Research Article

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Super-Exponentially Convergent Parallel Algorithm for Eigenvalue Problems with Fractional Derivatives

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Abstract: A new algorithm for eigenvalue problems for linear differential operators with fractional derivatives is proposed and justified. The algorithm is based on the approximation (perturbation) of the coefficients of a part of the differential operator by piecewise constant functions where the eigenvalue problem for the last one is supposed to be simpler than the original one. Another milestone of the algorithm is the homotopy idea which results at the possibility for a given eigenpair number to compute recursively a sequence of the approximate eigenpairs. This sequence converges to the exact eigenpair with a super-exponential convergence rate. The eigenpairs can be computed in parallel for all prescribed indexes. The proposed method possesses the following principal property: its convergence rate increases together with the index of the eigenpair. Numerical examples confirm the theory.

Keywords: Fractional Differential Operator, Eigenvalue Problem, Homotopy Idea, Parallel Algorithm, Super-Exponentially Convergent Algorithm

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1 Introduction

It was recognized over the last decades that realistic models of various physical phenomena can be better described with fractional calculus; see, e.g., [8, 20, 25, 31–33, 44, 47, 50], just to mention a few. A very good overview of applications of fractional calculus is given in [45, 49].

There are various definitions of fractional derivatives; see, e.g., [1, 31, 32, 34, 36–38, 49]. In [46], a fractional derivative was defined in various ways, especially through integer derivatives using the Taylor and Fourier series. For example, one can introduce fractional derivatives for periodic functions using the Fourier series and the elementary relations

$$^{F}D^{n}(\sin x) = (\sin x)^{(n)} = \sin\Big(x + n\frac{\pi}{2}\Big), \quad {}^{F}D^{n}(\cos x) = (\cos x)^{(n)} = \cos\Big(x + n\frac{\pi}{2}\Big), \quad n \in \mathbb{Z},$$

and replacing here the integer n by a real number. Alternatively one can use the Fourier or Laplace transform with the relations

$$\mathcal{F}(D^{\alpha}f)(\xi) = i^{|\alpha|}\xi^{\alpha}\mathcal{F}(f)(\xi), \quad \mathcal{L}(D^{\alpha}f)(\xi) = s^{\alpha}\mathcal{L}(f) - \sum_{k=0}^{n-1} f^{(k)}(0)s^{\alpha-k-1}$$

and declare them valid for all real values of α.

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The very popular definitions of the fractional calculus are the Riemann–Liouville and Caputo fractional derivatives and integrals; see, e.g., [10].

The Riemann–Liouville integral is defined by

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} f(t)(x-t)^{\alpha-1}\,dt,$$

where $\Gamma(\alpha)$ is the gamma function and a is a fixed base point. Another notation, which emphasizes the base point, is

$$ {}_{a}D_{x}^{-\alpha} f(x) = {}_{a}I_{x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} f(t)(x-t)^{\alpha-1}\,dt. \tag{1.1}$$

The following fundamental relations hold:

$$\frac{d}{dx} I^{\alpha+1} f(x) = I^{\alpha} f(x), \quad I^{\alpha}(I^{\beta} f) = I^{\alpha+\beta} f,$$

the latter of which is the semigroup property. These properties make possible not only the definition of fractional integration, but also the definition of fractional differentiation, by taking enough derivatives of $I^{\alpha} f(x)$. Computing the n-th order derivative over the integral of order $(n-\alpha)$, the α order derivative is obtained:

$${}_{a}^{RL}D_{t}^{\alpha} f(t) = \frac{d^n}{dt^n}\, {}_{a}D_{t}^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n}\, {}_{a}I_{t}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{a}^{t} f(\tau)(t-\tau)^{n-\alpha-1}\,d\tau,$$

where n is the nearest integer bigger than α, i.e. $n-1 \le \alpha < n \in \mathbb{Z}^{+}$ or $n = \lceil \alpha + 1 \rceil$.

The Caputo fractional derivative is another option for computing fractional derivatives. It was introduced by M. Caputo in 1967 in the following way:

$${}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau,$$

where $n-1 \le \alpha < n \in \mathbb{Z}^{+}$. From these definitions, one can see that ${}_{a}^{C}D_{t}^{\alpha} f^{(n)}(t) = {}_{a}^{C}D_{t}^{n+\alpha} f(t)$ and, for sufficient smooth f,

$${}_{a}^{C}D_{t}^{\alpha} f(t) = {}_{a}^{RL}D_{t}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha} f^{(k)}(a)}{\Gamma(k-\alpha+1)}, \tag{1.2}$$

i.e., both derivatives coincide for functions $f(t)$ with $f^{(k)}(a) = 0$, $k = 0, 1, \dots, n-1$. One can also rewrite this formula as

$${}_{a}^{C}D_{t}^{\alpha} f(t) = {}_{a}^{RL}D_{t}^{\alpha}\Big(f(t) - \sum_{k=0}^{n-1} \frac{t^{k} f^{(k)}(a)}{k!}\Big).$$

Both definitions differ only in the order of evaluation: whereas in the Caputo definition we first compute an ordinary derivative, then a fractional integral, in the Riemann–Liouville definition the operators are reversed. Comparing these definitions, we can see that functions which are derivable in the Caputo sense are much "fewer" than those which are derivable in the Riemann–Liouville sense.

Note that the fractional derivatives introduced above do not satisfy many properties of the classical differential calculus. But there are other definitions of fractional derivative in the literature which obey classical properties including: linearity, product rule, quotient rule, power rule, chain rule, vanishing derivatives for constant functions, Rolle's theorem and the Mean Value Theorem; see, e.g., [22].

Using the definitions above, one can consider differential operators of fractional order, boundary and eigenvalue problems for these operators, etc., as well as various approximation methods for them; see, e.g., [11, 14, 21, 31].

The eigenvalue problem (EVP) is the problem of finding eigenpairs (eigenvalues and eigenfunctions or, in the language of mechanicians, frequencies and vibration shapes). It plays an important role in various applications concerned with vibrations and wave processes [4, 19, 41]. Popular methods such as the finitedifference method (FD), the finite element (FEM) and other variational methods, as well as spectral methods allow one to compute efficiently some lower eigenvalues only. At the same time there are applied problems requiring the computation of a great number (hundreds of thousands) of eigenvalues and eigenfunctions including eigenpairs with great indexes; see, e.g., [41, p. 273].

Over the last decade, it has been demonstrated that also eigen-oscillations of many systems in science and engineering can be modeled more accurately by employing fractional-order rather than integer-order derivatives [3, 13, 35]. In most of the fractional Sturm–Liouville formulations presented recently, the ordinary derivatives in a traditional Sturm–Liouville problem are replaced with fractional derivatives, and the resulting problems are solved using some numerical schemes such as the Adomian decomposition method [3], the fractional differential transform method [13], or using the method of the Haar wavelet operational matrix [35]. Some of the proposed algorithms are given in the literature without sufficient theoretical justification or possess the same drawbacks as the corresponding algorithms for the classical Sturm–Liouville problem. It turns out that the Sturm–Liouville problem with fractional derivatives can possess similar qualitative properties as a traditional Sturm–Liouville problem [12, 51], but some qualitative properties differ from the traditional ones.

In [9], Chen, Shen and Wang consider a spectral approximation of fractional differential equations (FDEs). A new class of generalized Jacobi functions (GJFs) is defined, which are the eigenfunctions of some fractional Jacobi-type differential operator and can serve as natural basis functions for properly designed spectral methods for FDEs. The efficient GJF Petrov–Galerkin methods for a class of prototypical fractional initial value problems (FIVPs) and fractional boundary value problems (FBVPs) of general order are constructed and analyzed.

In order to solve numerically an eigenvalue problem for fractional differential operators, we propose a new approach described below which we will refer to as the FD-method (from "functional-discrete method", following [5, 6, 26–28]). Note that this approach for eigenvalue problems for nonlinear differential equations was applied for the first time in [28] and then continued in [15].

The FD-method is based on the perturbation and homotopy ideas. The perturbation in the case of ODE operators can be similar to that of the butt method (metodo dei tronconi; see, e.g., [7]), the Pruess method for BVPs [39–41] for the second-order ODEs, or of some methods for EVP from [2], where the coefficients of the differential equation are replaced by their piecewise constant approximations. This approach has been applied also to EVPs with multiple eigenvalues in [17].

