Research Article

Darío Galante Bardín* and Miguel A. Astiz Suárez

Design method for curved stayed cable bridges deck directrices for different cable systems

Abstract: In the specific case of curved cable-stayed bridges, the horizontal component of the load introduced by the stay cables on the deck is variable, concentric and dependent on the connection configuration between the tower and the cables, becoming a challenge in the design of these type of bridges. Hitherto, designers have dealt with this challenge in different ways, either by optimizing the position of the tower and its geometric characteristics, or by modifying the morphology of the stay cable system. This paper proposes the use of funicular and anti-funicular curves of the horizontal concentric load, introduced by the stay cables, to design the curved deck directrix, reducing lateral forces on the deck under the self-weight hypothesis. For the design of the deck directrix, two different formulations are considered: one discrete by means of summations and the other continuous by means of non-linear differential equations. Both formulations study the two possible signs of the axial force which will govern the design (funicular and anti-funicular curves). A least squares approximation is developed to facilitate the implementation of these formulations. The paper introduces a method to liberate the deck from its most important lateral loads, i.e., the concentric loads introduced by the stay cables. This way, it develops a deck dominated by axial forces instead of lateral ones (Bending moment with vertical axis, M_z, and lateral shear force, V_y), which can be critical for its design and decrease the stay-cable system efficiency. It explains, by different methods, how this directrices vary with different design decisions, so that the designer can develop the directrix that suits his design. Finally, two examples of directrices are given as a conclusion.

Keywords: curved cable-stayed bridge; funicular deck; anti-funicular deck; concentric load; axial force; optimal bridge design procedure; curved bridge directrices

1 Introduction

The nature of the increasingly complex geometric constraints posed by present-day pedestrian and road systems is very diverse, ranging from a combination of natural obstacles and greater environmental awareness, through to rights of way in urban areas and the development of major multi-height road junctions with link roads between them. Curved bridges have become a useful tool for engineers to solve these restraints. Deck on multiple pier supports is the type that has been most developed by researchers [1], engineers [2], and national standards [3]. Adapting the environment to eliminate these restraints, instead of investigating other design configurations, has been the usual response to the event of restraints preventing this type of construction. Engineers have been discouraged from designing more complex curved bridges, due to the lack of experience, of sufficient technical documentation, of regulatory coverage or of clear-cut design methodologies, staying in standard designs [4]. Thanks to new technologies [5, 6] there has been a certain development in different types of curved structures, such as cable-stayed, suspension or arch bridges. Pedestrian [7] use has almost monopolized this progress, where the freedom of design is greater and engineers such as Schlaich have designed very successful systems with suspension and arch bridges. Also, scientific works have been developed with load-adapted arches such as those in Refs. [8] or [9].

A deck dominated by axial forces presents many benefits over a deck dominated by transverse forces, reducing the vertical axis bending moment (the subject of this paper). From a resistive perspective, the vertical axe bending moments produced by the horizontal loads of the cables can be critical for most curved bridges decks, so reducing them will bring immediate benefits ([10, 11, 12]). Also, deck behaviour governed by axial forces, in most cases, are stiffer, improving the efficiency of the stay cable system [13].

*Corresponding Author: Darío Galante Bardín:** Innovación en el diseño de la ingeniería S.L., Calle Altamirano 36 Bajo D, 28008 Madrid, Spain, E-mail:

Miguel A. Astiz Suárez: School of Civil Engineering, Technical University of Madrid (UPM), Campus Ciudad Universitaria, Calle del Prof. Aranguren, 3, 28040 Madrid, Spain

The two-dimensionality force balance of straight cable stayed bridges [14] disappears in the curved cable-stayed bridges, instead torque, bending moment are coupled with radial axis, axial, shear forces and the bending moment of the vertical axis [15, 16, 17]. An appropriate design of the deck, the tower, and the cables as a whole, will allow to exclusively produce axial force on the deck and tangential forces to the directrix reactions on the abutments. A clear example of system optimization, with a concentric load as the main load, is the design of a bicycle wheel, where the curvature of the rim transforms the radial load into axial load, until it is compensated when the directrix of the rim is closed [18].

Based on the design of the bicycle wheel, some approaches of curved cable-stayed bridges have been designed by positioning the tower near the centre of the arc created by the deck [19]. This design has two major drawbacks: it is only feasible with very small arc radius (otherwise, the distance between the tower and the deck is excessive) and the tower is not positioned near the mass the centre of the deck (unless the arc covers the whole circumference), creating a horizontal component at the head of the tower that somehow must be anchored [20, 21]. To solve these problems, some engineers have evolved the design of the tower and the stay cables to transmit a constant radial force to the deck (as when the tower is placed in the centre of the arc) without having to position the tower in the centre of the arc. At the forefront of these developments is Schlaich with his suspension bridges [7], where the suspenders are distributed radially and collected by a main cable, supported by the tower. Positioning the tower in the symmetry axis of the arc allows a natural balance between the main suspension cables.

