Research Article

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Application of generalized equations of finite difference method to computation of bent isotropic stretched and/or compressed plates of variable stiffness under elastic foundation

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Abstract: The computation of bent isotropic plates, stretched and/or compressed, is a topic widely explored in the literature from both experimental and numerical point of view. We expose in this work an application of the generalized equations of Finite difference method to that topic. The strength of the proposed method is the ability to reconstruct the approximate solution with respect of eventual discontinuities involved in the investigated function as well as its first and second derivatives, including the right-hand side of the equilibrium equation. It is worth mentioning that by opposition to finite element methods our method needs neither fictitious points nor a special condensation of grid. Well-known benchmarks are used in this work to illustrate the efficiency of our numerical and the high accuracy of calculation as well. A comparison of our results with those available in the literature also shows good agreement.

Keywords: rectangular plate, elastic foundation, generalized equations of finite difference method, discontinuity

1 Introduction

One can start by recalling that a plate is a structure, which thickness is small beside its length and width. What happens when you crumple up a sheet of paper? How does a general raft supporting a building behavior? Can the futuristic roofs that adorn our most glorious buildings (shopping centers, airport buildings. . .), resist the wind? In which way the carrosseries of a car is distorted in an accident? Here are some of many scenarios of structure behavior that arise in real life problems. Characterizing the deformations undergone under certain constraints is our aim in this work. The issues related to this study are diverse: it will sometimes be a question for the engineer of designing resistant and / or aesthetic plates or shells. Sometimes covertly, the plate specifications must provide the deformations distributed

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over its mid surface. The designed structures aim is to absorb shocks, for example the front part of a vehicle or even submerged radiators.

During their exploitations, plate structures are subjected to the transversal loads (statics and dynamics). From computational point of view, efficient numerical tools are necessary for modeling sophisticated mechanical behavior of such structures, accounting with their specificities. Despite the abundance of literature on computations of plate structures [1, 2, 3], several questions remain topical here.

The behavior of those structures is governed by a linear partial differential equation (PDE, for short) of 4^{th} order which is not obvious to be solved using analytical methods [5, 6, 7]. Hence, to call out for numerical methods easy to implement and less onerous from computational point of view. Among popular numerical methods used for this topic, the finite element method (FEM, for short) is the most popular [4, 8, 9, 10]. However, the FEM presents a certain number of drawbacks as indicated in [9]: (a) The FEM is poorly adapted to a solution of the so-called singular problems like plates with cracks, corner points, discontinuity internal actions, and of problems for unbounded domains. (b) This method requires the use of powerful computers of considerable speed and storage capacity. (c) The method presents many difficulties associated with problems of C^1 continuity and nonconforming elements in plate (and shell) bending analysis. Note that the mathematical theory of FEM is exponentially increasing. So the above drawbacks could be addressed in a near future by the FEM.

More recent numerical methods have been developed for addressing singular problems [10, 11, 12, 13, 14, 15, 16]. Among the large variety of those methods for addressing singular problems, the Generalized Differential Quadrature (GDQ, for short) is proposed to solve different kinds of structural problems. In many applications present in literature [17, 18, 19, 20], the GDQ method has shown superb accuracy, efficiency, convenience, and great potential in solving differential equations. The generalized equations of the Finite Difference Method (GE-FDM, for short) are part of these recent numerical methods [21, 22, 23].

The aim of this work is to show that the generalized equation of the finite difference method (GE-FDM, for short) could be used to address the computation of bent, stretched and/or compressed rectangular plates of variable stiffness under elastic foundations. One of the main features of this method (GE-FDM) is the ability of dealing with finite discontinuities of the investigated solution and that of its first and second derivatives, including discontinuities of the right hand side of the primary PDE. According to [22] and [23, 25], the computation of these plates with GE-FDM leads to satisfactory approximate solutions faster than the successive approximation method (SAM, for short).

The work is organized as follows: after the introduction which poses the problematic of the subject, we unfold the methodology of the implementation of the generalized equations of the finite difference method which is broken down into three points. Subsequently we transform the new differential equation by the FDM. The last part of this framework will be devoted to the validation of our approach through the numerical resolution of test- problems.

2 Tools and techniques

In several works related to the numerical calculation of plates and shells, the authors use various approaches to solve the partial differential equations which govern these structures. As far as we are concerned in this frame work, our resolution methodology is as follows:

(a) transformation of the partial derivatives 4^{th} order deformation equation of a rectangular plate of variable thickness into a system of two differential equations of 2^{nd} order partial derivative;
(b) introduction of new dimensionless parameters in the system of equations obtained and in the equation describing the boundary conditions;
(c) transformation of new differential equations by the generalized equations of the finite difference method, these permits a system of algebraic equation to be obtain;
(d) transformation of boundary conditions;
(e) elaboration of a calculation algorithms;
(f) resolution of the system of algebraic equation in order to obtain the bending moment and the maximum displacement.

2.1 Equation of deformation of a bent rectangular plate stretched and /or compressed with variable rigidity on elastic foundation

In the following paragraphs, only bent, stretched and/or compressed plates of variable thickness will be analyzed. So it is convenient to express the governing differential equation of this plate. Figure 1 illustrates of the equilibrium of a sample plate stretched or compressed element:

Let us consider an infinitesimal element d_X, d_Y as indicated in Figure 1 and projections of membrane forces along

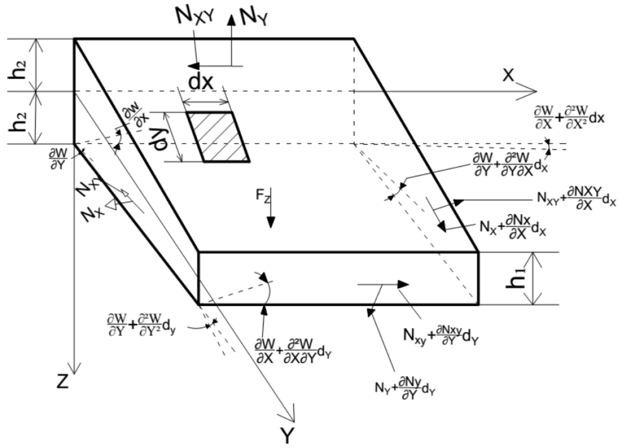


Figure 1: Element of a bent plate stretched and /or compressed.