The article is organized as follows. In Section 2, we describe the algorithm of the FD-method for EVPs in an abstract setting. Section 3 is devoted to the Fourier fractional derivative and some of its properties. In Section 4, we apply the FD-method for a differential equation with an integer highest derivative and a subordinated fractional derivative. Here we prove the superexponential convergence rate of our method independent of the definition of fractional derivatives in use and discuss two various algorithmic realizations: 1) solving the differential problems for the corrections of the FD-method, and 2) using a recursive procedure for the expansion coefficients of these corrections. Section 5 deals with the FD-method for a differential equation with a highest Riemann–Liouville fractional derivative, which is similar to the differential equation defining the generalized Jacobi functions (GJF). We prove the super-exponential convergence rate of our method. For the practical implementation we test the FD-algorithm directly and, besides, propose some new recursive procedure. Numerical examples are given to support the theoretical results.

2 The Homotopy-Based Method for EVPs in an Abstract Setting

Let us briefly explain the ideas of perturbation and homotopy for the eigenvalue problem

$$(A + B)u_n - \lambda_n u_n = \theta, \tag{2.1}$$

in a Hilbert space X with the scalar product $(\cdot\,,\cdot)$ and with the null-element θ under the assumption that the spectrum of the operator $A + B$ is discrete. We are looking for the eigenpair with a given fixed index n.

Let $\overline{B}$ be an approximating operator for B in the sense that the eigenvalue problem

$$(A + \overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{2.2}$$

is "simpler" than problem (2.1).

Formally, a homotopy between two problems P_1 and P_2 with solutions u_1 and u_2 from some topological space X is defined to be a parametric problem $P_H(t)$ with a solution $u(t)$ continuously and smoothly depending on the parameter $t \in [0, 1]$ and such that $u(0) = u_1$ and $u(1) = u_2$ (compare http://en.wikipedia.org/wiki/homotopy).

Following the homotopy idea for a given eigenpair number n, we embed our problem into the parametric family of problems

$$(A + W(t))u_n(t) - \lambda_n(t)u_n(t) = \theta, \quad t \in [0, 1] \tag{2.3}$$

with $W(t) = \overline{B} + t\varphi(B)$, $\varphi(B) = B - \overline{B}$ containing both problems (2.1) and (2.2), so that we obviously have

$$u_n(0) = u_n^{(0)}, \quad \lambda_n(0) = \lambda_n^{(0)}, \quad u_n(1) = u_n, \quad \lambda_n(1) = \lambda_n.$$

This suggests the idea to look for the solution of (2.3) in the form

$$\lambda_n(t) = \sum_{j=0}^{\infty} \lambda_n^{(j)} t^j, \quad u_n(t) = \sum_{j=0}^{\infty} u_n^{(j)} t^j, \tag{2.4}$$

where

$$\lambda_n^{(j)} = \frac{1}{j!} \frac{d^j \lambda_n(t)}{dt^j}\bigg|_{t=0}, \quad u_n^{(j)} = \frac{1}{j!} \frac{d^j u_n(t)}{dt^j}\bigg|_{t=0}. \tag{2.5}$$

Setting $t = 1$ in (2.4), we obtain

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \quad u_n = \sum_{j=0}^{\infty} u_n^{(j)},$$

provided that the series in (2.4) converge for all $t \in [0, 1]$.

The identities given in (2.5) are not suitable for a numerical algorithm, therefore we need another way to compute the corrections $\lambda_n^{(j)}, u_n^{(j)}$ which we describe below.

Substituting (2.4) into (2.3) and matching the coefficients in front of the same powers of t, we arrive at the following recurrence sequence:

$$(A + \overline{B})u_n^{(j+1)} - \lambda_n^{(0)} u_n^{(j+1)} = F_n^{(j+1)}, \quad j = -1, 0, 1, \dots, \tag{2.6}$$

with $F_n^{(0)} = 0$ and

$$\begin{aligned} F_n^{(j+1)} &= F_n^{(j+1)}(\lambda_n^{(1)}, \dots, \lambda_n^{(j+1)}; u_n^{(0)}, \dots, u_n^{(j)}) \\ &= -\varphi(B)u_n^{(j)} + \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} \\ &= \lambda_n^{(j+1)} u_n^{(0)} - \varphi(B)u_n^{(j)} + \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}, \quad j = 0, 1, \dots. \end{aligned} \tag{2.7}$$

For the pair $\lambda_n^{(0)}, u_n^{(0)}$ corresponding to the index $j = -1$ we get the so-called base eigenvalue problem

$$(A + \overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta,$$

which for simplicity is assumed to have no multiple eigenvalues, to be "simpler" than the original one, and to produce the initial data for problems (2.6), (2.7). The case of multiple eigenvalues of the base problem was studied in [17].

Problems (2.6) for higher indices $j \geq 0$ are solvable provided that

$$(F_n^{(j+1)}, u_n^{(0)}) = 0, \quad j = 0, 1, \dots,$$

from where we obtain

$$\lambda_n^{(j+1)} = (\varphi(B)u_n^{(j)}, u_n^{(0)}), \quad j = 0, 1, \dots. \tag{2.8}$$

Under this condition the general solution of the inhomogeneous equation (2.6) with the singular operator can be represented by

$$u_n^{(j+1)} = Cu_n^{(0)} + \sum_{p=1,\, p\neq n}^{\infty} \frac{(F_n^{(j)}, u_p^{(0)})}{\lambda_p^{(0)} - \lambda_n^{(0)}} u_p^{(0)}$$

with an arbitrary constant C. We choose the particular solution

$$u_n^{(j+1)} = \sum_{p=1,\, p\neq n}^{\infty} \frac{(F_n^{(j)}, u_p^{(0)})}{\lambda_p^{(0)} - \lambda_n^{(0)}} u_p^{(0)}$$

satisfying the condition

$$(u_n^{(j+1)}, u_n^{(0)}) = 0, \quad j = 0, 1, \dots.$$

The start values $\lambda_n^{(0)}$, $u_n^{(0)}$ for the recursion (2.6), (2.8) are the solutions of the base problem.

The truncated series

$$\overset{m}{u}_n = \sum_{j=0}^{m} u_n^{(j)}, \quad \overset{m}{\lambda}_n = \sum_{j=0}^{m} \lambda_n^{(j)} \tag{2.9}$$

represent an algorithm to find the approximate solution $\overset{m}{\lambda}_n$, $\overset{m}{u}_n$ (of rank m) to the solution of problem (2.1).

Below we give the error estimates of this method in the cases of a “dominated” fractional derivative and a “subordinated” fractional derivative.

3 The Fourier Fractional Derivative

The following differentiation and integration formulas can be easily proved for $n \in \mathbb{N}$, $a \in \mathbb{R}$:

$$^F D^n \sin(ax) = \sin^{(n)}(ax) = (a)^n \sin\Big(ax + \frac{\pi n}{2}\Big), \tag{3.1}$$

$$^F D^{-n} \sin(ax) = \underbrace{\int \cdots \int}_{n \text{ times}} = (a)^{-n} \sin\Big(ax - \frac{\pi n}{2}\Big). \tag{3.2}$$

We can generalize these formulas in a natural way for real n and introduce the differentiation and integration operators of fractional order $\alpha \in \mathbb{R}$ (the Fourier fractional derivative) by

$$^F D^\alpha \sin(ax) = \sin^{(\alpha)}(ax) = (a)^\alpha \sin\Big(ax + \frac{\pi \alpha}{2}\Big),$$

$$^F D^{-\alpha} \sin(ax) = (a)^{-\alpha} \sin\Big(ax - \frac{\pi \alpha}{2}\Big),$$

cf. [23, 24]. One can see, especially for $a = n\pi$, that

$$|^F D^\alpha \sin(n\pi x)| \le (n\pi)^\alpha. \tag{3.3}$$

The functions $\sqrt{2}\sin(n\pi x)$, $n = 1, 2, \dots$, build an orthonormal basis in the space $L_{2,0}(0,1)$ of functions vanishing at the ends of the interval with the norm $\|f\|_{L_2} = (\int_0^1 f^2(x)dx)^{1/2}$. An arbitrary function $f(x)$ from this space can be represented by the Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \tag{3.4}$$

with $a_n = \sqrt{2}\int_0^1 f(x)\sin(n\pi x)dx$. The fractional derivative for such functions can be defined by

$$^F D^\alpha f(x) = \sum_{n=1}^{\infty} a_n (n\pi)^\alpha \sin\Big(n\pi x + \frac{\pi\alpha}{2}\Big).$$

Given the Fourier representation (3.4), the inverse operator ${}^F D^{-\alpha}$ is defined by

$$ {}^F D^{-\alpha} f(x) = \sum_{n=1}^{\infty} a_n (n\pi)^{-\alpha} \sin\Big(n\pi x - \frac{\pi\alpha}{2}\Big). $$

One can also define the fractional derivative of a 2π-periodic function using the exponential form of the Fourier series (see, e.g., www.xuru.org/fc/exponentials.asp):

$$ f(x) = \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-int}\,dt \quad \Rightarrow \quad {}^F D^{\alpha} f(x) = \sum_{n=-\infty}^{\infty} \frac{(n\pi)^{\alpha} e^{i(nx+\alpha\pi/2)}}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-int}\,dt. $$