This paper sets the principles to change the standard starting point of a curved cable-stayed bridge design. The type of curvature developed by the deck will not be a starting point, but a consequence from the needs and constraints that the bridge has and the designer-controlled design decisions. This curvature will be used to find the balance of horizontal forces from the chosen load hypothesis (usually permanent loads, that are determinant in medium and large bridges), regardless of each component stiffness. The reason for this, is that the stay cables are to be preloaded to withstand the structure self-weight, which may include a percentage of the super dead loads, so the vertical load supported by each cable is known in advance [22, 23]. Later stages should analyse the system behaviour under other load cases, where the stiffness of the different elements must be considered. Each cable load is passed through the deck mass and shear centre, which are deemed to be coincident. The latter may not be optimal for subse-

quent torsional forces on the deck [24], but this analysis is beyond the scope of this paper.

There are basically three types of cable arrangements for cable-stayed bridges [14]: fan, harp and semi-harp. The proposed directrices calculation will allow the designer to use any of the three types of cable arrangement as a design choice adapted to his needs.

2 Design principles

The so-called "Axial Deck" will be a deck designed to operate primarily under axial force, without the necessity of radial stay cables layout. The directrix of the deck will be set by the direction of the axial force, and in turn, the value of this axial force and its direction will be set by the type and characteristics of the stay cable arrangement and the geometric characteristics of the tower (height and its plan position). The directrix may follow the anti-funicular (compression) or funicular (tension) shape of the horizontal forces introduced by the stay cables. The anti-funicular directrix will be concave in relation to the tower, rotating around the tower, while the funicular directrix will be convex, with an oblique asymptote.

To achieve a continuous directrix, it was decided not to consider the point axial force changes produced by the stay cables that support the deck. The problem is solved as if the whole deck was continuously connected to the tower, following the stay cable arrangement, like the calculation of a continuous catenary.

For a more intuitive understanding of how such directrices are designed, two types of calculations will be presented: one discrete and one continuous.

2.1 Directrix using a discrete calculation

In this calculation, the deck is going to be discretized into equal length straight segments, concentrating their mass in the nodes. All horizontal forces acting in the node are composed, obtaining the resultant force angle. Then the next segment and the resultant force directions are matched, repeating this procedure in the next node. It is obvious that the more nodes used into the discrete calculation, the closer it will be to a continuous calculation (later it will be verified that the only way to solve the continuous problem, because of its inherent nonlinearity, is by means of a numerical calculation, which as in the discrete calculation, provides the situation of the next node from the previous node available information) [25]. The stay cables are anchored to the

head of the tower along a straight line (another geometric function could be used if deemed appropriate) contained on the XZ plane. The coordinates of the connection points between stay cable and the tower (h(s) in the Z axis and xc(s) in the X axis) are parameterized by the variable "s", linking them to the deck length.

The discrete problem is described in Figure 1.

If node "i" in Figure 1 is analysed (after having analysed the i-1) :

N_{i-1} : Axial force on the deck prior to the node. Known data.

α_{i-1} : Angle of the axial force prior to the node. Known data.

P_i : Horizontal component introduced by the stay cable on the node. Data to be obtained.

$P_{Y\,i}$, $P_{X\,i}$: Components "Y" "X" of the horizontal force P_i introduced by the stay cable supporting the vertical concentrated load. Data to be obtained.

N_i : Axial force on the deck after the node. Data to be obtained.

α_i : Angle of the axial force after the node. Data to be obtained.

$h_{[i]}$: Height at which the stay cable is anchored to the head of the tower. Known data.

If the forces on the node are composed:

(1) $N_{i-1} \cos \alpha_{i-1} + P_{Y\,i} = N_i \cos \alpha_i$, $N_{i-1} \sin \alpha_{i-1} + P_{X\,i} = N_i \sin \alpha_i$

While the horizontal forces introduced by the stay cable on the node:

(2) $P_{Y\,i} = W_z \frac{y_{[i]}}{h_{[i]}}$, $P_{X\,i} = W_z \frac{x_{[i]} - xc_{[i]}}{h_{[i]}}$

This way, starting from known boundary conditions (such as a reaction on the abutment or the axial force carried by the deck when it crosses the x-x axis), it is easy to obtain the "axial force" directrix of the deck discretely from summations.

