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Research Article

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Static Perturbation Analysis of Inclined Shallow Elastic Cables under general 3D-loads

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Abstract: Inclined, shallow, elastic cables under static 3D loads, arbitrarily distributed, are studied. Cables having natural length both larger or smaller than the distance between the supports (*i.e.* suspended or taut cables, respectively), are considered. Kinematically exact equations are derived, and projected onto an orthonormal basis intrinsic to the chord. A perturbation procedure is proposed, which extrapolates the solution relevant to the taut string, up to the desired order, and leads to a closed-form solution. Lower-order solutions are consistent with the hypotheses normally adopted in technical environment. Emphasis is given to the mechanical interpretation of the cable behavior. The asymptotic solution is compared to the explicit (not in closed-form) solution of the literature.

Keywords: Statics of cables, inclined shallow cables, suspended vs. taut cables, elastic cables, perturbation methods

1 Introduction

Cable are efficient structural elements, widely used in civil [1, 2], offshore [3, 4], aerospace [5, 6] and mechanical [7, 8] engineering. The description of the mechanical behavior of cables is then an interesting scientific problem, which fascinated scientists since XVII century (see [9] for a historical review). Most of the existing literature is devoted to horizontal cables, *i.e.* cables fixed at the ends at the same level (see *e.g.* [10, 11]), sometimes even considering the bending and torsional stiffness for consistent evaluation of the aerodynamic forces (see *e.g.* [12–14]); however, inclined cables (different levels) also received attention. About them, innumerably examples exist in literature, concerning statics [15], linear dynamics [16], nonlinear dynamics [17, 18], including studies on aeroelastic effects [19–21] and vibration absorber devices [22–24]. An exhaustive review can be found in [25].

Usually, the dynamic analysis of cables is performed in two steps. First, the static profile assumed by the cable under steady-state forces is found, then the dynamics around this configuration is studied. The separation in two steps is dictated by the fact that, for stability reasons, the dynamic stress (being of both signs) cannot overcome in modulus the static stress (only of tensile nature). Therefore, it is found convenient for algorithmic reasons, to superimpose small (although finite) amplitude motions to a static configuration in which a large static tension already exists. Such a procedure would, in principle, require an accurate description of the static configuration, whose approximation, of course, influences the subsequent dynamics. In spite of this, it is customary in literature to adopt simplified models for the static profile, which lead to dynamic models still tractable by analytical methods. A classical example is given by the Irvine's theory of the shallow horizontal cable, in which the static loads are limited to the easiest case of self-weight, and where the exact catenary shape is approximated by a parabola, whose curvature and tension are taken constant.

On the other hand, there exist a general solution for sagged cables loaded by general forces, as described in [26]. However, this procedure leads to an explicit solution in integral form, whose primitive is generally unknown even for simple load laws. Therefore, it is not a closed-form solution, if we give to this locution the generally accepted meaning of combination of elementary functions. Thus, the exact solution, although explicit, remains substantially numerical. For horizontally suspended cables, in the specific case of uniformly distributed loads, combined to concentrated loads as well, a closed form solution is evaluated in [27], where distinction is given on small or large concentrated applied forces.

It appears, therefore, desirable, to implement a procedure able to give approximated solutions for inclined ca-

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bles, in closed-form form at least for simple load distributions. Moreover, such a solution should be susceptible to be improved up to the desired accuracy requested by the analyst. Such a goal can, indeed, be reached, if we assume that the cable is shallow, *i.e.* it is sufficiently close to the chord which connects the supports. In such a case, a perturbation analysis, aimed at extrapolating the solution relevant to the taut string, is viable. Moreover, by accounting for elasticity, suspended as well as taut cables can be analyzed, whose natural length is larger or smaller than the chord, respectively. In a previous paper [15] we tackled this problem by limiting ourselves to vertical loads, and adopted the Cartesian representation. Here we consider more general loads, while adopting the (more natural) parametric representation.

The paper is organized as follows. In Section 2 the equations governing the static problem are formulated. Their general solution is sketched in the Appendix A. In Section 3 a perturbation procedure is implemented; there, a discussion is carried out on the mechanical behavior of the cable at the various orders of the perturbation scheme. In Section 4 some sample case studies are worked out to illustrate the method; the asymptotic results are discussed for comparison with the exact (numerical) results.

2 Model

The cable is modeled as a 1D Cauchy continuum embedded in a 3D space (Figure 1). Let $\mathbf{x}(s)$ be the position of the material point P of (unstretched) abscissa $s \in [0, l]$ occupied at time t. We introduce the unit extension:

$$\varepsilon := \left\| \frac{d\mathbf{x}}{ds} \right\| - 1 \tag{1}$$

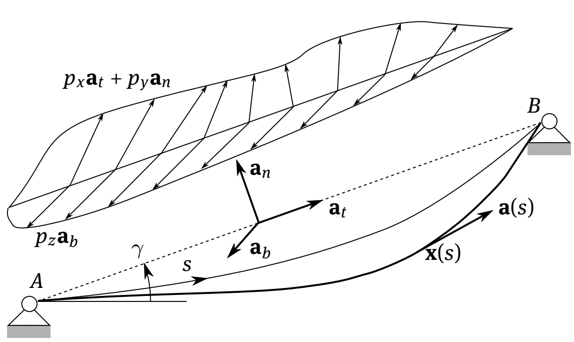

Figure 1: Shallow cable under general 3D-forces, referred to a basis intrinsic to the chord. Dashed line: chord; thin line: cable with no loads; thick line: equilibrium position of the cable with loads.

as a measure of strain, and the tension:

$$\mathbf{t} := T(s)\mathbf{a}(s) \tag{2}$$

as a measure of stress. Here T is the modulus of the tension, and $\mathbf{a} := \frac{1}{1+\varepsilon}\frac{d\mathbf{x}}{ds}$ the unit vector tangent at s to the current profile of the cable. When dealing with cables, T is greater than 0, being such a structure able to sustain tensile forces only. Anyway it might have some interest to consider possible cases of $T < 0$, which are of course not directly related to cable admissible solutions, but are relevant to the evaluation of the shape of the funicular of the applied loads ([28]). Equilibrium requires that:

$$\frac{d\mathbf{t}}{ds} + \mathbf{p} = \mathbf{0} \tag{3}$$

where $\mathbf{p}(s)$ are static loads, and where inertial effects have been neglected.

