DE GRUYTER Curved and Layer. Struct. 2021; 8:259–270

Research Article

Giulio Mirabella Roberti*, Giuseppe Ruscica, and Vittorio Paris

From the herringbone dome by Sangallo to the Serlio floor of Emy (and beyond)**

https://doi.org/10.1515/cls-2021-0023
Received Oct 01, 2020; accepted Mar 09, 2021

Abstract: The research starts from an analogy found between two apparently very different structural solutions: the double spiral pattern of the herringbone brick courses in the domes built by Antonio da Sangallo the Younger (1484-1546) during the Renaissance, and the particular pattern of a wooden floor 'à la Serlio', described by Amand Rose Emy in his Treatise at the beginning of 19th century, made by diagonal beams reciprocally sustained. The diagonal pattern of the floor has a geometrical relationship with the cross-herringbone pattern, so that the latter can be obtained by some geometrical transformations of the former. This pattern was also used in thin shells built by Nervi, from the destroyed airplane hangars in Tuscany to the *Palazzetto dello sport* in Rome, and even by Piacentini in 1936 and earlier in some neoclassical domes.
Thus the construction tool, useful for building domes without expensive scaffolding, could have a structural role at the completed construction stage. Within the research different structures were investigated, in order to observe the relevance of this peculiar structural scheme particularly in the construction of modern domes.

Keywords: conceptual design, masonry domes, herringbone pattern, reciprocally supported beams, 'Serlio' floors, metal spatial structures, membrane stiffening

***Corresponding Author: Giulio Mirabella Roberti:** University of Bergamo, 5, G. Marconi Street, 24044 Dalmine BG, Italy; Email: giulio.mirabella@unibg.it
Giuseppe Ruscica, Vittorio Paris: University of Bergamo, 5, G. Marconi Street, 24044 Dalmine BG, Italy
** Paper included in the Special Issue entitled: Shell and Spatial Structures: Between New Developments and Historical Aspects

1 Introduction

The starting point of this research is the similarity between two very different structural solutions such as the cross-spiral pattern of the herringbone brick course in the domes built by Antonio da Sangallo the Younger (1484-1546) during the Renaissance [1–3], and the particular pattern of wooden beams in the 'à la Serlio' floor described by Amand Rose Emy in his Treatise at the beginning of 19th century [4], made by diagonal beams reciprocally sustained.

The secret building process of the dome of the *Santa Maria del Fiore* cathedral in Florence (1418-1471) has fascinated researchers for around six hundred years, but the procedure for the construction of this impressive structure remains a mystery. The origins of the herringbone construction technology have been hidden by the fame of the dome of Santa Maria del Fiore. It is unreasonable to think that *Filippo di Ser Brunellesco Lapi* (1377-1446), commonly known as *Brunelleschi*, built his masterpiece without having a full grasp of the spiraling herringbone technology, that allowed a relatively fast building process of the dome without the need of a complete centering. Some researchers trace its origins to the East, based on the cultural contact that was evident between the Florentine, Byzantine and Arab cultures. After Brunelleschi, up until the second half of the 16th century, the spiraling herringbone technique was used systematically by the Florentine architects Sangallo [5]. The Sangallo perfected this building technology, developing a double pattern of herringbone bricks: the cross-herringbone [6] (see Figure 1).

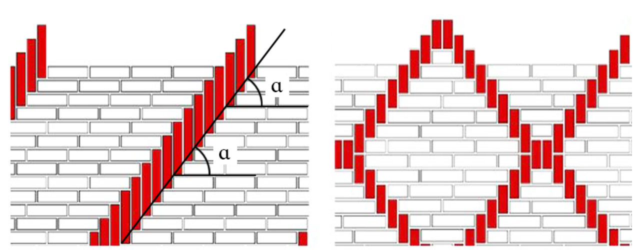

Figure 1: Right: herringbone pattern. Left: crossed-herringbone pattern

As shown in Figure 1, the spiraling herringbone consists of an arrangement of stretcher brick courses (laid horizontally at the springing) interrupted by soldier bricks (laid vertically), forming a herringbone pattern with the above and underlying course. As shown in Figure 2, throughout the different courses, the soldier bricks describe a peculiar path: a loxodromic curve, forming a constant angle with meridian lines, also called rhumb line (from marine science or Mercator track, in Geography).

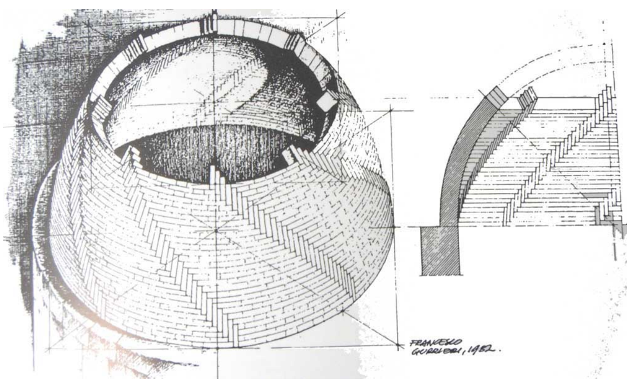

Figure 2: Herringbone pattern of a hemispherical dome during construction works. Original drawing by F. Gurrieri, 1982

Antonio da Sangallo the Younger developed a particular arrangement by crossing two herringbone patterns starting at the base from the same points, but one clockwise and the other anticlockwise [3]. Rowlock bricks were used instead of soldiers, composing a very regular (and symmetrical) pattern on the surface that could also balance some stability problems highlighted by Brunelleschi's dome [7]. Indeed, the dome of Santa Maria del Fiore displays a crack pattern that is affected by herringbone pattern [8].

This solution can still be observed in two similar domes built by Sangallo: the dome of *Santa Maria in Ciel d'Oro* [9], in Montefiascone, and the dome of *Simon Mago*, one of the eight 'octagonal rooms' around the central dome in the St. Peter's Basilica in Rome, located over the barrel vaults connecting the lateral arms of the transept with the choir by means of the round corner chapels.

We had the opportunity to survey both, combining laser-scanning and photogrammetric techniques, under the direction of prof. A. Pizzigoni [10, 11]. The geometry and the masonry texture provided by this survey constituted the basis for the numerical simulations described later.