To simplify the calculation, it has been assumed that the directrix of the deck starts on the X axis and perpendicular to it. This initial point coordinates will be considered a boundary condition, "x=xo". The axial force of the deck at this initial point will also be considered as a boundary condition ("FHT" axial force on the deck at the initial point "0"). The direction of this initial axial force ("FHT"), coincident with the deck directrix (deck directrix and axial force direction will always be coincident), will be perpendicular to the x-x axis. Also, for simplicity reasons, it was decided to position the foot of the tower in the centre of coordinates (Figure 1). As for the line that defines the position of the connection points between stay cables and the tower, it will be in the XZ plane and it will cross the Z axis (x=0) at a height which has been named "ho". The inclination of this line will be defined by the angle formed with the Z axis,

which has been named as "β". This arrangement will allow to develop the deck in one direction only (in Figure 1 case the "-y" direction has been chosen), knowing that the deck directrix will be symmetrical with respect to the x-x axis.

With all the boundary conditions described in the preceding paragraph, the discrete problem is defined as follows. The first two points coincide in the two types of decks (compression and tension deck).

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(3) Boundary conditions (first two points). Compression deck and tension deck

$$x[0] = x_0$$
$$x[1] = x[0]$$
$$y[0] = 0$$
$$y[1] = -\frac{d}{2}$$

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(4) $h[i] = ho + \left(\frac{d}{2} + (i-1)\,d\right) \rho hs \cos[\beta]$

(5) $x[i] = \left(\frac{d}{2} + (i-1)\,d\right) \rho hs \cos[\beta]$

Compression deck. For i>2:

(6)

$$x[i] = x[i-1] +$$
$$+ d \frac{W_z \sum_{j=1}^{i-1} \frac{x[j]-xc[j]}{h[j]}}{\sqrt{\left(W_z \sum_{j=1}^{i-1} \frac{x[j]-xc[j]}{h[j]}\right)^2 + \left(FHT + W_z \sum_{j=1}^{i-1} \frac{y[j]}{h[j]}\right)^2}}$$

$$y[i] = y[i-1] +$$
$$+ d \frac{FHT + W_z \sum_{j=1}^{i-1} \frac{y[j]}{h[j]}}{\sqrt{\left(W_z \sum_{j=1}^{i-1} \frac{x[j]-xc[j]}{h[j]}\right)^2 + \left(FHT + W_z \sum_{j=1}^{i-1} \frac{y[j]}{h[j]}\right)^2}}$$

Tension deck. For i>2:

(7)

$$x[i] = x[i-1] +$$
$$+ d \frac{-W_z \sum_{j=1}^{i-1} \frac{x[j]-xc[j]}{h[j]}}{\sqrt{\left(-W_z \sum_{j=1}^{i-1} \frac{x[j]-xc[j]}{h[j]}\right)^2 + \left(-FHT - W_z \sum_{j=1}^{i-1} \frac{y[j]}{h[j]}\right)^2}}$$

$$y[i] = y[i-1] +$$
$$+ d \frac{-FHT - W_z \sum_{j=1}^{i-1} \frac{y[j]}{h[j]}}{\sqrt{\left(-W_z \sum_{j=1}^{i-1} \frac{x[j]-xc[j]}{h[j]}\right)^2 + \left(-FHT - W_z \sum_{j=1}^{i-1} \frac{y[j]}{h[j]}\right)^2}}$$
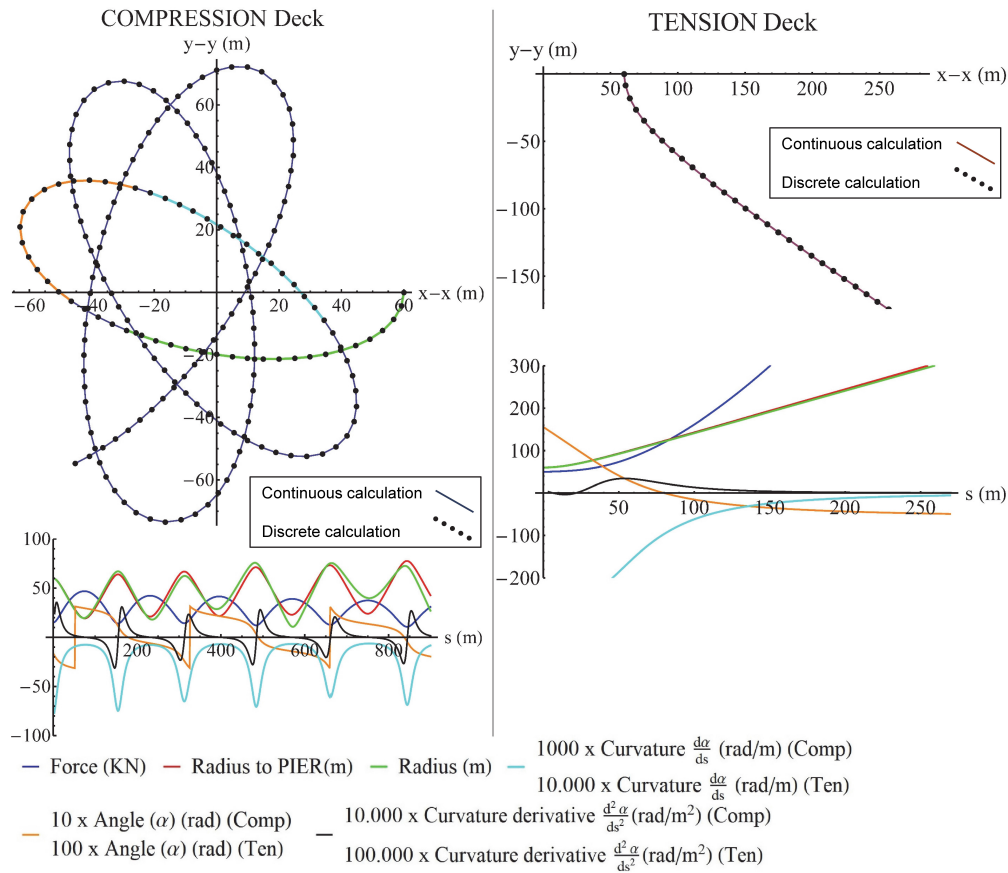
Where:

xo: Directrix calculation starting point (first point) "x" coordinate (boundary condition).

FHT: Axial force in the deck at the first point (x[0]) (positive for the tension deck and negative for the compression deck) (boundary condition).

Figure 1: Discrete calculation of directrix, boundary conditions. Funicular and anti-funicular.

d: Distance between nodes; minimising this distance takes the result of the discrete problem closer to the continuous problem (boundary condition).

s: Deck length (in the discrete problem, sum of each section "d" distance)

Wz: Vertical load supported by the stay cable at each node (boundary condition). If using self-weight as design load, this would be the weight of a "d" section of deck.

h[i]: Height of the connection point between the stay cable, which supports the point "i" and the head of the tower (in Figure 1, coordinate "z") (boundary condition).

xc[i]: Coordinate "x" of the connection point between the stay cable, which supports point "i", and the head of the tower (boundary condition).

ρhs: Ratio between the distance covered by the deck and the distance covered by the head of the tower (boundary condition).

β: Angle between the line that defines the head of the pile directrix and the Z axis (Positiva as shown in Figure 1). (boundary condition).

x[i]: Coordinate "x" of point "i".

y[i]: Coordinate "y" of point "i".

x[i-1]: Coordinate "x" of previous point.

y[i-1]: Coordinate "y" of previous point.

nt: Total number of nodes that will be studied; with distance "d" between them (Boundary condition).

To end with the discrete calculation, two examples will be presented.

Two radii are shown; the Radius to PIER is the length of the stay cable projection in the XY plane, while Radius

(8) Boundary conditions.	
Compression: Figure 2	Tension: Figure 2
$nt = 1000$	$nt = 300$
$ho = 50$ m	
$Wz = -1$ KN	
$x_0 = 60$ m	
$d = 1$ m	
$FHT = -15$ KN	$FHT = 50$ KN
$\rho hs = 0.07$ m/m	
$\beta = 0.1\,\pi$ rad	

is the distance from the deck to the centre of coordinates. The Radius to PIER will provide information about the behaviour of the directrix, as well as its relationship with the axial force of the deck, while the Radius will provide useful information about the geometry of the directrix. The angles and curvatures are always taken with respect to the centre of coordinates.

Figure 2 the completely different shape nature of the decks that will work with tension and with compression.

In the case of the compression deck (anti-funicular), the directrix has a cyclical shape (Figure 2). The maximum Radius to PIER coincides with the lower axial force and vice versa (Figure 2), when the directrix tangent perpendicular crosses the tower. While the angle value is divergent. The axial force in the deck and its Radius to PIER will be bounded between certain maximum and minimum values if the value of h[s] is constant for all the stay cables (Fan arrangement). Instead, if h[s] increases with "s", the force

Figure 2: Compression (anti funicular) and Tension (funicular) decks. Discrete and continuous calculations.

value will gradually decrease, whereas the Radius to PIER value will increase and vice versa. As for xc[s], it will cause a translation of the deck in the X axis, without affecting the Radius to PIER, in the same direction as xc[s].

The cyclicality of the directrix is the result of the fact that from the points of maximum and minimum Radius to PIER, the external forces invert their relationship with the axial force of the deck. If it is a point of maximum axial force and minimum radius, the external forces, introduced by the cables, go from increasing the axial force to decreasing it, and vice versa. For a possible deck directrix, any section of the curve can be taken, 3 hypothetic sections have been chosen in Figure 2 (COMPRESSION Deck). The boundary conditions (actions on the abutments, deck angle, etc.), for the different sections, can be obtained from the diagrams in Figure 2.

