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On Asymptotic Theory of Beams, Plates and Shells

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Abstract: Bases of asymptotic theory of beams, plates and shells are stated. The comparison with classic theory is conducted. New classes of thin bodies problems, for which hypotheses of classic theory are not applicable, are considered. By the asymptotic method effective solutions of these problems are obtained. The effectiveness of the asymptotic method for finding solutions of as static, as well as dynamic problems of beams, plates and shells is shown.

Keywords: Asymptotic Theory; Beams; Plates; Shells; Static Analysis

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Introduction

Such thin bodies of beams, plates and shells type are characteristic, as one of their dimensions sharply differs from the others. This circumstance permits us by passing to dimensionless coordinates and dimensionless displacements to form small geometrical parameter in the equations and correlations of statics and dynamics of elasticity theory. It seemed that using the ordinary method of decomposition of all the required values in the degree series by small parameter, it would be possible to solve the problem. But it turned out that the corresponding system of the equations was singularly perturbed by small parameter, which did not allow to solve the problem by a decomposition.

In order to illustrate the principal difference between the regularly and singularly perturbed equations, let's consider two model equations:

$$\text{a) } u'' + \varepsilon u' = 0, \quad u = u(x), \quad x \in [0, 1], \tag{1}$$

$$\text{b) } \varepsilon u'' + u' = 0, \tag{2}$$

where ε is the small parameter. It is required to find the solutions of Equations (1) and (2) under the boundary conditions

$$u(0) = \alpha, \quad u(1) = \beta. \tag{3}$$

$u = C_1 + C_2 \exp(-x\varepsilon)$, is solution of Equation (1), which is the continuous function from small parameter ε . Satisfying conditions (3) the constant integrations C_1, C_2 are uniquely determined. As long as Equation (1) contains small parameter, it is natural to use the asymptotic method and to seek the solution in the form of the degree series

$$u = \varepsilon^s u_s, \; s = \overline{0, \infty}, \tag{4}$$

where the notation $s = \overline{0, \infty}$ means summing by dummy (repeating) index s from zero up to $(+\infty)$, (Einstein symbolic). Substituting (4) into (1) to determine the coefficients u_s we get iterative equation

$$u_s'' + u_{s-1}' = 0, \quad u_m \equiv 0 \; \text{ for } \; m < 0 \tag{5}$$

and conditions (3) have the form of

$$u_0(0) = \alpha, \quad u_0(1) = \beta, \quad u_s(0) = 0, \quad u_s(1) = 0, \quad s \geqslant 1. \tag{6}$$

At $s = 0$ Equation (5) has the form of $u_0'' = 0$. I.e. in case of regularly-perturbed equation (small parameter is not the coefficient of the higher derivative) the shortened (nonperturbed) equation has got the same order, the initial Equation (1) does. This important property permits the given boundary conditions (3) to be satisfied. For example, at $s = 0$ we have the solution $u_0 = (\beta - \alpha)x + \alpha$. At $s = 1$ from (5) $u_1'' = -u_0'$ follows and satisfying Equation (6) we obtain $u_1 = \frac{1}{2}(\alpha - \beta)x(x - 1)$. The iterative process may be continued and obtained the solution for arbitrary s. Then the question of convergence of series (4) is considered. As a rule, the convergence is asymptotic, i.e. the error of the order of the series first rejected member. This property is general for all regularly perturbed equations, among them for the equations in the partial derivatives. Therefore, the equations like this may be solved by one decomposition of (4) type.