The second suggestion is a particular grid floor described by A.R. Emy [4] in his Treatise on the art of carpentry at the beginning of 19th century, in the chapter concerning the so-called 'à la Serlio' floors, that are floors composed by beams shorter than the span, sustaining each other in a reciprocal way. The floor described in table 34 of the Treatise, see Figures 3 and 4, was built in Corbeil, France, at the end of 18th century for a flour warehouse. It has the particularity of being a square grid made by diagonal beams, each one

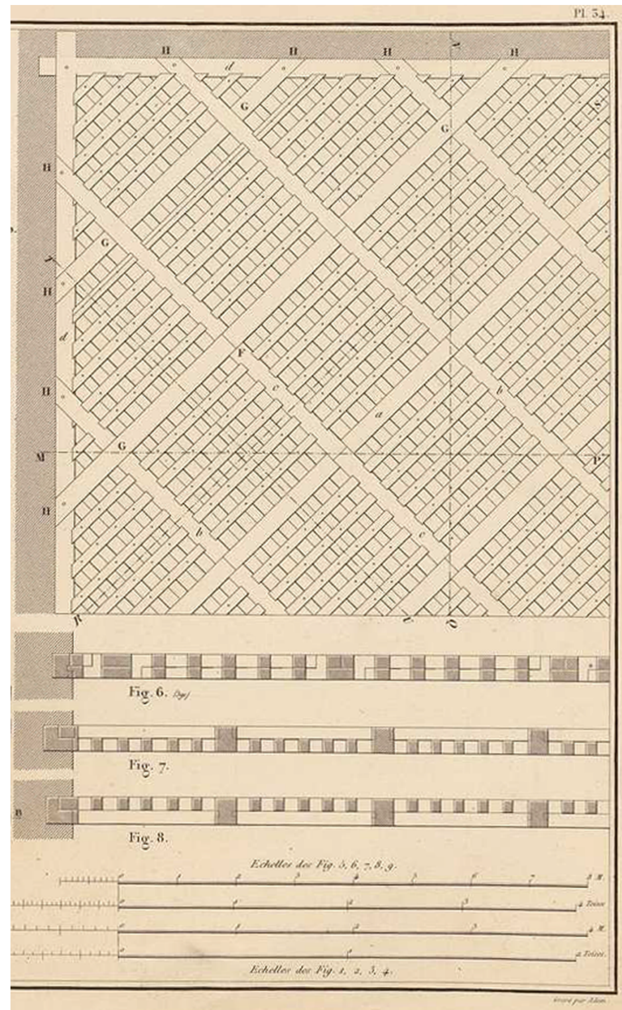

Figure 3: Table 34 of Emy's Treatise: the Corbeil floor

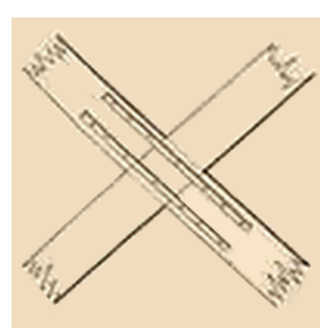

Figure 4: Detail of intrados connection of the Corbeil floor

covering two panels, strongly connected at the end by tie rods in tension to ensure flexural continuity.

So, a question arises: is there a relationship between the two solutions? Or else, provided that a geometrical transition from the flat floor to the hemispherical dome can be demonstrated, is there the possibility that the cross-herringbone bricks can involve a structural function also when the dome is completed?

Many intermediate solutions appear to confirm this possibility: the well-known Orbetello hangars by P.L. Nervi (Figure 14b), for example, used a crossed loxodromic pattern of diagonal arches having the same span on a barrel vault, keeping it strong and light; the flat dome of the *Palazzetto dello Sport* in Rome (Figure 6d) also uses loxodromic paths for the shell ribs, creating a beautiful and vibrating effect on the surface.

In order to investigate the reliability of our hypotheses, a few analyses have been performed using FEM Straus7 software [12, 13]. Moreover, Discrete Element Method (DEM) [14, 15] simulations have also been performed for the Simon Mago dome. The choice of two different numerical approaches is due to the very different characteristics of structures analysed, DEM simulation is adopted to investigate masonry structures [16], where discontinuities are assumed to be relevant to the mechanical behaviour, while beam and truss systems are better described through Finite Element Method analyses [17].

2 Geometry and history

2.1 Historical references

The literature on the application of herringbone spiralling technology for Brunelleschi's dome: Santa Maria del Fiore in Florence, is quite extensive [1, 18]. Although it is still not very clear how Brunelleschi gained his knowledge on this technology, his work has highly influenced the diffusion and the affirmation of the application of the herringbone spiralling technique in central Italy during the Renaissance. Undoubtedly, well before Brunelleschi's Masterpiece was built, this technology was seen applied within the Arabic region [19] as testified by domes of Isfahan Friday Mosque (1088) or Ardestan Friday Mosque (1158) in Iran.

In Europe, the most significant contribution to the knowledge and development of the herringbone spiralling technology was provided by a Florentine family of architects and master masons: the Sangallo. Despite a large spread at the time of this method of construction, it disappeared quite quickly, and today only two original documents remain to describe the herringbone technique, that are currently preserved at the Uffizi Museum. The first is the drawing 900A (n. 639051) GDSU (*Gabinetto dei Disegni e Stampe Uffizi*) and the second drawing 1330 (n. 594469) GDSU [9]. Both documents are associated with the Sangallo architects, and illustrate the herringbone or the cross-herringbone spiralling technologies.

Among the domes built by the Sangallo which are still standing, two are particularly interesting, the dome of Santa Maria in Ciel d'Oro in Montefiascone (Viterbo) [20] and the dome of Simon Mago room in San Pietro cathedral in Rome (see Figure 5): they show how the Sangallos – and especially Antonio the Younger – have contributed to the

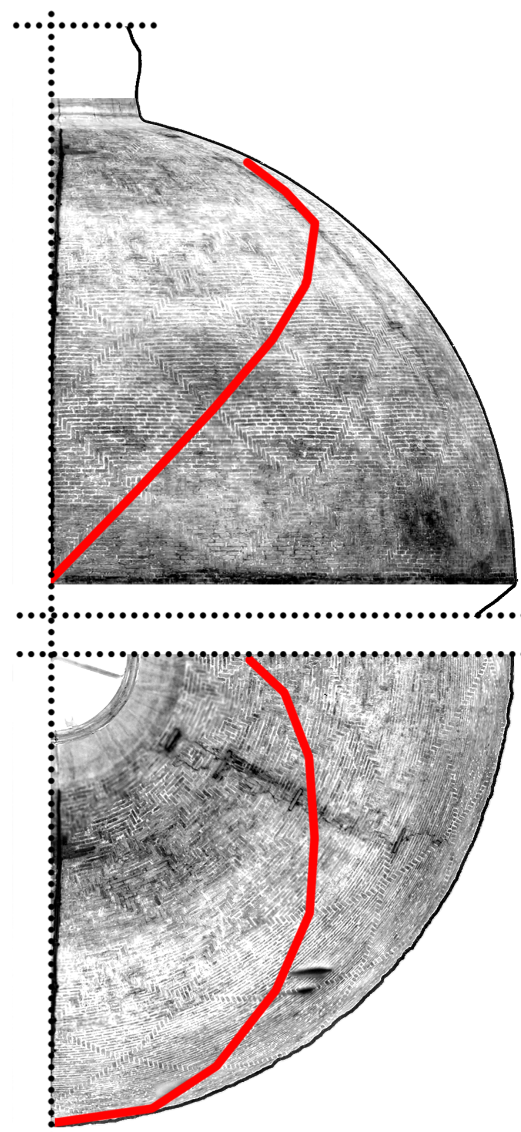

Figure 5: Survey of Simon Mago dome (San Pietro basilica), the loxodromic curve highlighted in red. Orthographic projections, top: half-section view, bottom: one-quarter hyposcopic view