the Z - axis. N_X, N_Y and N_{XY} are respectively the horizontal components of the normal and shear forces which are exerted on the various facets. The normal unit vector to the facets is tangent to the neutral plane in each direction (X, Y). By neglecting the forces of volume along the X and Y directions, we obtain according to Z- axis:

$$
\begin{aligned}
&\frac{\partial^2 D}{\partial X^2}\frac{\partial^2 W}{\partial X^2}+\mu\frac{\partial^2 D}{\partial X^2}\frac{\partial^2 W}{\partial Y^2}+2\frac{\partial D}{\partial Y}\frac{\partial^3 W}{\partial X^3}+2\frac{\partial D}{\partial X}\frac{\partial^3 W}{\partial X\partial Y^2}+\\
&2\frac{\partial D}{\partial Y}\frac{\partial^3 W}{\partial X^2\partial Y}+2\frac{\partial^2 D}{\partial X\partial Y}\frac{\partial^2 W}{\partial X\partial Y}-2\mu\frac{\partial^2 D}{\partial X\partial Y}\frac{\partial^2 W}{\partial X\partial Y}+\frac{\partial^2 D}{\partial Y^2}\frac{\partial^2 W}{\partial Y^2}\\
&+\mu\frac{\partial^2 D}{\partial Y^2}\frac{\partial^2 W}{\partial X^2}+D\Delta\Delta W=N_X\frac{\partial^2 W}{\partial X^2}+N_Y\frac{\partial^2 W}{\partial X^2}-2N_{XY}\frac{\partial^2 W}{\partial X\partial Y}\\
&-RW+F_Z
\end{aligned}
\tag{1}
$$

with

$$
D\Delta\Delta W=\frac{\partial^2 D}{\partial Y^2}\frac{\partial^2 W}{\partial X^2}-\frac{\partial^2 D}{\partial Y^2}\frac{\partial^2 W}{\partial Y^2}+2\frac{\partial^2 D}{\partial X\partial Y}\frac{\partial^2 W}{\partial X\partial Y} \tag{2}
$$

where

$$
D=\frac{EH^2}{12(1-\mu^2)} \tag{3}
$$

Eq. (1) is called the deformation equation of a bent rectangular plate stretched and /or compressed with variable rigidity on elastic foundation;

where $W = W(X, Y)$ is the transversal displacement of the plate (searched function); $D = D(X, Y)$ stiffness of a variable plate; μ is Poisson's coefficient; N_X, N_Y are normal membrane forces; N_{XY} denotes membrane shear forces ; R is the ground stiffness in N/m^3; $F_Z$represents the volume force along the Z - axis, while $H = H(X, Y)$ is the variable thickness of the plate.

The Eq. (1) can be transformed as a system of 2nd order partial derivative equations (pde):

$$
\begin{cases}
\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}=-\frac{M}{D}\\
\frac{\partial^2 M}{\partial X^2}+\frac{\partial^2 M}{\partial Y^2}=\left(F_Z+N_X\frac{\partial^2 W}{\partial X^2}+N_Y\frac{\partial^2 W}{\partial Y^2}-2N_{XY}\frac{\partial^2 W}{\partial X\partial Y}-RW\right)+\\
+(1-\mu)\left(\frac{\partial^2 D}{\partial X^2}\frac{\partial^2 W}{\partial Y^2}+\frac{\partial^2 D}{\partial Y^2}\frac{\partial^2 W}{\partial X^2}-2\frac{\partial^2 D}{\partial X\partial Y}\frac{\partial^2 W}{\partial X\partial Y}\right)
\end{cases}
\tag{4}
$$

where

$$
M_X=\frac{\partial^2 W}{\partial X^2}+\mu\frac{\partial^2 W}{\partial Y^2};\quad M_Y=\frac{\partial^2 W}{\partial Y^2}+\mu\frac{\partial^2 W}{\partial X^2} \tag{5}
$$

$$
M=\frac{M_X+M_Y}{1+\mu} \tag{6}
$$

M denotes the resultant moment, so $M_X$and M_Y are bending moment following X and Y directions, respectively.

2.2 Introductions to dimensionless parameters

Rewriting the Eq. (4) using dimensionless parameters [1], [14]:

$$
\begin{aligned}
\eta=\frac{Y}{l};\ \xi=\frac{X}{l};\ F=\frac{F_Z}{F_0};\ m=\frac{M}{F_0 l^2};\ v=\frac{WD_0}{F_0 l^4};\\
m^{(\xi)}=\frac{M_X}{F_0 l^2};\quad m^{(\eta)}=\frac{M_Y}{F_0 l^2};\ g=\frac{D(X;Y)}{D_0}
\end{aligned}
\tag{7}
$$

$$
l=\max\left(|l_X|,|l_Y|\right);\ k=\frac{Nl}{D_0};\ \bar{\alpha}=\frac{N_X}{N};\ \bar{\gamma}=\frac{N_Y}{N};\ \bar{\beta}=\frac{-2N_{XY}}{N} \tag{8}
$$

where $N = \max(|N_X|, |N_Y|, |N_{XY}|)$; $-1 \le \overline{\alpha} \le 1$, $-1 \le \overline{\beta} \le 1$, $-1 \le \overline{\gamma} \le 1$; $(\eta\,;\xi)$ are Cartesian coordinates without units; F is Load factor; m is Moment coefficient; μ is the coefficients of deflection; D_0 denotes the cylindrical stiffness of any section of the slab; g is the stiffness coefficient; l is plate length; $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and k are coefficients without unit.

Introducing the parameter in Eqs. (7) and (8) into the system of Eq. (4), we obtain:

$$
\begin{cases}
\frac{\partial^2 v}{\partial\xi^2}+\frac{\partial^2 v}{\partial\eta^2}=-\frac{m}{g}\\
\frac{\partial^2 m}{\partial\xi^2}+\frac{\partial^2 m}{\partial\eta^2}-k\left(\bar{\alpha}\frac{\partial^2 v}{\partial\xi^2}+\bar{\gamma}\frac{\partial^2 v}{\partial\eta^2}+\bar{\beta}\frac{\partial^2 v}{\partial\xi\partial\eta}\right)+\\
(1-\mu)\left(\frac{\partial^2 g}{\partial\xi^2}\frac{\partial^2 v}{\partial\eta^2}+\frac{\partial^2 g}{\partial\eta^2}\frac{\partial^2 v}{\partial\xi^2}-2\frac{\partial^2 g}{\partial\xi\partial\eta}\frac{\partial^2 v}{\partial\xi\partial\eta}\right)-\bar{\lambda}v=F
\end{cases}
\tag{9}
$$

where $\overline{\lambda} = \frac{Rl^4}{D}$.

