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Experimental Study on Nonlinear Vibrations of Fixed-Fixed Curved Beams

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Abstract: Nonlinear dynamic behavior of fixed-fixed shallow and deep curved beams is studied experimentally using non-contact type of electromagnetic shaker and acceleration measurements. The frequency response obtained from acceleration measurements is found to be in fairly good agreement with the computational response. The travelling wave phenomenon along with participation of higher harmonics and softening nonlinearity are observed. The experimental results on the internal resonance of curved beams due to direct excitation of anti-symmetric mode are reported for the first time. The deep curved beam depicts chaotic response at higher excitation amplitude.

Keywords: Nonlinear Vibration; Experimental; Curved Beam, Traveling Wave; Internal Resonance

1 Introduction

The vibration problem of thin structural members with amplitude of the order of their thickness cannot be adequately addressed by linear models which at best can be treated as first approximation to the actual solution. The solutions based on the geometrically nonlinear models predict a rich and varied response which includes the amplitude dependence of the resonant frequencies, amplitude and temporal dependence of deformation shapes, multi-valued region in the nonlinear frequency response curve and jump phenomenon, internal and parametric resonance, sub- or super-harmonics, quasi periodic response, etc.

Curved structures are more efficient in carrying and transferring load to the supports as compared to the straight counterparts due to the combined action of bending and stretching. For a wide range of geometrical parameters of shallow curved beams, the linear free vibration frequency of the first symmetric mode is approximately twice of the first anti-symmetric mode. The finite amplitude forced response of such beams exhibits internal resonance between first symmetric and first anti-symmetric modes. A number of analytical/numerical studies have been carried out on the nonlinear dynamic behavior of straight/curved beams undergoing large amplitude vibrations using commonly available methods of harmonic balance, method of averaging, Galerkin's method, perturbation method and finite element method.

Ribeiro and Carneiro [1] and Ribeiro *et al.* [2] conducted experimental and numerical studies to investigate the amplitude dependence of resonant frequencies, mode shapes, and the presence of internal resonance for hinged-hinged and clamped-clamped straight beams. The interaction of first and third, first and second modes was predicted for excitation frequencies close to first mode linear resonance frequency for both the beams [1, 2] whereas the interaction of second and fourth modes for excitation close to second mode linear resonance frequency for clamped-clamped beam [2]. Huang *et al.* [3] have studied nonlinear vibration of curved beams under harmonic base excitation with quadratic and cubic nonlinearities to compare the response under symmetric and anti-symmetric mode of excitation. The Galerkin method was employed to discretize the governing equations, and incremental harmonic balance (IHB) method was used to obtain the steady state response. Period doubling response was predicted for symmetric mode excitation and quasi-periodic one for anti symmetric mode excitation. Nonlinear behaviour in the symmetric mode of excitation near the first natural frequency was reported to be of softening type and that near the third natural frequency was reported to be of hardening type. Ibrahim *et al.* [4] investigated the large amplitude response of curved beams under periodic excitation using a 3-noded beam element based on higher order shear deformation theory. For solving second order differential equations, a modified shooting technique coupled with Newmark time marching was developed. The anti-symmetric and symmetric modes participate with odd and even harmonics of half of the forcing

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frequency, respectively, in the coupled response due to 1:2 internal resonance between excited symmetric mode and first anti-symmetric mode. For the excitation in the neighbourhood of second anti-symmetric mode, the type of nonlinearity softening or hardening is governed by the participation of second symmetric mode. A shallow arch modelled as a two DOF system was analyzed by Malhotra and Namachchivaya [5, 6] employing the Kovacic and Wiggins's perturbation technique to study the dynamic behaviour near 1:1 and 1:2 internal resonances between symmetric and anti-symmetric modes. The nonlinear dynamic behaviour of a fixed-fixed buckled beam is investigated experimentally/analytically by Kreider and Nayfeh [7], Emam and Nayfeh [8] and Lacarbonara *et al.* [9] for single mode response under transverse harmonic support excitation. It was found that for low level of buckling, supercritical period doubling occurs during an amplitude sweep whereas at higher buckling level, the period doubling bifurcation changes from supercritical to subcritical. The anti-symmetric response of clamped–clamped buckled beam [10], curved panel [11] and hinged-hinged curved beam [12, 13] subjected to symmetric sinusoidal excitation is numerically and experimentally investigated. It is observed that the autoparametric response occurs due to the nonlinear modal interaction between the first symmetric and first anti-symmetric modes when the natural frequency of the former is approximately twice of the latter. The nonlinear vibration of clamped-clamped beam with initial deflection/initial axial displacement [14] and buckled beam [15] subjected to symmetric base excitation is numerically and experimentally studied. The small amplitude first mode vibrations near buckled position depicted softening and that with large amplitude depicted hardening spring behaviour.

The three-to-one internal resonance of axially loaded hinged-clamped straight beam under transverse harmonic excitation is theoretically investigated based on two-mode approximation by Chin and Nayfeh [16]. It was found that for a range of axial loads, the natural frequency of third mode is approximately three times of the first mode leading to coupled mode response due to cubic non linearity produced by mid plane stretching when first mode is directly excited. For the primary resonance of second mode, both single and coupled mode response was predicted. The 1:1 internal resonance between first and second modes, and 1:3 internal resonance between first and third modes of fixed-fixed buckled beam were analytically/numerically investigated when first mode was directly excited [17]. It was also brought out that the first mode may or may not be activated when the third mode is directly excited but when the first mode is directly excited then both the modes participate in coupled mode response.

It can be concluded from the literature review that the available experimental studies on nonlinear response of curved beams dealt only with the internal resonance between symmetric and anti-symmetric modes with the direct excitation of symmetric mode. In this paper, we report the experimental results on the internal resonance between the first anti-symmetric and first symmetric modes of fixed-fixed curved beam with the direct excitation of symmetric and anti-symmetric modes under concentrated harmonic force excitation. To the best of the author's knowledge, the experimental results on the internal resonance of curved beams due to direct excitation of anti-symmetric mode are reported for the first time. For the comparison purpose, frequency response of the beams is also obtained using curved beam finite element, Newmark's time integration and shooting technique.

2 Experimental Setup

The experimental set up developed for the study consists of curved beams of two different span angles, excitation system, accelerometers, data acquisition and processing system as shown in Fig. 1. Two different curved beams with geometrical parameters: thickness (h) = 1 mm, radius (R) = 35.5 cm, width (b) = 25 mm and span angle (θ_s) = 80°, 130^o are fabricated from aluminum strip (Young modulus = 70 GPa, Poisson's ratio = 0.33, density = 2700 kg/m^3). The ends of the beams are fixed on a wooden plate having the desired radius of curvature at the ends with the help of three rows of screws. The wooden plate is clamped to the fixture through nut and bolts and then to a rigid table with the help of c-clamp.