In the case of the tension deck (funicular), its directrix shows how the distance from the tower (Radio and Radio to PIER) diverges with an oblique asymptote (Figure 4). It is the angle (α) which is bounded and approaches a stable value, matching the angle of the oblique asymptote, while the axial foce and radius are those divergent (Figure

2, TENSION Deck). If h[s] increases with "s", the angle of the asymptote will open, increasing its absolute value, and vice versa. While xc[s], as in the case of the Compression Deck, will produce a slight translation on the deck, which will separate it from the asymptote, this effect will only be appreciable with values of xc[s] not applicable in the design of bridges.

The curvature and its derivative are included in Figure 2. The goal is to check the possibilities of developing a directrix compatible with a road layout, composed of circumferences and clothoids [26]. If the boundary conditions imposed are changed (Figure 2), it is possible to match the derivative of the curvature (Figure 2), stringing together clothoids (with constant derivative of the curvature) and circumference (with zero derivative of the curvature), without the directrix undergoing major changes. The development of these approaches is beyond the scope of this work. For footbridges, there is freedom to develop the directrix without any modifications.

2.2 Directrix using a continuous calculation

2.2.1 Problem approach

In order to approach the continuous problem, with the aim of obtaining an analytical solution which solves the directrix of the deck, we need to analyse a differential section of the deck (Figure 3).
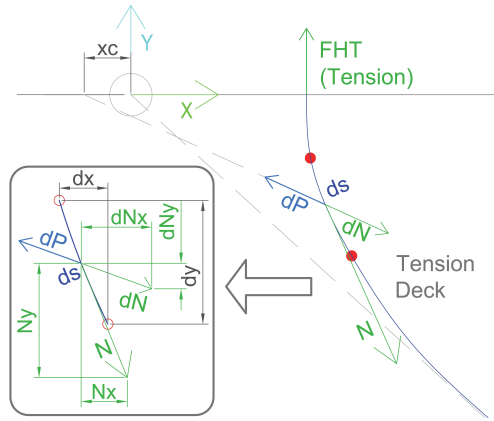

Figure 3: Continuous calculation of the directrix.

(9) $\frac{dx}{ds} = \frac{N_x}{N}$

(10) $\frac{dy}{ds} = \frac{N_y}{N}$

(11) $\frac{N_x}{N_y} = \frac{dx}{dy}$

(12) $dN_x = -\frac{x - xc[s]}{h[s]} \rho_t \, ds$

(13) $dN_y = -\frac{y}{h[s]} \rho_t \, ds$

(14) $\frac{dN_x}{dN_y} = \frac{x - xc[s]}{y}$

(15) $h[s] = ho + s \, \rho hs \cos[\beta]$

(16) $xc[s] = s \, \rho hs \, \sin[\beta]$

Where:

xo: Directrix calculation starting point (first point) "x" coordinate (boundary condition).

FHT: Axial force in the deck at the first point (x[0]) (positive for the tension deck and negative for the compression deck) (boundary condition).

ρ_t: Vertical load per linear meter of the deck (boundary condition).

s: Length of the deck. Variable of the problem

h[s]: A function of "s" which gives the height of the connection point between the stay cable and the tower head (in this case, coordinate "z").

xc[s]: A function of "s" representing the coordinate "x" of the connection point between the stay cable and the tower head.

x[s]: Deck "x" coordinate.

y[s]: Deck "y" coordinate.

N[s]: Deck resultant horizontal force, coincident with directrix tangent (same value as deck axial force).

$N_x[s]$: Component in "x" of the deck resultant horizontal force.

$N_y[s]$: Component in "y" of the deck resultant horizontal force.

Figure 3 shows the relation between the position of the deck and its axial force. The vector of the Deck resultant horizontal force N[s] influences the directrix in two ways: First, its direction will indicate always the direction that the deck must follow, and second, its module will indicate how stable this direction is. This direct relation between axial modulus and directrix stability (in other words, an inverse relation between curvature and axial force modulus) is an important issue when the deck undergoes the transverse force discontinuity introduced by the supports in the abutments. Logically, the more stable the directrix, the lower the effect of this discontinuity.

To simplify the calculation, we have assumed similar boundary conditions as in the discrete problem.

(17) Boundary conditions (first two points). Compression deck and tension deck
$N_x[0] = 0$
$x[0] = x_0$
$N_x'[0] = -x_0 \frac{\rho_T}{hp}$
$x'[0] = 0$
$N_y[0] = -FHT$
$y[0] = 0$
$N_y'[0] = 0$
$N_x[0] = 0$

With these boundary conditions, the equations that solve the problem are:

Compression deck:

(18) $N_x[s] = \sqrt{N_x[s]^2 + N_y[s]^2} \; (N_x''[s] \, h[s] + N_x'[s] \, h'[s] + xc'[s]) \frac{1}{\rho_T}$

$N_y[s] = \sqrt{N_x[s]^2 + N_y[s]^2} \; (N_y''[s] \, h[s] + N_y'[s] \, h'[s]) \frac{1}{\rho_T}$