A linear elastic constitutive law is adopted:

$$T = EA\varepsilon \tag{4}$$

where EA is the axial stiffness, constant along the cable.

The governing equations are conveniently projected onto an orthonormal $(\mathbf{a}_t, \mathbf{a}_n, \mathbf{a}_b)$ basis intrinsic to the chord $\overline{AB}$, which connects the end-points and is inclined of an angle γ with respect to the horizontal plane. This choice, in conjunction with the fact that no privileged directions of the loads exist (contrary to what occurs for gravitational forces), makes unimportant the distinction between horizontal and inclined cables, which is usually made in the technical literature. The unit vectors of the triad are taken as follows: $\mathbf{a}_t$ is directed along the chord; $\mathbf{a}_n$ is a normal direction to the chord, arbitrarily chosen (*e.g.* spanning with $\mathbf{a}_t$ the vertical plane); $\mathbf{a}_b$ is the binormal, completing the basis. A system of coordinates is taken with origin at the left end-point A, so that $\mathbf{x} = x\mathbf{a}_t + y\mathbf{a}_n + z\mathbf{a}_b$; consistently $\mathbf{p} = p_x\mathbf{a}_t + p_y\mathbf{a}_n + p_z\mathbf{a}_b$. When the governing equations are expressed in this basis, they read:

$$\begin{aligned}
&\left(\frac{T}{1+\frac{T}{EA}}x'\right)' + p_x(s) = 0 \\
&\left(\frac{T}{1+\frac{T}{EA}}y'\right)' + p_y(s) = 0 \\
&\left(\frac{T}{1+\frac{T}{EA}}z'\right)' + p_z(s) = 0 \\
&\frac{T}{EA} = \sqrt{x'^2 + y'^2 + z'^2} - 1
\end{aligned} \tag{5}$$

where a dash denotes s-differentiation, and where the constitutive law has been used. The equations above must be equipped with boundary conditions, expressing fixed supports:

$$x(0)=0,\quad y(0)=0,\quad z(0)=0 \qquad (6)$$
$$x(l)=l_0,\quad y(l)=0,\quad z(l)=0$$

where l_0 is the length of the chord. Usually $l > l_0$ (suspended cable); however, we will account also for the case in which $l < l_0$ (taut cable).

3 Perturbation analysis

Since the cable, by hypothesis, is shallow, there exist an intrinsic perturbation parameter ϵ, which is of the order of the sag-to-chord ratio, when the cable is deployed according to a continuous planar curve between the supports. More effectively, we can refer to the *length-to-chord relative difference* $\Delta \gtrless 0$, which is defined by:

$$l = l_0(1+\Delta) \qquad (7)$$

in which $\Delta = O\left(\epsilon^2\right)$. In these cables, for equilibrium reasons, it is well known that the tension T is large with respect to the total load. To avoid that the tension diverges to infinity when $\epsilon \to 0$, we rescale the loads as $p_\alpha \to \epsilon p_\alpha$, $\alpha = x, y, z$, so that $T = O(1)$. Finally, we account for small strains of the same order of Δ, by performing the rescaling $\frac{T}{EA} \to \epsilon^2 \frac{T}{EA}$.

The rescaled field equations (5) therefore read:

$$\left(\frac{T}{1+\epsilon^2\frac{T}{EA}}x'\right)' + \epsilon p_x(s) = 0 \qquad (8)$$
$$\left(\frac{T}{1+\epsilon^2\frac{T}{EA}}y'\right)' + \epsilon p_y(s) = 0$$
$$\left(\frac{T}{1+\epsilon^2\frac{T}{EA}}z'\right)' + \epsilon p_z(s) = 0$$
$$\epsilon^2\frac{T}{EA} = \sqrt{x'^2+y'^2+z'^2} - 1$$

The boundary conditions (6), with Eq. (7) and $\Delta \to \epsilon^2\Delta$, become:

$$x(0)=0,\qquad y(0)=0,\quad z(0)=0 \qquad (9)$$
$$x(l)=\frac{l}{1+\epsilon^2\Delta},\quad y(l)=0,\quad z(l)=0$$

Perturbation equations

We note that, when $\epsilon \to 0$, the rescaled equilibrium equations (8-a,b,c) become homogeneous and admit the solution:

$$x_0 = s,\qquad y_0 = 0,\qquad z_0 = 0 \qquad (10)$$

with T_0 an arbitrary constant. This (generating) solution represents a rectilinear cable, lying along the chord and taut by an arbitrary tension T_0. Note, moreover, that, in the limit process for $\epsilon \to 0$, the right support moves along the chord of a small quantity $|l - l_0| \simeq \epsilon^2 l_0\Delta$, to the left if $l > l_0$ (suspended cable), to the right if $l < l_0$ (taut cable).