Remark 3.1. It is easy to see that the eigenpairs of the operator

$$ Au = -\frac{d^2 u}{dx^2} \quad \text{for all } u \in D(A) := \{u \in H^2(0,1) : u(0) = 0,\ u(1) = 0\}, $$

are $\lambda_k = k^2\pi^2$, $u_k = \sin(k\pi x)$ for $k = 1, 2, \dots$. Therefore we obtain the following connection between the fractional powers A^{α} of this operator and the Fourier fractional derivatives on the elements (3.4):

$$ \begin{aligned} {}^F D^{\alpha} f(x) &= \sum_{n=1}^{\infty} a_n (n\pi)^{\alpha} \sin\Big(n\pi x + \frac{\pi\alpha}{2}\Big) \\ &= \cos\Big(\frac{\pi\alpha}{2}\Big) \sum_{n=1}^{\infty} a_n (n\pi)^{\alpha} \sin(n\pi x) + \sin\Big(\frac{\pi\alpha}{2}\Big) \sum_{n=1}^{\infty} a_n (n\pi)^{\alpha} \cos(n\pi x) \\ &= \cos\Big(\frac{\pi\alpha}{2}\Big) A^{\alpha/2} f + \sin\Big(\frac{\pi\alpha}{2}\Big) \frac{d}{dx}\, {}^F D^{\alpha-1} f. \end{aligned} $$

That is, we have

$$ A^{\alpha/2} = \cos^{-1}\Big(\frac{\pi\alpha}{2}\Big)\, {}^F D^{\alpha} - \tan\Big(\frac{\pi\alpha}{2}\Big) \frac{d}{dx}\, {}^F D^{\alpha-1}, \quad \alpha \neq 1. $$

Example 3.2. To obtain the fractional derivative of the product of functions $f(x) = x$ and $g(x) = \sin(n\pi x)$, let us represent this product by the Fourier series

$$ x \sin(k\pi x) = \sum_{p=1}^{\infty} a_p \sin(p\pi x), $$

where

$$ a_k = \frac{1}{\sqrt{2}} \int_0^1 x \sin(n\pi x) \sin(k\pi x)\,dx = \begin{cases} -\dfrac{4nk[1 - (-1)^{n+k}]}{\pi^2 (k^2 - n^2)^2} & \text{if } k \neq n, \\ \dfrac{1}{2} & \text{if } k = n. \end{cases} $$

Then using the definitions (3.1) and (3.2), we have

$$ {}^F D^{\alpha}[x \sin(n\pi x)] = \sum_{k=1}^{\infty} a_k (k\pi)^{\alpha} \sin\Big(k\pi x + \frac{\pi\alpha}{2}\Big), $$

in particular

$$ {}^F D^{1/2}[x \sin(n\pi x)] = \sum_{k=1}^{\infty} a_k (k\pi)^{1/2} \sin\Big(k\pi x + \frac{\pi}{4}\Big). $$

Let us consider the asymptotic behavior with respect to n of the Caputo and Riemann–Liouville fractional derivatives of the function $\sin(n\pi x)$. We have

$$ {}^C_a D^{\alpha}_t \sin(n\pi t) = \frac{n\pi}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \cos(n\pi\tau)\,d\tau = \frac{n\pi}{\Gamma(1-\alpha)} \varphi_n^{(\alpha)}(t), $$

where

$$ \varphi_n^{(\alpha)}(t) = \int_t^1 \frac{\cos(n\pi\xi)}{(\xi - t)^{\alpha}}\,d\xi, \quad t \in [0,1],\ \alpha \in (0,1). $$

The asymptotic behavior of $\varphi_n^{(\alpha)}$ with respect to n gives the next simple lemma.

Lemma 3.3. *There exists a constant $c(\alpha)$ independent of n such that*

$$n^{1-\alpha} \max_{t\in[0,1]} |\varphi_n^{(\alpha)}(t)| \leq c(\alpha),$$

where $c(\alpha) = \frac{1}{1-\alpha}$.

Proof. By change of the variable $\xi = \zeta/n$, we obtain

$$\varphi_n^{(\alpha)}(t) = \int_t^1 \frac{\cos(n\pi\xi)}{(\xi - t)^\alpha} d\xi = \frac{1}{n^{1-\alpha}} \int_{nt}^n \frac{\cos(\pi\zeta)}{(\zeta - nt)^\alpha} d\zeta.$$

Further, for $t \in [0, 1]$, we have

$$n^{1-\alpha} |\varphi_n^{(\alpha)}(t)| \leq \int_{nt}^n \frac{d\zeta}{(\zeta - nt)^\alpha} d\zeta = \frac{(\zeta - nt)^{-\alpha+1}}{-\alpha+1}\bigg|_{nt}^n = \frac{(n - nt)^{-\alpha+1}}{-\alpha+1} \leq \frac{1}{-\alpha+1} = c(\alpha). \qquad \square$$

The lemma yields the estimate

$$|{}_a^C D_t^\alpha \sin(n\pi t)| \leq \frac{c(\alpha)\pi}{\Gamma(1-\alpha)} n^\alpha, \tag{3.5}$$

which is of the same order in n as the Fourier fractional derivative (3.3). Due to (1.2) we have the same estimate for the Riemann–Liouville derivative too.

4 The Sturm–Liouville Problem with a Subordinated Fractional Derivative

Let us consider the following Sturm–Liouville problem:

$$\begin{cases} \dfrac{d^2 u(x)}{dx^2} + k(x) D^\alpha u(x) + (\lambda - q(x)) u(x) = 0, \quad x \in (0, 1), \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{4.1}$$

where D^α denotes the Fourier, Caputo or Riemann–Liouville fractional derivatives. For shortness, we will use in this section the Fourier derivative ${}^F D^\alpha u(x)$, since the main property in use is the asymptotic (3.3) and (3.5) which are of the same order in n for all three derivatives.

If we approximate the coefficients $k(x)$, $q(x)$ by the constant 0 on the whole interval (the simplest variant), then the FD-method for (4.1) consists of the following sequence of recursive problems:

$$\begin{cases} \dfrac{d^2 u_n^{(j+1)}(x)}{dx^2} + \lambda_n^{(0)} u_n^{(j+1)}(x) = F_n^{(j+1)}(x), \quad x \in (0, 1), \\ u_n^{(j+1)}(0) = 0, \quad u_n^{(j+1)}(1) = 0, \quad j = -1, 0, \dots, \end{cases} \tag{4.2}$$

where

$$\begin{cases} F_n^{(j+1)}(x) = -\displaystyle\sum_{s=0}^{j} \lambda_n^{(j+1-s)} u_n^{(s)}(x) - k(x) D^\alpha u_n^{(j)}(x) + q(x) u_n^{(j)}(x), \quad x \in (0, 1), \\ F_n^{(0)}(x) = 0, \\ u_n^{(0)}(x) = \sqrt{2} \sin(n\pi x), \quad \lambda_n^{(0)} = (n\pi)^2. \end{cases} \tag{4.3}$$

The solvability condition

$$\int_0^1 F_n^{(j+1)}(\xi) \sin(n\pi\xi) d\xi = 0$$

implies

$$\lambda_n^{(j+1)} = -\int_0^1 k(\xi) D^\alpha u_n^{(j)}(\xi)\sqrt{2}\sin(n\pi\xi)\,d\xi + \int_0^1 q(\xi) u_n^{(j)}(\xi)\sqrt{2}\sin(n\pi\xi)\,d\xi. \tag{4.4}$$

The particular solution of problem (4.2) satisfying the orthogonality condition

$$\int_0^1 u_n^{(j+1)}(\xi)\sin(n\pi\xi)\,d\xi = 0$$

can be represented by

$$u_n^{(j+1)}(x) = 2\sum_{p=1,\,p\neq n}^{\infty} \frac{\int_0^1 F_n^{(j+1)}(\xi)\sin(p\pi\xi)\,d\xi}{\pi^2(n^2-p^2)}\sin(p\pi x) \tag{4.5}$$

and we have

$$D^\alpha u_n^{(j+1)}(x) = 2\sum_{p=1,\,p\neq n}^{\infty} \frac{\int_0^1 F_n^{(j+1)}(\xi)\sin(p\pi\xi)\,d\xi}{\pi^2(n^2-p^2)}(p\pi)^\alpha \sin\Big(p\pi x + \frac{\pi\alpha}{2}\Big).$$