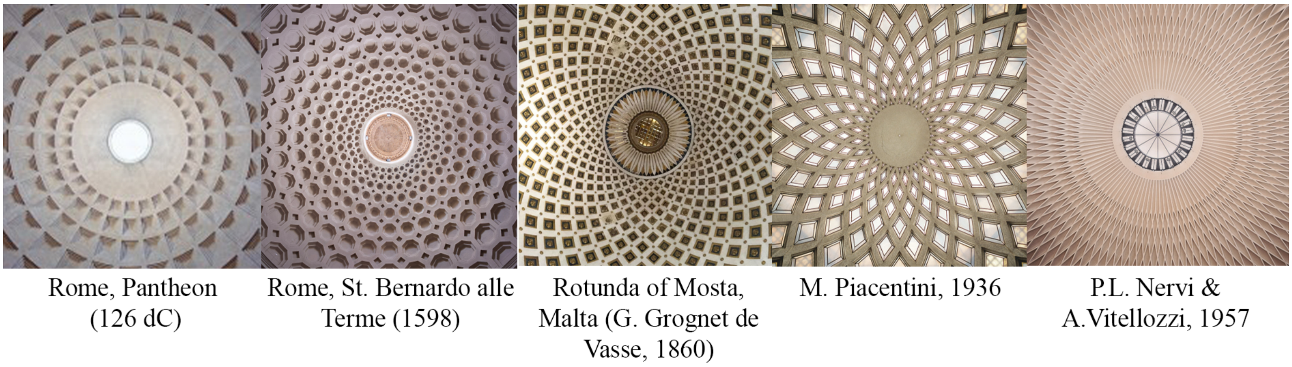

Figure 6: Comparison between different rib patterns in domes from antiquity to present

diffusion of the herringbone spiralling technique. They not only systematically applied this technique for the construction of domes, but they also perfected it. This research of the Sangallo family then further led to developing an even more advanced construction system: the cross-herringbone spiralling technology. Both of these methods allow to build masonry domes without any centering to support the dead load during the construction, but even today, the role and long term advantages of cross-herringbone construction technique in terms of its structural behaviour is not explored.

As already mentioned, the peculiar topology visible in Simon Mago dome of Saint Peter Basilica (Rome), illustrated in Figure 5, can also be found in other shell structures after the Renaissance, but not in the classical ones of the Pantheon or the thermal dome of St. Bernardo in Rome.

In Figure 6 a quick comparison of five different domes is presented. In the Roman domes (like the Pantheon or the thermal dome of St. Bernardo, rebuilt in 1598) a classical meridian-parallel structure is highlighted: but in the second one the octagonal shape of the 'lacunari' underlines the need of a diagonal stiffening, like that depicted in the 'Rotunda' of Mosta. Here the basket-like structure, made by crossed loxodromic ribs, is only painted on the surface of a hemispherical dome, but the examples of Piacentini and Nervi show two shallow shells, where these ribs have a relevant structural role.

2.2 Geometries

Surfaces such as planes, cylinders, cones or spheres are undoubtedly different; their mathematical description involves different equations. For example, planes are defined by a linear equation and cylinders' geometry by a quadratic one but, precisely as cones and spheres, all geometries can be generated by rotating a planar curve around a fixed axis. Hence, denoting the matrix of rotation by $\mathbf{R}$, all surfaces Σ

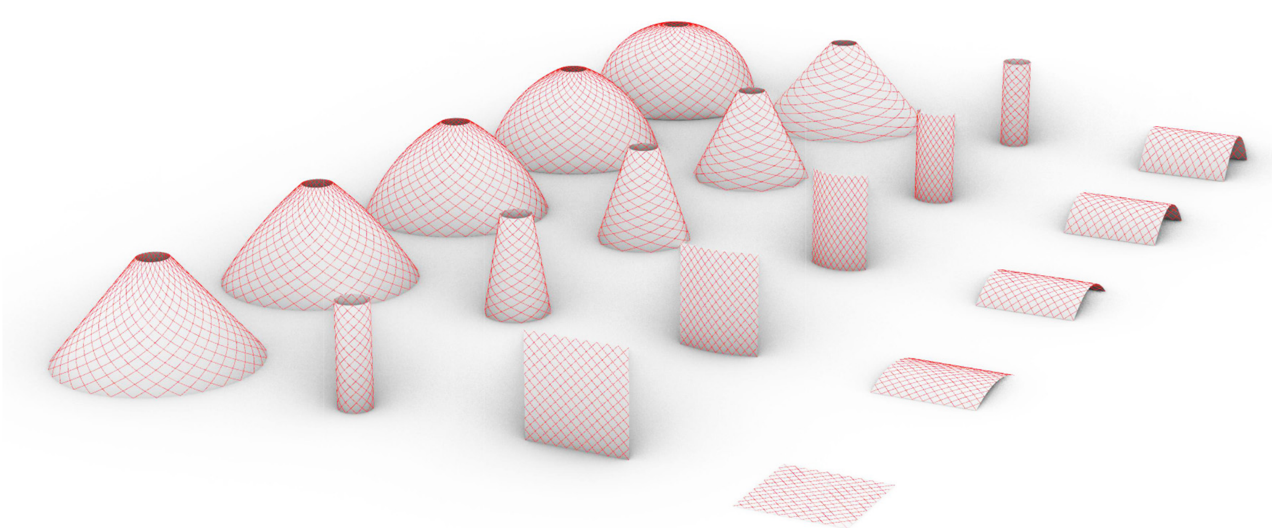

Figure 7: Different geometries and relative loxodromic curves

shown in Figure 7 can be described through the rotation of a plane curve, called generatrix Γ_g, around an axis r, see (1).

$$\Sigma = \mathbf{R}\Gamma_\mathbf{g} \quad (1)$$

All geometries drawn in Figure 7 and many others can be obtained through the revolution operation (1). For such class of surfaces, it is possible to write the equation of loxodromic curves explicitly. Loxodromic curves Γ_l, also called rhumb lines, constitute one-dimensional entities which lay on revolution surfaces and are strictly related to them [21].

Their main property is the following: for any point $\overline{P}$ belonging to the loxodromic curve, the angle α between the tangent versor of the generatrix curves of the related surface and the tangent versor of the loxodromic itself remains constant, that is:

$$\frac{\Gamma'_g\left(\overline{P}\right)}{\left\|\Gamma'_g\left(\overline{P}\right)\right\|} \cdot \frac{\Gamma'_l\left(\overline{P}\right)}{\left\|\Gamma'_l\left(\overline{P}\right)\right\|} = k\,(costant)\,\forall \overline{P} \in \Sigma \wedge \Gamma'_l \quad (2)$$

Undoubtedly, due to eq. (2), the curve used for tracing routes in the navigation field [22] is a loxodrome of the sphere representing the Earth. Nevertheless, along the history, the geometry of the rhumb lines found several applications in defining the shape of architectural or structural elements. When this occurs, they may enable to confer interesting peculiarity to the structures in which they were applied, *e.g.* in the case of Santa Maria del Fiore in Florence [6]. Furthermore, structures, where the loxodromes geometries are used as the structural scheme, are composed of structural elements that belong to the same structural hierarchy level. Within this research, different loxodromic trajectories are assumed as primary structural systems for different geometries, but all obtained by eq. (2).