The Eq. (9) can be written as follows:

$$
\begin{cases}
\frac{\partial^2 v}{\partial\xi^2}+\frac{\partial^2 v}{\partial\eta^2}=-\frac{m}{g}\\
\frac{\partial^2 m}{\partial\xi^2}+\frac{\partial^2 m}{\partial\eta^2}+\left(-k\bar{\alpha}+(1-\mu)\frac{\partial^2 g}{\partial\eta^2}\right)\frac{\partial^2 v}{\partial\xi^2}+\\
+\left(-k\bar{\gamma}+(1-\mu)\frac{\partial^2 g}{\partial\xi^2}\right)\frac{\partial^2 v}{\partial\eta^2}+\left(-k\bar{\beta}-2(1-\mu)\frac{\partial^2 g}{\partial\xi\partial\eta}\right)\frac{\partial^2 v}{\partial\xi\partial\eta}=F+\bar{\lambda}v
\end{cases}
\tag{10}
$$

Eq. (10) is called the generalized algebraic equation of the finite difference method which replaces Eq. (4). Note that v and its partial derivative can be discontinuous when the plate has ball points, while m will be discontinuous if external point bending moments are applied in one of the directions of the coordinate axes.

2.3 Boundary conditions

Several boundary conditions are discussed in this work in accordance with practical and site works constraints in foundations design. We are going to emphasize on displacement and moments conditions on the borders.

2.3.1 Articulated supports

If the articulated edge is parallel to the X - axis, in other words, $Y = 0$, then:

$W = 0$; $M_Y=0$.

If the above conditions are imposed on the edge of the plate: $W = W_0(X)$ and $M_Y = M^{\circ}{}_Y(X)$, therefore from formulas (10), we obtain:

$$M = M_0^Y(X) - D(1-\mu)\frac{\partial^2 W_0(X)}{\partial X^2} \tag{11}$$

If the edge is parallel to the Y - axis, in other words, $X = 0$, then:

$$W = \mathbf{0}; \quad M_X = \mathbf{0}. \tag{12}$$

2.3.2 Embedded edges

If the embedded edge is parallel to the Y - axis, then:

$$\left(\frac{\partial W}{\partial Y}\right)_{y=a}; (W)_{y=a} = 0 \tag{13}$$

If on the other hand it is parallel to the X - axis, then:

$$(W)_{x=a} = 0; \left(\frac{\partial W}{\partial x}\right)_{x=a} = 0 \tag{14}$$

All the ingredients are gathered to find out the generalized equations of the finite difference method.

3 Formulation of the Generalized Equation of the Finite Difference Method (GE- FDM)

The technique exposed here for obtaining the generalized equations of the finite difference has been first introduced in [14].

3.1 Finite difference mesh over the structure

A square mesh of size h is defined over the structure as indicated in Figure 2 where the roman numerals are used for mesh element numbers and *(i,j)* represents mesh nodes

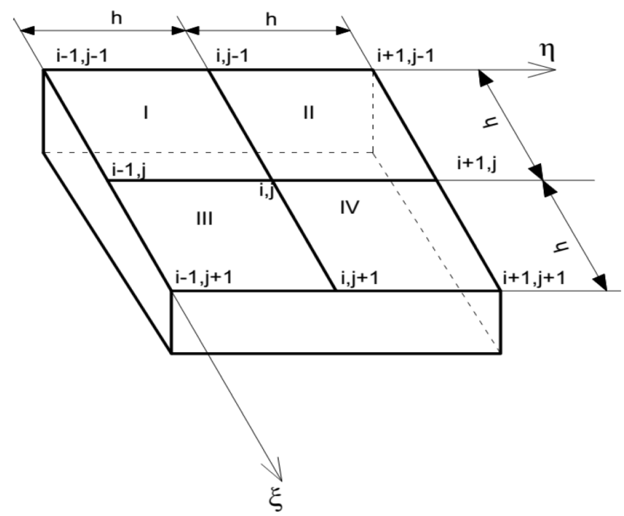


Figure 2: Square mesh for GE-FDM.

Taking into consideration Eq. (6) established in [14] and identifying Eq. (10) with Eq. (6), we obtained:

(a)

$$\mathcal{V}_{i,j-1} + \mathcal{V}_{i-1,j} + \mathcal{V}_{i+1,j} + \mathcal{V}_{i,j+1} - 4\boldsymbol{v}_{i,j} = -h^2\frac{\boldsymbol{m}_{ij}}{g_{ij}} \tag{15}$$

with $P = -\frac{m}{g}$ $\omega = \mathbf{v}$; $\alpha = \gamma = 1$; $\delta = \beta = \sigma = o$; also considering that m and $v$are continuous, as well as their first and second order derivatives

(b)

$$m_{i,j-1}+m_{i-1,j}+m_{i+1,j}+m_{i,j+1}-4m_{i,j}+ \frac{h}{2}\left(\Delta^{I-II}m^{\xi}_{ij}+\Delta^{III-IV}m^{\xi}_{ij}+\Delta^{I-III}m^{\eta}_{ij}+\Delta^{II-IV}m^{\eta}_{ij}\right)+ -\frac{k}{4}\left[\begin{array}{l}\beta\left(v_{i-1,j-1}-v_{i-1,j+1}-v_{i+1,j-1}+v_{i+1,j+1}\right)+\\ 4\bar{\alpha}\left(v_{i-1,j}+v_{i+1,j}\right)+4\bar{\gamma}\left(v_{i,j-1}+v_{i,j+1}\right)-8(\bar{\alpha}+\bar{\gamma})v_{i,j}\end{array}\right]+ +(1-\mu)\left[\begin{array}{c}-\frac{1}{2h^2}g^{\xi\eta}\left(v_{i-1,j-1}-v_{i+1,j+1}-v_{i+1,j-1}-v_{i-1,j+1}\right)+\\ \frac{1}{h^2}g^{\eta\eta}\left(v_{i-1,j}+v_{i+1,j}-2v_{i,j}\right)+\frac{1}{h^2}g^{\xi\xi}\left(v_{i,j-1}+v_{i,j+1}-2v_{i,j}\right)\end{array}\right] =-h^2P_{i,j} \tag{16}$$

where: $P_{i,j}=F_{i,j}+\bar{\lambda}v_{i,j}.$; $g^{\eta\eta}=\frac{\partial^2 g}{\partial\eta^2}$; $g^{\xi\xi}=\frac{\partial^2 g}{\partial\xi^2}$ and $g^{\xi\eta}=\frac{\partial^2 g}{\partial\xi\partial\eta}$

With: $\omega = m$; $\alpha_1 = -k\overline{\alpha}+(1-\mu)\frac{\partial^2 g}{\partial\eta^2}$; $\gamma_1 = -k\overline{\gamma}+(1-\mu)\frac{\partial^2 g}{\partial\xi^2}$; $\beta_1 = -k\overline{\beta}+(1-\mu)\frac{\partial^2 g}{\partial\xi\partial\eta}$ and $\sigma_1=\delta_1=0$, then also consider that P is constant within inside of each element but can vary abruptly from one element to another.