Now consider a singularly perturbed Equation (2), i.e. when small parameter is the coefficient of the higher operator (derivative). The solution of this equation is

$$u = A_1 + A_2 e^{-x/\varepsilon} \tag{7}$$

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which is not the continuous function from ε yet.
Satisfying conditions (3), determine A_1, A_2 and the precise solution

$$u = \frac{1}{1-e^{-1/\varepsilon}}\left[\beta - \alpha e^{-1/\varepsilon} + (\alpha-\beta)e^{-x/\varepsilon}\right]. \quad (8)$$

From solution (8) it follows, that at $\varepsilon << 1$, $u \approx \beta$ outside of the dependence of the values x, with the exception of some small area near $x = 0$, which is called boundary layer. Consider the possibility of the boundary problem solution (2), (3) by an asymptotic method. Seek the solution in the form of (4). For the determination u_s we get the equation

$$u''_{s-1} + u'_s = 0. \quad (9)$$

At $s = 0$ we have $u'_0 = 0$ or $u_0 = C_1 = const$, i.e. the nonperturbed equation has smaller order, than the basic Equation (2), therefore it is impossible to satisfy two conditions (3). The question slands, which of these conditions should be satisfied. From the analysis of the exact solution, brought above, the satisfaction to the second condition (3) follows naturally, i.e. the condition on the edge near which the boundary layer is absent. Then for C_1 we obtain $C_1 = \beta$, and $u_0 = \beta$. It turns out, that the first condition (3) can be satisfied as well, if to built a new solution – a solution of the boundary layer for the edge $x = 0$. For this change of the variable $\gamma = -x/\varepsilon$ is input and such a solution of the transformed Equation (2), which has damped character and can remove rising at $x = 0$ incoordination [1, 2], is sought. This solution will be $u_b = C_2 e^{\gamma} = C_2 \exp(-x/\varepsilon)$. Demanding that at $x = 0$ ($\gamma = 0$)

$$u_0(x=0) + u_b(\gamma=0) = \alpha \quad (10)$$

we determine $C_2 = \alpha - \beta$. As a result the solution

$$u^{(0)} = u_0 + u_b = \beta + (\alpha-\beta)e^{-x/\varepsilon} \quad (11)$$

which at small ε practically coincides with the exact solution (8), will correspond to the initial approximation.
If with the determination u_0 we satisfy the first condition (3), we get $u_0 = \alpha$. Then it is impossible to satisfy condition (3) at $x = 1$, as long as there is no boundary layer there. One can be convinced in it, if we input a new change of the variables $\eta = (x-1)/\varepsilon$. Then $-1/\varepsilon \leqslant \eta \leqslant 0$, Equation (2) is transformed into $u''_\eta + u'_\eta = 0$, which does not have damping solution in the internal $-1/\varepsilon \leqslant \eta \leqslant 0$. The iterative process may be continued. Thus, in this way the procedure of finding the asymptotic solution is succeeded to justify mathematically.
From the above built asymptotic solution the following general for singularly-perturbed differential equations conclusions run out: the solution is impossible to obtain in the form of one decomposition by small parameter of (4) type. It is combined of the outer solution $\left(I^{out}\right)$ and the solutions for the boundary layers (I_b). Depending on the problem and the order of the perturbed operator several boundary layers can exist. The solution I^{out} and I_b may be built separately and perform their conjugation (joining) with the help of the boundary conditions.
In the problems of elasticity theory for beams, bars, plates and shells in the equations, the formed small parameter is the coefficient of not the whole higher operator, but only of its part, yet the structure of the solution remains unchangeable:

$$I = I^{out} + I_b. \quad (12)$$

The nonperturbed equation has got less space dimension, and the boundary functions compose a denumerable set.
Classic theory of beams and bars is built on the base of Bernoulli-Coulomb-Euler hypotheses of plane sections. Kirchhoff generalized this hypothesis (hypothesis of nondeformable normals) for the conclusion of two-dimensional equations of plates, and Love did it for the conclusion of the equations of shells. Classic theory of plates and shells got a final shape thanks to Timoshenko S.P., Flugge V., Vlasov V.Z., Goldenveizer A.L., Lourier A.I., Novozhilov V.V. monographs. Classic theory of anisotropic plates, among them layered ones, is built by Lekhnitskii S.G., and Ambartsumyan S.A. did it for the anisotropic shells. With the very sequence we consider the problem of reduction of the corresponding two-dimensional and three-dimensional problems of elasticity theory to one-dimensional and two-dimensional problems of mathematical physics, show the relation of such reduction with classic theory of beams, bars, plates and shells [3–10]. By the asymptotic method we solve new classes of problems for thin bodies, which is not possible to solve on the base of the hypothesis of classical theory.