A low mass electromagnetic coil (12 grams) is attached to the curved beam with the help of screws at the excitation location. This coil freely moves in the air gap of a permanent magnet of the custom made electromagnetic shaker to provide non-contact type sinusoidal excitation. The APLAB multi-waveform function generator is used to provide input signal to the electromagnetic shaker through the Bruel and Kjaer power amplifier (Model No. 2718). Low weight (0.2 gram) PCB accelerometers (Model No. 353C23, sensitivity = 5 mV/g) are used to measure acceleration. The accelerometers are affixed with the help of adhesive on the surface of curved beam at three different locations A, B and C as shown in Figure 2. The distances of points A, B, C are 15 cm, 26 cm, 32.5 cm and 27.5 cm, 41.0 cm, 56.7 cm for θ_s = 80° and 130°, respectively, mea-

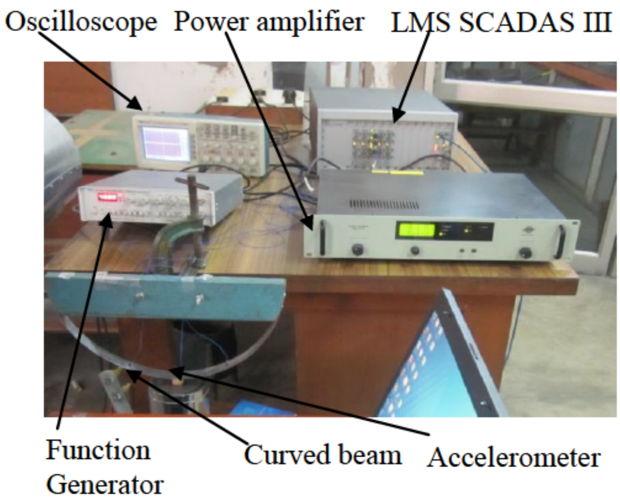


Figure 1: Experimental Setup.

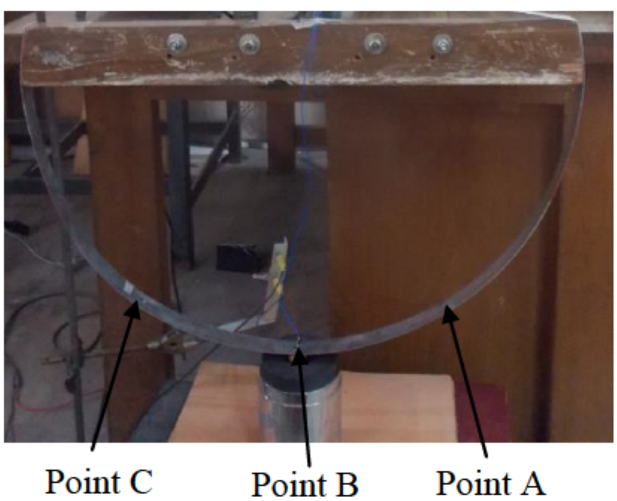


Figure 2: Photograph of curved beam with exciter.

sured along the curved length of the beam from left end. The LMS SCADAS III and LMS test lab software are used for data acquisition and signal processing. Beams are excited in the neighbourhood of resonance frequencies of the first symmetric and anti-symmetric modes. For excitation, a sinusoidal input signal generated by the function generator is fed to the electromagnetic shaker through the power amplifier. The shaker can exert a harmonic force of constant amplitude at different frequencies of interest. At each incremental value of the forcing frequency, the structure is allowed to reach steady state before the responses are measured.

3 Finite Element Modelling

For the frequency response analysis, a three-noded curved beam element with four degrees of freedom (u_o, w, $w_{,s}$, θ) at end nodes and two degrees of freedom (u_o, θ) at centre node where u_o, w are mid surface displacements and θ is the independent rotation of the normal in *sz* plane, is used as detailed in Ibrahim *et al.* [4]. The element formulation includes nonlinear strain-displacement relations with small strains and moderately large rotation. Based on the convergence study, the beam is discretized into 20 elements along length with 116 as the total number of degrees of freedom after imposing the boundary conditions. The equations of motion involving geometric nonlinearity are integrated using Newmark's time marching with shooting technique and arc-length continuation for the prediction of forced periodic response. The detailed description of the finite element formulation and solution methodology are available in Ibrahim *et al.* [4].

4 Results and Discussion

The curved beams of two different span angles are first symmetrically excited by attaching the electromagnetic coil at centre point (B) and the location of node points (A and C) are marked along the length of the beams. To excite the beams in anti-symmetric mode, the electromagnetic coil is attached at point A. The experimentally obtained natural frequencies of the beams using impact hammer are given in Table 1. The forced response of the beams under symmetric and anti-symmetric excitations is discussed next.

4.1 Symmetric Excitation of curved beam (span angle = 80°)

For the symmetrically excited beam at location B, the acceleration is measured at three different locations A, B, and C (Fig. 2). The steady state experimental frequency response curves for excitation amplitudes of 1.2, 2.4 and 3.5 volts are shown in Fig. 3. For comparison purpose, the corresponding computational frequency response curves obtained using the methodology briefly described under Finite Element Modelling are also shown in the figure. The amplitude of harmonic force ($F = F_0 \text{Cos}\omega_F t$) in computations is taken based on the comparison of experimental and computational acceleration amplitudes at point B for forcing frequency of 80 Hz and is found to be 0.354 N,

Table 1: First two linear free vibration frequencies of beams.

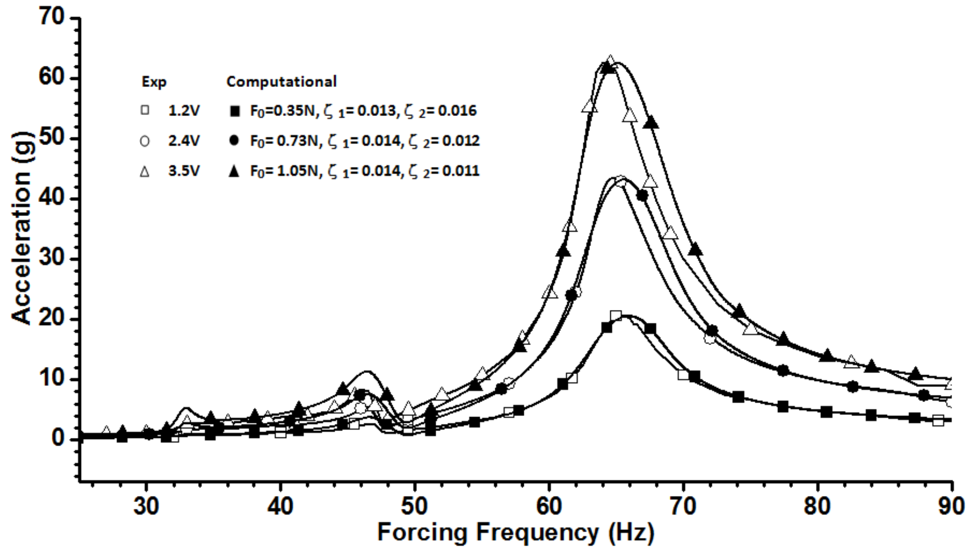
Span Angle (θ_s)	Mode	Coil at B		Coil at A	
		Experimental	Computational	Experimental	Computational
80°	First anti-symmetric	48.60 Hz	47.23 Hz	40.40 Hz	38.10 Hz
	First symmetric	65.28 Hz	64.94 Hz	89.60 Hz	89.89 Hz
130°	First anti-symmetric	14.45 Hz	16.71 Hz	13.09 Hz	14.45 Hz
	First symmetric	25.00 Hz	25.64 Hz	31.84 Hz	33.72 Hz
	Second anti-symmetric	80.27 Hz	83.38 Hz	74.41 Hz	79.68 Hz