Eqs. (15) and (16) are called the generalized equations of the finite difference method for a bent rectangular plate stretched and /or compressed with variable stiffness on elastic foundation, which substitute's Eq. (10), where, = 2, 3… n-1; = 2, 3, …,-1.

(i) At the end, $F+\bar{\lambda}v$ will be replaced by $F_{i,j}+\bar{\lambda}v_{i,j}$.

We write the equation for a regular mesh:

$$\mathrm{h}_i = \tau_i = \mathrm{h}_{i+1} = \tau_{i+1}$$

hence, by combining Eqs. (15) and (16) we obtain for Eq. (10):

$$\left\{\begin{array}{l} v_{i-1,j}+v_{i,j-1}-4v_{i,j}+v_{i+1,j}+v_{i,j+1}=-h^2\frac{m_{ij}}{g_{ij}}\\ m_{i,j-1}+m_{i-1,j}+m_{i+1,j}+m_{i,j+1}-4m_{i,j}+\\ \frac{h}{2}\left(\Delta^{I-II}m^{\xi}_{ij}+\Delta^{III-IV}m^{\xi}_{ij}+\Delta^{I-III}m^{\eta}_{ij}+\Delta^{II-IV}m^{\eta}_{ij}\right)+\\ -\frac{k}{4}\left[\begin{array}{l}\overline{\beta}(v_{i-1,j-1}-v_{i-1,j+1}-v_{i+1,j-1}+v_{i+1,j+1}\Big)+\\ 4\bar{\alpha}\left(v_{i-1,j}+v_{i+1,j}\right)+4\bar{\gamma}\left(v_{i,j-1}+v_{i,j+1}\right)-8(\bar{\alpha}+\bar{\gamma})v_{i,j}\end{array}\right]+\\ +(1-\mu)\left[\begin{array}{c}-\frac{1}{2h^2}g^{\xi\eta}\left(v_{i-1,j-1}-v_{i+1,j+1}-v_{i+1,j-1}-v_{i-1,j+1}\right)+\\ \frac{1}{h^2}g^{\eta\eta}\left(v_{i-1,j}+v_{i+1,j}-2v_{i,j}\right)+\frac{1}{h^2}g^{\xi\xi}\left(v_{i,j-1}+v_{i,j+1}-2v_{i,j}\right)\end{array}\right]\\ =-h^2P_{i,j}\end{array}\right. \tag{17}$$

where: $\Delta^{I-II}m^{\xi}_{i,j}={}^{I}m^{\xi}_{i,j}-{}^{\mathrm{II}}m^{\xi}_{i,j}$; $\Delta^{III-IV}m^{\xi}_{i,j}={}^{\mathrm{III}}m^{\xi}_{i,j}-{}^{\mathrm{IV}}m^{\xi}_{i,j}$; $\Delta^{I-III}m^{\xi}_{i,j}={}^{I}m^{\xi}_{i,j}-{}^{\mathrm{III}}m^{\xi}_{i,j}$; $\Delta^{II-IV}m^{\xi}_{i,j}={}^{\mathrm{II}}m^{\xi}_{i,j}-{}^{\mathrm{IV}}m^{\xi}_{i,j}$, $i = 2,3,\ldots..,n-1$; $j=2,3,\ldots..,n-1$.

h - Mesh spacing; *i* - measuring along the Axis; *j* - measuring along the Axis

Eq. (17) is called the generalized algebraic equation of the finite difference method for a bent rectangular plate stretched and /or compressed with variable stiffness on elastic foundation, which substitute Eq. (10). This equation is solved taking into consideration the transformed boundary conditions.

3.2 Transformation of the boundary conditions by the generalized equation of the finite difference method

In this section, we establish equations called boundary conditions, which are combined to the Eq. (17) lve the differential Eq. (2). The said boundary conditions are presented below.

3.2.1 Articulated sides

If all the edges of the plate are articulated, it is enough to solve the Eqs. (15) and (16) Those equations are written for each point inside the field. Thus, we have:

$$m^{(\xi)}=0 \quad ; \; v=v_0(\xi); \qquad m=m_0{}^{(\eta)}(\xi)-(1-)v_0^{\xi\xi}(\xi); \tag{18}$$

where $v_0(\xi)$; $m_0{}^{(\eta)}(\xi)$; $v_0^{\xi\xi}(\xi)$ are all known.

3.2.2 Embedded edges

We remind that if an edge is embedded, the deflection and rotation on that edge are zero.

Edges parallel to ξ- axis:

if the edge $\eta=0$ is embedded, then:

$(v_{i,j})_\eta=0$ either $v_{i-1,j}=v_{i+1,j}=v_{i,j}=0$ and $(\frac{\partial v}{\partial y})_{\eta=0}=0$. Hence, $v^{\eta}_{i,j}=0$. By substituting the first equation of the system (17) with the generalized Eq. (3) giving in [14] and by noting that $\alpha=\gamma=1$; $\delta=\beta=\sigma=o$ we obtain:

$2v^{\eta}_{i,j}=0=\frac{1}{h^2}(v_{i,j-1}+v_{i-1,j}+v_{i+1,j}+v_{i,j+1}-4v_{i,j}-h^2\frac{m_{ij}}{g_{ij}})$, either $m^{\xi}_{i,j}=\frac{2}{h^2}v_{i,j+1}$ from where:

$$\left\{\begin{array}{l} v_{i,j}=\mathbf{0}\\ m^{\xi}_{i,j}=\frac{-2g_{i,j}}{h^2}\boldsymbol{V}_{i,2}\end{array}\right. \tag{19}$$

where $i=2,3,\ldots..,n-1$; $j=2$.

if the edge $\eta=$ is embedded then:

by proceeding in the same way as previously and by considering the pair of elements (a-c), and according to [14], we obtain:

$$\left\{\begin{array}{l} \boldsymbol{v}_{i,\eta}=\mathbf{0}\\ m^{\xi}_{i,\eta}=\frac{-2g_{i,\eta}}{h^2}\boldsymbol{v}_{i,\eta-1}\end{array}\right. \qquad i=2,3,\ldots\ldots,n-1 \tag{20}$$

Edges parallel to η- axis:

If the edge ξ = is embedded:

By considering the pair of element (c-d) we have:

$$2v^{\xi}_{i,j} = {}_{i,j-1} + v_{i-1,j} + v_{i+1,j} + v_{i,j+1} - 4v_{i,j} - h^2 \frac{m_{ij}}{g_{ij}}$$

and noticing that in [14], $\alpha = \gamma = 1$; $\delta = \beta = \sigma = o$ we obtain: $m^{\eta}_{i,j} = \frac{2g_{1,j}}{h^2}\boldsymbol{V}_{i+1,j}$,

from where:

$$\begin{cases} v_{1j} = \mathbf{0} \\ m^{\eta}_{1,j} = \frac{2g_{1,j}}{h^2}\boldsymbol{V}_{2,j} \end{cases}, \quad j = 1, 2, \ldots.n-1 \tag{21}$$

