

## Research Article

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# An analytical formulation to evaluate natural frequencies and mode shapes of high-rise buildings

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**Abstract:** In this paper, an original analytical formulation to evaluate the natural frequencies and mode shapes of high-rise buildings is proposed. The methodology is intended to be used by engineers in the preliminary design phases as it allows the evaluation of the dynamic response of high-rise buildings consisting of thin-walled closed- or open-section shear walls, frames, framed tubes, and diagrid systems. If thin-walled open-section shear walls are present, the stiffness matrix of the element is evaluated considering Vlasov's theory. Using the procedure called General Algorithm, which allows to assemble the stiffness matrices of the individual vertical bracing elements, it is possible to model the structure as a single equivalent cantilever beam. Furthermore, the degrees of freedom of the structural system are reduced to only three per floor: two translations in the  $x$  and  $y$  directions and a rigid rotation of the floor around the vertical axis of the building. This results in a drastic reduction in calculation times compared to those necessary to carry out the same analysis using commercial software that implements Finite Element models. The potential of the proposed method is confirmed by a numerical example, which demonstrates the benefits of this procedure.

**Keywords:** high-rise buildings, dynamic modal analysis, general algorithm, Vlasov's theory

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## 1 Introduction

In the recent decades there has been a rapid increase in the number of high-rise buildings [1] and many of these are built in highly seismic areas. As a result, the study of their dynamic behavior has aroused the interest of engineers around the world. The main objective of designers is to increase structural safety and reduce the amplitude of the oscillations in order to guarantee the integrity and comfort of the occupants [2, 3].

Considering that seismic forces are proportional to the mass of the structure itself, and that the response of the structure depends on its rigidity [4], it is deduced, in a somewhat counterintuitive way, the more rigid structural elements are subject to greater inertial forces than more flexible ones. In addition, high-rise buildings, as well as long-span bridges, are generally not very susceptible to the effects of earthquakes [5], due to the mismatch between their fundamental resonance frequency and characteristic frequency of earthquakes. However, this does not mean that the dynamic study should be omitted or underestimated. In the design process, particular attention must be paid to the determination of the center of stiffness of the entire building. In fact, considering that the forces of inertia are applied in the center of mass of each floor, the further this point is away from the center of stiffnesses the greater the torsional effects due to the action of horizontal loads [6].

Since many tall buildings have an irregular floor plan, which often changes geometry from one floor to another, it is very complex to match the stiffness center and the geometric center of gravity of each floor [7]. For this reason, it is essential to study the dynamic response of the building by carrying out three-dimensional flexural and torsional coupled analysis [8].

Today, most engineers use commercial Finite Element Model (FEM) software to perform structural analysis even in the preliminary design stages. This software, although simple to use, involves long calculation times and results difficult to interpret. Alternatively, the analytical methods present in the literature can be used, although these often

involve complex and difficult calculations to carry out manually. The analytical procedure proposed in this paper is inserted in such context and implemented in a calculation code. It allows to evaluate with a good approximation the frequencies of vibration and the modal shapes of a high-rise building in short times and with little outlay of resources [9]. For this reason, the proposed method can be an excellent tool for designers who can use it in the preliminary design phases.

In the literature, there are two types of study methods: the former defines mathematical models for analyzing the behavior of a single beam (local effects), whereas the latter is intended to define analytical models for the dynamic behavior of the entire building (global effects). The proposed analytical formulation, neglecting local effects, is mainly addressed to the evaluate the building global dynamic behavior.

## 2 Literature review of the analysis of an high-rise building

In the preliminary stages of the design of a high-rise building, it is essential to estimate with a good approximation the possible vibrating mode shapes of the structure in order to guarantee the necessary conditions of comfort to the future occupants. Among the first studies on the dynamic behavior of a tall building, we find the work of Mallick [10] who analyzed a shear walls structure. A similar model was taken up in 1981 by Paulay [11]. The case in which the core is composed by thin-walled open-section shear walls connected by rigid floors was treated by Mendelson [12, 13], who examined the structural response of a non-symmetrical multi-storey structure with or without damping effects. Approximate analytical methods for estimating the vibration periods of high-rise buildings composed by several structural elements were presented by Stafford Smith and Crowe [14], and some years later by Wang [15] and Zalka [16].

In the mid-1990s, Pekau published two works [17], [18], in which the Finite Story Method is used, that is the global behaviour of the building depends on the nodal displacements of two-storey substructures into which the whole construction is split. For the dynamic analysis of coupled shear walls, Swaddiwudhipong [19, 20] used the continuum medium technique that can model the tip connection generated by the floors as a continuum having equivalent geometric properties and also considers the axial deformation by the Galerkin Method [21]. In 2007, Meftah [22] formulated a simple analytical method that uses the Galerkin technique, for the evaluation of the free vibrations of build-

ings braced by shear walls and open-section elements. In 2009 Bozdogan presented a formulation that, using the transfer matrix method [23], allows an approximated dynamic analysis of a symmetric wall-frame building [24] and thin-walled open-section structures [25]. In the following years, this procedure was optimized and the multi-storey structure was modeled as an equivalent cantilever beam [26] and the shear walls and frames were assumed to be flexural and shear-deformable cantilever beam structures [27]. Finally, applying the Differential Transform Method to convert the differential equation into an algebraic equation, the same Author considered the free vibration analysis of tube-in-tube tall buildings [28].