We will pursue the solution to the non-homogeneous problem by perturbing that of the unloaded taut string. To this end, we expand all the unknowns in series of ϵ, as:

$$x = s + \epsilon x_1(s) + \epsilon^2 x_2(s) + \epsilon^3 x_3(s) + \epsilon^4 x_4(s) + \ldots \qquad (11)$$
$$y = \epsilon y_1(s) + \epsilon^2 y_2(s) + \epsilon^3 y_3(s) + \epsilon^4 y_4(s + \ldots$$
$$z = \epsilon z_1(s) + \epsilon^2 z_2(s) + \epsilon^3 z_3(s) + \epsilon^4 z_4(s) + \ldots$$
$$T = T_0 + \epsilon T_1(s) + \epsilon^2 T_2(s) + \epsilon^3 T_3(s) + \epsilon^4 T_4(s) + \ldots$$

and derive the following perturbation equations with relevant boundary conditions:

Order ϵ:

$$T_1' = -p_x - T_0 x_1'' \qquad (12)$$
$$T_0 y_1'' = -p_y$$
$$T_0 z_1'' = -p_z$$
$$x_1' = 0$$

$$x_1(0)=0,\quad y_1(0)=0,\quad z_1(0)=0 \qquad (13)$$
$$x_1(l)=0,\quad y_1(l)=0,\quad z_1(l)=0$$

Order ϵ^2:

$$T_2' = -\left(T_1 x_1'\right)' - T_0 x_2'' \qquad (14)$$
$$T_0 y_2'' = -\left(T_1 y_1'\right)'$$
$$T_0 z_2'' = -\left(T_1 z_1'\right)'$$
$$x_2' = -\frac{1}{2}\left(y_1'^2 + z_1'^2\right) + \frac{T_0}{EA}$$

$$x_2(0)=0,\qquad y_2(0)=0,\quad z_2(0)=0 \qquad (15)$$
$$x_2(l)=-l\Delta,\quad y_2(l)=0,\quad z_2(l)=0$$

Order ϵ^3:

$$T_3' = -\left(T_1 x_2' + T_2 x_1'\right)' - T_0 x_3'' + 2\frac{T_0}{EA}T_1' + \frac{T_0^2}{EA}x_1'' \qquad (16)$$
$$T_0 y_3'' = -\left(T_1 y_2' + T_2 y_1'\right)' + \frac{T_0^2}{EA}y_1''$$
$$T_0 z_3'' = -\left(T_1 z_2' + T_2 z_1'\right)' + \frac{T_0^2}{EA}z_1''$$
$$x_3' = \frac{1}{2}x_1'\left(y_1'^2 + z_1'^2\right) - y_1' y_2' - z_1' z_2' + \frac{T_1}{EA}$$

$$x_3(0)=0,\quad y_3(0)=0,\quad z_3(0)=0 \qquad (17)$$
$$x_3(l)=0,\quad y_3(l)=0,\quad z_3(l)=0$$

Order ϵ^4:

$$\begin{aligned}
T_4' &= -\frac{T_1}{EA}\left(EAx_3''-2T_0x_1''-2T_1'\right) \qquad (18)\\
&+T_2'\left(x_2'-\frac{2T_0}{EA}\right)+T_0x_4''+T_1'x_3'+T_2x_2''+(T_3x_1')'\\
&-\frac{T_0}{EA}\left(T_0x_2''+2T_1'x_1'\right)\\
T_0y_4'' &= \frac{T_0^2y_2''}{EA}+\frac{2T_0T_1'y_1'}{EA}+\frac{2T_0T_1y_1''}{EA}\\
&-(T_1y_3'+T_2y_2'+T_3y_1')'\\
T_0z_4'' &= \frac{T_0^2z_2''}{EA}+\frac{2T_0T_1'z_1'}{EA}+\frac{2T_0T_1z_1''}{EA}\\
&-(T_1z_3'+T_2'z_2+T_3z_1')'\\
x_4' &= \frac{T_2}{EA}+x_1'y_1'y_2'-\frac{1}{2}x_1'^2y_1'^2+x_1'z_1'z_2'-\frac{1}{2}x_1'^2z_1'^2\\
&+\frac{1}{2}x_2'y_1'^2+\frac{1}{2}x_2'z_1'^2-y_1'y_3'+\frac{1}{4}y_1'^2z_1'^2+\frac{1}{8}y_1'^4\\
&-\frac{1}{2}y_2'^2-z_1'z_3'+\frac{1}{8}z_1'^4-\frac{1}{2}z_2'^2
\end{aligned}$$

$$x_4(0)=0,\quad y_4(0)=0,\quad z_4(0)=0 \qquad (19)$$
$$x_4(l)=l\Delta^2,\quad y_4(l)=0,\quad z_4(l)=0$$

in which we preferred to rearrange Eqs. (12-a), (14-a), (16-a), (18-a) to make evident their meaning, as discussed soon. The transverse equilibrium equations describe the equilibrium of a slightly deflected string (as in the linearized theory), taut by the tension T_0, under the action of known transverse loads (determined at the previous steps). The longitudinal equilibrium equations describe the equilibrium, in terms of stress only, of an undeflected truss under distributed loads, which depend on quantities of the same order, and therefore still unknown. The equilibrium is combined to elasto-geometric conditions which express, at the different orders: (a) linear inextensibility; (b) elastic second-order extension equal to a constant, $\frac{T_0}{EA}$; (c) elastic third-order extension equal $\frac{T_1}{EA}$ (and $\frac{T_2}{EA}$ at the fourth order), variable in the domain. As a further remark we observe that the influence of the strain on equilibrium is weak, since it appears firstly only in the equations for x_3. In contrast, elasticity appears in the elasto-geometric equation already at the ϵ^2-order, where, will we see, the leading-order arbitrariness, related to the value of T_0, is resolved.

First-order solution

Let us start by expressing the first-order transverse displacements from the chord. They assume the form:

$$y_1=\frac{1}{T_0}f_y(s) \qquad (20)$$
$$z_1=\frac{1}{T_0}f_z(s)$$

where T_0 is an arbitrary constant and $f_\alpha(s)$, $\alpha=y,z$, are solutions to the following linear boundary value problems:

$$f_\alpha''=-p_\alpha(s) \qquad (21)$$
$$f_\alpha(0)=0,\qquad f_\alpha(l)=0$$

expressing the linear response to a transverse loads of a string taut by a unitary tension.