Using the orthonormality of the system $\sqrt{2}\sin(p\pi x)$ for the corrections of the eigenfunction, we obtain the estimates

$$\begin{aligned}\|u_n^{(j+1)}\| &\le \frac{1}{\pi^2(2n-1)}\|F_n^{(j+1)}\| \\ &\le \frac{1}{\pi^2(2n-1)}\Big\{\sum_{s=0}^{j}|\lambda_n^{(j+1-s)}|\|u_n^{(s)}\| + \|k\|_\infty\|D^\alpha u_n^{(j)}\| + \|q\|_\infty\|u_n^{(j)}\|\Big\}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\|D^\alpha u_n^{(j+1)}\| &\le \frac{\sqrt{2}}{\pi^{2-\alpha}}\max\Big(\frac{(n-1)^\alpha}{2n-1}, \frac{(n+1)^\alpha}{2n+1}\Big)\|F_n^{(j+1)}\| \\ &\le M_n^{(1)}\Big\{\sum_{s=0}^{j}|\lambda_n^{(j+1-s)}|\|u_n^{(s)}\| + \|k\|_\infty\|D^\alpha u_n^{(j)}\| + \|q\|_\infty\|u_n^{(j)}\|\Big\}\end{aligned} \tag{4.7}$$

with

$$M_n^{(1)} = \frac{\sqrt{2}}{\pi^{2-\alpha}}\max\Big(\frac{(n-1)^\alpha}{2n-1}, \frac{(n+1)^\alpha}{2n+1}\Big) \le \frac{2^\alpha}{\pi^{2-\alpha}}\cdot\frac{(n+1)^\alpha}{2n-1} \le \overline{M}_n^{(1)} = \frac{2^{\alpha+0.5}}{\pi^{2-\alpha}}n^{\alpha-1}. \tag{4.8}$$

The corrections to the eigenvalues are estimated by

$$|\lambda_n^{(j+1)}| \le \|k\|_\infty\|D^\alpha u_n^{(j)}\| + \|q\|_\infty\|u_n^{(j)}\|. \tag{4.9}$$

Introducing the majorants U_j, V_j and Λ_j by

$$\|u_n^{(j+1)}\| \le U_{j+1}, \quad \|D^\alpha u_n^{(j+1)}\| \le V_{j+1}, \quad |\lambda_n^{(j+1)}| \le \Lambda_{j+1}$$

and replacing the inequality signs in (4.6)–(4.9) by equal signs, we obtain the following majorant system of equations:

$$\begin{cases} U_{j+1} = \dfrac{2}{\pi^2(2n-1)}\displaystyle\sum_{s=0}^{j}\Lambda_{j+1-s}U_s, \quad \Lambda_{j+1} = \|k\|_\infty V_j + \|q\|_\infty U_j, \\ V_{j+1} = 2M_n^{(1)}\displaystyle\sum_{s=0}^{j}\Lambda_{j+1-s}U_s, \quad j = 0, 1, \dots, \quad U_0 = 1, \quad V_0 = (\pi n)^\alpha. \end{cases} \tag{4.10}$$

A consequence of (4.10) is

$$V_{j+1} = \kappa_V U_{j+1}, \quad \Lambda_{j+1} = \kappa_\Lambda U_j, \quad U_{j+1} = \kappa_U\sum_{s=0}^{j}U_{j-s}U_s, \quad j = 0, 1, \dots, \tag{4.11}$$

where

$$\kappa_V = \frac{\pi^2(2n-1)}{2} M_n^{(1)}, \quad \kappa_\Lambda = \left[\|k\|_\infty \frac{\pi^2(2n-1)}{2} M_n^{(1)} + \|q\|_\infty\right], \quad \kappa_U = \left[\|k\|_\infty M_n^{(1)} + \frac{2\|q\|_\infty}{\pi^2(2n-1)}\right].$$

The last equation in (4.11) is a recurrence equation of convolution type which can be solved by the method of generating functions (see, e.g., [6, 42]). We successively obtain

$$U_j = (r_n)^j 2 \frac{(2j-1)!!}{(2j+2)!!}, \quad V_j = \kappa_\Lambda (r_n)^j 2 \frac{(2j-1)!!}{(2j+2)!!}, \quad \Lambda_{j+1} = \kappa_\Lambda (r_n)^j 2 \frac{(2j-1)!!}{(2j+2)!!},$$

where

$$r_n = 4\kappa_U = 4\left[\|k\|_\infty M_n^{(1)} + \frac{2\|q\|_\infty}{\pi^2(2n-1)}\right]$$

and $r_n \in (0,1)$ for n large enough.

From the definition of the majorant sequences we obtain the accuracy estimates

$$\|u_n - \overset{N}{u}_n\| \le \left\| \sum_{k=N+1}^{\infty} U_k \right\| \le 2 \sum_{k=N+1}^{\infty} (r_n)^k \frac{(2k-1)!!}{(2k+2)!!} \le 2\kappa_U \cdot \frac{(2N+1)!!}{(2N+4)!!} \cdot \frac{r_n^{N+1}}{1-r_n}.$$

It is easy to see that

$$n!! = \begin{cases} \prod_{i=1}^{\frac{n}{2}} 2i = 2^{\frac{n}{2}} \cdot \left(\frac{n}{2}\right)! & \text{if } n \text{ is even}, \\ \prod_{i=0}^{\frac{n-1}{2}} (2i+1) = \dfrac{n!}{2^{\frac{n-1}{2}} \cdot (\frac{n-1}{2})!} & \text{if } n \text{ is odd}. \end{cases}$$

Therefore,

$$\frac{(2N+1)!!}{(2N+4)!!} = \frac{(2N+1)!}{2^{2N+2} \cdot N!(N+2)!} = \frac{\Gamma(2N+2)}{2^{2N+2}\Gamma(N+1)\Gamma(N+3)}.$$

Now, the well-known Stirling's asymptotic formula $\Gamma(t+1) \asymp \sqrt{2\pi t}(t/e)^t$ implies

$$\begin{aligned} \frac{(2N+1)!!}{(2N+4)!!} &\asymp \frac{\sqrt{2\pi(2N+1)}(\frac{2N+1}{e})^{2N+1}}{2^{2N+2}\sqrt{2\pi N}(\frac{N}{e})^N \sqrt{2\pi(N+2)}(\frac{N+2}{e})^{N+2}} \\ &\asymp \frac{1}{2^{2N+2}\sqrt{2\pi N}} \cdot \frac{(\frac{2N+1}{e})^{2N} \cdot (\frac{2N+1}{e})}{(\frac{N}{e})^{2N} (\frac{N+2}{e})^2} \\ &\asymp N^{-3/2}\left(1 + \frac{1}{2N}\right)^{2N} \\ &\asymp N^{-3/2}. \end{aligned}$$

Analogously we obtain the corresponding estimate for the eigenvalues.

Thus, we come to the following assertion.

Theorem 4.1. *Let for $\alpha \in [0,1)$ the following condition be fulfilled:*

$$r_n = 4\left[\|k\|_\infty \frac{2^{\alpha+0.5}}{\pi^{2-\alpha}} n^{\alpha-1} + \frac{2\|q\|_\infty}{\pi^2(2n-1)}\right] < 1.$$

Then the FD-method for (4.1) is super-exponentially convergent with the error estimates

$$\|u_n - \overset{N}{u}_n\| \le cN^{-3/2} r_n^{N+1}, \quad \|\lambda_n - \overset{N}{\lambda}_n\| \le cN^{-3/2} r_n^{N+1},$$

where c is a constant independent of N.

j	$\lambda_5^{(j)}$
0	$25\pi^2$
1	$\sqrt{10\pi}/2$
2	0.04646
3	−0.00153
4	0.00007
5	$-0.37408 \cdot 10^{-5}$
6	$0.21009 \cdot 10^{-6}$
7	$-0.12374 \cdot 10^{-7}$
8	$0.75412 \cdot 10^{-9}$
9	$-0.47155 \cdot 10^{-10}$
10	$0.30082 \cdot 10^{-11}$

Table 1. Correction $\lambda_5^{(j)}$ vs. j for $M = 2048$.

M	$\overset{10}{\lambda}_5$
8	243.99009
16	243.98510
32	243.98375
64	243.98323
128	243.98297
256	243.98282
512	243.98272
1024	243.98265
2048	243.98260

Table 2. The approximation $\overset{10}{\lambda}_5$ vs. M.