In 2015 Piccardo et al. [29] formulated an equivalent nonlinear one-dimensional shear-shear torsional beam model, which reproduces the dynamic behavior of tower-buildings through a heuristic identification method. Subsequently, taking inspiration from fluid-elastic models, this method was extended to take account of non-linearities generated by the stretching of the columns [30], and the mechanical non-linearities [31], which are often believed to be unimportant but do affect the amplitude of motion. Recently, these same Authors have proposed a model for static and dynamic analysis of tall buildings based on the Timoshenko equivalent column [32]. There are also techniques in the literature that can evaluate the dynamic behavior of tall buildings with not only shear walls, but also braced frame systems [33, 34], outrigger [35, 36], and tube-in-tube structures [37, 38, 39]. Eventually, it is interesting to note how the seismic performance of the building can vary according to the actual rigidity of the floors [40, 41]; see also the case of buildings with a large bottom podium [42].

In the present paper, a formulation for the three-dimensional analysis of a high-rise building, based on the General Algorithm [43], is proposed in the dynamic regime. The basic formulation has already been published in [44, 45, 46], whereas in the present paper it is extended to framed-tube or diagrid systems [47, 48], as well as to structures in which there are simultaneously cores of different heights, and to irregular buildings. The proposed analytical formulation has been implemented in a computational code by using Mathworks Matlab. The entire calculation code, consisting of 37 functions for a total of more than 8,000 command lines, has a user-friendly graphics interface that allows a simple and intuitive input and output procedures. The input phase consists of entering global data of the entire building, *i.e.* total number of storeys, size and mass of the slabs (which may also have a geometry different from another); geometrical and mechanical data of each individual vertical stiffening, which can be of the following different typologies:

- thin-walled closed- or open-section shear walls;
- plane frames and braced frames;
- framed tubes;
- 2D or 3D diagrid systems.

With this analytical code it is possible to compute easily and in short time the deformations, the vibration frequencies, and the mode shapes of the building. The comparison between the results obtained using the analytical algorithm and those obtained using a commercial FEM software allows verifying the accuracy of the proposed formulation.

### 3 General algorithm

The analytical procedure, called General Algorithm [43], is a general formulation that allows to determine the external lateral loading distribution between the vertical bracings, the floor displacements, and the stresses acting on the vertical stiffeners of the high-rise buildings.

Based on the Capurso's method [49], it was presented in 1985 by A. Carpinteri [43]. In the last few years, this method of calculation has been further developed and improved [50, 51, 52] with the purpose of analyze widest typologies of high-rise buildings, for example structures in which there are simultaneously cores of different heights [53], irregular buildings in floor plan or in height [54], as well as to framed tube or diagrid structures [55]. In the case of thin-walled open-section shear walls, Vlasov's Theory [56], and consequently the warping deformation [57] are applied.

Such a formulation proves to be general and can be adopted with any kind of structural elements, provided that its own stiffness matrix is known [58]. The main advantage of this approach is that it considers only three degrees of freedom (DOF) for each storey, therefore it requires a shorter computation time if compared to the commercial FEM softwares, which are mesh-dependent and are characterized by six DOF for each node.

On the other hand, the greatest strength of this analytical code is the reduced processing time, which makes it suitable for use in preliminary analysis and structural optimization [59, 60, 61]. In these phases, in order to find the best compromise between safety and construction costs, it is necessary to vary one parameter at a time (thickness of each shear-wall, size and shape of the cross-section of bracings, etc.) making it necessary to create thousands of different models. The effectiveness of this analytical formulation has been verified in some case studies carried out by the Authors [62, 63]. Moreover, the proposed model produces results which are rather close to those achieved by FEM, with differences generally less than 10%.

### 4 Notes on the stiffness matrix evaluation

In the scientific literature there are different calculation procedures for stiffness matrices, but with the aim of developing a simple calculation code that at the same time provides acceptable results, the following analytical formulations are applied. Let us consider the generic vertical bracing containing  $N$  different floors. In the local reference coordinate system, with origin in its shear center, the  $3N$ -vector  $\{F^*\}$ , in which  $2N$  shearing forces  $\{p_x\}$ ,  $\{p_y\}$ , and  $N$  torsional moments  $\{m_z\}$  are included, is connected to the  $3N$  floor displacement vector  $\{\delta^*\}$ , constituted by  $2N$  translations  $\{\xi\}$ ,  $\{\eta\}$ , and  $N$  rigid rotations about the  $z$  axis,  $\{\theta\}$ , by the  $3N \times 3N$  stiffness matrix of the entire shear wall  $[K^*]$

$$[K^*] = \begin{bmatrix} [K_x] & 0 & 0 \\ 0 & [K_y] & 0 \\ 0 & 0 & [K_\theta] \end{bmatrix} \quad (1)$$

where each term is a  $N \times N$  sub-matrix.

The analytical evaluation of the stiffness matrices has already been extensively described in previous works by the Authors of this paper. For this reason, in the following, only a brief note is made.

In particular, for rectangular or closed-section shear walls, transverse displacements and floor rotations are evaluated using Saint Venant's Theory and the  $[K^*]$  stiffness matrix is assembled using Betti's Theorem [64].

In the case of thin-walled open-section shear walls, Vlasov's Theory [56] is applied to evaluate the torsional stiffness. In this way it is also possible to consider the stiffening effect due to the bimoment and secondary torsional moment. This theory allows a more accurate estimate for torsional stiffness if compared to Saint Venant's Theory. Finally, the stiffness matrix is assembled using the formulation based on the Capurso's method [49]. The detailed procedure has been published in [51, 52].