3 Numerical analyses

3.1 Analysis of the Sangallo dome

Herringbone and cross-herringbone technologies played an essential role during the construction stages of masonry domes [6]: due to their pattern, the rowlock bricks work linking two different masonry courses. Proceeding to laying of completed brick courses, the masons built the dome without using temporary support. In this sense, the role of the double system of loxodromic curves does not change with respect to the single herringbone trajectory, indeed, to reach a self-balanced state, the two construction systems involve the same resistant structure: the plate bande confined by soldier (or rowlock) bricks.

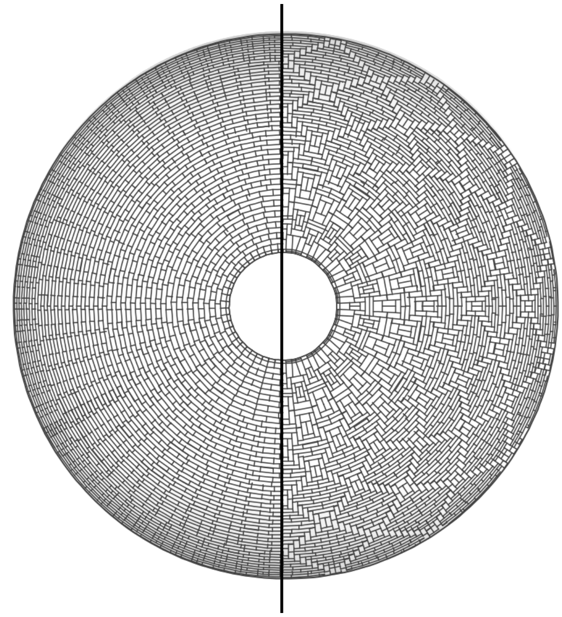

Figure 8: Left half dome: top view of common masonry pattern. Right half dome: top view of cross-herringbone masonry pattern

In order to estimate the effect of herringbone pattern on hemispherical masonry dome, DEM analyses were performed using the 3DEC software [23]. Through this numerical approach masonry structures are described by a system of rigid bodies, and phenomena as sliding, overturning and even collisions between blocks are considered [14, 15].

In the analyses carried out, the geometry of each block has been defined by the same volume occupied by one brick plus a portion of mortar that surrounds it. Hence, the system of rigid bodies is composed by the same number of bricks illustrated in domes modelled in Figure 8. Furthermore, all simulations were performed considering a finite value of friction angles $\phi = 25°$, and the interfaces between the blocks are governed by the Mohr-Coulomb failure criterion, without cohesive or tension capacity. The normal JK_n and shear JK_s stiffness parameters are defined as [24], and they are related respectively to the difficulty of pressing and slipping of the blocks with respect to each other. As shown in Figure 9, DEM analyses have been carried out on two hemispherical domes, whose diameter is 9,50 m, thickness equal to 0,28 cm and loaded only by self-weight (18.00 kN/m^3); so, according to Heyman's theory [25], they can easily stand up. Thus, the two models present the same geometrical parameters and the same shell thickness. The only difference is the masonry pattern: one has the cross-herringbone pattern which model is based on the survey executed on the dome of Simon Mago in San Peter Basilica (Rome) [11], and the other shows a common masonry

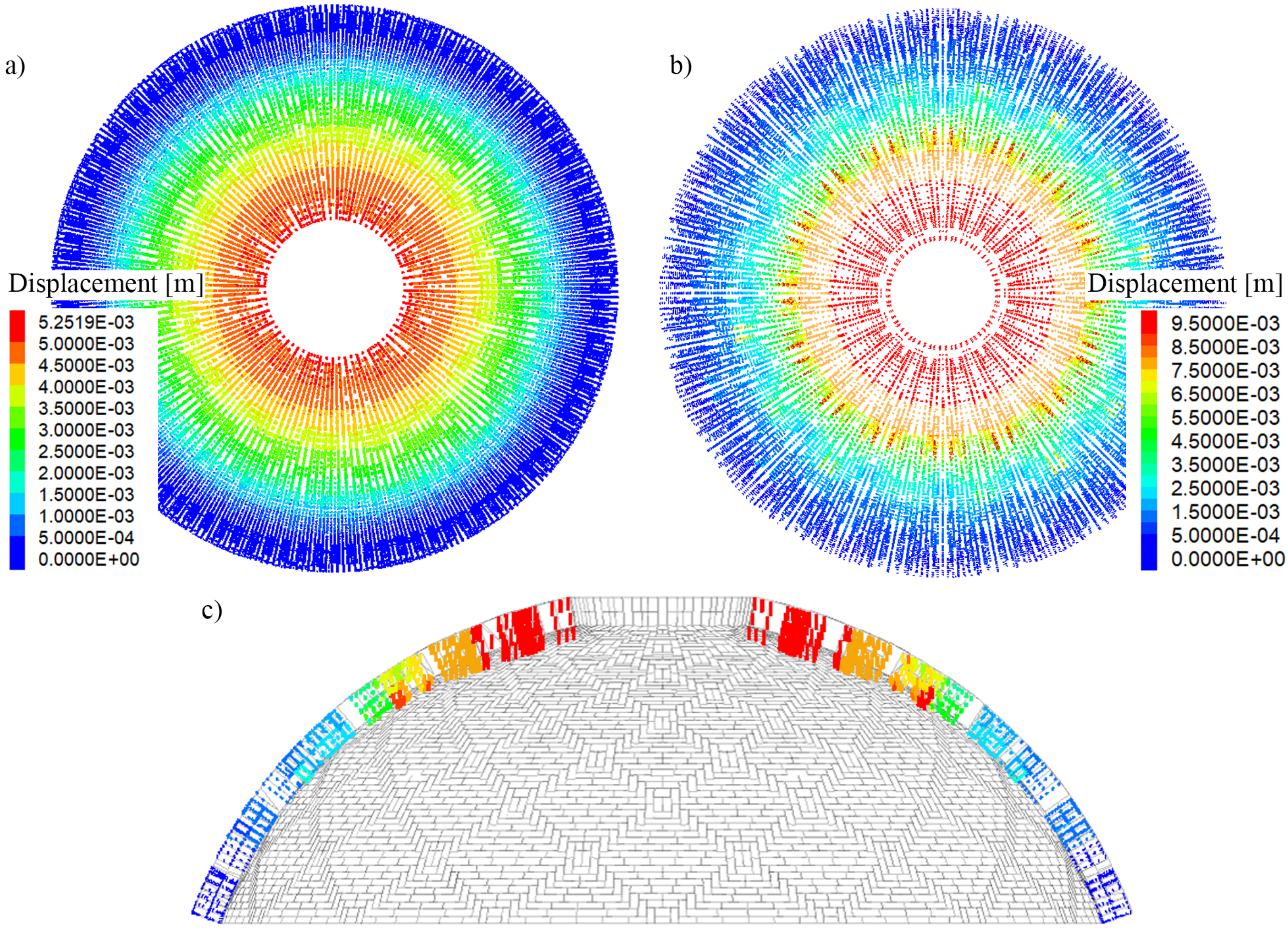

Figure 9: Map of displacement distributions: a) displacements of a dome with common masonry pattern. b) and c) displacements of a dome with cross-herringbone. The complex geometry leads to the sliding of some blocks (displacement 1.0×10^{-2} meters)