If edge ξ = is embedded:

here, by considering the pair of elements (c-d), and according to [14], we obtain:

$$\begin{cases} \boldsymbol{v}_{\eta,j} = 0 \\ m^{\eta}_{\eta,j} = \frac{2g_{\eta,j}}{h^2}\boldsymbol{V}_{\eta-1,j} \end{cases}, \quad j = 1, 2, \ldots.n-1 \tag{22}$$

4 Numerical results and discussion

Section one: square plate of variable thickness articulated on all edges

In this first section, the case of a square plate of variable thickness articulated on all edges and on the unit side is examined. It's a benchmark widely use in literature to test numerical models. Some examples of calculation of said plate subjected to simple bending, then to bending combined with compressed and /or traction using the boundary conditions, are presented. The calculations consist in determining the maximum values of the coefficients of the deflection and the moment of the plate, according to different meshes.

Since the plate is in contact with the ground, then we choose for (those cases) the stiffness of the soil, (according to geotechnical reconnaissance of the state of Cameroon).

The plate will have a variable stiffness:

$g_{i,j}(\eta;\xi) = a_{01}\eta^2_{i,j} + b_{01}\xi^2_{i,j} + a_{02}\eta_{i,j} + b_{02}\xi_{i,j} + c_{01}\eta\xi + d_{01}$ where a_{01} ; b_{01} ; b_{02} ; c_{01} ; d_{01} are known constants.

The Young's modulus is $E = 4 \times 10^9$MPa

These values will be introduced into the Eq. (17) to obtain a new system corresponding to each request. The other parameter will be defined according to the type of stress.

4.1 Example 1: Case of bending combined with unidirectional compression

For this first example, it is a square plate subjected to bending combined with compression. The four sides of which are articulated. It is also subjected to the action of a load uniformly distributed over its entire surface. Moreover, the compression loads are applied uniformly and parallel to -axis. A mesh of such a plate is shown in the Figure 3.

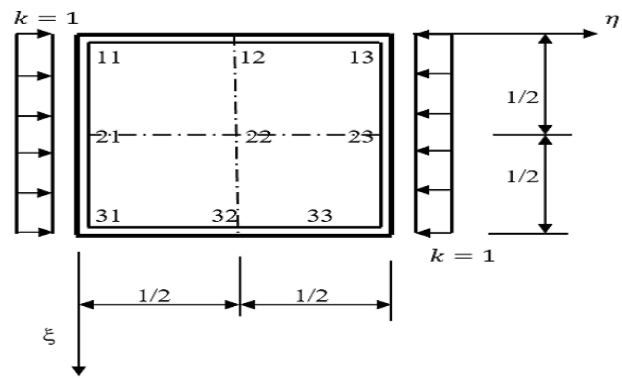


Figure 3: Square plate of variable thickness hinged on all edges.

In this first example, we look at two cases:

(a) Case where the plate is subjected to a constant thickness:

For g = const, the differential equations of bending plates of variable stiffness are specials cases of the equations for plates of constant stiffness.

Using those equations, a computer program was compiled for calculating plates with stiffness continuously constant according to an arbitrary law for the action of breaking static and dynamic loads. The program takes into consideration all types of boundary conditions; it has been introduced into the practice of engineering calculations. Table1 gives bending momentum and deflection coefficients of the plate resulting from this calculation:

In order to evaluate the convergence of the solutions or to check the error, we determine the speed of convergence in the form: $e_{\max} = \alpha h^r \left| v^{\exp}_{\max} - v^{GE-FDM}_{\max} \| \right|$, with r>0, α >0 and where r is the order of convergence, α a constant, $v^{\exp}_{\max}$ and $v^{GE-FDM}_{\max}$ denote respectively the maximum value of deflection obtained in [1], [12] and by the Generalized equations of the finite difference method. Eq. (23) can be written as: $y = rx + \rho$ where $y = \text{Log}e_{\max}$; $x = \text{Log}h$ and $\rho = \text{Log}\alpha$

We will define r and ρ by the least squares' method. For that, we introduce the function Φ defined by:

$\Phi(r,\rho) = \sum_i^6 \left[y_i + (rx_i + \rho\right]^2$. The least square problem consists in finding$(\widehat{r}, \widehat{\rho})$ such that:

Table 1: Moment and deflection (cm) coefficients for bending combined with compression for constant thickness (BCCT).

CaseI:BCCT	*GE-FDM*	$\bar{\alpha} = -1; \bar{\beta} = 1; \bar{\gamma} = 0; k = 1; \mu = 0,16; N < 0; F = -1;$ and $R = 0$					Other researchers		
							[12]	[1]	[11]
Mesh	4 x 4	8 x 8	16 x 16	20 x 20	24 x 24	32 x 32			
va^4PD_{max}	0,00411	0,00413	0,00415	0,00416	0,00418	0,00418	0,00417	0,00417	0,00490
$m_{max}(a^2P)$	0,07238	0,07486	0,07551	0,07581	0,07582	0,07582	0,07580	0,07580	/

Table 2: Values of the relative error of v_{max}.

h	v_{max}^{GE-FDM}	v_{max}^{exp}	e_{max}	*i*	*x*	*y*
4 x 4	0,00411	0,00417	0,00006	1	-0,60206	-4,22184
8 x 8	0,00413	0,00417	0,00004	2	-0,90309	-4,39794
16 x 16	0,00415	0,00417	0,00002	3	-1,20412	-4,69897
20 x 20	0,00416	0,00417	0,00001	4	-1,30103	-5,0000
24 x 24	0,00418	0,00417	0,00001	5	- 1,38021	-5,0000
32 x 32	0,00418	0,00417	0,00001	6	- -1,50515	-5,0000

$\text{Min}\Phi(r,\rho) = \Phi(\hat{r},\hat{\rho}) \in \mathbb{R}_+^\star \times \mathbb{R}$

This leads us to the system:

$$\begin{cases} \frac{\Phi(\hat{r},\hat{\rho})}{\partial r} = 0 \\ \frac{\Phi(\hat{r},\hat{\rho})}{\partial \rho} = 0 \end{cases} \text{that is to say:} \begin{cases} \sum_i^6 x_i^2 r) + \sum_i^6 x_i)\rho = \sum_i^6 x_i y_i \\ \sum_i^6 x_i)r + 6\rho = \sum_i^6 y_i \end{cases}$$

The resolution of this system leads to:

$r = 1,01$ and $\rho = 3,572$;

hence the regression line of y as a function of x is given by:

$$y = 1,01x - 3,572;$$

with this error obtained we can say that the convergence of the results towards those obtained in is established.