In the case of plane frames and braced frames, the stiffness matrix is evaluated considering a simplified method that allows to drastically reduce the calculation time while providing results comparable to those obtained using FEM software. From a static point of view, a multi-storey plane frame is a many-time redundant structure and, using the traditional equilibrium method, its resolution can be laborious and the computational time very long. In addition, to automate the computational formulation with numerical codes, it becomes fundamental to define a matrix method, limiting the subjective intervention of the operator. In order to do this, the following hypotheses are introduced:

- the frame has no flexural stiffness outside the plane containing it;
- the frame has no torsional stiffness;
- all nodes belonging to the same plane translate by the same amount.

Under these hypotheses, the multi-storey plane frame is assimilated to a continuous equivalent cantilever beam having flexural rigidity only in the plane that contains it, and torsional elastic constraints at each floor level. The  $3N \times 3N$  stiffness matrix expressed in the local reference system, shall take the following form:

$$[K^*] = \begin{bmatrix} [K_x] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

Since there are diagonal braces in the frame, the method illustrated can still be used with the foresight to add, at the level of each floor, additional effective elastic constraints in the equivalent cantilever beam. For the complete procedure, the reader can refer to [44, 46].

If the building consists of several plane frames connected to each other along the edges to create a closed framed tube, it is necessary to consider the three-dimensional problem because the phenomenon of shear lag [65] occurs. For this type of buildings, the  $3N \times 3N$  stiffness matrix of the entire building is evaluated using the method introduced by Nitti [66].

Finally, for diagrid structures, i.e. buildings in which the vertical resistant system to horizontal actions is composed of columns and beams not orthogonal to each other arranged along the perimeter [67], the stiffness matrix is evaluated using the matrix-based method proposed by Lacidogna et al. [55].

## 5 Analytical formulation for dynamic investigation

An  $N$ -storey high-rise building with  $M$  vertical bracings is considered. An arbitrary global reference system and a local reference system with origin in the shear center of each resistant element are introduced; the vectors  $\{\xi\}$ ,  $\{\eta\}$ , and  $\{\vartheta\}$  containing the displacements of the origin of the global reference system are selected. It is possible to define the vectors containing the displacements of the shear center  $\{\xi_{C,i}\}$  and  $\{\eta_{C,i}\}$  of the  $i$ -th resistant element as:

$$\{\xi_{C,i}\} = \{\xi\} - y_i \{\vartheta\} \quad (3a)$$

$$\{\eta_{C,i}\} = \{\eta\} + x_i \{\vartheta\} \quad (3b)$$

where  $x_i$  and  $y_i$  are the coordinates of the shear center of the  $i$ -th bracing, referring to the global reference system.

In a similar way, using the General Algorithm, the entire building, Figure (1a), is modeled as an equivalent cantilever beam, in which the masses of the floors are concentrated in the gravity center of their own, as shown in Figure (1b).

Considering that the dynamic equation of the system in absence of viscous dissipations can be written as:

$$[m]\{\ddot{\delta}\} + [k]\{\delta\} = \{0\} \quad (4)$$

where  $\{\delta\} = \{\{\xi\}; \{\eta\}; \{\vartheta\}\}$  is the vector of floor displacement,  $[m]$  and  $[k]$  are mass and stiffness matrices, respectively, for the  $j$ -th floor it is possible to write the dynamic equilibrium equation in the  $x$ -direction:

$$m_j \ddot{\xi}_{G,j} + \sum_{i=0}^M (\{k_{x,i,j}\} \{\xi_{C,i}\}) = 0 \quad (5)$$

where  $m_j$  and  $\ddot{\xi}_{G,j}$  are mass and acceleration in the  $x$ -direction of the  $j$ -th storey, respectively,  $\{k_{x,i,j}\}$  is the vector corresponding to the  $j$ -th row of the local stiffness matrix (estimated for the  $x$ -direction only) of the  $i$ -th resistant element, and  $\{\xi_{C,i}\}$  is the vector containing the  $i$ -th vertical bracing displacements in the  $x$ -direction of the shear center of all floors. Considering Equation (3a), the term  $\ddot{\xi}_{G,j}$  can be written as:

$$\ddot{\xi}_{G,j} = \ddot{\xi}_j - y_{m,j} \ddot{\vartheta}_j \quad (6)$$

where  $\xi_j$  and  $\vartheta_j$  are respectively the  $x$ -direction displacements and the rotation angle of the global reference system origin of the  $j$ -th floor, while,  $y_{m,j}$  is the coordinate of the center of mass of the  $j$ -th floor with respect to the origin of the global reference system. Replacing Equations (3a) and (6) into Equation (5), we obtain:

$$m_j (\ddot{\xi}_j - y_{m,j} \ddot{\vartheta}_j) + \sum_{i=0}^M [\{k_{x,i,j}\} (\{\xi\} - y_i \{\vartheta\})] = 0 \quad (7)$$

Considering all  $N$  storeys of the building, the following system of equations is obtained:

$$\begin{cases} m_1 (\ddot{\xi}_1 - y_{m,1} \ddot{\vartheta}_1) + \sum_{i=0}^M [\{k_{x,i,1}\} (\{\xi\} - y_i \{\vartheta\})] = 0 \\ m_2 (\ddot{\xi}_2 - y_{m,2} \ddot{\vartheta}_2) + \sum_{i=0}^M [\{k_{x,i,2}\} (\{\xi\} - y_i \{\vartheta\})] = 0 \\ \vdots \\ m_N (\ddot{\xi}_N - y_{m,N} \ddot{\vartheta}_N) + \sum_{i=0}^M [\{k_{x,i,N}\} (\{\xi\} - y_i \{\vartheta\})] = 0 \end{cases} \quad (8)$$