0.73 N and 1.05 N corresponding to experimental excitation amplitudes of 1.2 V, 2.4 V and 3.5 V, respectively. The damping factors corresponding to anti-symmetric (ζ_1) and symmetric (ζ_2) modes are obtained by comparing the peak responses corresponding to different excitation amplitudes.

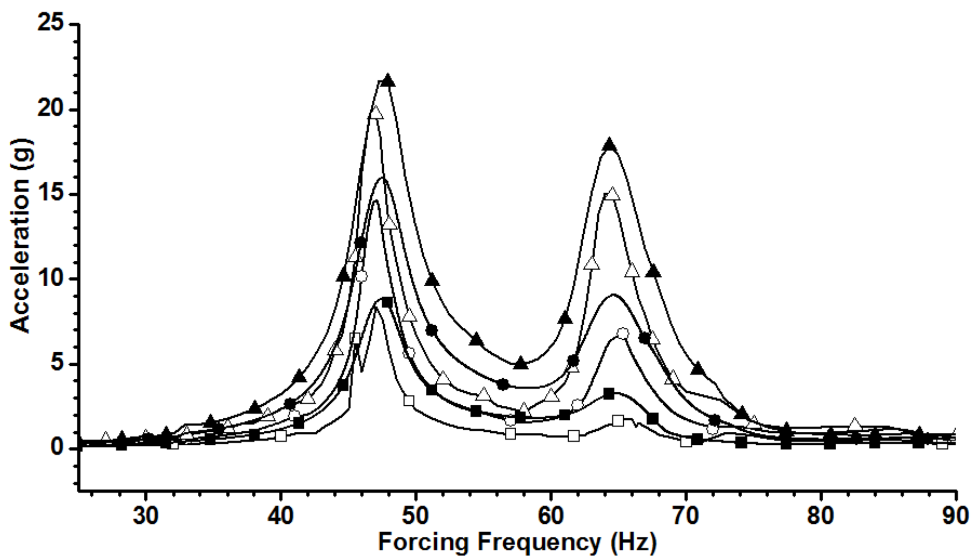
It can be inferred from the figure that the beam vibrates predominantly in the first anti-symmetric mode for forcing frequency in the neighborhood of 47 Hz due to internal resonance between the directly excited symmetric mode and anti-symmetric mode. The response for forcing frequency in the neighborhood of 65 Hz involves both symmetric and anti-symmetric modes with significantly greater participation of the former. The response depicts slight softening nonlinearity. It can also be seen from the figure that the experimental and computational results are in fairly good agreement. The frequency spectrum and time history response for the excitation amplitude of 3.5 volts is shown in Fig. 4 for different locations and forcing frequencies. The participation of higher harmonics is smaller for the forcing frequency close to natural frequency of first anti symmetric mode. The participation of second and third harmonics is dominant for forcing frequency close to first symmetric mode natural frequency especially in the response at location A. The steady state response history depicts the significant differences between amplitudes of positive and negative half cycles.

4.2 Anti-Symmetric Excitation of curved beam (span angle = 80°)

The experimental and computational frequency response curves of anti-symmetrically excited curved beam (span angle = 80°) through excitation coil at location A are shown in Fig. 5 for excitation amplitudes of 1, 2 and 3 volts. The coupled mode response involving the participation of symmetric and anti-symmetric modes is observed for the forcing frequency in the neighbourhood of natural frequency of anti-symmetric mode (39.2 Hz) due to internal resonance between directly excited anti-symmetric mode and symmetric mode. In the response of curved beam at location B, the first and second peaks around 38 Hz correspond to anti-symmetric mode natural frequency and half of the natural frequency of symmetric mode, respectively. It can also be observed from Fig. 5 that with the increase in excitation amplitude, the relative participation of symmetric mode compared to anti-symmetric mode increases leading to saturation of response with almost same participation of symmetric and anti-symmetric modes for forcing frequency in the range of 34 to 50 Hz. The response for forcing frequency around 89 Hz is predominantly in symmetric mode excited due to nonlinear coupling with the directly excited anti-symmetric mode. The mild softening nonlinearity is observed with the increase in the excitation amplitude. Further, the experimental and computational results are in fairly good agreement. The response history and frequency spectrum of accelerations measured at points A and B are shown in Fig. 6 for excitation amplitude of 3 volts. It can be inferred from Fig. 6 that the symmetric and anti-symmetric modes participate with dominant second and first harmonics respectively, for forcing frequency around 38 Hz. Whereas the excitation close to 88 Hz leads to response with the dominant participation of first and second harmonics of symmetric and anti-symmetric modes, respectively. Further, the response history depicts greater negative half cycle acceleration amplitudes for forcing frequency in the neighbourhood of 38 Hz and smaller one for forcing frequency around 88 Hz compared to positive half cycle acceleration amplitudes. The analysis of variation of radial displacement along length of the beam (Fig. 7) at different time instants of a cycle in the neighbourhood of 45 Hz forcing frequency depicts traveling wave from left to right ends and vice versa due



(a) Response at location B



(b) Response at location A

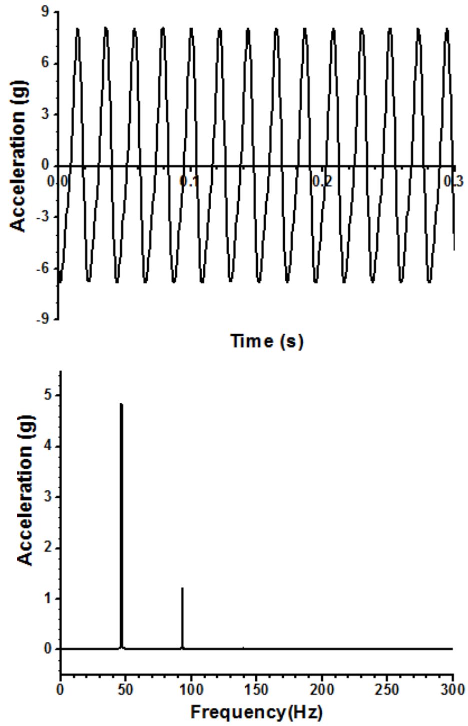
Figure 3: Acceleration versus forcing frequency curves of curved beam (span angle = 80°) excited at location B with excitation amplitudes of 1.2, 2.4 and 3.5 Volts: (a) Response at B, (b) Response at A.

to the participation of both symmetric and anti-symmetric modes.