(b) Case where the plate is subjected to a variable thickness and under elastic foundation:

Using Eq. (17), a computer program was compiled for calculating plates with stiffness continuously varying according to an arbitrary law ($\eta;\xi$) for the action of breaking static and dynamic loads. The program takes into consideration all types of boundary conditions; it has been introduced into the practice of engineering calculations. Table 3 gives the corresponding values of maximum moment and deflection (coefficients) of the plate resulting from this computer program for this case:

Since solutions for this problem do not exist anywhere, we checked the error of the results obtained by using the principle of static equilibrium of the plate (see Table 2). In this view we have determined the sum of the projections of all the reactions on the axis perpendicular to the average plane of the plate. Under the symmetry property, we can consider half of the plate. The resultant of external loads applied to this portion is equal to 1.

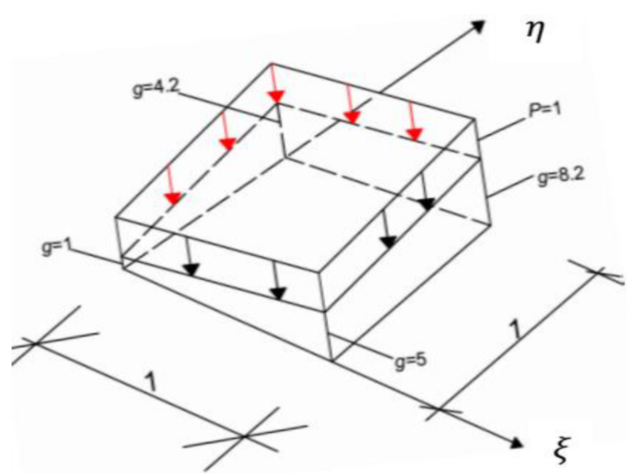


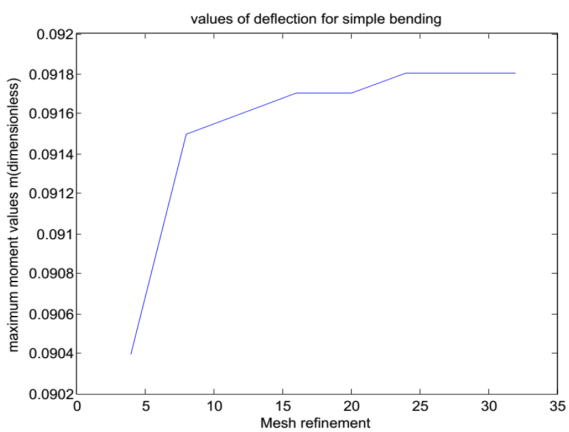
Figure 4: Square plate of variable stiffness with loading.

4.2 Example 2: Simple bending plate

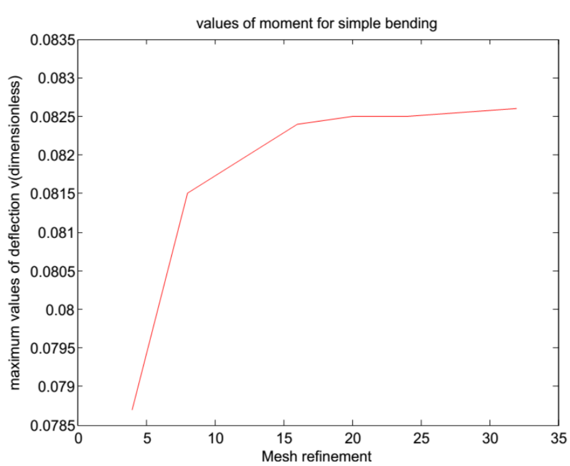
In this example, the plate is subjected to uniformly distributed load over its entire surface as shown in Figure 4. Moreover, the compression loads are applied uniformly and parallel to the axis. The value of the compression loads is much lower than the critical value. The dimensionless value of distributed load is $P = 1$. The thickness of the plate varies along η and ξ as shown in the Figure 4.

Table 3: Moment and deflection coefficients for bending combined with compression for variable thickness (BCVT).

Case2: BCVT	*GE-FDM*	$\bar{\alpha} = -1; \bar{\beta} = 1; \bar{\gamma} = 0; k = 1; \mu = 0,16; F = -1;$ and $R = 0$				
Meshes	4 x 4	8 x 8	16 x 16	20 x 20	24 x 24	32 x 32
va^4PD_{max}	0,001961	0,001966	0,001967	0,001968	0,001968	0,001969
$m_{max}(a^2P)$	0,07316	0,07365	0,07382	0,07390	0,07395	0,07399



(a) Graphical representation of the coefficients of moment.



(b) Graphical representations of the coefficients of deflection.

Figure 5

The goal is to compare our results to the reference values available in literature for the same element in order to better quantify the influence of the flexible foundation, the variable stiffness and the influence of the flexible foundation and the influence of membrane forces.

The computation results on various meshes for a square slab hinged along the contour, the rigidity of which changes in two directions, on the action of the load evenly distributed over the entire area in Figure 4 are compared with the numerical solution of [24].

Table 4 also illustrates the convergence of the numerical solution.

While proceeding as in the case of BCCT, the speed of convergence is an order of convergence equal to $r = 0,99$and the regression line as a function of x is given by**:** $y = 0,99x - 2,64$**.** So the convergence of the results towards those obtained in [24] is established. The figures 5(a) and 5(b) illustrate perfectly this convergence.

4.3 Example 3: Case of flexion combined with unidirectional traction

Figure 6 shows a square slab of length 1 pivotally supported along its contour, the stiffness and distributed load of which in the direction y vary linearly. The results of the computation when half of the plate is loaded make it possible to obtain a solution in the case of loading the entire plate with the same load.

The values of the largest bending moments and deflections are obtained by us on a 36x36 square bit.

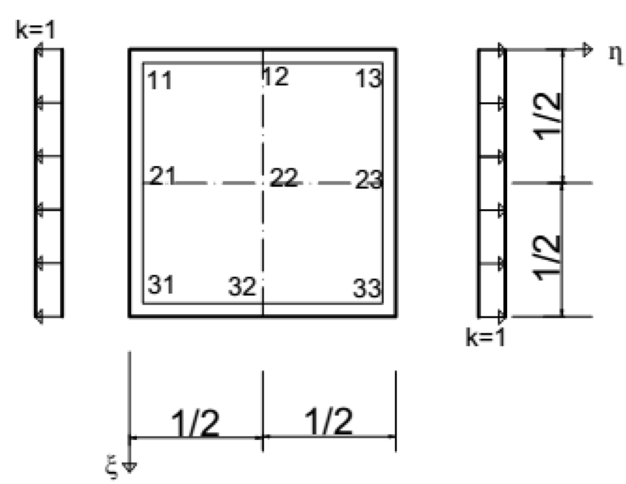


Figure 6: Square plate under elastic foundation.