Tension deck:

(19) $N_x[s] = -\sqrt{N_x[s]^2 + N_y[s]^2} \; (N_x''[s] \, h[s] + N_x'[s] \, h'[s] + xc'[s]) \frac{1}{\rho_T}$

$N_y[s] = -\sqrt{N_x[s]^2 + N_y[s]^2} \; (N_y''[s] \, h[s] + N_y'[s] \, h'[s]) \frac{1}{\rho_T}$

The Eqs (18) and (19) only show the force functions (Nx[s] and Ny[s]) and the tower geometrical functions (xc[s] and h[s]), but what we actually want are the directrix func-

tions ($x[s]$ and $y[s]$). The reason lies in the impossibility to leave a system with only the two directrix functions.

It can be seen how in the event of a fan-type stay cable arrangement, $h[s]$ being constant, differential equations would be simplified. However, they would still be non-linear, as it is essential to know the path covered to a point to obtain the following one.

To obtain the position functions, two different options have been developed.

First, making a change of variable to clear the force functions, isolating the directrix functions. Similar to a dynamic problem, changing the space variable, to the time variable: $dt_T = \frac{ds}{N}$. But "t_T" has a non-linear relation with the main variable ("s") ,so the directrix functions become very unstable on the tension deck from a given "$t_T Final$".

So, it has been decided to derivate the force functions obtained by their differential functions (Eqs. (18), compression, and (19), tension), to obtain directrix functions using Eqs. (20):

(20) $x = -\frac{dN_x}{ds} \frac{h[s]}{\rho_t} + xc[s]$

$y = -\frac{dN_y}{ds} \frac{h[s]}{\rho_t}$

The main advantage of this option is that the function obtained are dependent on the essential variable of the problem, "s", while its disadvantage is that the axial force functions of the deck must be differentiated in order to obtain the directrix. The next section shows that the forces differential equations system numerical solution (Eqs. (18) and (19)) will directly give us the solution for the forces and the coordinates.

The conclusion is that the second option is much simpler from all points of view. Therefore, only this route will be developed from here on.

AQUÍ

2.2.2 Results of the differential equation and calculations conclusion

The differential equations forces system depending on the variable "s" is second-order non-linear, although it is a system of ordinary differential equations due to its exclusive dependence on one variable (the functions $h[s]$ and $xc[s]$ are initial data, as their values are known). For this reason, a numerical approximation using the Runge-Kutta method has been used. To solve the problem, the two second-order equations system will be transformed into a four first-order equations system, with two new variables, given by the Eqs. (20).

Compression deck is defined by the Eqs. (20) and (21):

(21) $N_x[s] = \sqrt{N_x[s]^2 + N_y[s]^2} \, x'[s]$

$N_y[s] = \sqrt{N_x[s]^2 + N_y[s]^2} \, y'[s]$

Tension deck is defined by the Eqs. (20) and (22):

(22) $N_x[s] = -\sqrt{N_x[s]^2 + N_y[s]^2} \, x'[s]$

$N_y[s] = -\sqrt{N_x[s]^2 + N_y[s]^2} \, y'[s]$

As shown in Figure 2 identical directives are obtained with the discrete calculation and the continuous calculation. Therefore, the analyses conducted for curves by means of discrete method are valid for this methodology.

The conclusion is that the more realistic approach is by numerical problem-solving. In any case, after analysing the results, it has been found that these numerical approximations are similar and completely satisfactory. For a possible implementation and use of such directrices in the actual design of bridges, a least squares approximation has been made of the solutions obtained numerically for the case of fan-type cable-stayed systems:

(23) $h[s] = hp$

$xc[s] = 0$.

2.3 Least squares for the directrix ($h[s] = hp$).

2.3.1 Compression deck

The closest shape found to the directrix obtained with the differential equations, Eqs. (20) and (21), for the specific case of a fan-type cable-stayed system, Eqs. (23), is a hypotrochoid:

(24) $x(t) = (a - b) \cos[t] + c \, \cos\left[\frac{a-b}{b} t\right]$

$y(t) = (a - b) \sin[t] - c \, \sin\left[\frac{a - b}{b} t\right]$

As the problem has been set up, the directrix of the deck starts (s=0) with a singular point. In Figure 2, it's a maximum radius (R_{max}) and a minimum force (N_{min}), as can be seen in the radius and force diagrams (also Figure 4). There are certain boundary conditions, Eq. (25), for which the developed directrix is a circle (with $h[s] = hp$). That is the principle followed by a bicycle wheel. In that case, the radius and the axial force would remain constant throughout the alignment.