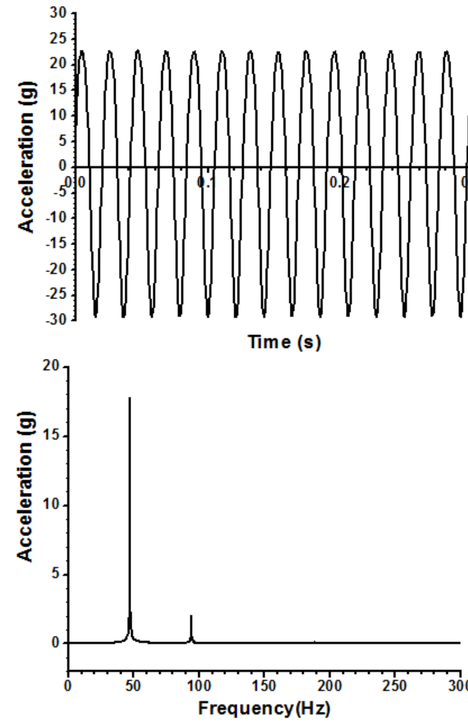
4.3 Symmetric Excitation of curved beam (span angle = 130°)

The steady state experimental frequency response curves of the symmetrically excited curved beam (span angle = 130°, excitation coil at B) are shown in Fig. 8 for excitation amplitudes of 0.5, 1.0 and 1.5 volts. The participation of symmetric and anti-symmetric modes is observed due to nonlinear coupling.

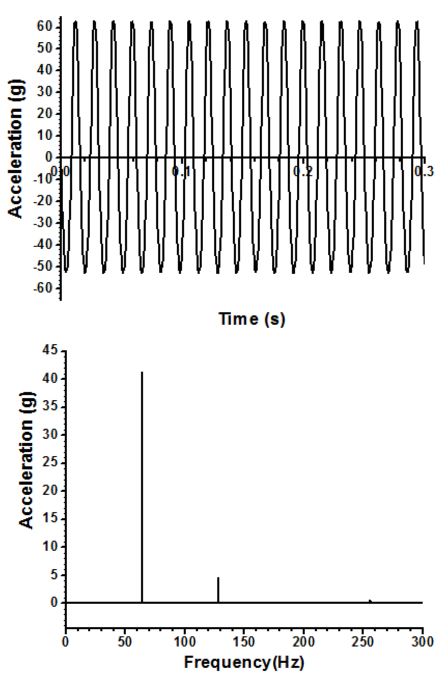
The participations of the first anti-symmetric, first symmetric and second anti-symmetric modes are dominant for forcing frequency in the neighbourhood of 14 Hz, 25 Hz and 80 Hz, respectively. The peak response amplitude at the anti-node point of first symmetric mode (location B) is 3 times for forcing frequency around 24 Hz and about 0.5 times for 79 Hz compared to those at location A. The frequency spectrum and time history response for the excitation amplitude of 1.5 volts are shown in Fig. 9 for dif-