This makes it possible to note the good behavior of the method. It should be noted that the algorithm was developed with the aim of writing a code of calculation on the basis of generalized equations of the finite difference

Table 4: Maximum moment and deflection coefficients for simple bending (SB).

Case3: SB	*GE-FDM*	$\bar{\alpha}=0; \bar{\beta}=0; \bar{\gamma}=0; k=0; N=0; \mu=0,16; F=-1;$ and $R=0$					[24]
Meshes	4 X 4	8x8	16 X 16	20 X 20	24 x 24	32 X 32	
$v2_{00}{}^{4}_{max}$	0,0904	0,0915	0,0916	0,0916	0,0918	0,0918	0,0917
$m(\eta)^2_{X_0max}$	0,0457	0,0469	0,0475	0,0475	0,0476	0,0476	0,0477
$m(\eta)^2_{Y_0max}$	0,0456	0,0472	0,0478	0,0478	0,0480	0,0482	0,0481

Table 5: Moment and deflection coefficients for bending combined with unidirectional traction (BCT).

Case 4: BCT	*GE-FDM* $\bar{\alpha}=0; \bar{\beta}=0; \bar{\gamma}=1; k=1; \mu=0,16; F=-1;$ (Normal forces tend to lengthen the plate: N>0); $R=22754.0$					
Meshes	4 X 4	8 X 8	16x16	20 X 20	24 X 24	32 X 32
va^4PD_{max}	0,001986	0,001992	0,001993	0,001994	0,001994	0,001994
$m_{max}(a^2P)$	0,07418	0,07467	0,07484	0,07492	0,07497	0,07501

Table 6: Deflection (cm) and moment of the plate with variable thicknesses under a uniformly distributed load (BCVT).

Case 1: BCVT	*GE-FDM*	$\bar{\alpha}=-1; \bar{\beta}=-1; \bar{\gamma}=0; k=1; N<0; F=-1$ and R=0					[11]	FEM	EXPERIMENTAL
Meshes	4 x 4	8x8	16 X 16	20 X 20	24x24	32 X 32			
va^4PD_{max}	0,0049	0,0054	0,0058	0,0062	0,0063	0,0063	0,0069	0,0070	0,0059
$m_{max}(a^2P)$	0,07238	0,07486	0,07551	0,07581	0,07582	0,07582	/	/	/

method. Thus, we can say that with a mesh course, the generalized equations give good results. The refinement of the mesh makes it possible to observe the convergence of the results. Tables 1 to 5 above illustrate the Convergence well.

to evaluate the impact which the application of the normal forces of membranes causes on a bent, tended and or compressed plate of variable rigidity, we will refer to Tables 1 to 4 then the curves of Figures 7a and 7b 7; hence :

a) In all cases when the mesh is increased, the moments and arrows increase and are almost monotonic from a certain mesh pitch.

b) The decrease in bending forces caused by the elastic foundation with the normal forces of membranes acting in traction in one direction is equal to the increase in forces caused in the same conditions compared to those acting in compression in the same direction. The combined effects of these two forces cancel each other out.

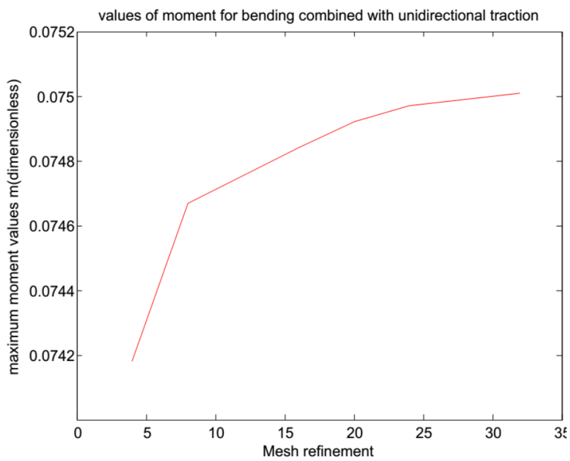
One can deduct from these interpretations that the deflection of a plate of variable rigidity on a flexible foundation and subjected to bending combined with traction or compression is less important or even negligible when the two stresses are combined simultaneously.

Section two: rectangular plate of variable thickness freely supported at two opposites edges and the other two edges fixed

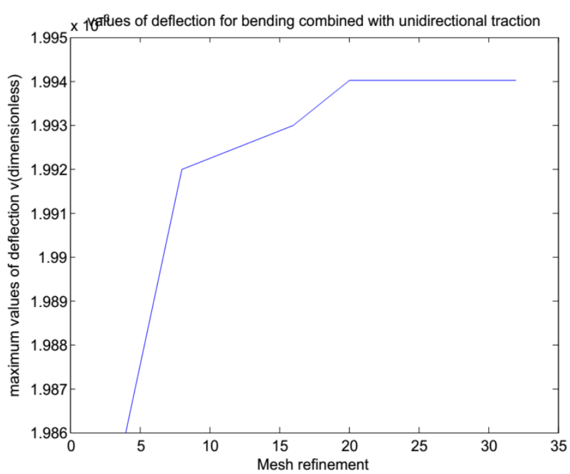
Figure 8 shows a rectangular plate with variable thicknesses, freely supported at two opposite edges $y=0, y=b$ and two fixed edges x 0, $x=a$. It should be noted that here the plate is not under elastic foundation (R=0).

In order to solve the Eq. (17) and to obtain the numerical results of deflection and moment, a computer program was used. The results are presented by the values of deflection for the case of a plate with variable thicknesses ($h_1 = 6$ *mm*, $h_2 = 8$ *mm*, Figure 8) loaded by the uniformly distributed load. This choice is in order to compare our results with the experiments ones obtain by [11], using a tensile test machine with additional equipment.

The total load is equal to *24* kPa for the uniformly distributed load. The calculations were made for plates with dimensions of *180 mm* in width and *400 mm* in length loaded by the uniformly distributed load. The steel grade NVA with yield stress *235 N/mm*^{2} is used.



(a)



(b)

Figure 7: a) Graphical representation of the moment coefficients; b) Ggraphical representations of coefficients of deflection.