4.1 Recursive Implementation of the Fourier Derivative

In this subsection we show that the corrections for eigenpairs can be computed without use of (4.4) and (4.5), i.e., we can avoid the computation of the integrals included. For the sake of simplicity let us consider problem (4.1) with $k(x) = 1$ and $q(x) = 0$ (otherwise one should expand these functions in Fourier series or approximate by a trigonometric polynomial). Substituting the Fourier representation

$$u_n^{(j)}(x) = \sum_{k=1}^{\infty} a_{n,k}^{(j)} \sqrt{2} \sin(k\pi x)$$

with unknown coefficients $a_{n,k}^{(j)}$ into the formulas (4.3), (4.4) and (4.5), we obtain the recurrence relations

$$\begin{aligned}
\lambda_n^{(0)} &= (n\pi)^2, \quad \lambda_n^{(1)} = -(n\pi)^\alpha \cos\Big(\frac{\alpha\pi}{2}\Big),\\
a_{nk}^{(0)} &= \delta_{n,k}, \quad a_{nk}^{(1)} = -\frac{2n^\alpha}{\pi^{3-\alpha}} \sin\Big(\frac{\alpha\pi}{2}\Big) \frac{k[(-1)^{n+k} - 1]}{(k^2 - n^2)^2}, \quad k = 1, \dots, \infty,\ k \neq n,\\
\lambda_n^{(j+1)} &= -\sum_{k=1,\, k\neq n}^{\infty} a_{n,k}^{(j)} (n\pi)^\alpha 2 \int_0^1 \sin\Big(k\pi x + \frac{\alpha\pi}{2}\Big) \sin(n\pi x)\,dx\\
&= -\frac{2n}{\pi} \sin\Big(\frac{\alpha\pi}{2}\Big) \sum_{k=1,\, k\neq n}^{\infty} (k\pi)^\alpha \frac{[(-1)^{k+n} - 1]}{k^2 - n^2} a_{nk}^{(j)}, \quad j = 1, 2, \dots,\\
a_{nk}^{(j+1)} &= \frac{-1}{\pi^2(n^2 - k^2)} \Bigg(\sum_{p=1}^{j} \lambda_n^{(j+1-p)} a_{nk}^{(p)} + \frac{2k}{\pi} \sin\Big(\frac{\alpha\pi}{2}\Big) \sum_{t=1, t\neq n}^{\infty} a_{nt}^{(j)} (t\pi)^\alpha \frac{[(-1)^{k+t} - 1]}{t^2 - k^2}\\
&\qquad + a_{nk}^{(j)} \cos\Big(\frac{\alpha\pi}{2}\Big)(k\pi)^\alpha \Bigg), \quad k = 1, \dots, \infty,\ k \neq n,\ j = 1, 2, \dots .
\end{aligned}$$

Note that the pairs $\lambda_n^{(j+1)}$, $a_{nk}^{(j+1)}$ for all $j = 1, 2, \dots$ can be computed simultaneously in a loop with respect to j with an included loop with respect to k. For practical computation one can truncate the series keeping M summands. The behavior of the corrections to the eigenvalue λ_5 computed with the computer algebra tool Maple for our example with $\alpha = \frac{1}{2}$ and $M = 2048$ is illustrated in Table 1.

Thus, our method of the rank 10 provides the approximation

$$\overset{10}{\lambda}_5 = 243.9826068784193.$$

Table 2 demonstrates the dependence of the eigenvalue from M.

As appears from Table 1, the FD-method of rank 5 instead of 10 would provide the same accuracy.

4.2 Direct Implementation of the Riemann–Liouville Derivative

Let us consider the following eigenvalue problem with the Riemann–Liouville derivative:

$$\begin{cases} u''(x) + ({}^{RL}_{\ 0}D^{1/2}_{x} u)(x) + \lambda u(x) = 0, \quad x \in (0,1), \\ u(0) = 0, \quad u(1) = 0, \\ ({}^{RL}_{\ 0}D^{1/2}_{x} u)(x) = \dfrac{1}{\sqrt{\pi}} \dfrac{d}{dx} \displaystyle\int_0^x \dfrac{u(t)}{\sqrt{x-t}} dt. \end{cases} \tag{4.12}$$

We apply to problem (4.12) the simplest variant of the FD-method by setting the coefficient in front of the fractional derivative equal to zero. We obtain the base problem

$$\begin{cases} \dfrac{d^2 u_n^{(0)}(x)}{dx^2} + \lambda_n^{(0)} u_n^{(0)}(x) = 0, \quad x \in (0,1), \\ u_n^{(0)}(0) = 0, \quad u_n^{(0)}(1) = 0. \end{cases}$$

The solution of the base problem is

$$u_n^{(0)}(x) = \sqrt{2}\sin(n\pi x), \quad \lambda_n^{(0)} = (n\pi)^2, \quad n = 1, 2, \ldots .$$

The next corrections are the solutions of

$$\begin{cases} \dfrac{d^2 u_n^{(1)}(x)}{dx^2} + \lambda_n^{(0)} u_n^{(1)}(x) = -\lambda_n^{(1)} u_n^{(0)}(x) - {}^{RL}_{\ 0}D^{1/2}_{x}(u_n^{(0)}), \quad x \in (0,1), \\ u_n^{(1)}(0) = 0, \quad u_n^{(1)}(1) = 0, \\ {}^{RL}_{\ 0}D^{1/2}_{x}(u_n^{(0)}) = 2\sqrt{\pi n}[\sin(n\pi x)S(\sqrt{2nx}) + \cos(n\pi x)C(\sqrt{2nx})], \end{cases} \tag{4.13}$$

where $S(z)$, $C(z)$ are the Fresnel's integrals; see, e.g., [43]. The solvability condition implies

$$\lambda_n^{(1)} = \sqrt{\frac{n}{2\pi}}\left[2\pi S(\sqrt{2n}) - \frac{1}{n}C(\sqrt{2n}) - (-1)^n\sqrt{\frac{2}{n}}\right].$$

The general solution of (4.13) is

$$\begin{aligned} u_n^{(1)}(x) = c_n^{(1)} u_n^{(0)}(x) + \frac{1}{2\pi\sqrt{n}}\Big[& S(\sqrt{2nx})\Big(2x\pi\cos(n\pi x) - \frac{1}{n}\sin(n\pi x)\Big) \\ & - C(\sqrt{2nx})\Big(2x\pi\sin(n\pi x) + \frac{1}{n}\cos(n\pi x)\Big) + \sqrt{\frac{2x}{n}}\Big] - \frac{\lambda_n^{(1)}}{\sqrt{2}\pi^2 n}\Big[-\frac{1}{n}\sin(n\pi x) + \pi x\cos(n\pi x)\Big] \end{aligned}$$

with an arbitrary constant $c_n^{(1)}$. We choose the particular solution satisfying the orthogonality condition

$$\int_0^1 u_n^{(1)}(x) u_n^{(0)}(x) dx = 0$$

and obtain

$$c_n^{(1)} = -\frac{1}{16\pi^{7/2}\sqrt{\pi}}\left[S(\sqrt{2n})\frac{4\sqrt{2}\pi^2}{n} + C(\sqrt{2n})\Big(-4\sqrt{2}\pi^3 - \frac{\sqrt{2}\pi}{n^2}\Big)\right].$$

We find the next corrections within a guarantied accuracy by the corresponding choice of the parameter "Digits" in the computer algebra tool Maple.

To evaluate the accuracy of the results obtained, we find the exact first eigenvalue of problem (4.12) using the Laplace transform which provides the correspondence $U(p) = \mathcal{L}\{u(x)\}$. The solution of the ODE (4.12) satisfying the conditions

$$u(0) = 0, \quad u'(0) = 1$$

is then given by

$$
\begin{aligned}
u(x) &= \mathcal{L}^{-1}\Big(\frac{1}{p^2+\sqrt{p}+\lambda}\Big)\\
&= -\frac{\frac{1}{\sqrt{\pi x}} + y_1 e^{y_1^2 x}\operatorname{erf}(-y_1\sqrt{x})}{(y_4-y_1)(y_3-y_1)(y_2-y_1)} - \frac{\frac{1}{\sqrt{\pi x}} + y_2 e^{y_2^2 x}\operatorname{erf}(-y_2\sqrt{x})}{(y_4-y_2)(y_3-y_2)(y_1-y_2)}\\
&\quad + \frac{\frac{1}{\sqrt{\pi x}} + y_3 e^{y_3^2 x}\operatorname{erf}(-y_3\sqrt{x})}{(-y_4+y_3)(-y_3+y_2)(y_1-y_3)} - \frac{\frac{1}{\sqrt{\pi x}} + y_4 e^{y_4^2 x}\operatorname{erf}(-y_4\sqrt{x})}{(-y_4+y_3)(y_2-y_4)(y_1-y_4)},
\end{aligned}
\tag{4.14}
$$

where $y_i = y_i(\lambda)$, $i = 1, \dots, 4$ are the roots of the equation $y^4 + y + \lambda = 0$, and $\operatorname{erf}(z)$ is the complementary error function. Setting $x = 1$ in (4.14), we obtain a transcendent equation defining the eigenvalues of (4.12). In particular, we obtain

$$\lambda_1 = 8.8857068923\dots.$$

4.3 Recursive Implementation of the Riemann–Liouville Derivative

Let us consider another algorithm similar to that of Section 4.1 and based on the recurrence formulas for the expansion coefficients of the corrections.