To complete the first-order solution, we notice that, since, from kinematics, $x_1'=0$, it is $x_1=$ const; however, due to the boundary conditions, $x_1=0$. Therefore, due to the lowest-order inextensibility approximation, the string does not undergo longitudinal displacements.

Concerning equilibrium, the along-chord equation (12-a) governs the statics of a truss under distributed forces, namely:

$$T_1'=-p_x(s) \qquad (22)$$

whose solution is:

$$T_1=\tau_1-\int_0^s p_x(\bar{s})d\bar{s} \qquad (23)$$

where $\tau_1 := T_1(0)$ is a hyperstatic unknown which cannot be determined at this order, since no elasticity nor curvature of the cable appears.

Second-order solution

By going to the ϵ^2-order, we notice that, with y_1, z_1 now determined, we can use first the geometric condition (14-d), to derive, by integration, the along-chord second-order displacements:

$$x_2=-\frac{1}{2T_0^2}\int_0^s\left(f_y'^2(\bar{s})+f_y'^2(\bar{s})\right)d\bar{s}+\frac{T_0s}{EA} \qquad (24)$$

where $x_2(0)=0$ has been enforced. By requiring that $x_2(l)=-l\Delta$ (from Eqs. (15)), an algebraic condition for the unknown T_0 follows:

$$\frac{T_0^3l}{EA}+l\Delta T_0^2-\frac{1}{2}\int_0^l\left(f_y'^2(\bar{s})+f_z'^2(\bar{s})\right)d\bar{s}=0 \qquad (25)$$

This is a cubic equation in T_0, admitting just one positive root, according to the Descartes rule of sign, irrespectively of the sign of Δ (as it happens for the self-weight case, see [9]); moreover, applying the corollary of the Descartes rule of sign on the negative roots, it turns out that, if $\Delta < 0$ (taut cable), the remaining two roots are complex conjugate while, if $\Delta > 0$ (suspended cable), they can be either real negative or complex conjugate.

If we are interested to a higher-order solution, first we have to determine y_2, z_2. With the previous results, using Eq. (12-b,c), (22) and (23), Eqs. (14-b,c) become:

$$T_0 y_2'' = \frac{1}{T_0}\left[p_x(s)f_y'(s) + p_y(s)\left(\tau_1 - \int_0^s p_x(\bar{s})d\bar{s}\right)\right] \quad (26)$$
$$T_0 z_2'' = \frac{1}{T_0}\left[p_x(s)f_z'(s) + p_z(s)\left(\tau_1 - \int_0^s p_x(\bar{s})d\bar{s}\right)\right]$$

whose solution reads:

$$y_2 = \frac{1}{T_0^2}\left[g_y(s) - \tau_1 f_y(s)\right] \quad (27)$$
$$z_2 = \frac{1}{T_0^2}\left[g_z(s) - \tau_1 f_z(s)\right]$$

where f_y, f_y have already been introduced in Eqs. (21) and g_y, g_z are solutions to:

$$g_\alpha'' = p_x(s)f_\alpha'(s) - p_\alpha(s)\int_0^s p_x(\bar{s})d\bar{s} \quad (28)$$
$$g_\alpha(0) = 0, \qquad g_\alpha(l) = 0$$

To complete the solution at the second order, we need to determine T_2 from Eq. (14-a), not used yet. By using the expression for x_2 and integrating, we obtain:

$$T_2 = \tau_2 + \frac{1}{T_0}\int_0^s \left(f_y'(\bar{s})f_y''(\bar{s}) + f_z'(\bar{s})f_z''(\bar{s})\right) d\bar{s} \quad (29)$$

where $\tau_2 := T_2(0)$ is a new arbitrary constant.

Third-order solution

Now we are in the same situation of the previous step, namely, with y_2, z_2 determined to within a constant stress (T_0 at the ϵ-order, τ_1 at the ϵ^2-order). Thus, by going to the ϵ^3-order, we need to integrate the elasto-geometric condition (16-d), to get:

$$x_3 = -\frac{1}{T_0^3}\int_0^s \left(f_y'(\bar{s})g_y'(\bar{s}) + f_z'(\bar{s})g_z'(\bar{s})\right) d\bar{s} \quad (30)$$
$$- \frac{\tau_1}{T_0^3}\int_0^s \left(f_y'^2(\bar{s}) + f_z'^2(\bar{s})\right) d\bar{s} + \frac{\tau_1}{EA}s - \int_0^s d\tilde{s}\int_0^{\tilde{s}} \frac{p_x(\bar{s})}{EA}d\bar{s}$$

in which $x_3(0) = 0$ has been used. By requiring that $x_3(l) = 0$, a linear algebraic equation for the unknown τ_1 is derived:

$$\tau_1\left[\frac{l}{EA} - \frac{1}{T_0^3}\int_0^l \left(f_y'^2(s) + f_z'^2(s)\right) ds\right] \quad (31)$$
$$- \frac{1}{T_0^3}\int_0^l \left(f_y'(s)g_y'(s) + f_z'(s)g_z'(s)\right) ds - \int_0^l d\tilde{s}\int_0^{\tilde{s}} \frac{p_x(\bar{s})}{EA}d\bar{s} = 0$$

Note that the value of τ_1 which resolves the hyperstatic nature of the problem (22), depends not only on the axial elasticity, but also on the geometric nonlinearities, which account for the curvature of the cable. Thus, even in the inextensible case, the hyperstatic unknown is determined.

Then, by Eqs. (16-b,c), y_3, z_3 are evaluated to within τ_2 (their expression is not shown here because they are too large), and integrating Eq. (16-d) provides the expression for $T_3(s)$ to within a new constant τ_3, undetermined so far.

Fourth-order solution

At the ϵ^4-order, used as final perturbation step, just the elasto-geometric condition Eq. (18-a) is used to supply x_4, whose (non homogeneous) boundary condition in l, (19), determines τ_2. The relevant expressions are not written here for the sake of brevity. We chose to conclude the perturbation procedure at this step, without evaluation of y_4 and z_4.