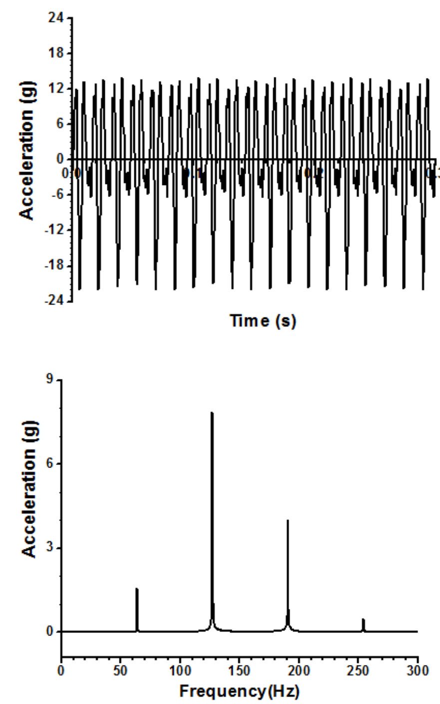
(a) Response at B, ω_F = 46 Hz



(b) Response at A, ω_F = 47 Hz

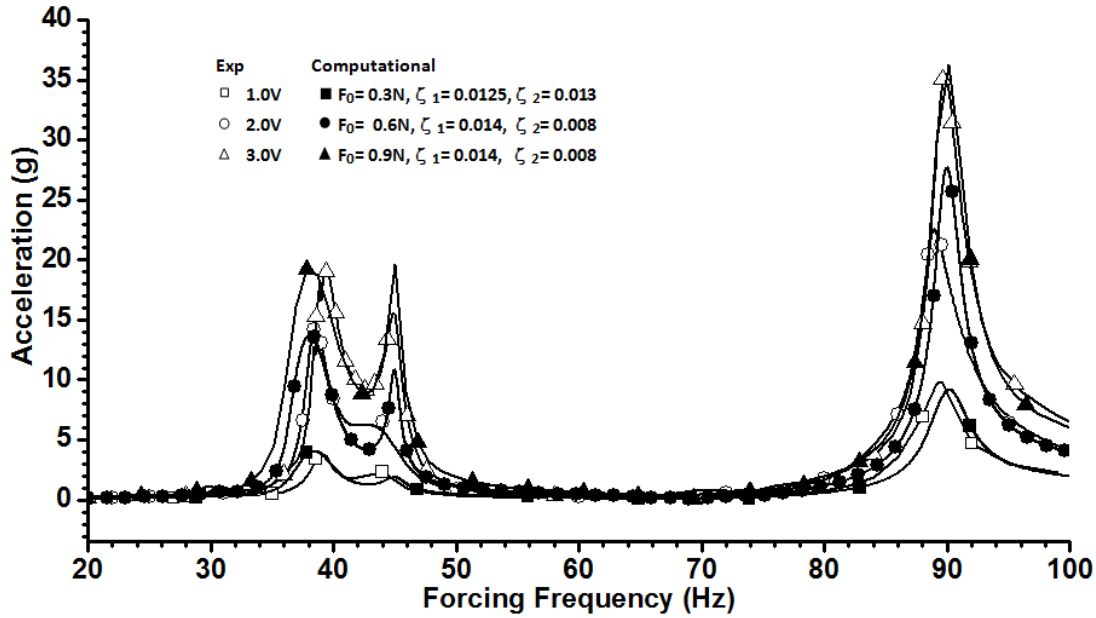


(c) Response at B, ω_F = 64 Hz

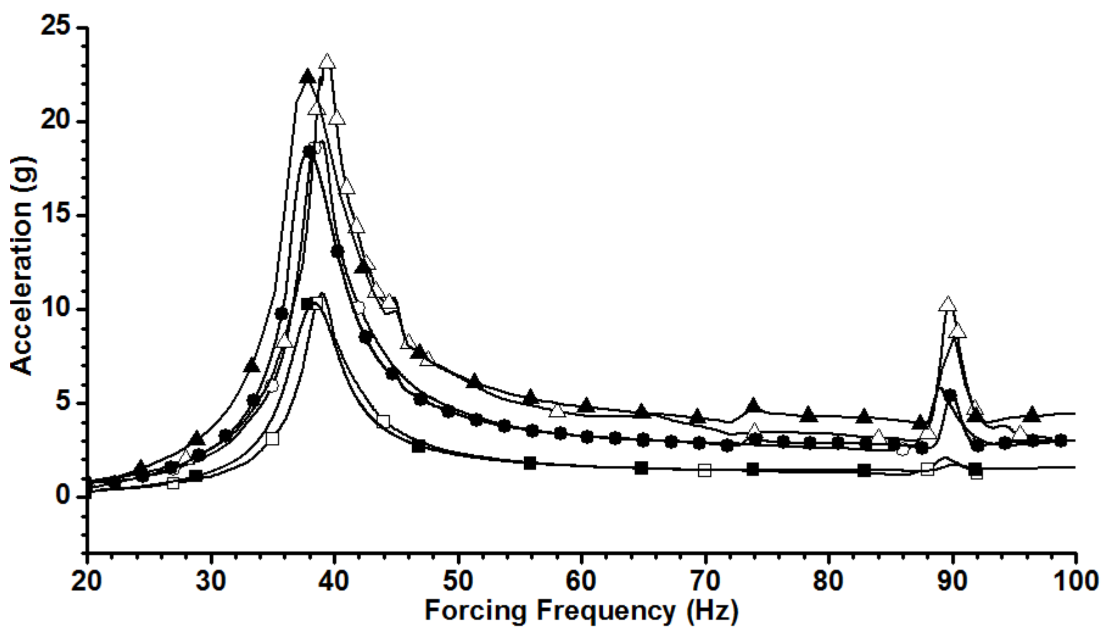


(d) Response at A, ω_F = 64 Hz

Figure 4: Steady state response history and frequency spectra of curved beam of span angle 80° excited at location B (excitation amplitude = 3.5 V).



(a) Response at location B



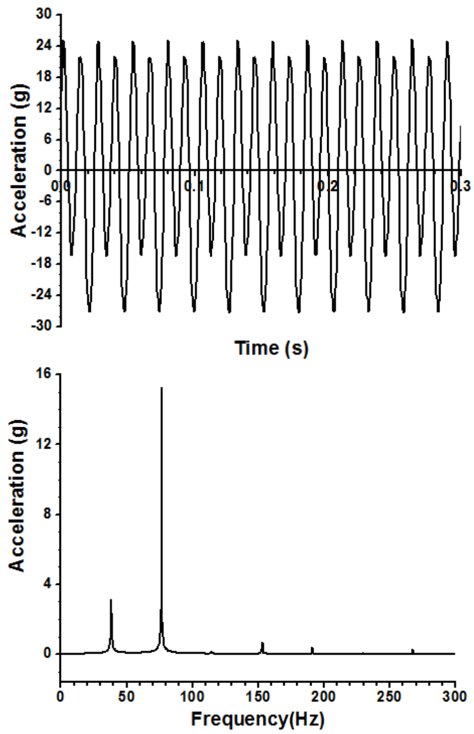
(b) Response at location A

Figure 5: Acceleration versus forcing frequency curves of curved beam (span angle = 80°) excited at location A with excitation amplitudes of 1, 2 and 3 Volts: (a) Response at B, (b) Response at A.

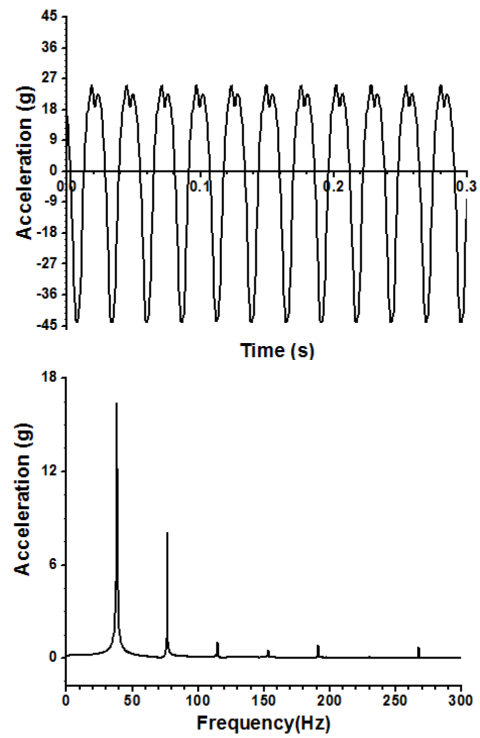
ferent locations and forcing frequencies. The participation of second harmonic is significant in response at location B for ω_F = 14 Hz and that at location A for ω_F = 23.8 Hz. The steady state response history depicts the significant differences between amplitudes of positive and negative half cycles.

4.4 Anti-Symmetric Excitation of curved beam (span angle = 130°)

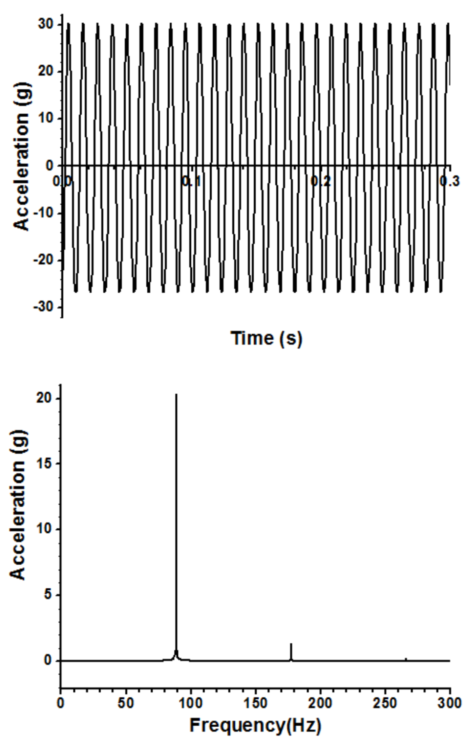
The experimental frequency response curves of anti-symmetrically excited curved beam (span angle = 130°) through excitation coil at location A are shown in Fig. 10 for excitation amplitudes of 0.2, 0.4 and 0.6 volts. The coupled mode response involving the participation of sym-