We look for the corrections of the eigenfunctions in the form

$$u_n^{(j)}(x) = \sum_{k=1,\,k\neq n}^{\infty} a_{n,k}^{(j)}\sqrt{2}\sin(k\pi x). \tag{4.15}$$

Note that the solvability condition $\int_0^1 u_n^{(j)}(x)u_n^{(0)}(x)dx = 0$ is automatically satisfied. We will use the representation

$${}_0^{RL}D_x^{1/2}(u_n^{(j)})(x) = \sum_{k=1,\,k\neq n}^{\infty} a_{n,k}^{(j)}\varphi_k(x) \tag{4.16}$$

with

$$
\begin{aligned}
\varphi_k(x) &= 2\sqrt{k\pi}\big[\sin(k\pi x)S(\sqrt{2kx}) + \cos(k\pi x)C(\sqrt{2kx})\big]\\
&= \sum_{p=1,\,p\neq k}^{\infty} \mu_{p,k}\sqrt{2}\sin(p\pi x) + \mu_{k,k}\sin(k\pi x),
\end{aligned}
$$

where

$$S(x) = \int_0^x \sin(t^2)dt, \quad C(x) = \int_0^x \cos(t^2)dt$$

are the Fresnel integrals, and

$$
\mu_{p,k} = \begin{cases}
-2\sqrt{\dfrac{2kp}{\pi}}\,\dfrac{(-1)^{k+p}C(\sqrt{2p})\sqrt{k} - C(\sqrt{2k})\sqrt{p}}{-p^2+k^2} & \text{if } p \neq k,\\[2ex]
\sqrt{\dfrac{k}{2\pi}}\big(-C(\sqrt{2k})\sqrt{2k} + 2S(\sqrt{2k}) + 2k\cos(k\pi)\big) & \text{if } p = k.
\end{cases}
$$

After substitution of (4.15) and (4.16) into the expression

$$F_n^{(j+1)}(x) = -\sum_{p=0}^{j} \lambda_n^{(j+1-p)}u_n^{(p)}(x) - {}_0^{RL}D_x^{1/2}(u_n^{(j)})(x)$$

for the right-hand side of the equations for corrections

$$\frac{d^2 u_n^{(j+1)}(x)}{dx^2} + (n\pi)^2 u(x) = F_n^{(j+1)}(x), \tag{4.17}$$

j	$\lambda_1^{(j)}$
0	9.86960440108935861883447
1	−1.01447613638170169201166
2	0.0297977106497561210442917
3	0.000748595098364490732963004
4	0.0000307247268252977828544031
5	$0.151034209083836378732232 \cdot 10^{-5}$
6	$0.817967082430059455248254 \cdot 10^{-7}$
7	$0.470596353098774231258105 \cdot 10^{-8}$
8	$0.282203827854256565319426 \cdot 10^{-9}$
9	$0.174411370055653495235000 \cdot 10^{-10}$
10	$0.110294240120233228374921 \cdot 10^{-11}$

Table 3. The corrections of the FD-method for problem (4.12) computed by truncated series with $M = 32$ summands.

we obtain

$$F_n^{(j+1)}(x) = -\lambda_n^{(j+1)} \sqrt{2}\sin(n\pi x) - \sum_{k=1,\, k\neq n}^{\infty} f_{n,k}^{(j+1)} \sqrt{2}\sin(k\pi x),$$

where

$$f_{n,k}^{(j+1)} = \sum_{p=1}^{j} \lambda_n^{(j-p+1)} a_{n,k}^{(p)} + \sum_{s=1}^{\infty} a_{n,s}^{(j)} \mu_{s,k}.$$

The solution of (4.17) is then given by (4.15) with

$$\begin{aligned} a_{n,k}^{(j+1)} = \frac{1}{\pi^2(n^2-k^2)} \Bigg\{ &\sum_{p=1,\, p\neq k,\, p\neq n}^{\infty} \frac{2\sqrt{2p}\, k a_{n,p}^{(j)} [-C(\sqrt{2k})\sqrt{p} + (-1)^{p+k} C(\sqrt{2p})\sqrt{k}]}{\pi^{1/2}(p^2-k^2)} \\ &+ \sum_{p=0}^{j} \lambda_n^{(j+1-p)} a_{n,k}^{(p)} + \frac{1}{2\sqrt{\pi k}} [-C(\sqrt{2k})\sqrt{2k} + (2k)^{3/2}\pi S(\sqrt{2k}) + 2k(-1)^k] a_{n,k}^{(j)} \Bigg\}. \end{aligned} \tag{4.18}$$

For the eigenvalue corrections we have

$$\lambda_n^{(j+1)} = -\int_0^1 {}_0^{RL}D_x^{1/2}(u_n^{(j)})(x) u_n^{(0)}(x)\,dx = -\sum_{k=1,\, k\neq n}^{\infty} a_{n,k}^{(j)} \frac{2\sqrt{2kn}[(-1)^{n+k}C(\sqrt{2k})\sqrt{n} - C(\sqrt{2n})\sqrt{k}]}{\sqrt{\pi}(-n^2+k^2)}. \tag{4.19}$$

The formulas (4.18), (4.19) with the initial conditions

$$a_{n,k}^{(0)} = \delta_{nk}, \quad \lambda_n^{(0)} = (n\pi)^2,$$

where δ_{nk} is the Kronecker delta, represent a recursive algorithm for the corrections which contrary to the one of Section 4.2 avoids the solution of differential problems.

In practical computations we used truncated sums with M summands instead of infinite series and then computed the N-th approximation to the eigenpair according to (2.9) (the FD-method of rank N). The corrections $\lambda_1^{(j)}$, $j = 0, 1, \ldots, 10$, of the series with $M = 32$ of the FD-method for the lowest eigenvalue of problem (4.12) are given in Table 3. Thus, we have

$$\overset{10}{\lambda}_1 = 8.88570689232811335600142,$$

where the first ten digits after the decimal point coincide with the exact ones.

For the third eigenvalue ($n = 3$) and various M we obtained the results given in Table 4. The numerical results for the eigenvalue λ_1 obtained by the recursive algorithm with $M = 32$ coincide with the ones from Section 4.2. Table 5 shows the behavior of corrections for the third eigenvalue. The FD-method of the rank 10 provides the approximation

$$\overset{10}{\lambda}_3 = 86.7795885027336720205576,$$

M	$\overset{10}{\lambda}_3$	$\lambda_3^{(10)}$
8	86.7794941481798320121360	$.2361983069100882194049 34 \cdot 10^{-14}$
16	86.7795885027336720205576	$.2323034642353150436061 28 \cdot 10^{-14}$
32	86.7795973153306097826733	$.2321262201861665428936 69 \cdot 10^{-14}$

Table 4. The FD-approximations $\overset{N}{\lambda}_n$ and the eigenvalue corrections $\lambda_n^{(j)}$ vs. M with $n = 3$, $N = 10$, $j = 10$ for problem (4.12).

j	$\lambda_3^{(j)}$
0	88.826439609804227569 5102
1	−2.0497251080435899968 4540
2	0.0029021299189264822539 1892
3	−0.0000305277301362658818404213
4	$0.2226531261229442938802 27 \cdot 10^{-5}$
5	$0.1648830329607008031238 70 \cdot 10^{-6}$
6	$0.7118605182018374766054 41 \cdot 10^{-8}$
7	$0.2441356275005306957166 62 \cdot 10^{-9}$
8	$0.7043296379340840892173 46 \cdot 10^{-11}$
9	$0.1636124440995954862638 64 \cdot 10^{-12}$
10	$0.2323034642353150436061 28 \cdot 10^{-14}$

Table 5. The FD-corrections $\lambda_n^{(j)}$ with $M = 32$, $n = 3$ for example (4.12).

where the first nine digits after the decimal point coincide with the exact eigenvalue obtained by the Laplace transform method. One can observe the principal characteristic of the FD-method: the convergence rate increases together with the eigenvalue index.

5 Application to a Jacobi-Type ODE with a Dominated Fractional Derivative

Let us consider the following problem:

$$\begin{cases} \mathcal{L}_1^\mu u(x) + q(x)u(x) + \lambda w(x)u(x) = 0, \quad x \in (-1, 1), \\ u(-1) = 0, \quad {}^{RL}_{\;x}I_1^{1-\mu}[{}^{\;C}_{-1}D_x^\mu u(x)]_{x=1} = 0, \\ \mathcal{L}_1^\mu u(x) = {}^{RL}_{\;x}D_1^\mu({}^{\;C}_{-1}D_x^\mu u(x)), \end{cases} \tag{5.1}$$

where $\mu \in (0, 1)$, $w(x) = (1-x)^{-\mu}(1+x)^{-\mu}$, ${}^{RL}_{\;x}D_1^\mu$ is the Riemann–Liouville derivative, ${}^{\;C}_{-1}D_x^\mu$ the Caputo fractional derivative, and ${}^{RL}_{\;x}I_1^{1-\mu}$ the Riemann–Liouville integral (1.1). Note that the operator defined by $\mathcal{L}_1^\mu$ and by the given boundary conditions is self-adjoint. This can be easily derived analogously to [9, Corollary 3.1, formula (3.25)], where one can easily show that $\alpha = s$, $\beta = -s$ is permissible since the Caputo and the Riemann–Liouville derivatives coincide on the functions vanishing at $x = 1$. Besides we can set $s = \mu$.