4 Case study

As a case study, a harmonically distributed force is applied to the cable in the vertical direction, namely $P\sin\left(\frac{\pi s}{l}\right)$, giving rise to the two projections $p_x(s), p_y(s)$ in the tangent and normal to the chord directions; furthermore, an uniformly distributed force is applied in the binormal direction. In particular, the three force components are:

$$p_x(s) = P_x \sin\left(\frac{\pi s}{l}\right) \quad (32)$$
$$p_y(s) = P_y \sin\left(\frac{\pi s}{l}\right)$$
$$p_z(s) = P_z$$

where $P_x = P\sin\gamma$, $P_y = P\cos\gamma$.

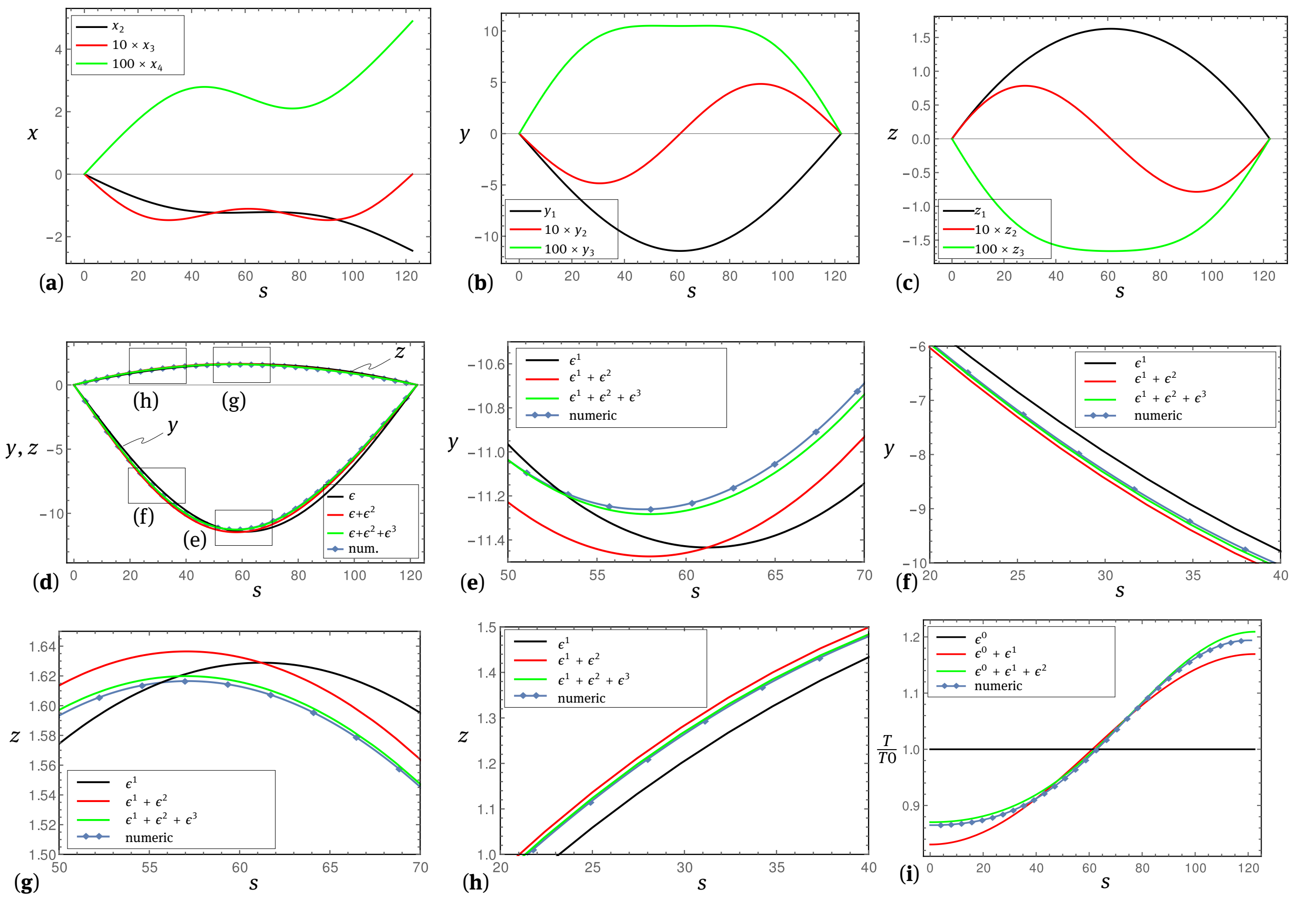

Figure 2: Case study, positive T_0: **(a)** solution $x(s)$; **(b)** solution $y(s)$; **(c)** solution $z(s)$; **(d)** reconstituted y and z, complete view and zoom windows for subfigures (e), (f), (g), (h); **(e)** reconstituted y close to the mid-span; **(f)** reconstituted y close to quarter-span; **(g)** reconstituted z close to the mid-span; **(h)** reconstituted z close to quarter-span; **(i)** force ratio $T(s)/T_0$; $[s, y, z] = m$.