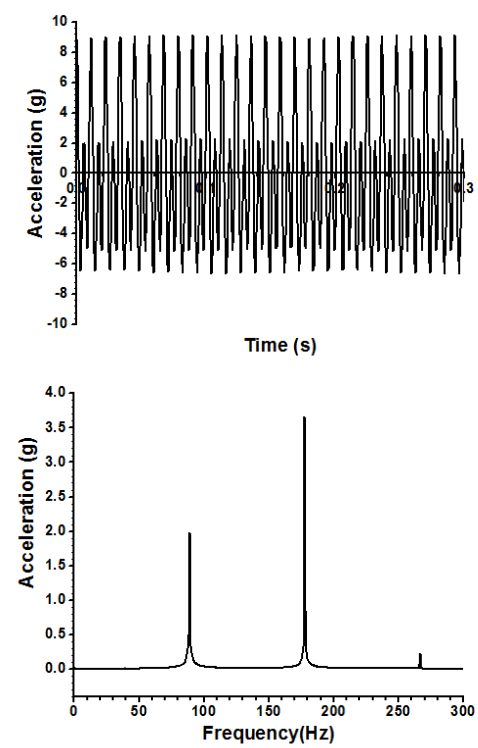
(a) Response at B, ω_F = 38.2 Hz



(b) Response at A, ω_F = 38.2 Hz



(c) Response at B, ω_F = 88.5 Hz



(d) Response at A, ω_F = 88.8 Hz

Figure 6: Steady state response history and frequency spectra of curved beam of span angle 80° excited at location B (excitation amplitude = 3.0 V).

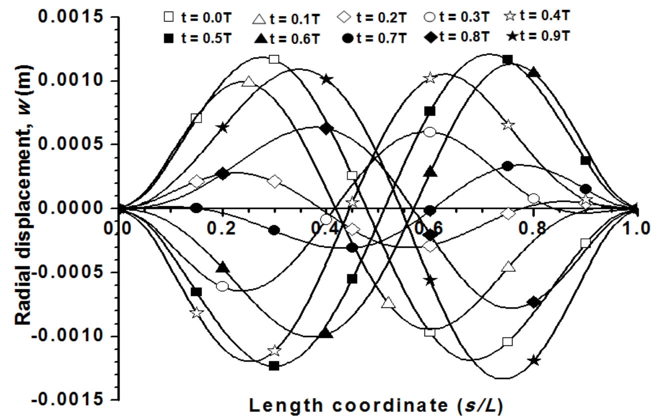
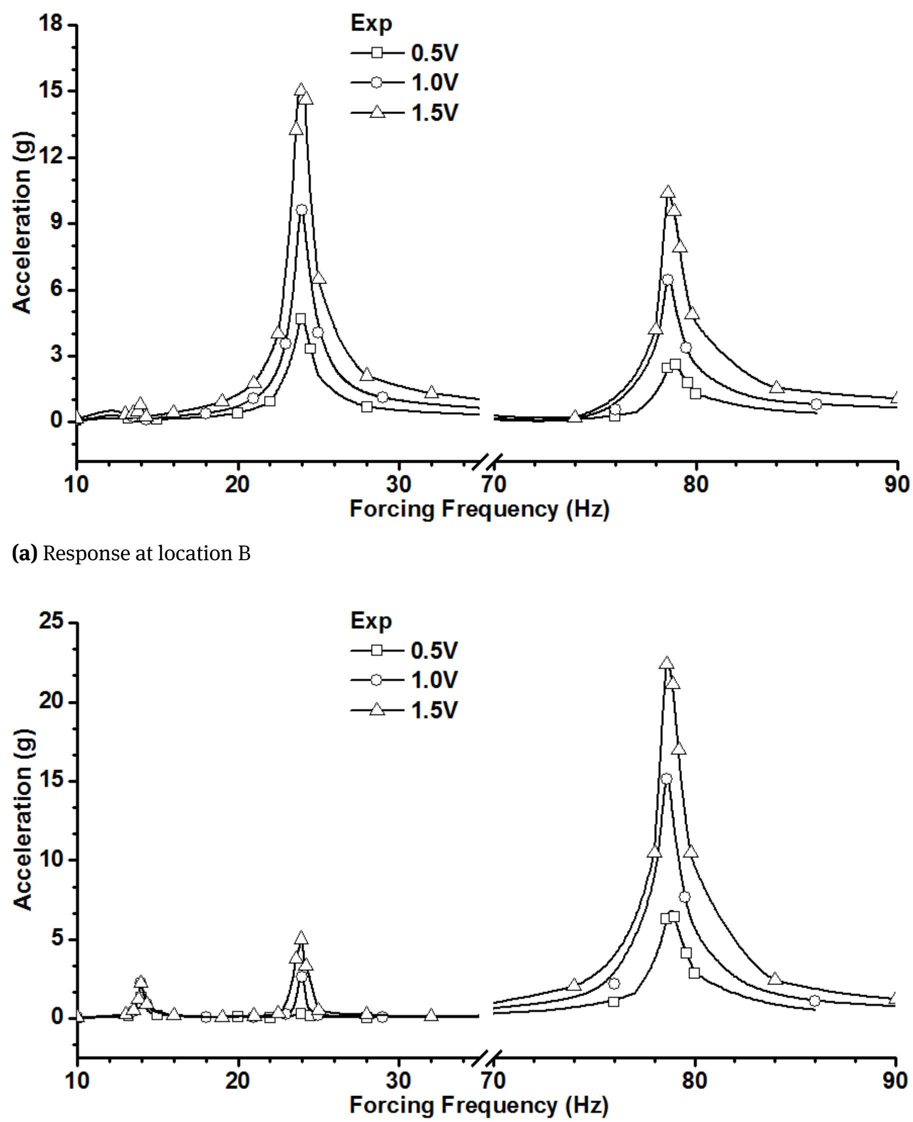