5 Conclusion

In this work we have exposed a mathematical technique that transforms the PDE of 4th order governing the deformation equation of a rectangular plate into a system of PDE of 2nd order. An analyst process leading to a dimensionless parameters and unknown functions have been implemented. Then after generalized equations of finite difference method are derived from a dimensionless system of 2nd order PDE previously obtained. The resolution takes into account the boundary conditions which was done using the iterative method of Gauss-Seidel.

This numerical approach has been tested on benchmark problems found in the literature. The provided numerical solutions are satisfactory and are in accordance with those found in the literature. It is worth mentioning

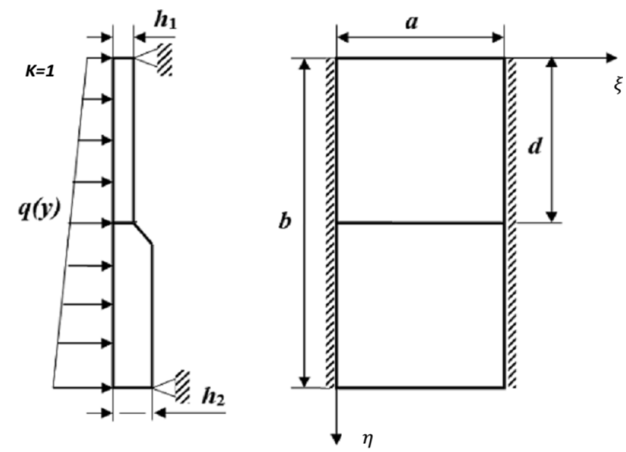


Figure 8: Rectangular plate with variable thicknesses with two opposite edges freely supported and two fixed edges [11].

that the GE – FDM has displayed ability to yield accurate solutions on relatively coarse grid, with an order of convergence equal to 0, 99.

This shows as well the stability of the method.

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References

[1] Timoshenkol S, Woinowsky-Krieger S. Theory of Plates and Shells. Ed. McGraw-Hill, New York. 1966;591.

[2] eontiv NN, Leontiv AN, Sobolev DN, Anohin NN. Fundamentals of the theory of beams and plates on a deformed base. Moscow: Ed. MISI. 1982;119.

[3] Nzengwa R, Tagne BH. A two-dimensional model for linear elastic thick shells. Int J Solids Struct. 1999;34(36):5141-5176.

[4] Nkongho AJ, Nzengwa R, Amba JC, Ngayihi CVA. Approximation of linear elastic shell by curved triangular finite element base on elastic thick shells theory. Math Probl Eng. 2016.

[5] Blassov BZ, Leontiv NN. Plates and shells on anelastic base. Moscow: Ed. GIFML; 1960;208.

[6] Leontiv NN, Leontiv ANm Sobolev DN, Anohin NN. Fundamentals of the theory of beams and plates on a deformed base. Moscow: Ed. MISI. 1982;119.

[7] Koreneva EB. Analytical methods for calculating variable thickness plates and their practical applications. Ed. ABS publisher, Moscow. 2009.

[8] Sharma S, Gupta US, And Singhal P. Vibration Analysis of Non-Homogeneous Orthotropic Rectangular Plates of Variable Thickness Resting on Winkler Foundation. J Appl Sci Eng. 2012;15(3):291-300.

[9] Ventsel E, Krauthammer T. Thin Plates and Shells Theory, Analysis and Applications. The Pennsylvania State University and University Park, Pennsylvania. 2001;651.

[10] Gabbassov RF, Filatov VV, Ovarova NB, Mansour AM. Dissection Method Applications for complex shaped membranes and plates. Procedia Eng. 2016;153:444–9.

[11] Aryassov G, Gornostajev D, and Penkov I. Calculation method for plates with discrete variable thickness under uniform loading or hydrostatic pressure Int J of Appl Mech Eng. 2018;23(4):835-853.

[12] Gabbassov RF, Gabbassov AR, Filatov VV. Numerical construction of discontinuous solutions to problems of structural mechanics. Moscow: Ed. ABC. 2008;280.

[13] Komlev AA, Makaev SA. The calculation of rectangular plate on elastic foundation: the finite difference method. J Phys Conf Ser. 2018;944:012056.

[14] Gabbassov RF, Moussa S. Generalized Equations of Finite Difference Method and their Application for Calculation of Variable Stiffness Curved Plates. Ed. News of Higher Educational Institutions Construction. 2004;(17–22):5.

[15] Moussa S, Lontsi F, Hamandjoda O, Raidandi D. Calculation of plates on elastic foundation by the generalized equations of finite difference method. Int J Eng Sci. 2018;7(8):32- 38.

[16] Moussa S, Njifenjou A, Youssoufa S. Équations généralisées de la méthode des différences finies pour le calcul des plaques minces isotropes d'épaisseur constant soumises à une flexion, compression et ou/traction. AfriqueSCIENCE. 2019;15(3):49 - 63.

[17] Tornabene F, Fantuzzi N, Ubertini F, Viola E. Strong formulation finite element method based on differential quadrature: a survey. Appl Mech Rev. 2015;67(2):20801.

[18] Fazzolari FA, Viscoti M, Dimitri R, Tornabene F. 1D – Hierarchical Ritz and 2D-GDQ Formulations for free vibration analysis of circular/elliptical cylindrical Shells and beam structures. Compos Struct. 2021;158:113338.

[19] Tornabene F, Viscoti M, Dimitri R, Reddy JN. Higher order theories for the vibration study of doubly – curved anisotropic shells with a variable thickness and isogeometric mapped geometry. Compos Struct. 2021;267:113829.

[20] Dimitri R, Fantuzzi N, Tornabene F, Zavarise G. Innovative numerical methods based on SFEM and IGA for computing stress concentrations in isotropic plates with discontinuities. Int J Mech Sci. 2016;118:166-187.

[21] Gabbasov RF. Difference equations in problems of strength and stability of plates. Soviet Appl Mech. 1982;18:820-824.

[22] Uvarova N, Gabbasov RF. Calculation of plates in a geometrically nonlinear setting with the use of generalized equations of finite difference method, Theoretical foundation of civil engineering. Web Conf. 2018;196:01024.

[23] Gabbasov RF. Generalized equations of the finite difference method in polar coordinates on problems with discontinuous solutions. Resistance of materials and theory of structures. Kiev: Budivelnik. 1984;45:55-58.

[24] Smirnov VA. Calculation of plates of complex shape. M. Stroyizdat; 1978;300.

[25] Nkongho AJ, Amba JC, Essola D, Ngayihi Abbe VC, Bodol Momha M, Nzengwa R. Generalised assumed strain curved shell finite elements (CSFE-sh) with shifted-Lagrange and applications on N-T's shells theory. Curved Layar Struct. 2020;7(1):125-138.