4.1 Perturbation solution

From Eqs. (20), the first order solution is

$$y_1 = \frac{P_y l^2}{T_0 \pi^2} \sin\left(\frac{\pi s}{l}\right) \qquad (33)$$

$$z_1 = \frac{P_z s}{2T_0}(s - l)$$

and from Eq. (23) one has:

$$T_1 = \tau_1 - \frac{P_x l}{\pi}\left(1 - \cos\left(\frac{\pi s}{l}\right)\right) \qquad (34)$$

Then, from Eq. (24), the expression for x_2 is:

$$x_2 = \frac{T_0}{EA}s - \frac{P_y^2 l^2}{8\pi^2 T_0^2}\left(2s + \frac{l}{\pi}\sin\left(\frac{2\pi s}{l}\right)\right) - \frac{P_z^2}{T_0^2}\left(\frac{s^3}{6} - \frac{l s^2}{4} + \frac{l^2 s}{8}\right) \qquad (35)$$

Consequently, Eq. (25) becomes:

$$\frac{T_0^3 l}{EA} + l\Delta T_0^2 - \frac{l^3}{4}\left(\frac{P_y^2}{\pi^2} + \frac{P_z^2}{8}\right) = 0 \qquad (36)$$

Equation (36) provides only one positive root for T_0. Following the perturbation steps, the solutions for $y_2(s)$, $z_2(s)$ are consequently obtained from Eqs. (27):

$$y_2 = -\frac{l^2 P_y}{2\pi^3 T_0^2}\sin\left(\frac{\pi s}{l}\right)\left[l P_x\left(\cos\left(\frac{\pi s}{l}\right) - 2\right) + 2\pi\tau_1\right] \qquad (37)$$

$$z_2 = \frac{P_z}{2T_0^2}\left[\frac{l P_x}{\pi^3}\left(2l^2\cos\left(\frac{\pi s}{l}\right) - 2l^2 + ls(\pi^2 + 4) - \pi l(l - 2s)\sin\left(\frac{\pi s}{l}\right) - \pi^2 s^2\right) - s\tau_1(l - s)\right]$$

and Eq. (29) provides the following expression for T_2:

$$T_2 = \tau_2 - \frac{P_y^2 l^2}{2\pi^2 T_0}\sin\left(\frac{\pi s}{l}\right)^2 + \frac{P_z^2}{2T_0}\left(s^2 - ls\right) \qquad (38)$$

The expression for x_3 given by Eq. (30) is not shown here for the sake of brevity, but the consequent Eq. (31) becomes:

$$\left[\frac{l}{EA} + \frac{l^3}{T_0^3}\left(\frac{P_y^2}{2\pi^2} + \frac{P_z^2}{12}\right)\right]\left(\tau_1 - \frac{P_x l}{\pi}\right) = 0 \qquad (39)$$

which provides $\tau_1 = P_x l/\pi$. The further expressions for y_3, z_3, x_4 and τ_2 are omitted here because very large.

4.2 Numerical results

For the numerical values $l_0 = 120$ m, $\gamma = \pi/6$ rad, $\varDelta = 2\%$ (suspended cable), $EA = 2.9704 \times 10^7$ N, $P = -518$ N/m, and assuming $P_z = \frac{P}{10}$, the positive root for Eq. (36) is $T_0 = 59550.8$ N, and the values $\tau_1 = -10090.3$ N and $\tau_2 = 2368.7$ N are obtained. At the various perturbation orders, the corresponding plots of x_{i+1}, y_i, z_i, $i = 1, 2, 3$, are shown in Figure 2-a,b,c. It can be observed from Figure 2-b,c that, for the specific loads of the example, the contributions at odd orders are symmetric while the ones at the even order are anti-symmetric. In Figure 2-d, the reconstituted solutions $y(s)$ and $z(s)$ at various orders are shown, compared to the numerical solution (evaluated by Eqs. (A4) described in the Appendix A); zoomed plots close to the mid-span and quarter-span for $y(s)$ and $z(s)$ are shown in Figure 2-e,f,g,h, where it can be observed how the solution up to the highest order (in green) always better fits the numeric one (in blue), giving rise to a very good agreement. Moreover, the reconstituted plot of the force ratio $T(s)/T_0$ is shown in Figure 2-i: it appears that the highest reconstituted order (in green) is essential to fit the numerical solution (in blue), so that the significant modification of the modulus of the tension along the span (up to 36% of difference from section A to B) are caught. Existence of other possible solutions is discussed in Appendix B.

For the suspended cable ($\varDelta > 0$), the effect of the modification of $\varDelta$ on the modulus of tension $T(s)$ and tension ratio $T(s)/T_0$ is shown in Figure 3-a,b, respectively, where it can be observed how the growth of $\varDelta$ causes a general reduction of the average values of $T(s)$, asymptotically tending to a minimum (Figure 3-a), and a concurrent increasing in the difference between the tension at the two supports (Figure 3-b); this means that the first order elastic strain decreases its contribution as the cable is more slack, while the higher order contributions become more important. For the taut cable ($\varDelta < 0$), Figure 4 shows the effect of the modification of $\varDelta$ on the modulus of tension $T(s)$, where an almost proportional increasing of the values of $T(s)$ occurs as the cable becomes more taut, indicating that, when $\varDelta < 0$, the most relevant contribution to the modulus of tension is due to the elastic strain directly induced by $\varDelta$.

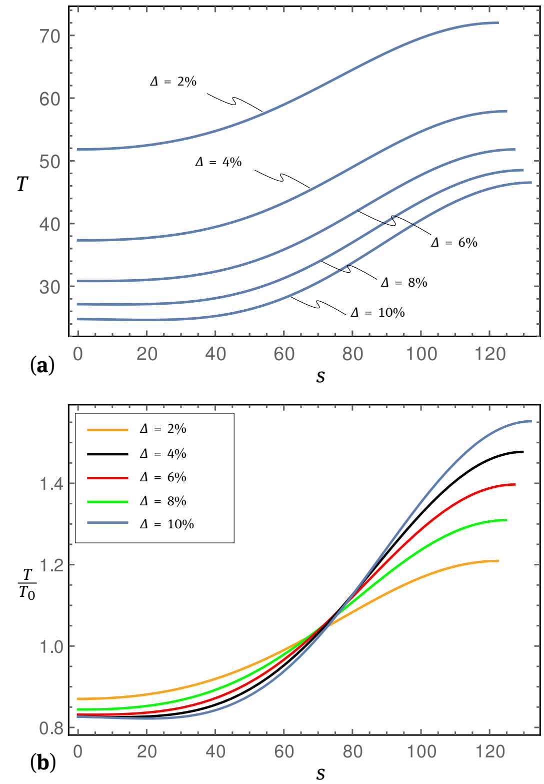

Figure 3: Suspended cable: **(a)** Modulus of the tension T and **(b)** tension ratio $T(s)/T_0$, for different values of $\varDelta$; $[s] = m$, $[T] = kN$.