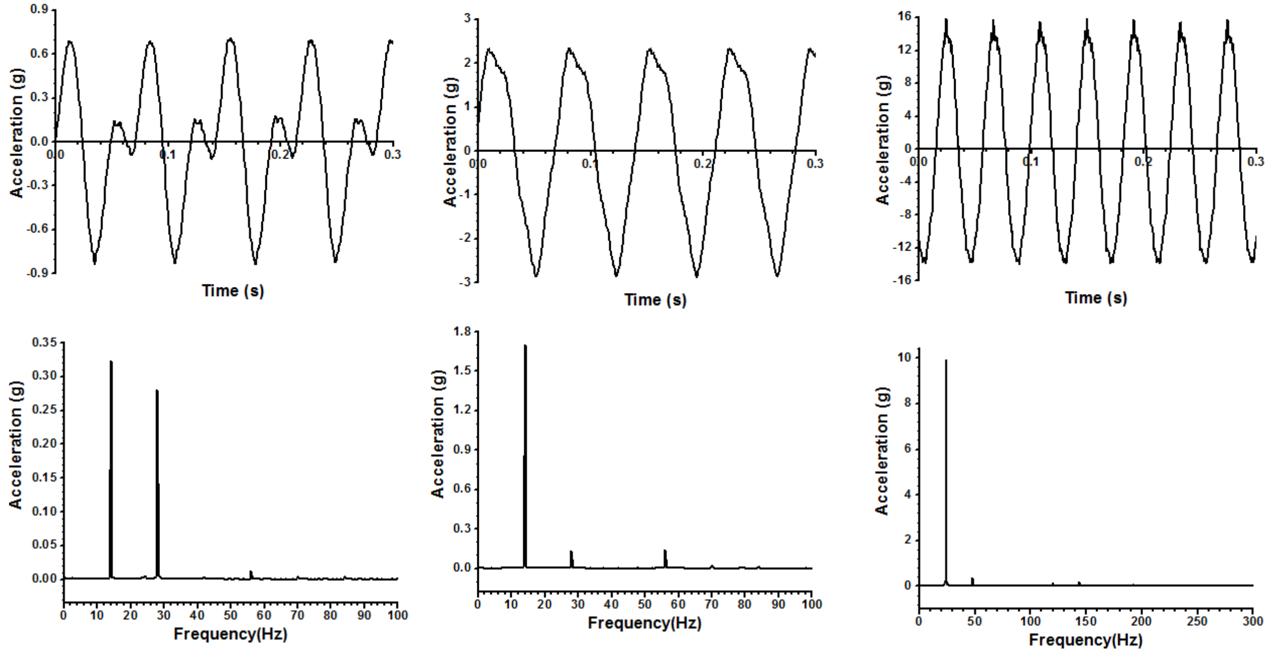
Figure 7: Variation of radial displacement along length at different time instants of a cycle. (excitation amplitude = 3.0 V).



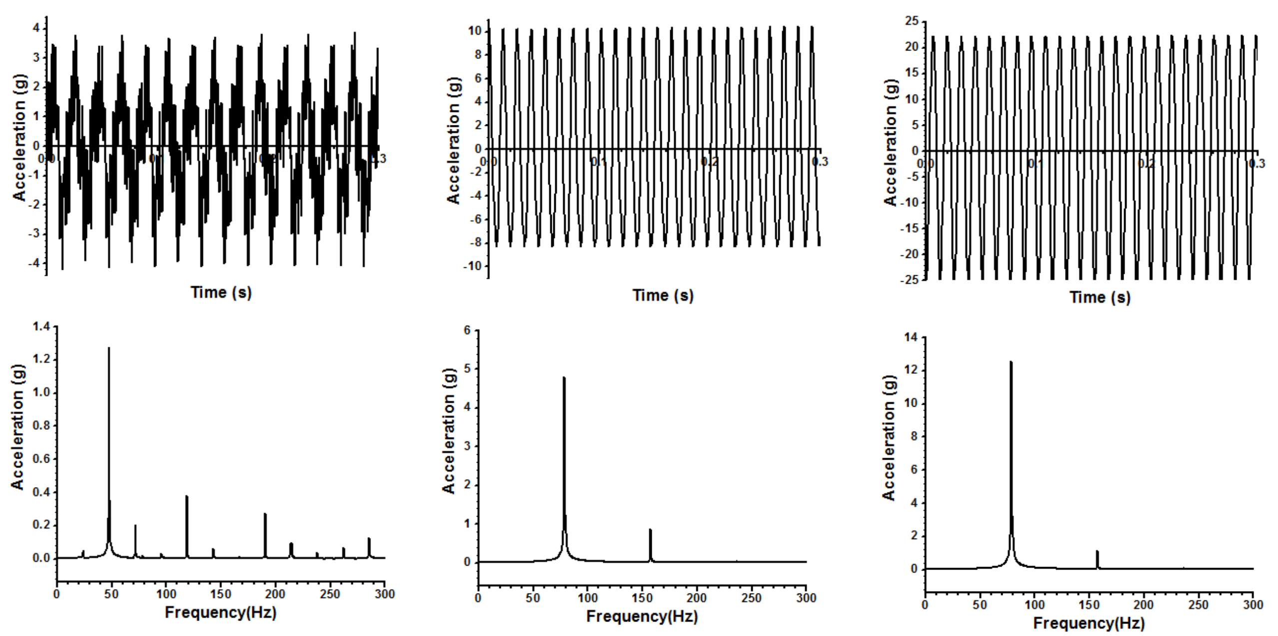
(a) Response at location B

(b) Response at location A

Figure 8: Acceleration versus forcing frequency curves of curved beam (span angle = 130°) excited at location B with excitation amplitudes of 0.5, 1.0 and 1.5 volts: (a) Response at B, (b) Response at A.

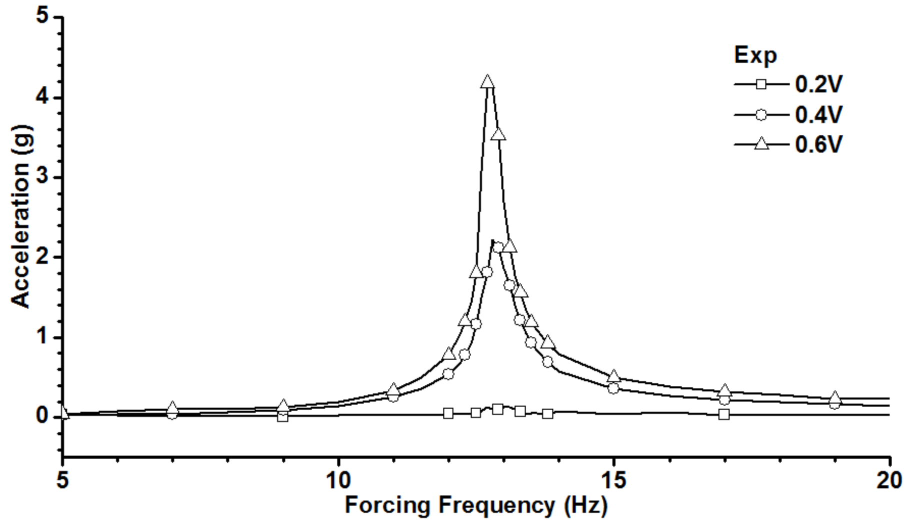


(a) Response at B, ω_F = 14Hz **(b)** Response at A, ω_F = 14Hz **(c)** Response at B, ω_F = 24 Hz