1 The asymptotic solution of the first boundary problem of elasticity theory for thermoelastic strip. The relation of classic theory of beams and bars.

Set the problem: to find the solution of the first static boundary problem of thermoelasticity for an isotropic rect-

angle $D = \{(x, y) : 0 \leqslant x \leqslant \ell, -h \leqslant y \leqslant h, h \leqslant \ell\}$ taking into account the volume forces and the temperature by Duhamel-Neuman model. It is necessary to find the solution:
of the equilibrium equations

$$\frac{\partial \sigma_{jx}}{\partial x} + \frac{\partial \sigma_{jy}}{\partial y} + F_j(x, y) = 0, \quad j = x, y \tag{13}$$

of state equations (Hook general law)

$$\frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) + \alpha_{11}\theta, (u, v; x, y; 1, 2), \tag{14}$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{G}\sigma_{xy} + \alpha_{12}\theta$$

where α_{ik} are the coefficients of heat conductivity. E, G are Young and shear modules, ν is Poisson coefficient, $\theta = T(x, y) - T_0(x, y)$ is the change of the temperature. The solution should satisfy the boundary conditions on the longitudinal edges $y = \pm h$:

$$\sigma_{xy}(x, \pm h) = \pm X^{\pm}(x), \quad \sigma_{yy}(x, \pm h) = \pm Y^{\pm}(x) \tag{15}$$

and the conditions at $x = 0, \ell$ (conditions of fastening), which are still considered to be arbitrary. As volume forces the weight of the strip, for example ($F_x = 0, F_y = -\rho(x, y)g, \rho$ is the density) can arise; the reduced seismic force by Mononobe model ($F_x = \beta k_s\delta, F_y = 0, 75F_x, k_s$ is the seismicity coefficient; δ is the specific weight, β is the agility coefficient).
In order to solve the stated problem, we pass to dimensionless coordinates and dimensionless displacements

$$\xi = x/\ell, \quad \zeta = y/h, \quad U = u/\ell, \quad V = v/\ell. \tag{16}$$

Equations (13) and (14) will have the form of

$$\frac{\partial \sigma_{xx}}{\partial \xi} + \varepsilon^{-1}\frac{\partial \sigma_{xy}}{\partial \zeta} + \ell F_x(\ell\xi, h\zeta) = 0, \quad \varepsilon = h/\ell, \tag{17}$$
$$\frac{\partial \sigma_{xy}}{\partial \xi} + \varepsilon^{-1}\frac{\partial \sigma_{yy}}{\partial \zeta} + \ell F_y(\ell\xi, h\zeta) = 0,$$
$$\frac{\partial U}{\partial \xi} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{xy}) + \alpha_{11}\theta,$$
$$\varepsilon^{-1}\frac{\partial V}{\partial \zeta} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) + \alpha_{22}\theta,$$
$$\varepsilon^{-1}\frac{\partial U}{\partial \zeta} + \frac{\partial V}{\partial \xi} = \frac{1}{G}\sigma_{xy} + \alpha_{12}\theta.$$

System (17) is singularly perturbed by small (big) parameter ε. It solution has the form of (12). The solution of the outer problem is sought in the form of

$$I^{out} = \varepsilon^{q_I+s} I^{(s)}(\xi, \zeta), s = \overline{0, N} \tag{18}$$

where q_I characterizes the order (intensivity) of the corresponding required magnitude. Their values should be determined so, that after having substituted (18) into (17) and equated in each equation the coefficients under the same degrees of ε, we obtained a noncontradictory system relative to $I^{(s)}$. This is the most difficult moment in the asymptotic method [11]. The values q_I in each problem are determined only in one way. For our problem we have [12]

$$q_{\sigma_{xx}} = -2, \; q_{\sigma_{xy}} = -1, \; q_{\sigma_{yy}} = 0, \; q_u = -2, \; q_v = -3. \tag{19}$$