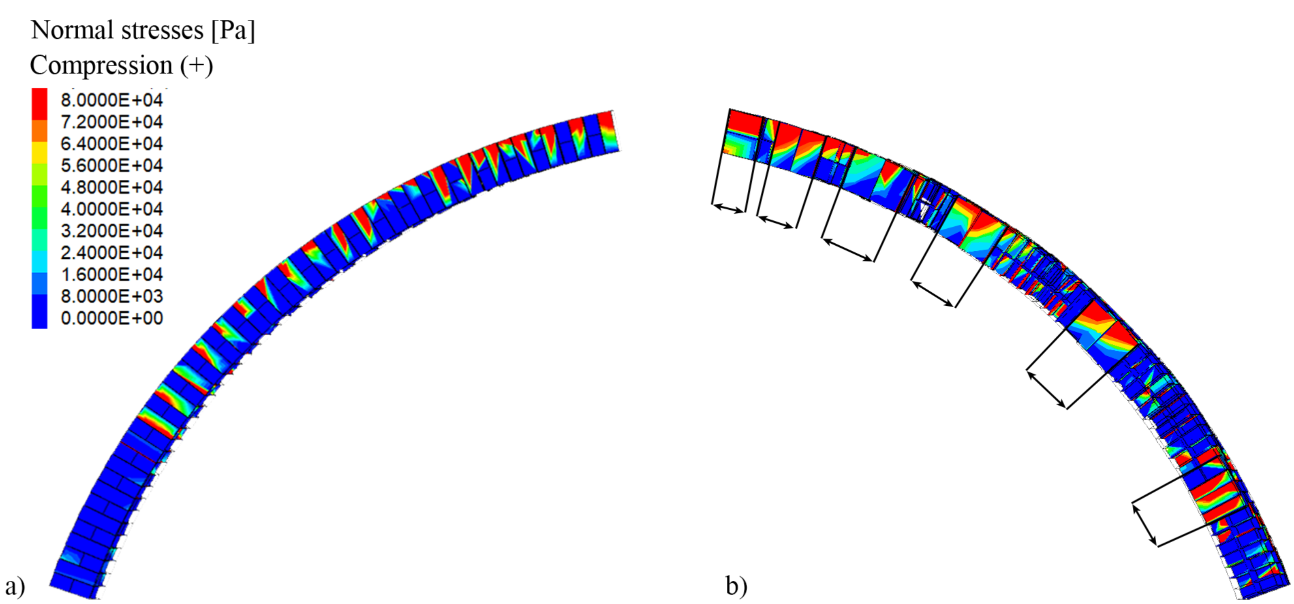

Figure 10: Hoop stresses on vertical joints. a) Common tessellation, the magnitude of stresses increases from the bottom to the crown. b) Cross-herringbone model. The magnitude of stresses increases from bottom to the crown, but in correspondence of rowlock blocks (delimited by black quotes) the normal stress is high regardless of the position of the block

tessellation. The simulations were performed only at the completed construction stage.

As illustrated in Figure 9, the two models exhibit the same displacement map: the movements increase from the base to the top, but in the case of the cross-herringbone model, the magnitude is about two times bigger than the common tessellation. Furthermore, near the crown, the cross-herringbone model shows some irregularity in terms of displacement map.

The diversity of masonry patterns cannot entirely explain the difference of magnitude and the irregularities in the displacement map. DEM allows considering rigid bodies constituted only by planar faces; thus, due to the complexity of the cross-herringbone tessellation and the dome's double curvature, geometrical incongruences near the crown are determined. This approximation is even highlighted in the cross-herringbone model, where the tessellation is more complex and leads to local sliding phenomena. Indeed, in the lower portion of the dome and even in the central one, the difference of displacements between the two models is negligible.

From the stresses point of view the two models do not show large differences, at least in the direction of the meridian. The normal stresses distribution on the vertical joints (those relating to the hoop forces) instead show differences: even if the magnitude is equal, the simulations highlight perturbations, see Figure 10. The cross-herringbone model, Figure 10b), shows alterations especially in correspondence with herringbone bricks, where the hoop forces are concentrated.

3.2 Analysis of a lattice dome

A second investigation was conducted on the same hemispherical geometry adopted in Section 3.1. In these simulations, truss elements have been disposed along the loxodromic patterns, and membrane elements have been used to represent the continuity of the shell. The final goal is to verify if the same pattern adopted in Renaissance domes can be used to build modern lattice domes: as a first attempt FEM analyses were executed with pinned joints and reticular elements with the purpose to understand the influence of such a structural choice.

Usually in lattice domes, such as the geodesic domes built by Buckminster Fuller, the geometrical solutions are all based on triangular faces formed by truss elements, so that pinned joints (connecting six or five elements) can be used. Moreover, the truss elements connected to a joint do not belong to the same plane. Here, the loxodromic pattern defines rhomboidal meshes and, as expected, even if the nodes of each rhomboid do not lie on the same plane, the dome is not statically determined. Thus, as shown in Figure 11 and 12, additional stiffness must be provided by tension and compression rings, or by membrane elements; the two solutions together were adopted.

Therefore, a FEM model has been set up using different connection solutions to reach a satisfying deformation/stress response to simple vertical loads, such as dead loads (see Figure 11 and 12). A more extended analysis would be requested to assess the structural performance, since these are only preliminary considerations to explore this solution's possibilities, especially in comparison to a standard 'meridians-and-parallels' structural system.

But more interesting is the possibility to apply the same solution adopted in the *Serlio* floor of Emy treatise, that is to make the elements (spanning two meshes) reciprocally sustained, like those of the Zollinger roofs, as described later in Section 3.4.

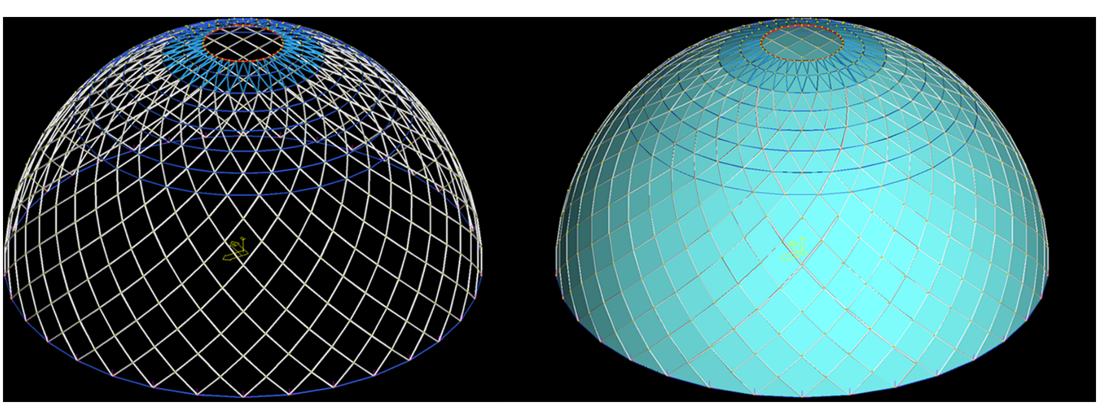

Figure 11: The lattice dome. Left: truss elements with upper compression rings and lower tension ring. Right: the same with membrane elements