Approximating $q(x)$ by the constant zero, we obtain the base problem of the FD-method with the solution

$$u_n^{(0)}(x) = (1-x)^\mu P_{n-1}^{\mu,-\mu}(x), \quad \lambda_n^{(0)} = -\frac{\Gamma(n+\mu)}{\Gamma(n-\mu)}, \quad n = 1, 2, \ldots,$$

where $P_n^{\alpha,\beta}(x)$ denotes the standard Jacobi polynomials. The eigenfunctions build an orthogonal basis in the space $L_w^2[-1, 1]$ of quadratic integrable functions with weight w, and the recurrence sequence of problems for the corrections is

$$\begin{cases} L_1^{\mu} u_n^{(j+1)}(x) + \lambda_n^{(0)} w(x) u_n^{(j+1)}(x) = -\sqrt{w(x)} F_n^{(j+1)}(x), & x \in (-1,1), \\ u_n^{(j+1)}(-1) = 0, \quad {}^{RL}_{\ x}I_1^{1-\mu}\big[{}^{C}_{-1}D_x^{\mu} u_n^{(j+1)}(x)\big]_{x=1} = 0, & j = 0,1,\dots, \end{cases} \tag{5.2}$$

where

$$F_n^{(j+1)}(x) = \frac{1}{\sqrt{w(x)}}\Big[\sum_{s=0}^{j} \lambda_n^{(j+1-s)} w(x) u_n^{(s)}(x) + q(x) u_n^{(j)}(x)\Big].$$

The solvability condition

$$\int_{-1}^{1} w(x) F_n^{(j+1)}(x)(x) u_n^{(0)}(x)\,dx = 0$$

for the singular problem (5.2) implies

$$\lambda_n^{(j+1)} = \frac{\int_{-1}^{1} q(x) u_n^{(j)}(x) u_n^{(0)}(x)\,dx}{\int_{-1}^{1} w(x)[u_n^{(0)}(x)]^2\,dx}. \tag{5.3}$$

Using the condition

$$\int_{-1}^{1} w(x) u_n^{(j+1)}(x) u_n^{(0)}(x)\,dx = 0,$$

we single out from the general solution (i.e. from the set of all possible solutions of (5.2)) the following particular solution:

$$u_n^{(j+1)}(x) = \sum_{p=1,\,p\neq n}^{\infty} \frac{\int_{-1}^{1} F_n^{(j+1)}(\xi) u_p^{(0)}(\xi)\,d\xi}{\lambda_n^{(0)} - \lambda_p^{(0)}} u_p^{(0)}(x).$$

This equality together with the orthogonality of the system $\{u_p^{(0)}(x)\}_{p=1,\dots,\infty}$ yields

$$\|u_n^{(j+1)}\|_w \le M_n^{(2)} \|F_n^{(j+1)}\|_w \le M_n^{(2)}\Big[\sum_{s=1}^{j} |\lambda_n^{(j+1-s)}| \|u_n^{(s)}\|_w + \max_{x\in[-1,1]}\Big(\frac{|q(x)|}{w(x)}\Big) \|u_n^{(j)}\|_w \frac{\|u_n^{(0)}\|_w}{\|u_n^{(0)}\|_w}\Big], \tag{5.4}$$

where

$$\|u\|_w = \Big(\int_{-1}^{1} w(x) u^2(x)\,dx\Big)^{1/2}, \quad \|u_n^{(0)}\|_w^2 \le 2^{2\mu} \|P_{n-1}^{-\mu,\mu}(x)\|_w^2 = 2^{2\mu} \frac{2\Gamma(n-\mu)\Gamma(n+\mu)}{(2n-1)\Gamma^2(n)},$$

$$M_n^{(2)} = \Big[\frac{\Gamma(n+\mu)}{\Gamma(n-\mu)} - \frac{\Gamma(n-1+\mu)}{\Gamma(n-1-\mu)}\Big]^{-1} \|u_{n-1}^{(0)}\|_w = \Big[\frac{2\mu}{n-1-\mu} \frac{\Gamma(n-1+\mu)}{\Gamma(n-1-\mu)}\Big]^{-1} \|u_{n-1}^{(0)}\|_w.$$

It was shown in [48] that, for $z \to \infty$,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\Big[1 + \frac{(\alpha-\beta)(\alpha-\beta-1)}{2z} + o(|z|^{-2})\Big],$$

which yields the existence of a constant c independent of n such that

$$\frac{\Gamma(n+1-\mu)}{\Gamma(n+1+\mu)} \le cn^{-2\mu}, \quad \|u_n^{(0)}\|_w^2 \le 2^{2\mu} \frac{2\Gamma(n-\mu)\Gamma(n+\mu)}{(2n-1)\Gamma^2(n)}.$$

Using [18], one can obtain

$$\frac{\Gamma(n+1-\mu)}{\Gamma(n+1+\mu)} \le n^{-2\mu}\Big(\frac{6}{\sqrt{13}-1}\Big)^{2\mu}, \quad n \ge 3.$$

Thus, we have

$$M_n^{(2)} \le n^{1-2\mu} \|u_{n-1}^{(0)}\|_w \frac{1}{2\mu((\sqrt{13}-1)/6)^{2\mu}} \le n^{1-2\mu} \|u_{n-1}^{(0)}\|_w \frac{1}{2\mu((\sqrt{13}-1)/6)^{2\mu}} \le cn^{1/2-2\mu} \tag{5.5}$$

and can see that $M_n^{(2)} \to \infty$ as $\mu \to 0$ but $M_n^{(2)} \to 0$ as $n \to \infty$ for each fixed $\mu > \frac{1}{4}$.

Introducing the new variable

$$\bar{u}_n^{(j)}(x) = \frac{u_n^{(j)}(x)}{\|u_n^{(0)}\|_w},$$

we obtain from (5.4)

$$\|\bar{u}_n^{(j+1)}\|_w \le M_n^{(2)} \sum_{s=0}^{j} \|\bar{u}_n^{(j-s)}\|_w \|\bar{u}_n^{(s)}\|_w, \quad j = 0, 1, \ldots, \quad \|\bar{u}_n^{(0)}\|_w = 1.$$

We solve this recurrence system of inequalities of convolution type analogously as above by switching to the majorant system and using the method of generating functions; see, e.g., [6, 16, 26, 27]. Then we obtain

$$\|\bar{u}_n^{(j)}\|_w \le \frac{\|u_n^{(j)}\|_w}{\|u_n^{(0)}\|_w} \le q_n^j \frac{2(2j-1)!!}{(2j+2)!!} \le \frac{q_n^j}{(j+1)\sqrt{\pi j}}, \quad q_n = 4M_n^{(2)}. \tag{5.6}$$

Using (5.6), we have from (5.3)

$$|\lambda_n^{(j+1)}| \le \max_{x\in[-1,1]} \left(\frac{|q(x)|}{w(x)}\right) \frac{q_n^j}{(j+1)\sqrt{\pi j}}.$$

Thus, we have proved the following assertion.

Theorem 5.1. *Let $q(x)/w(x) \in C[-1, 1]$, $\mu > \frac{1}{4}$ and $q_n = 4M_n^{(2)} < 1$. Then the FD-method converges super-exponentially with the estimates*

$$|\lambda_n - \overset{m}{\lambda}_n| = \left|\sum_{j=m+1}^{\infty} \lambda_n^{(j)}\right| \le \max_{x\in[-1,1]} \left(\frac{|q(x)|}{w(x)}\right) \frac{q_n^m}{(m+1)\sqrt{\pi m}} \frac{1}{1-q_n}, \tag{5.7}$$

$$\|u_n - \overset{m}{u}_n\|_w = \left\|\sum_{j=m+1}^{\infty} u_n^{(j)}\right\|_w \le \frac{q_n^{m+1}}{(m+2)\sqrt{\pi(m+1)}} \frac{\|u_n^{(0)}\|_w}{1-q_n}. \tag{5.8}$$

Remark 5.2. It follows from (5.5) that for $\mu > \frac{1}{4}$ there exists such n_0 that for all $n \ge n_0$ the assumptions of the theorem are fulfilled and the FD-method converges super-exponentially with the accuracy estimates (5.7) and (5.8). The condition $\mu > \frac{1}{4}$ in Theorem 5.1 means that the fractional derivative should be of order greater than $\frac{1}{2}$. The condition $\alpha < 1$ in Theorem 4.1 means that the convergence of the FD-method is guaranteed if the fractional derivative is of order less than 1, i.e., it does not dominate.