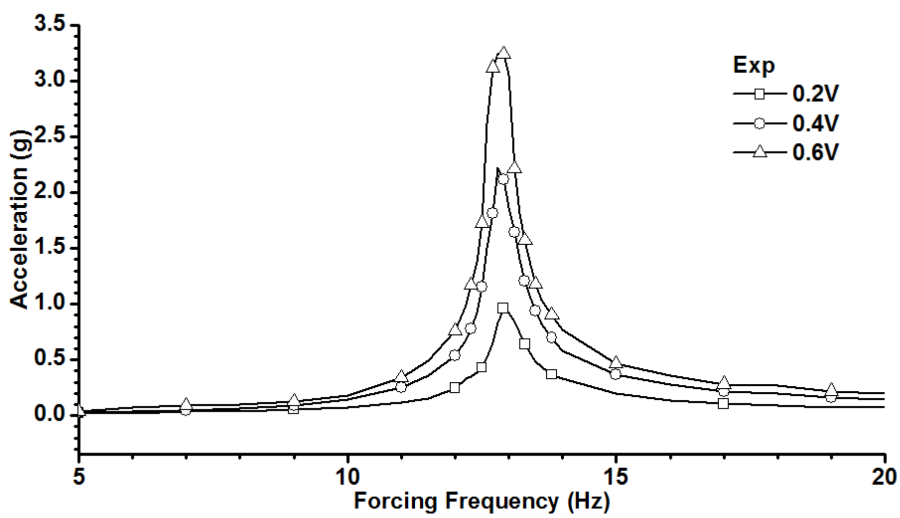


(d) Response at A, ω_F = 23.8 Hz **(e)** Response at B, ω_F = 78.7 Hz **(f)** Response at A, ω_F = 78.7 Hz

Figure 9: Steady state response history and frequency spectrum of curved beam of span angle 130° excited at location B (excitation amplitude = 1.5 V).



(a) Response at location B



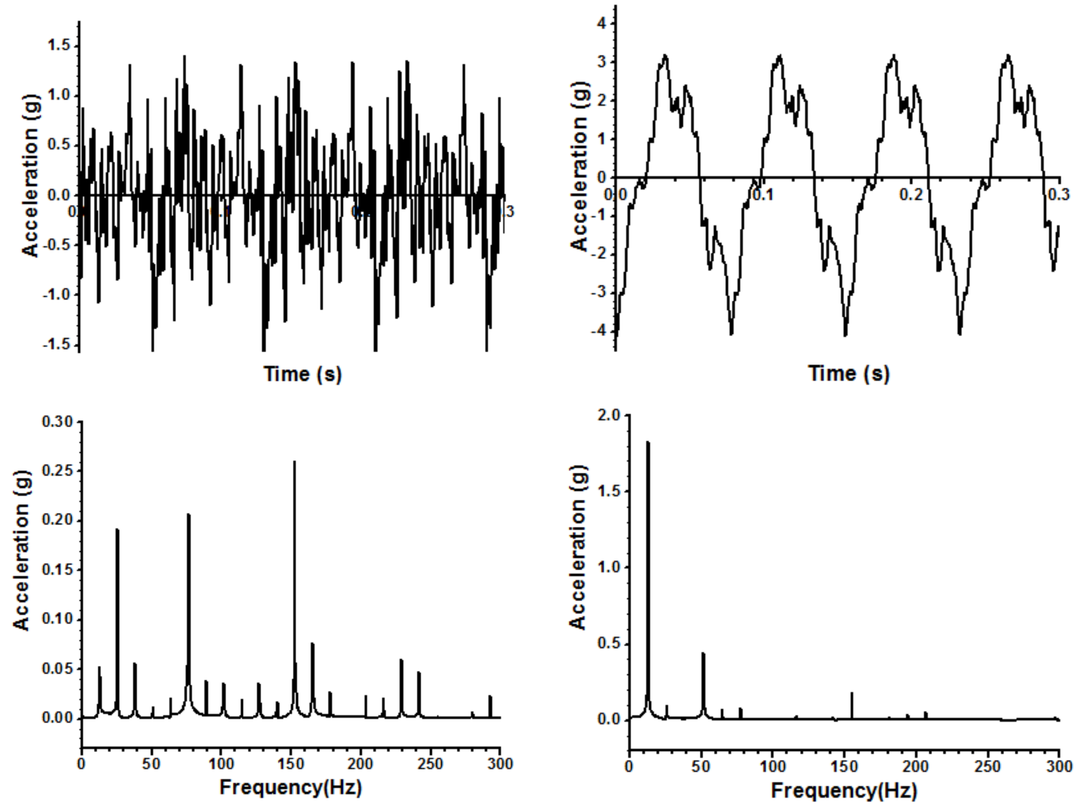
(b) Response at location A

Figure 10: Acceleration versus forcing frequency curves of curved beam (span angle = 130°) excited at location A with excitation amplitudes of 0.2, 0.4 and 0.6 Volts: (a) Response at B, (b) Response at A.

metric and anti-symmetric modes is observed for the forcing frequency in the neighbourhood of natural frequency of the first anti-symmetric mode (13 Hz) due to internal resonance between directly excited first anti-symmetric mode and first symmetric mode. It can be observed from Fig. 10 that with the increase in excitation amplitude, the relative participation of symmetric mode compared to anti-symmetric mode increases and becomes equal and greater for excitation amplitude of 0.4 volt and 0.6 volt, respectively due to increasing nonlinear coupling between the modes. The mild softening nonlinearity is observed with the increase in the excitation amplitude for this case also. The response history and frequency spectrum of accelerations measured at points A and B are shown in Fig. 11 for excitation amplitude of 0.6 volt. The response at location B depicts the increasing participation of second, sixth and twelfth harmonics and smaller participations of a number of other harmonics indicating chaotic response.

5 Concluding Remarks

The experimental results on the internal resonance between the first anti-symmetric and first symmetric modes of fixed-fixed curved beam with the direct excitation of symmetric and anti-symmetric modes under concentrated



(a) Response at B, ω_F = 12.7 Hz **(b)** Response at A, ω_F = 12.9 Hz

Figure 11: Steady state response history and frequency spectrum of curved beam of span angle 130° excited at location A (excitation amplitude = 0.6 V).

harmonic force excitation are investigated with the later case being reported for the first time. For shallow curved beam (span angle = 80°), frequency response is also obtained using curved beam finite element, Newmark's time integration and shooting technique, and the results are in fairly good agreement with the experimental ones. The coupled mode response involving symmetric and anti-symmetric modes is observed for the direct excitation of both symmetric and anti-symmetric modes due to nonlinear coupling. The mild softening nonlinearity is observed for the cases considered. The relative participation of symmetric mode in the direct excitation of anti-symmetric mode increases with the increase in the excitation amplitude and the response depicts traveling wave from left to right ends and vice versa. The deep curved beam (span angle = 130°) depicts chaotic response at higher excitation amplitude.

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