$FHT = FHT_{cir} = x_o{}^2 \frac{\rho_T}{hp}$

This starting force (FHT_{cir}) value at the beginning of the deck Eq. (25) is the frontier from which the type of singular starting point changes. If FHT is less than this value, the starting point will be a maximum radius (R_{max}) and minimum force (N_{min}) singular point and vice versa. Some

dimensionless variables have been defined to develop the approximation:

(26) $N_{\text{rel}} = \text{Abs}[\frac{FHT}{FHT_{\text{cir}}}]$, $Cf = \text{Abs}[\frac{N_{\min}}{R_{\max}^2 \frac{p_T}{hp}}]$, Cfm = $\text{Abs}[\frac{N\max}{R_{\min}^2 \frac{p_T}{hp}}]$, $R_{\min /\text{rel}} = \frac{R_{\min}}{x_0}$, $\beta_{\min}$

To obtain the approximation, the coefficients from the hypotrochoid, Eq. (24), are defined as:

(27) $a = \frac{\pi (R_{\min}-R_{\max})}{2 \beta_{\min}}$, $b = \frac{(\pi+ \beta_{\min}) (R_{\min}-R_{\max})}{2 \beta_{\min}}$, $c = \frac{(R_{\min}+R_{\max})}{2}$

Which can be solved with the Eqs. (28).

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(28) Coefficient's definition for Compression approximation directrix	
$N_{\text{rel}} \leq 1$	$N_{\text{rel}} > 1$
$R_{\max} = x_0$	$R_{\min} = x_0$
$N_{\min} = FHT$	$N_{\max} = FHT$
-	$Cf = \frac{101.35885+15.37035\ \text{Cfm}+0.100525\ \text{Cfm}^2}{-4.78159+117.93709\ \text{Cfm}+5.16712\ \text{Cfm}^2+0.0044895\ \text{Cfm}^3}$
	$\frac{R_{\min}}{R_{\max}} = 1.8687\ \text{Cf} - 2.0041\ \text{Cf}^2 + 1.8051\ \text{Cf}^3 - 0.6697\text{Cf}^4$
	$\beta_{\min} = \frac{-0.0008611-0.486716\ \text{Cf}-24.281\text{Cf}^2+192.151\ \text{Cf}^3- \text{Cf}^4}{0.0005482+0.3073\ \text{Cf}+14.6197\ \text{Cf}^2+103.169\ \text{Cf}^3+ \text{Cf}^4}$

2.3.2 Tension deck

In the case of the deck in tension (funicular), for the specific case of a fan-type cable-stayed system (23), there is an oblique asymptote passing through the centre of coordinates. There is only one singular point, which coincides with the starting point of the directrix, x_0 (Figure 4). Once the asymptote angle has been obtained (β_{asin}), the directrix approximation curve has been defined in cylindrical coordinates (β, R) using the method of least squares. Again, some dimensionless variables have been defined:

(29) $Cf = \text{Abs}[\frac{N_{\min}}{R_{\max}^2 \frac{p_T}{hp}}]$, $\beta f = \frac{ArcTan[\frac{y[s]}{x[s]}]}{\beta_{\text{asin}}}$, Rf = $\frac{\sqrt{x[s]^2+y[s]^2}}{x_0}$, $\beta_{\text{asin}} = \lim_{s\to\infty} \frac{N_y}{N_x}$.

From the analysis, it has been found that the asymptote angle (β_{asin}) is only dependant of Cf and that Rf is only dependant of βf and vice versa. So, using the method of least squares:

(30) $\beta_{\text{asin}} = \frac{-126.511\ Cf-656.321\ Cf^2+155.665\ Cf^3- \frac{\pi}{2}\ Cf^4}{32.7+514.791\ Cf+672.957\ Cf^2+111.067\ Cf^3+ Cf^4}$

(31) $\beta = \beta_{\text{asin}} \frac{-0.677747+1.28168\ Rf+0.46836\ Rf^2-2.07229\ Rf^3+ Rf^4}{-0.13952+0.09436\ Rf+1.20776\ Rf^2-2.16259\ Rf^3+ Rf^4}$

with $Rf \geq 1$

(32) $R =$

$x_0 \frac{1-1.76725\ \beta f+0.64759\ \beta f^2+017558\ \beta f^3+0.05433\ \beta f^4}{1-1.76726\ \beta f+0.17938\ \beta f^2+1.00042\ \beta f^3+ 0.41253\ \beta f^4}$

with $0 \leq \beta f \leq 1$

The results from the approximations developed using the method of least squares both, for compression and tension, have been tested in Figure 4.

3 Conclusions

The directrices of curved cable-stayed bridges decks optimization has been studied by removing its most harmful forces, those transverse to the directrix, introduced by the stay cables. The problem has been solved discretely, by means of summations, and continuously, by means of a differential equations system. The differential equations were solved numerically, due to the impossibility of an analytical solution. Finally, approximated formulas have been developed for the directrices of fan-type cable-stayed systems (Eqs.(23)), both for decks in compression (anti-funicular curve) and those in tension (funicular curve), Figure 4.