Substituting (18), (19) into system (17) and having solved once again obtained system, we shall have

$$V^{(s)} = v^{(s)}(\xi) + v_*^{(s)}(\xi, \zeta), \tag{20}$$
$$U^{(s)} = -\frac{dv^{(s)}}{d\xi}\zeta + u^{(s)}(\xi) + u_*^{(s)}(\xi, \zeta),$$
$$\sigma_{xx}^{(s)} = -E\frac{d^2v^{(s)}}{d\xi^2}\zeta + E\frac{dU^{(s)}}{d\xi} + \sigma_{xx*}^{(s)}(\xi, \zeta),$$

$$\sigma_{xy}^{(s)} = \frac{1}{2}E\frac{d^3v^{(s)}}{d\xi^3}\zeta^2 - E\frac{d^2u^{(s)}}{d\xi^2}\zeta + \sigma_{xy0}^{(s)}(\xi) + \sigma_{xy*}^{(s)}(\xi, \zeta),$$

$$\sigma_{yy}^{(s)} = -\frac{1}{6}E\frac{d^4v^{(s)}}{d\xi^4}\zeta^3 + \frac{1}{2}E\frac{d^3v^{(s)}}{d\xi^3}\zeta^2 - \frac{d\sigma_{xy0}^{(s)}}{d\xi}\zeta + \sigma_{yy0}^{(s)}(\xi) + \sigma_{yy*}^{(s)}(\xi, \zeta),$$

where

$$v_*^{(s)} = \int_0^\zeta \left[\frac{1}{E}\left(\sigma_{yy}^{(s-4)} - \nu\sigma_{xx}^{(s-2)}\right) + \alpha_{22}\theta^{(s-2)}\right] d\zeta, \tag{21}$$
$$u_*^{(s)} = \int_0^\zeta \left[\frac{1}{G}\sigma_{xy}^{(s-2)} + \alpha_{12}\theta^{(s-1)} - \frac{\partial v_*^{(s)}}{\partial \xi}\right] d\zeta,$$
$$\sigma_{xx*}^{(s)} = E\frac{\partial u_*^{(s)}}{\partial \xi} + \nu\sigma_{yy}^{(s-2)} - E\alpha_{11}\theta^{(s)},$$
$$\sigma_{xy*}^{(s)} = -\int_0^\zeta \left[F_x^{(s)} + \frac{\partial \sigma_{xx*}^{(s)}}{\partial \xi}\right] d\zeta,$$
$$\sigma_{yy*}^{(s)} = -\int_0^\zeta \left[F_y^{(s)} + \frac{\partial \sigma_{xy*}^{(s)}}{\partial \xi}\right] d\zeta,$$
$$F_x^{(0)} = \ell\varepsilon^2 F_x(\ell\xi, h\zeta), \quad F_y^{(0)} = \ell\varepsilon F_y(\ell\xi, h\zeta),$$
$$F_x^{(s)} = F_y^{(s)} = 0, \quad s \neq 0,$$
$$\theta^{(0)} = \varepsilon^2\theta, \quad \theta^{(s)} = 0, \quad s \neq 0, \quad I^{(m)} \equiv 0 \text{ at } m < 0.$$