Example 5.3. Let us consider the case $q(x) = w(x)x$, $\mu = \frac{4}{5}$. For $n = 1$ the solution of the base problem is

$$u_1^{(0)}(x) = (1+x)^{4/5}, \quad \lambda_1^{(0)} = -\frac{\Gamma(9/5)}{\Gamma(1/5)}.$$

Table 6 gives the corrections $\lambda_1^{(j)}$, $j = 0, 1, \ldots, 5$. Thus, we have

$$\overset{5}{\lambda}_1 = -0.8656637788992767$$

and one can observe practical convergence. Nevertheless, the sufficient convergence condition does not hold since $q_1 = 28.82\ldots > 1$. Only beginning with $n = 4$ we have $q_n < 1$. In order to get theoretically justified eigenvalues with smaller numbers, one should apply the general algorithm of the FD-method with a piecewise constant approximation of $q(x)$ on a partitioning of the interval fine enough.

5.1 Recursive Implementation of the FD-Method for a Jacobi-Type Differential Operator

Now, let us show that the algorithm above can be reformulated as a recurrence algorithm with respect to the coefficients of some expansions of corrections $\lambda_n^{(j)}$ and $u_n^{(j)}$.

j	$\lambda_1^{(j)}$
1	−0.2028785620703973
2	−0.7999999999999999
3	0.07587304555679625
4	0.03837812694709120
5	0.01714234558251600
6	0.005821265084716930

Table 6. The FD-corrections for (5.1) with $q(x) = w(x)x$, $\mu = 4/5$.

For the sake of simplicity we consider the problem

$$\begin{cases} {}_{-1}D_x^s\{(1-x^2)^s\,{}_xD_1^s u(x)\} + q(x)u(x) - \lambda u(x) = 0, \quad x \in (-1,1), \\ |u(-1)| < \infty, \quad |u(1)| < \infty, \\ s \in (0,1). \end{cases}$$

This problem is of type (5.1) and is a particular case of the problem from [9] (the first formula on the top with $\alpha = \beta = 0$). The proof of convergence of the FD-method for this problem can be done analogously to the one of Theorem 5.1. Note that for the case of a polynomial potential $q(x)$ and $s = 1$, a modification of the FD-method was proposed in [29] without solving boundary value problems in each iteration. A further modification of this procedure was proposed in [30].

Analogously to the case of the Fourier fractional derivative the algorithm for the coefficients of corrections of eigenfunctions for a polynomial potential $q(x)$ can be formulated as a recurrence procedure. But now these corrections are finite linear combinations of Legendre polynomials with the number of summands depending on the degree of the potential. For the sake of simplicity we illustrate this for $q(x) = x^2$. The base problem in this case is

$$\begin{cases} {}_{-1}D_x^s\{(1-x^2)^s\,{}_xD_1^s u_n^{(0)}(x)\} - \lambda_n^{(0)} u_n^{(0)}(x) = 0, \quad x \in (-1,1), \\ |u_n^{(0)}(-1)| < \infty, \quad |u_n^{(0)}(1)| < \infty, \\ u_n^{(0)}(x) = P_n(x), \quad \lambda_n^{(0)} = \dfrac{\Gamma(n+s+1)}{\Gamma(n-s+1)}, \end{cases}$$

where $P_n(x)$ are the Legendre polynomials. The recurrence sequence of problems for corrections of the FD-method is

$$\begin{cases} {}_{-1}D_x^s\{(1-x^2)^s\,{}_xD_1^s u_n^{(j+1)}(x)\} - \lambda_n^{(0)} u_n^{(j+1)}(x) = F_n^{(j+1)}(x), \quad x \in (-1,1), \\ |u_n^{(j+1)}(-1)| < \infty, \quad |u_n^{(j+1)}(1)| < \infty, \end{cases} \tag{5.9}$$

where

$$F_n^{(j+1)}(x) = \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}(x) - x^2 u_n^{(j)}(x), \quad j = 0, 1, \ldots.$$

Using the well-known property of the Legendre polynomials

$$\begin{aligned} x^2 P_n(x) &= \frac{x}{2n+1}((n+1)P_{n+1}(x) + nP_{n-1}(x)) \\ &= \frac{1}{2n+1}\Big(\frac{(n+1)(n+2)}{(2n+3)}P_{n+2}(x) + \Big(\frac{(n+1)^2}{(2n+3)} + \frac{n^2}{2n-1}\Big)P_n(x) + \frac{n(n-1)}{2n-1}P_{n-2}(x)\Big) \\ &= b_n P_{n+2}(x) + c_n P_n(x) + d_n P_{n-2}(x), \end{aligned}$$

the orthogonality condition and the mathematical induction, we can show the following representation:

$$u_n^{(j)}(x) = \sum_{p=-j,\, n+2p\ge 0}^{j} a_{n+2p}^{(j)} P_{n+2p}(x), \quad a_n^{(j)} = 0, \quad a_n^{(0)} = 1. \tag{5.10}$$

After substitution of (5.10) into (5.9) we arrive at the following recurrence system for the coefficients of (5.10):

$$a_{n+2p}^{(j+1)} = \frac{1}{\lambda_{n+2p}^{(0)} - \lambda_n^{(0)}} \Big[\sum_{k=0}^{j} \lambda_n^{(j+1-k)} a_{n+2p}^{(k)} - (a_{n+2p-2}^{(j)} b_{n+2p-2} + a_{n+2p}^{(j)} c_{n+2p} + a_{n+2p+2}^{(j)} d_{n+2p+2}) \Big] \tag{5.11}$$

for $j = 0, 1, \dots$ and $p = -\lfloor n/2 \rfloor, \dots, j+1$. The solvability condition for problem (5.9) yields

$$\lambda_n^{(j+1)} = a_{n-2}^{(j)} b_{n-2} + a_{n+2}^{(j)} d_{n+2}. \tag{5.12}$$

Now (5.11) and (5.12) build a self-contained algorithm.

For $n = 0$ we have

$$\begin{aligned}
\lambda_0^{(0)} &= \frac{1}{2}, & u_0^{(0)}(x) &= P_0(x),\\
\lambda_0^{(1)} &= \frac{1}{3}, & u_0^{(1)}(x) &= -\frac{1}{3} P_2(x),\\
\lambda_0^{(2)} &= -\frac{2}{45}, & u_0^{(2)}(x) &= \frac{11}{26} P_2(x) + \frac{1}{35} P_4(x),\\
\lambda_0^{(3)} &= \frac{11}{945}, & u_0^{(3)}(x) &= -\frac{481}{26460} P_2(x) - \frac{359}{32340} P_4(x) - \frac{1}{693} P_6(x),\\
\lambda_0^{(4)} &= -\frac{481}{198450}, & u_0^{(4)}(x) &= \frac{23693}{12224520} P_2(x) + \frac{5287031}{1942340400} P_4(x) + \frac{6547}{9604980} P_6(x) + \frac{1}{19305} P_8(x),\\
&\ \vdots\\
\lambda_0^{(27)} &= -0.1855129254 \dots \cdot 10^{-13}.
\end{aligned}$$

According to our theory the sequence $\overset{N}{\lambda_0} = \lambda_0^{(0)} + \cdots + \lambda_0^{(N)}$ converges to the exact eigenvalue

$$\lambda_0 = 0.79839 \dots .$$

Remark 5.4. The algorithm described above can be generalized to the case when the coefficient $r(x)$ in the front of the fractional derivative is not constant. In this case we cover the whole interval by a grid

$$\omega_h = \{x_0 = -1 < x_1 < x_2 < \cdots < x_{M-1} < x_M = 1\}$$

with $M - 1$ points and approximate it by a piecewise constant function $\overline{r}(x) = r(x_{i-1})$, $i = 1, \dots, M$. The crucial point is the solution of the base problem. To do this, we write down the general solution on each subinterval depending on two arbitrary constants. Two of these constants (on the edge subintervals) can be eliminated using the boundary conditions. Then stitching the solutions and their derivatives at the grid points, we obtain a system of $2M - 2$ linear homogeneous algebraic equations with $2M - 2$ unknowns with a matrix depending on the eigenvalue parameter $\lambda_n^{(0)}$. The solvability condition for this system leads to a transcendent equation with this parameter from where we obtain the eigenvalues of the base problem. Further we can iterate as usual for the FD-method exploiting the fact that the differential operator with piecewise coefficients of the problem for corrections remains the same.

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