This approach opens a new perspective when faced with a curved deck with the impossibility of introducing multiple support directly to the ground. The elimination of lateral forces for certain load hypothesis and the possibility of modifying the directrix by varying boundary conditions, give the bridge designer a great amount of freedom. This is also important from a financial point of view, considering the possibility of placing the different components (essentially abutments and piers) where they are more effective and the inclusion of more advantageous forces on the deck, increasing the tower-cable structural system efficiency. The study of the directrix curvature (Figure 2) opens the door to the possibility of using this type of design for roadways cable-stayed bridges (Figure 5).

The development of the designing tool has been shown step by step, so that the possible designer or researcher can take advantage of the process at any phase to fulfil his purposes: programming, designing directrices for different types of bridges, create new approximation formulas for new cable arrangements...

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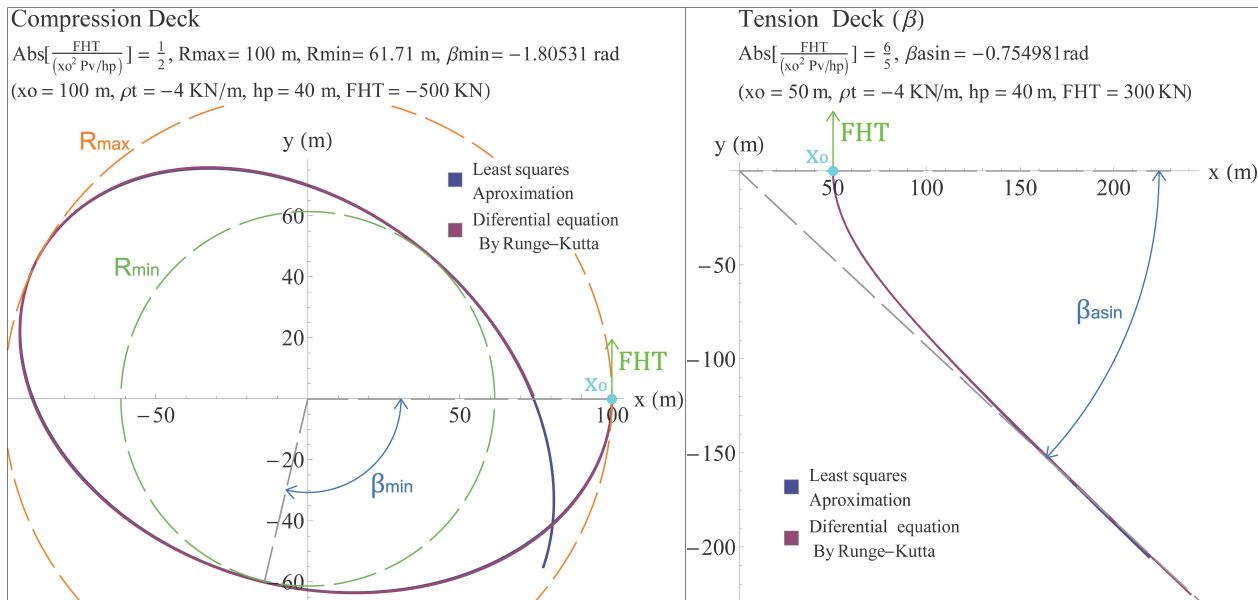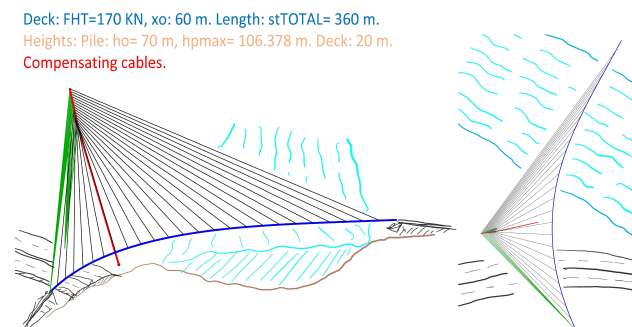**Conflict of interest:** The authors state no conflict of interest.

Compression Deck

$\text{Abs}[\frac{FHT}{(xo^2\, Pv/hp)}] = \frac{1}{2}$, $Rmax = 100$ m, $Rmin = 61.71$ m, $\beta min = -1.80531$ rad

$(xo = 100$ m, $\rho t = -4$ KN/m, $hp = 40$ m, $FHT = -500$ KN$)$

Tension Deck (β)

$\text{Abs}[\frac{FHT}{(xo^2\, Pv/hp)}] = \frac{6}{5}$, $\beta asin = -0.754981$ rad

$(xo = 50$ m, $\rho t = -4$ KN/m, $hp = 40$ m, $FHT = 300$ KN$)$

Figure 4: Least squares approximation test.

Figure 5: Roadway example.

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