Introducing the matrices:

$$[M_{xx}] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N \end{bmatrix} \quad (9a)$$

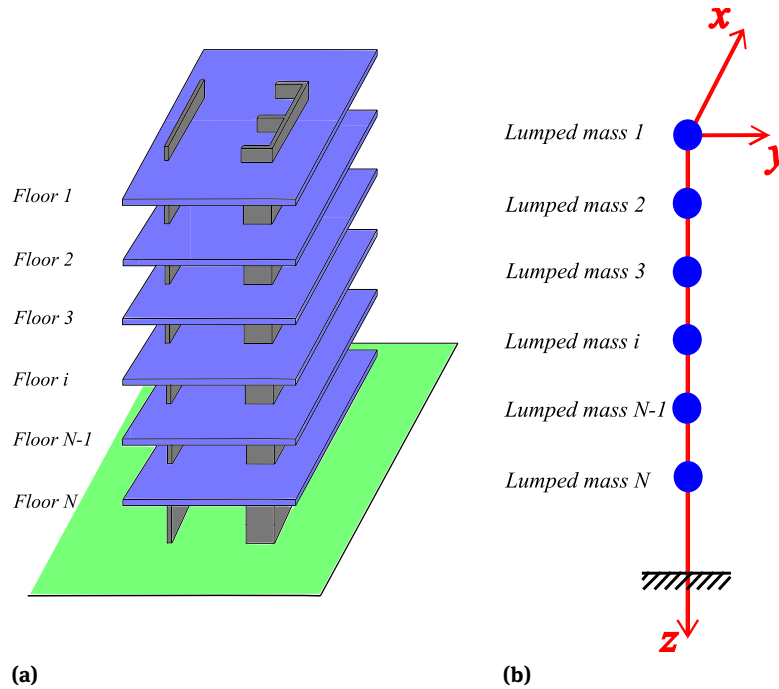


Figure 1: (a) 3D-model of the building; (b) Equivalent cantilever beam with lumped masses

$$[M_{xg}] = \begin{bmatrix} -m_1 y_{m,1} & 0 & \cdots & 0 \\ 0 & -m_2 y_{m,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -m_N y_{m,N} \end{bmatrix} \quad (9b)$$

$$[K_{xi}] = \begin{bmatrix} \{k_{xi,1}\} \\ \vdots \\ \{k_{xi,N}\} \end{bmatrix} \quad (9c)$$

the system of equations (8) can be written in compact form:

$$[M_{xx}]\{\ddot{\xi}\} + [M_{xg}]\{\ddot{\theta}\} + \sum_{i=0}^M [K_{xi}]\{\xi\} - \sum_{i=0}^M (y_i [K_{xi}])\{\theta\} = \{0\} \quad (10)$$

Defining the following matrices:

$$[K_{xx}] = \sum_{i=0}^M [K_{xi}] \quad (11a)$$

$$[K_{xg}] = - \sum_{i=0}^M (y_i [K_{xi}]) \quad (11b)$$

where  $[K_{xx}]$  indicates the stiffness matrix of the entire building evaluated for x-direction only, the equation of the dynamic equilibrium can be written as:

$$[M_{xx}]\{\ddot{\xi}\} + [M_{xg}]\{\ddot{\theta}\} + [K_{xx}]\{\xi\} + [K_{xg}]\{\theta\} = \{0\} \quad (12)$$

In a completely analogous way, the equation of dynamic equilibrium in the y-direction can be written as:

$$[M_{yy}]\{\ddot{\eta}\} + [M_{yg}]\{\ddot{\theta}\} + [K_{yy}]\{\eta\} + [K_{yg}]\{\theta\} = \{0\} \quad (13)$$

where  $\eta$  indicates the y-direction displacement vector of the global reference system origin, while the matrices are defined as follows:

$$[M_{yy}] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N \end{bmatrix} \quad (14a)$$

$$[M_{yg}] = \begin{bmatrix} m_1 x_{m,1} & 0 & \cdots & 0 \\ 0 & m_2 x_{m,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_N x_{m,N} \end{bmatrix} \quad (14b)$$

$$[K_{yy}] = \sum_{i=0}^M \begin{bmatrix} \{k_{yi,1}\} \\ \vdots \\ \{k_{yi,N}\} \end{bmatrix} = \sum_{i=0}^M [K_{yi}] \quad (14c)$$

$$[K_{yg}] = \sum_{i=0}^M x_i \begin{bmatrix} \{k_{yi,1}\} \\ \vdots \\ \{k_{yi,N}\} \end{bmatrix} = \sum_{i=0}^M x_i [K_{yi}] \quad (14d)$$



The equation of the dynamic rotational equilibrium for the  $j$ -th floor and for the  $i$ -th vertical bracing can be written as:

$$- (\{k_{x_{i,j}}\} \{\xi_{C,i}\}) y_{i,j} + (\{k_{y_{i,j}}\} \{\eta_{C,i}\}) x_{i,j} + (m_j \ddot{\xi}_{G,j}) y_{m,j} + (m_j \ddot{\eta}_{G,j}) x_{m,j} + I_{G,j} \ddot{\theta}_j = 0 \quad (15)$$

in which the terms  $(\{k_{x_{i,j}}\} \{\xi_{C,i}\})$  and  $(\{k_{y_{i,j}}\} \{\eta_{C,i}\})$  are the elastic forces in the  $x$  and  $y$  directions, the terms  $(m_j \ddot{\xi}_{G,j})$  and  $(m_j \ddot{\eta}_{G,j})$  are the inertia forces, while the term  $I_{G,j} \ddot{\theta}_j$  represents the relative angular moment of the  $j$ -th floor in which the polar moment of inertia appears to refer to the center of gravity of the floor.