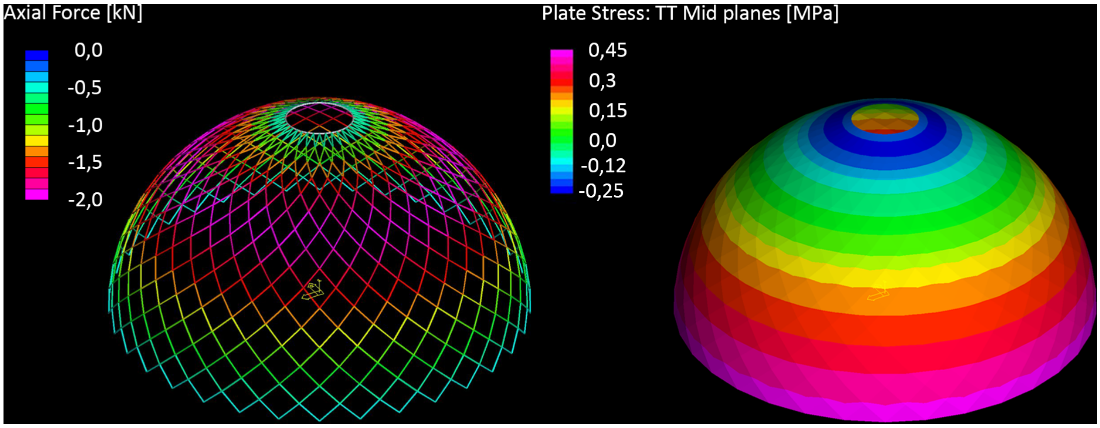

Figure 12: Shell displacements. Left: axial forces in truss elements. Right: Membrane stresses in hoop direction tangent to the surface (purple = tension, blue = compression)

3.3 Analysis of the Emy floor

Loxodrome geometries are found even in the model described by A.R. Emy. Here FEM analysis has been performed to investigate the efficiency of the beam arrangement described in his Treatise, both on the hypothesis of a full continuity reached by the connection ties disposed on the bottom of crossing elements (see the detail in Figure 4), and in the situation of a hinged connection among intersecting beams at their support, using the scheme of reciprocal sup-

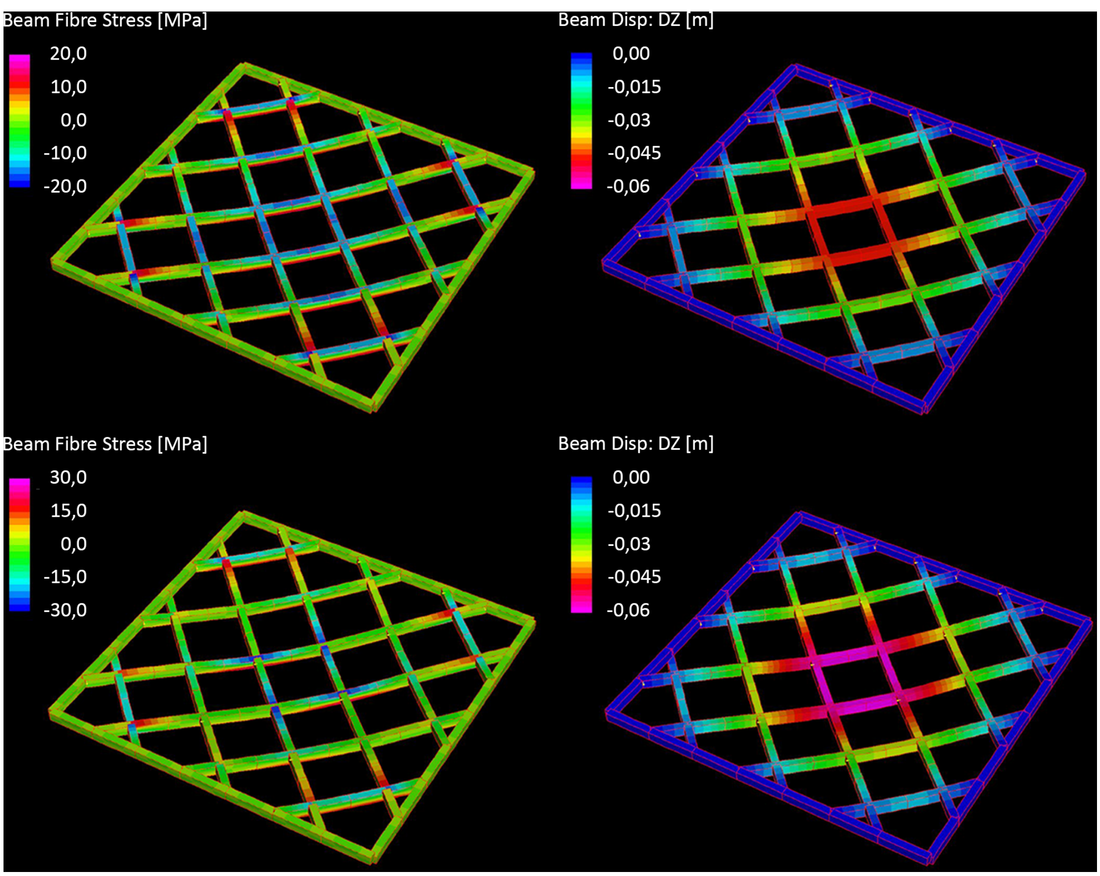

Figure 13: Top left: Stresses due to axial forces and bending moments (fibre stresses) in the continuous grid beams. Top right: beam displacements in vertical direction for the continuous beams grid. Bottom left: Stresses due to axial forces and bending moments (fibre stresses) in the reciprocal frame roof. Bottom right: beam displacements in vertical direction for the reciprocal frame roof

ported beams depicted by Emy neglecting the contribution of connection ties.

The two 7m × 7m frames were subjected to the same uniformly distributed load of 4 kN/m^2, and the beams had the same geometry, and the elastic properties of the material were equally the same. The results are quite interesting: for the reciprocal frame the vertical displacements increase only by 25% compared to the fully fixed one and the flexural stresses increase by 38%, showing not only the greater efficiency of the continuous frame, as expected, but also a quite acceptable solution for the discontinuous reciprocal frame (see Figure 13).

3.4 Examples of Emy-like shells

In the past, reciprocal frame structures were conveniently used thanks to the possibility to span large spaces with short elements. Beyond the 19th century roofs, described by A.R. Emy, in 1921 Friedrich Zollinger patented a “lamella roof” (also known as the “Zollinger roof”, see Figure 14), which consisted of a series of wooden boards [26], connected with each other according to the topology of a canonical reciprocal frame structure. It can be noticed that the geometry of Zollinger’s barrel vaults [27] resembles P.L. Nervi’s Orbetello and Orvieto hangars (see Figure 14).