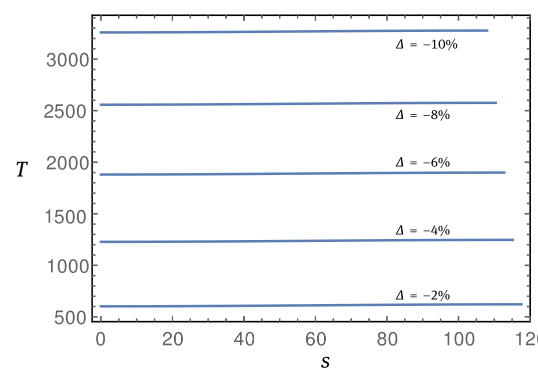

Figure 4: Taut cable: modulus of the tension T for different values of $\varDelta$; $[s] = m$, $[T] = kN$.

5 Conclusions

A perturbation method for solving the static problem of an inclined, shallow, elastic cable under general 3D-loads has been implemented. The method is not straightforward, since it requires to handle some arbitrariness arising at each level. The taut string lying along the chord is taken as generator system, from which the solution of the cable is then extrapolated. Perturbation equations are found to possess the following mechanical meaning:

1. The transverse equilibrium is governed by a linear taut string model, on which known distributed transverse forces, determined at the previous perturbation steps, act. The tension, however, is initially unknown and arbitrary.
2. The longitudinal equilibrium is governed by a truss model, on which, still unknown distributed forces act. Solution depends on hyperstatic unknowns, which are determined ahead in the algorithm.
3. The compatibility condition expresses inextensibility at the lowest order, and then accounts for elastic strains, first constant along the cable, then variable.

The perturbation method proposed has been applied to a sample problem, relevant to a combination of uniform and sinusoidal loads. The cases of both suspended and taut cable have been considered, and the effect of elasticity commented. Results (just for the suspended case) have been compared to an exact (numerical) solution available in the literature, obtaining good agreement when, particularly for a reliable evaluation of the module of the tension, the highest perturbation order is considered. Modification of the parameter Δ, describing the ratio between the initial length of the cable and the chord length, indicates that, for suspended cables, the elastic contribution of the tension decreases as the cable is more slack, whereas, in cases of taut cables, the tautness defines almost linearly the modulus of tension. Moreover, the existence of other multiple solutions is highlighted and discussed; they are not coherent with the physics of the cables, due to the negative value of the module of tension, but can be very useful in the process of design of arches.

In closing, the perturbation procedure implemented here is believed to be efficient, not only from an algorithmic point of view, since it leads to closed-form solutions, but even for throwing light on the mechanical behavior of the cable, and for giving a consistent asymptotic ground to the approximations usually introduced in literature.

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A The exact solution

We will resume here the exact solution to Eqs. (5), (6), following [26], but ignoring the transverse effect on the cross-section area, accounted for there.

First, we define the primitive functions of the loads, as:

$$\mathcal{P}_\alpha(s) := \int_0^s p_\alpha(\bar{s})d\bar{s}, \qquad \alpha = x, y, z \tag{A1}$$

Thus, integration of the equilibrium equations (5-a,b,c) supplies:

$$\frac{T}{1+\frac{T}{EA}}x' = C_x - \mathcal{P}_x(s) \tag{A2}$$
$$\frac{T}{1+\frac{T}{EA}}y' = C_y - \mathcal{P}_y(s)$$
$$\frac{T}{1+\frac{T}{EA}}z' = C_z - \mathcal{P}_z(s)$$

where C_α are arbitrary constants. By squaring and summing these equations and using the elasto-geometric relation Eq. (5-d), we find the tension (to within the arbitrary constants):

$$T = \sqrt{\left(C_x - \mathcal{P}_x(s)\right)^2 + \left(C_y - \mathcal{P}_y(s)\right)^2 + \left(C_z - \mathcal{P}_z(s)\right)^2} \tag{A3}$$

With T now known, the equations (A2) can be integrated, to furnish:

$$x = \int_0^s \left(\frac{1}{T(\bar{s})} + \frac{1}{EA}\right)\left(C_x - \mathcal{P}_x(\bar{s})\right) d\bar{s} \tag{A4}$$
$$y = \int_0^s \left(\frac{1}{T(\bar{s})} + \frac{1}{EA}\right)\left(C_y - \mathcal{P}_y(\bar{s})\right) d\bar{s}$$
$$z = \int_0^s \left(\frac{1}{T(\bar{s})} + \frac{1}{EA}\right)\left(C_z - \mathcal{P}_z(\bar{s})\right) d\bar{s}$$

where the boundary conditions at the left end have been taken into account. By requiring that the boundary conditions at the right end are also satisfied, a system of three algebraic equations for the constants C_α, coupled by the way of Eq. (A3), is finally drawn:

$$l_0 = \int_0^l \left(\frac{1}{T(\bar{s})} + \frac{1}{EA}\right)\left(C_x - \mathcal{P}_x(\bar{s})\right) d\bar{s} \tag{A5}$$
$$0 = \int_0^l \left(\frac{1}{T(\bar{s})} + \frac{1}{EA}\right)\left(C_y - \mathcal{P}_y(\bar{s})\right) d\bar{s}$$
$$0 = \int_0^l \left(\frac{1}{T(\bar{s})} + \frac{1}{EA}\right)\left(C_z - \mathcal{P}_z(\bar{s})\right) d\bar{s}$$

whose evaluation completes the solution to the problem.