By inserting Equations (3) and (6), and the analogous equations written for the  $y$ -direction in Equation (15), we obtain:

$$\begin{aligned} & - [\{k_{x_{i,j}}\} (\{\xi\} - y_i \{\theta\})] y_{i,j} + \\ & + [\{k_{y_{i,j}}\} (\{\eta\} + x_i \{\theta\})] x_{i,j} + \\ & - [m_j (\ddot{\xi}_j - y_{m,j} \ddot{\theta}_j)] y_{m,j} + \\ & + [m_j (\ddot{\eta}_j + x_{m,j} \ddot{\theta}_j)] x_{m,j} + I_{G,j} \ddot{\theta}_j = 0 \end{aligned} \quad (16)$$

and carrying out the calculations, the following equation is obtained:

$$\begin{aligned} & - y_{i,j} \{k_{x_{i,j}}\} \{\xi\} + y_{i,j}^2 \{k_{x_{i,j}}\} \{\theta\} + \\ & + x_{i,j} \{k_{y_{i,j}}\} \{\eta\} + x_{i,j}^2 \{k_{y_{i,j}}\} \{\theta\} + \\ & - m_j y_{m,j} \ddot{\xi}_j + m_j y_{m,j}^2 \ddot{\theta}_j + m_j x_{m,j} \ddot{\eta}_j + \\ & + m_j x_{m,j}^2 \ddot{\theta}_j + I_{G,j} \ddot{\theta}_j = 0 \end{aligned} \quad (17)$$

According to the Huygens-Steiner theorem, the angular moment which refers to the origin of the global reference system can be written as:

$$I_{O,j} = I_{G,j} + m_j (x_m^2 + y_m^2) = I_{G,j} + m_j r_m^2 \quad (18)$$

from which it follows that:

$$\begin{aligned} & - y_{i,j} \{k_{x_{i,j}}\} \{\xi\} + y_{i,j}^2 \{k_{x_{i,j}}\} \{\theta\} + x_{i,j} \{k_{y_{i,j}}\} \{\eta\} + \\ & + x_{i,j}^2 \{k_{y_{i,j}}\} \{\theta\} - m_j y_{m,j} \ddot{\xi}_j + m_j x_{m,j} \ddot{\eta}_j + I_{O,j} \ddot{\theta}_j = 0 \end{aligned} \quad (19)$$

Considering all the  $N$  storeys of the building, and all the  $M$  resistant vertical elements, we get a system of equations that, after introducing the matrices:

$$[M_{g\theta}] = \begin{bmatrix} I_{O,1} & 0 & \cdots & 0 \\ 0 & I_{O,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{O,N} \end{bmatrix} \quad (20a)$$

$$[K_{g\theta}] = \sum_{i=0}^M [K_{x_i}] y_i^2 + \sum_{i=0}^M [K_{y_i}] x_i^2 \quad (20b)$$

takes the following form:

$$\begin{aligned} & [M_{x\theta}] \{\ddot{\xi}\} + [M_{y\theta}] \{\ddot{\eta}\} + [M_{g\theta}] \{\ddot{\theta}\} + [K_{x\theta}] \{\xi\} + \\ & + [K_{y\theta}] \{\eta\} + [K_{g\theta}] \{\theta\} = \{0\} \end{aligned} \quad (21)$$

The three-dimensional problem can be described by a system of three differential equations, each of which relates to the dynamic equilibrium in the  $x$ -direction (12),  $y$ -direction (13), and in the rotation around the vertical axis of the structure (21).

Defining the mass matrix:

$$[M] = \begin{bmatrix} [M_{xx}] & [0] & [M_{x\theta}] \\ [0] & [M_{yy}] & [M_{y\theta}] \\ [M_{g\theta}] & [M_{g\theta}] & [M_{g\theta}] \end{bmatrix} \quad (22)$$

and the stiffness matrix:

$$[K] = \begin{bmatrix} [K_{xx}] & [0] & [K_{x\theta}] \\ [0] & [K_{yy}] & [K_{y\theta}] \\ [K_{g\theta}] & [K_{g\theta}] & [K_{g\theta}] \end{bmatrix} \quad (23)$$

the previous formulation can be written in compact form as:

$$[M] \{\ddot{\delta}\} + [K] \{\delta\} = \{0\} \quad (24)$$

Assuming that the displacement vector  $\{\delta\}$  is given by the product of a vector depending only on the spatial coordinate  $\{\zeta(z)\}$  by a scalar quantity depending only on time  $\psi(t)$ :

$$\{\delta\} = \{\zeta(z)\} \psi(t) \quad (25)$$

and inserting Equation (25) into Equation (24), we obtain:

$$[M] \{\zeta(z)\} \ddot{\psi}(t) + [K] \{\zeta(z)\} \psi(t) = \{0\} \quad (26)$$

Pre-multiplying both terms by  $\{\zeta(z)\}^T$ ,

$$\{\zeta(z)\}^T [M] \{\zeta(z)\} \ddot{\psi}(t) + \{\zeta(z)\}^T [K] \{\zeta(z)\} \psi(t) = \{0\} \quad (27)$$

and splitting the terms, it is possible to write:

$$\frac{\{\zeta(z)\}^T [K] \{\zeta(z)\}}{\{\zeta(z)\}^T [M] \{\zeta(z)\}} = \frac{\ddot{\psi}(t)}{\psi(t)} = \omega_n^2 \quad (28)$$

where the terms  $\omega_n$  are the angular frequencies of the system.