Recently, the use of reciprocal frame structures has come to a new life, thanks to the continuous architecture’s quest for efficient structural forms of the past. A few exam-

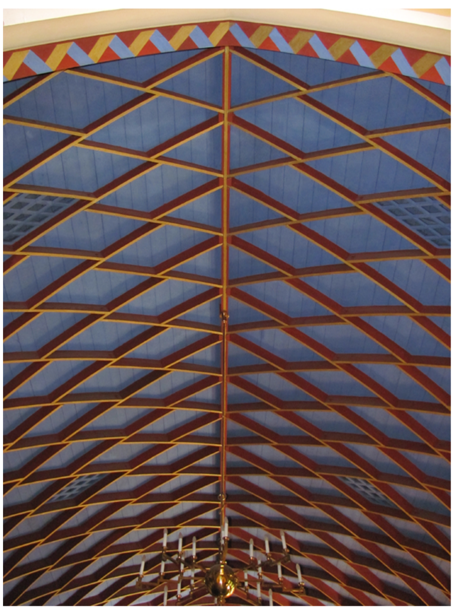


Figure 14: Left: Zollinger roof in the Lutherhaus in Kötzschenbroda, Germany (photograph by Radobyl, distributed under a CC-BY 2.0 license). Right: Pier Luigi Nervi’s Orvieto Airplane Hangar (https://quod.lib.umich.edu)

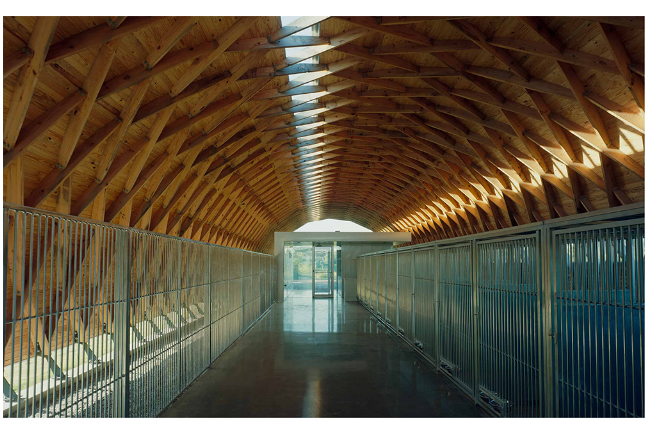
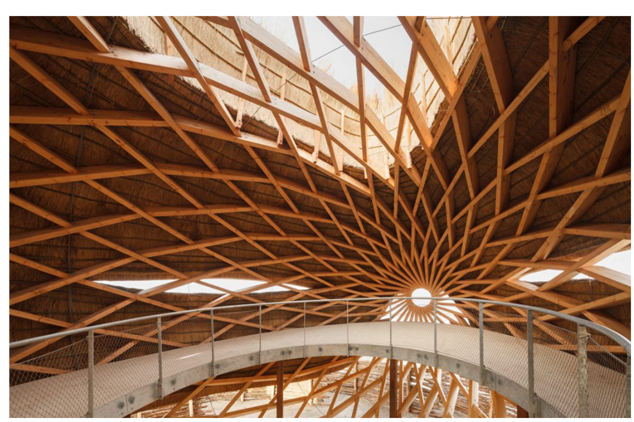

Figure 15: Left: Hale County Animal Shelter by Rural Studio (http://ruralstudio.org). Right: ”Tij” bird observatory by Geometria Architecture Ltd (https://geometria.fi)

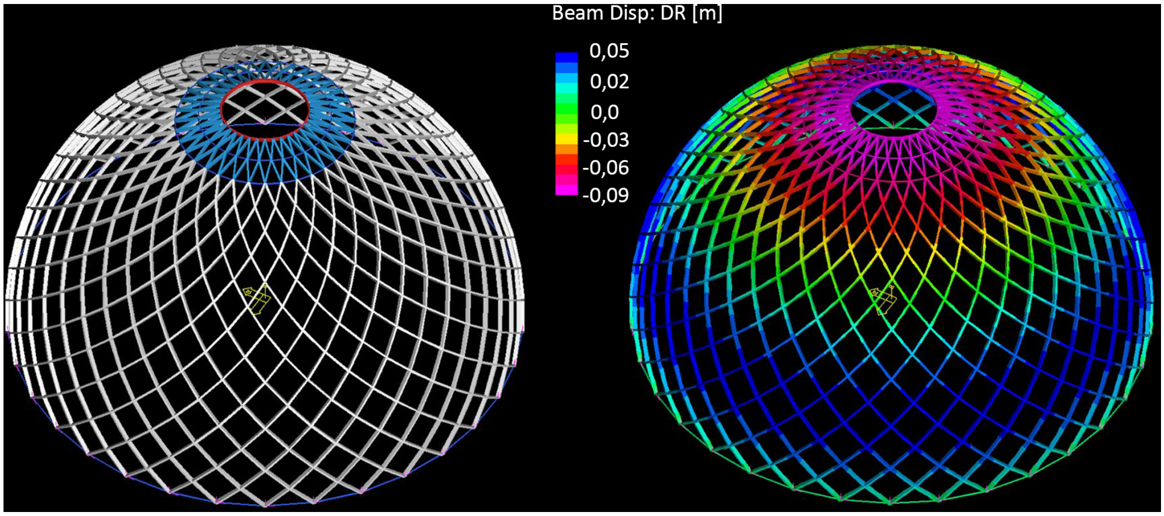

Figure 16: Left: Dome with reciprocal frame beam elements. Right: displacements in radial direction

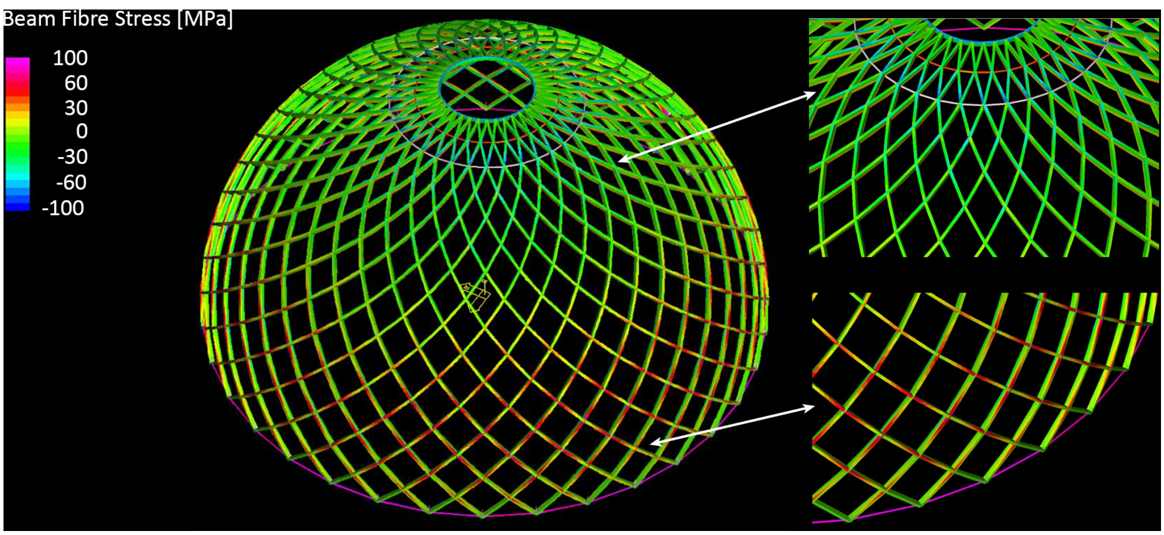

Figure 17: Left: Stresses due to axial forces and bending moments (fibre stresses) in the reciprocal frame beams. Top right: detail of element stresses in the upper part. Bottom right: detail of element stresses in the lower part

ples are constituted by some small pavilions of Geometria Architecture Ltd architectural practice and the Hale County Animal Shelter by Rural Studio (see Figure 15).