It should be noticed that, even when the loads assume simple expressions (*e.g.* polynomial or harmonic), the integrals in Eqs. (A4), and therefore those in Eqs. (A5), cannot be computed in terms of elementary functions. Therefore the solution remains essentially numerical.

B The funicular solutions

With the same numerical values used in the case study ($\Delta = 2\%$), Eq. (36) provides two additional real roots, which are negative: $T_0 = -66269.7$ N and $T_0 = -587360.0$ N. As previously discussed, two negative roots may exist just in cases where $\Delta > 0$ (otherwise they are complex conjugate) and, more importantly, they are not consistent with the physics of the cable. However they are interesting to be analyzed for the evaluation of possible funicular solutions, relevant for instance to the problem of the design of arches. Considering as an example the first negative root for T_0, the corresponding values for $\tau_1 = -10090.3$ N and $\tau_2 = -2376.31$ N are obtained, and the related results are shown in Figure B1, still in very good agreement with the numerical ones. Note that, in this case, besides the load in the normal direction $p_y(s)$ is non-positive, the reconstituted function y is non-negative, giving rise to the shape of an arch (see *e.g.* Figure B1-d). Moreover, the solution is not the opposite of the previously obtained one (Figure 2), due to the presence of the tangent load $p_x(s)$. The third possible solution, associated to the root $T_0 = -587360.0$ N and not shown here, describes an arch with a significantly reduced sag-to-chord ratio, due to very large in modulus, negative, strain.

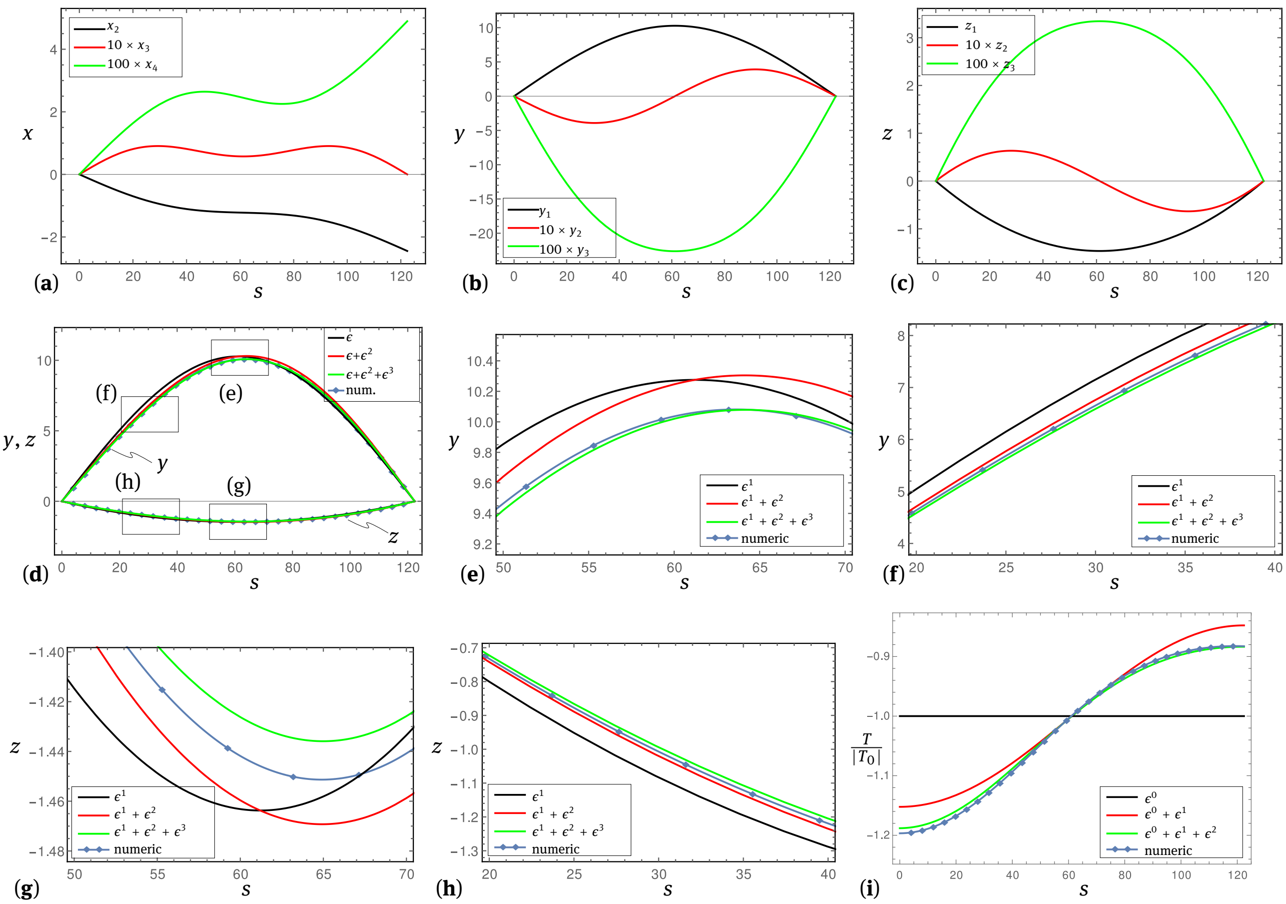

Figure B1: Case study, negative T_0: **(a)** solution $x(s)$; **(b)** solution $y(s)$; **(c)** solution $z(s)$; **(d)** reconstituted y and z, complete view and zoom windows for subfigures (e), (f), (g), (h); **(e)** reconstituted y close to the mid-span; **(f)** reconstituted y close to quarter-span; **(g)** reconstituted z close to the mid-span; **(h)** reconstituted z close to quarter-span; **(i)** force ratio $T(s)/|T_0|$; $[s, y, z] = m$.