \lesssim \|u\|_{D(\Delta, L^2(\Gamma_{G,R}))}$ and, from the definition of ℓ, shows that the mapping T can be extended from $H^2(\Gamma_{G,R})$ to $D(\Delta, L^2(\Gamma_{G,R}))$. □

Corollary A.5. *For all $u \in D(\Delta, L^2(\Gamma_{G,R}))$ and $v \in V^2_{00}(\Gamma_{G,R})$, we have*

$$\left\langle \gamma_{\mathrm{Dir}} \frac{\partial u}{\partial n}, \gamma_{\mathrm{Dir}} v \right\rangle - \left\langle \gamma_{\mathrm{Dir}} u, \gamma_{\mathrm{Dir}} \frac{\partial v}{\partial n} \right\rangle + \left\langle \gamma_{\mathrm{Diss}} \frac{\partial u}{\partial n}, \gamma_{\mathrm{Diss}} v \right\rangle - \left\langle \gamma_{\mathrm{Diss}} u, \gamma_{\mathrm{Diss}} \frac{\partial v}{\partial n} \right\rangle = \int_{\Omega} (\Delta \bar{u} v - \bar{u}\Delta v)\, dx, \tag{A.3}$$

where the first duality bracket has to be understood as a duality between

$$(\tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}) \cap V_0^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}))' \quad \text{and} \quad \tilde{H}^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}}) \cap V_0^{\frac{3}{2}}(\Gamma_{\mathrm{Dir}})$$

(note that this bracket is conjugate linear in u), and similarly for the other brackets.

Proof. By Lemma A.3, there exists a sequence of $u_n \in H^2(\Gamma_{G,R})$ such that

$$u_n \to u \text{ in } D(\Delta, L^2(\Gamma_{G,R})) \text{ as } n \to \infty.$$

We then apply Green's formula (A.1) to the pair (u_n, v) and can pass to the limit in n to find (A.3) since Tu is by definition the limit of Tu_n in X'. □

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