These considerations show that reciprocal frames and herringbone domes share the same principles of structural efficiency and attention to the load paths, both important aspects when considering historical constructions. Although the structures above seem far different from the ones reported in the previous sections, they all employ loxodromic curves as underlying geometry for the principal structural scheme.

In the attempt to apply these solutions to hemispherical domes, the same lattice shell of sect. 3.2 (with 20m diameter) was studied making use of reciprocal beams, each of them spanning two romboidal meshes, without the structural contribution of the membrane elements. Only the top and bottom annular ring were left in order to give the necessary stiffness in compression and tension, respectively. The dome was subjected to a distributed vertical load of 1 kN/m^2 on horizontal projection, giving the results shown in Figures 16 and 17. Both displacements and stresses appear to be adequate.

4 Conclusions and further developments

The research is still not concluded. However, loxodromic geometries show how fascinating these elements appear in any observed structures. Unlike a common frame-like structure, the system of crossed loxodromic ribs affects and involves a major quote of the shell surface. The orientation of spiral geometries leads to a structural behaviour more similar to that of the shell, giving to the surface an intrinsic in-plane stiffness.

From a merely structural point of view, their presence (see Section 3.1) in the masonry dome could appear ineffective, increasing the displacements at the top compared to the usual brick disposition; but observing the building and technological aspects, the cross-herringbone system offers enormous advantages. Indeed, these patterns are not conceived originally as stiffener ribs, but rather as constructive aids. At the beginning, they have been probably introduced as decorating elements, and then gradually became more and more structurally effective, as well as the comprehension of structural behaviour of shells became more and more deep.

That is why, throughout history, several architects conceived their structures taking into account the use of loxodromic geometries. If we look even further, this geometric pattern can adequately be used in reciprocal frames, merging the loxodromic curves into the "Emy floor" schemes or, as a further development, even into deployable structures, like the well-known Chuck Hoberman's "Iris Dome" [28, 29], which has been exhibited at the New York's Museum of Modern Art (MoMA) in 1994.

Acknowledgement: The fundamental help of Dr. P. Azzola in the survey operations and restitution process is gratefully acknowledged, as well as the work done by MSc student D. Pezzoli for his graduation thesis; the critical discussion with prof. A. Pizzigoni stimulated a widening of the research.

Funding information: The authors acknowledge the financial support from STaRs Programme 2020 of the University of Bergamo. Software 3DEC [23] was provided by Itasca C.G. under the Education Partnership Program, for which the authors also express their gratitude.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

References

[1] Di Pasquale S. La costruzione della cupola di Santa Maria del Fiore. Venezia: Marsilio; 2002.
[2] Acidini Luchinat C, editor. La cattedrale di Santa Maria del Fiore a Firenze. Vol. 2. Firenze: Cassa di Risparmio di Firenze; 1995.
[3] Giovannoni G. Antonio da Sangallo il Giovane. Vol. 1. Tipografia regionale; 1959.
[4] Emy AR. Traité de l'art de la charpenterie. Vol. 1. Paris; 1837.
[5] Zander G. Gli ottagoni di San Pietro riconosciuti nel dis. Arch. Uff. n. 1330. Palladio. 1988;1:67–82.
[6] Paris V, Pizzigoni A, Adriaenssens S. Statics of self-balancing masonry domes constructed with a cross-herringbone spiraling pattern. Engineering Structures. 2020;215:110440.
[7] Chiarugi A, Foraboschi P. Monitoraggio e identificazione della cupola di Santa Maria del Fiore. L'Edilizia. 1993;20–42.
[8] Bartoli G, Betti M, Borri C. Numerical modeling of the structural behavior of Brunelleschi's dome of Santa Maria del Fiore. International Journal of Architectural Heritage. 2015;9(4):408–29.
[9] Pizzigoni A, Paris V, Ruscica G. Herringbone technique: truth and history of a cutting-edge technology. Proceedings of IASS Annual Symposia, IASS 2018 Boston Symposium: Historical spatial structures. 2018;1–8.
[10] Paris V. On the equilibrium of self-balanced shells under construction [Doctoral dissertation]. University of Bergamo; 2020.
[11] Pezzoli D. Il dispositivo "spinapesce" per la costruzione di volte in muratura nella tecnologia rinascimentale dei Sangallo [Master thesis]. 2019.
[12] HSH s.r.l. Straus7. HSH s.r.l.; 2013.
[13] Cook RD, Malkus DS, Plesha ME, Witt RJ. Concepts and Applications of Finite Element Analysis. John Wiley & sons; 2007.
[14] Cundall PA. Formulation of a three-dimensional distinct element model—Part I. A scheme to detect and represent contacts in a system composed of many polyhedral blocks. International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts. 1988;25(3):107–16.
[15] Hart R, Cundall PA, Lemos J. Formulation of a three-dimensional distinct element model—Part II. Mechanical calculations for motion and interaction of a system composed of many polyhedral blocks. International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts. 1988;25(3):117–25.
[16] Lemos JV. Discrete element modeling of masonry structures. International Journal of Architectural Heritage. 2007;1(2):190–213.
[17] Bathe KJ. Finite element procedures. books.google.com; 2006.
[18] Sanpaolesi P. Brunelleschi. Milano: Edizioni per il Club del libro; 1962.
[19] Galdieri E. Da Gerusalemme a Dakha: mille anni di cupole islamiche. In: Conforti C, editor. Lo specchio del Cielo. Milano: Electa; 1997. p. 53–66.
[20] Paris V, Pizzigoni A, Ruscica G. Brunelleschi's herringbone hidden reciprocal structure and the form finding of its self-supporting bricks. Proceedings of IASS Annual Symposia, IASS 2017 Hamburg Symposium: Historic Shell & Spatial Structures. 2017;1–10.

[21] Loria G. Curve sghembe speciali, algebriche e trascendenti. Vol. I: Curve algebriche. Bologna: Zanichelli; 1925.
[22] Conti G, Corazzi R. Il segreto della Cupola del Brunelleschi a Firenze – The Secret of Brunelleschi's Dome in Florence. Firenze: Pontecorboli Editore; 2011.
[23] Itasca Consulting Group, Inc. 3DEC. Itasca Consulting Group, Inc.; 2020.
[24] Simon J, Bagi K. Discrete element analysis of the minimum thickness of oval masonry domes. International Journal of Architectural Heritage. 2016;10(4):457–75.
[25] Heyman J. The stone skeleton: structural engineering of masonry architecture. Cambridge University Press; 1995.
[26] Weller B, Tasche M, Baatz J. Lamella Roof Constructions by Hugo Junkers. Proceedings of the International Association for Shell and Spatial Structures (IASS) Symposium 2009, Valencia. 2009.
[27] Franke L, Stahr A, Dijoux C, Heidenreich C. How does the Zollinger Node really work? Proceedings of the IASS Annual Symposium 2017. 2017.
[28] Hoberman C. Reversibly expandable doubly-curved truss structure. United States; 4942700, 1990.
[29] Hoberman C. Radial expansion/retraction truss structures. United States; 5024031, 1991.