Equation (28) can be written as two decoupled equations:

$$\ddot{\psi}(t) + \omega_n^2 \psi(t) = 0 \quad (29a)$$

$$\{\zeta(z)\}^T ([K] - \omega_n^2 [M]) \{\zeta(z)\} = \{0\} \quad (29b)$$

The solution of Equation (29a) can be expressed in the form:

$$\psi(t) = A \cos(\omega_n t) + B \sin(\omega_n t) = 0 \quad (30)$$

in which  $A$  and  $B$  are constants whose values are deducted from the initial conditions of the problem. Equation (29b) instead represents an eigenvalue problem and is solved by imposing that:

$$\det([K] - \omega_n^2 [M]) = 0 \quad (31)$$

In this way, it is possible to determine the  $3N$  eigenvalues  $\omega_n$ , from which the natural frequencies ( $f_n = \omega_n/2\pi$ ) and periods of vibration of the building ( $T_n = 1/f_n$ ) are derived. Being known the  $n$  values of the angular frequencies, it is possible to determine the  $3N$  eigenvectors  $\{\zeta(z)\}$  that represent the building's deformed shapes related to every mode of vibration.

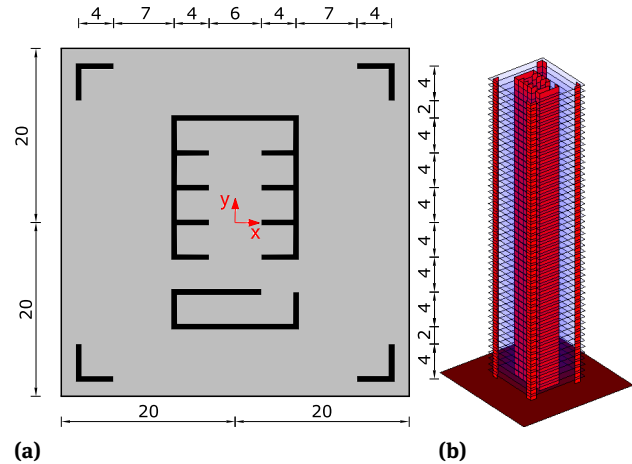
In doing so, for each period of vibration, a deformed configuration is determined related to the origin of the global reference system. By subsequently applying the General Algorithm, it is possible to define the displacements of all storeys of the  $i$ -th stiffening element and, therefore, the stresses acting on it.

## 6 Numerical example

To demonstrate the effectiveness of the analytical formulation, the results of the study of a simple model of high-rise building are presented.

The multi-storey building, measuring 40 m on each side of the square base, consists of 50 floors above the ground, each inter-storey distance is 4 m, therefore, considering all the storeys, the building reaches 200 m in height. The building is characterized in its entire height by a regular structural scheme as shown in Figure (2a). In particular, the floor plan embeds two internal thin-walled open-section shear walls, which allow to house stairwells and lift shafts, and four L-shaped shear-walls placed at the corners of the building as shown in Figure (2b).

All the vertical bracings have the same thickness (0.70 meters) and are made of high-performance concrete with normal elastic modulus and Poisson ratio of  $3.5 \times 10^7$  kN/m<sup>2</sup> and 0.20, respectively. The right-handed system  $XY$  represents the global coordinate system and has the origin coincident with the geometric center of the floor plan, as shown in Figure (2a). Each floor slab has a net area (excluding stairwells and lift shafts) equal to 1432 m<sup>2</sup>. With respect to the global coordinate system, the position of the stiffness center is  $x_c = -0.91$  m and  $y_c = 14.12$  m. The position of the gravity center of the slab is  $x_G = 5.59 \times 10^{-2}$  m and



**Figure 2:** (a) Floor plan of the building (dimensions in meters); (b) 3D-model of the building

$y_G = -7.82 \times 10^{-2}$  m. The polar moment of inertia, referred to the center of gravity of the slab, is equal to 413957.50 m<sup>4</sup>.

As far as the dynamic analysis is concerned, only the structural masses related to the slabs were considered. These were determined considering a specific weight of 150 kg/m<sup>2</sup>. Following the same assumptions adopted for the analytical model, in order to validate its results, a FEM model was also created using Midas Civil, a Finite Element Analysis software developed by MIDASoft. This model has 1,937 nodes and 1,550 shell elements.

Carrying out the dynamic analysis, natural periods of vibration of the high-rise building are determined, as shown in Table (1) and in Figure (3). The first six eigenvectors representing the normalized displacements of the origin of the global reference system are shown in Figures (4,5,6,7,8,9).

Eventually, Figure (10) shows the first six deformed shapes on the 3D-model and on the floor plan of the building created with Matlab analytical code. In the first three vibration modes, nodal sections in deformed shapes do not appear, whereas, in the following three, they manifest themselves. In particular, the first, third, fourth and sixth modal shapes are predominantly torsional, whereas the second and fifth are predominantly flexural modes.

## 7 Conclusions

The design of a high-rise building is a long and complex process that aims to define a structural system compatible with architectural restraints and, at the same time, capable of safely bearing gravitational and lateral loads. Furthermore, the designer must respect severe limits regarding lateral dis-

Table 1: Natural periods of the first ten modes

Mode	Analytical Model	FEM Model
1	2.682	2.655
2	2.186	2.166
3	1.214	1.260
4	0.444	0.456
5	0.349	0.367
6	0.197	0.258
7	0.160	0.173
8	0.125	0.142
9	0.082	0.114
10	0.071	0.094

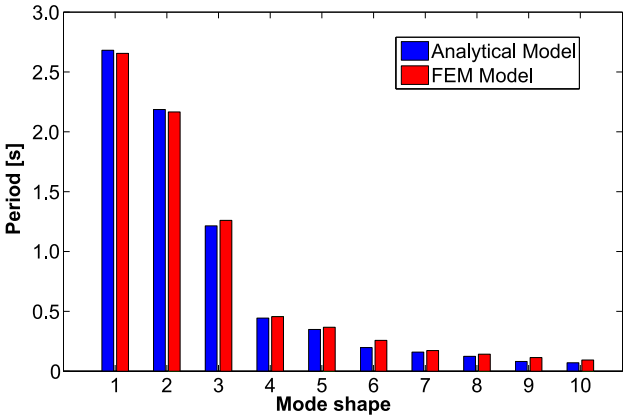


Figure 3: Natural periods of the first ten modes

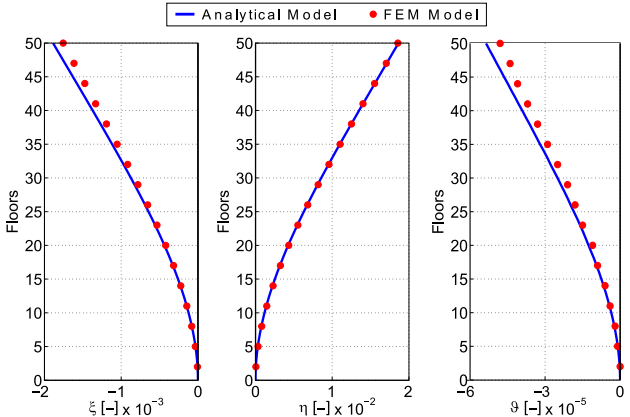


Figure 5: Normalized eigenvector components of the 2nd mode shape

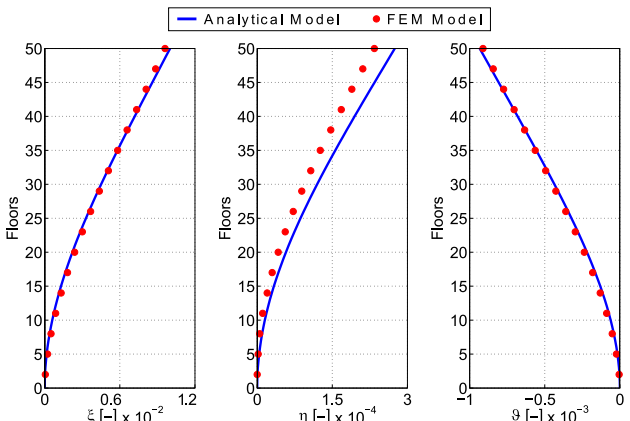


Figure 6: Normalized eigenvector components of the 3rd mode shape

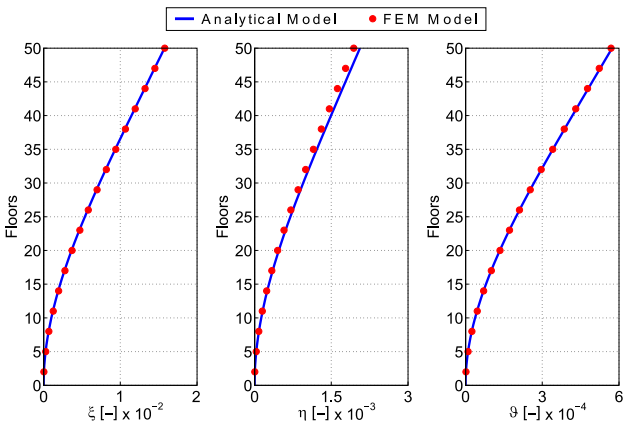


Figure 4: Normalized eigenvector components of the 1st mode shape

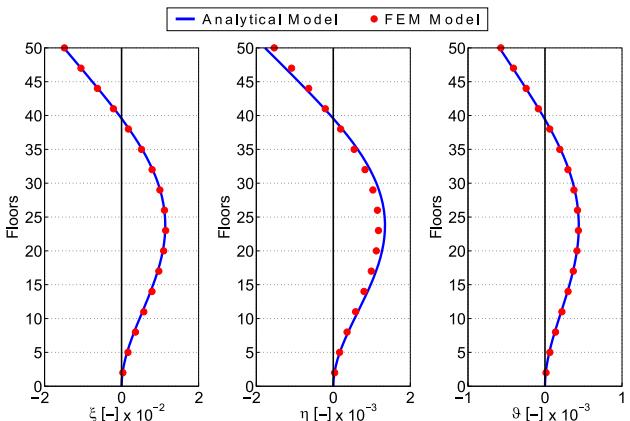
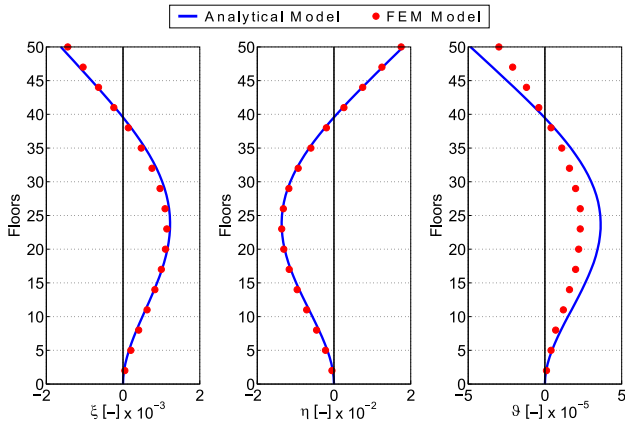
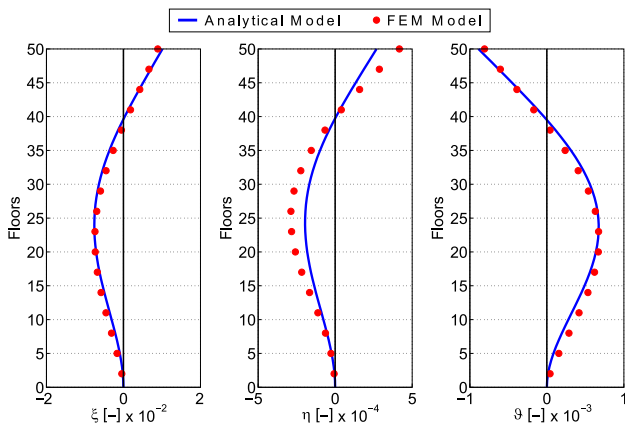


Figure 7: Normalized eigenvector components of the 4th mode shape





**Figure 8:** Normalized eigenvector components of the 5th mode shape



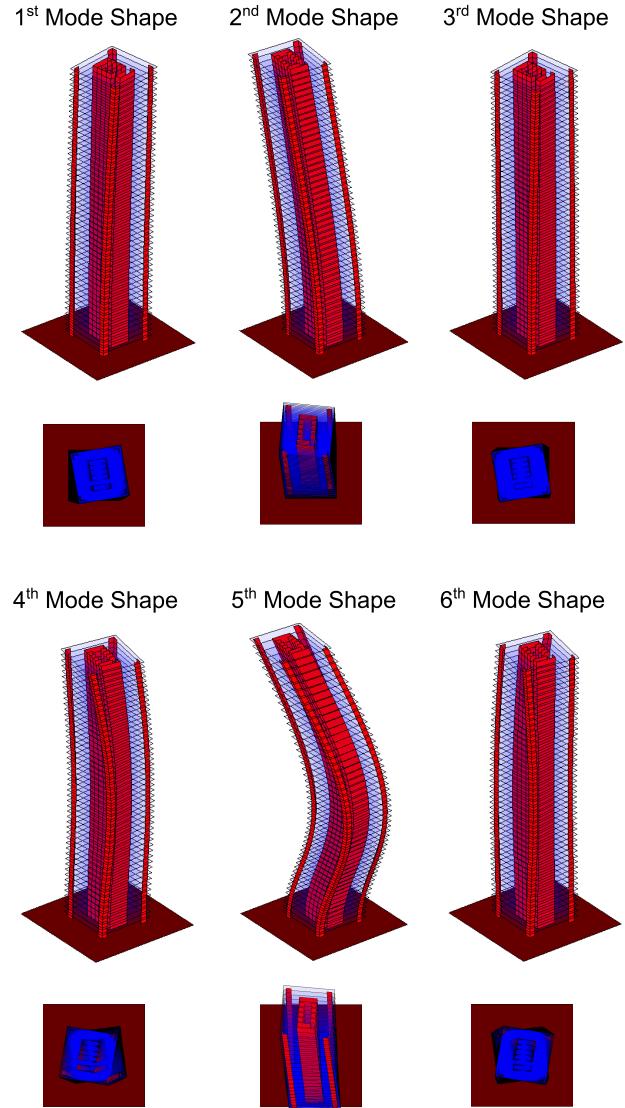
**Figure 9:** Normalized eigenvector components of the 6th mode shape

placements, as well as control vibrations in order to ensure comfort to the occupants. To achieve these goals, in the preliminary structural design stage of the high-rise buildings, it is convenient to use analytical algorithms based on simplified hypothesis instead of complex and expensive FEM software programs.

In the literature there are methods that allow to analyze the global dynamic behavior, and formulations addressed to evaluate the local effects due to the dynamic action on individual elements composing the building.

In this paper, a simplified formulation, which, implemented in a numerical software, allows determining the global natural frequencies, periods of vibration, and mode shapes of a high-rise building, is illustrated.

This computational tool combines the advantages of analytical formulations, like the absence of meshes, and a very reduced number of unknowns. With the advantages of computer software, such as the great potential for cal-



**Figure 10:** 3D-models and floor plans of the first six analytical modes

culation and the ability to visualize the results through video-graphics and three-dimensional models.

To demonstrate the effectiveness of the algorithm, the results of the analysis of a high-rise building are presented. The comparison between the results obtained using the analytical algorithm and those obtained using a commercial FEM software allows verifying the accuracy of the proposed formulation. Although the FEM model was characterized by a very large number of degrees of freedom of the structural mesh, and therefore by much longer computational times than the analytical model, the results between the two different approaches are almost completely comparable.

Clear knowledge of how the *structure works* makes it possible to understand how to optimize the system studied

and hence to continue a more detailed analysis with FEM software programs.

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