Solution (20) involves unknown yet functions $u^{(s)}, v^{(s)}, \sigma_{xy0}^{(s)}, \sigma_{yy0}^{(s)}$, which should be determined from conditions (15) and conditions at $x = 0, \ell$. Having satisfied conditions (15), $\sigma_{xy0}^{(s)}, \sigma_{yy0}^{(s)}$ will be expressed through $u^{(s)}, v^{(s)}$ and for the determination of the last, we get the equations

$$E\frac{d^2u^{(s)}}{d\xi^2} = q_x^{(s)}, \tag{22}$$

$$q_x^{(s)} = -\frac{1}{2}\left(X^{+(s)} + X^{-(s)}\right) + \frac{1}{2}\left(\sigma_{xy\star}^{(s)}(\xi,1) - \sigma_{xy\star}^{(s)}(\xi,-1)\right),$$

$$\frac{1}{3}E\frac{d^4 v^{(s)}}{d\xi^4} = q^{(s)},$$

$$\begin{aligned} q^{(s)} &= \frac{1}{2}\left(Y^{+(s)} + Y^{-(s)}\right) - \frac{1}{2}\left(\sigma_{yy\star}^{(s)}(\xi,1) - \sigma_{yy\star}^{(s)}(\xi,-1)\right) + \\ &+ \frac{1}{2}\frac{d}{d\xi}\left(X^{+(s)} - X^{-(s)} - \sigma_{xy\star}^{(s)}(\xi,1) - \sigma_{xy\star}^{(s)}(\xi,-1)\right), \end{aligned}$$

$$X^{\pm(0)} = \varepsilon X^{\pm},\quad Y^{\pm(0)} = Y^{\pm},\quad X^{\pm(s)} = Y^{\pm(s)} = 0,\quad s \neq 0.$$

Equations (22) have got direct connection with the classical theory equations of expansion-compression bars and beams bend. In order to reveal this connection we write them in dimensional coordinates. We have

$$\begin{aligned} u &= \ell\varepsilon^{-2+s}u^{(s)} = u_0 + \varepsilon u_1 + \ldots + \varepsilon^s u_s + \ldots, \\ v &= \ell\varepsilon^{-3+s}v^{(s)} = v_0 + \varepsilon v_1 + \ldots + \varepsilon^s v_s + \ldots, \end{aligned}$$

from where

$$u^{(s)} = \frac{1}{\ell}\varepsilon^2 u_s,\quad v^{(s)} = \frac{1}{\ell}\varepsilon^3 v_s \tag{23}$$

follows.

Substituting (23) into (22) we get

$$EF\frac{d^2 u_s}{dx^2} = q_{x0}^{(s)},\quad q_{x0}^{(s)} = 2\varepsilon^{-1}q_x^{(s)},\quad F = 2h\cdot 1, \tag{24}$$

$$EJ\frac{d^4 v_s}{dx^4} = q_0^{(s)},\quad q_0^{(s)} = 2q^{(s)},\quad J = \frac{2}{3}h^3\cdot 1, \tag{25}$$

where EF is the rigidity of the bar under the expansion – compression, EJ is the rigidity of the beam under the bend. At $s = 0$ $q_{x0}^{(0)} = -(X^+ + X^-)$, $q_0^{(0)} = Y^+ + Y^- + h\frac{d}{dx}(X^+ - X^-)$, Equation (24) coincides with the classic equation of the bar expansion – compression, and Equation (25) coincides with the classical equation of the beam bend. Moreover, the initial approximation of the outer problem asymptotic solution contains more information as by formulae (20) stresses σ_{xy}, σ_{yy} are calculated too, the latter is generally neglected in classical theory.

The solutions of the ordinary differential Equations (22) totally contain six arbitrary constants, three of which characterize a rigid displacement and do not affect on the stresses values, the other three constants are not capable to satisfy the boundary conditions in each point of the end-wall sections $x = 0, \ell$, for example, the conditions at $x = 0$ of $\sigma_{xx}(0,\zeta) = \phi(\zeta)$, $\sigma_{xy}(0,\zeta) = \psi(\zeta)$ type.

This problem can be solved with the help of the new type solution the solution of the boundary layer.

As the solution of the outer problem satisfies inhomogeneous Equations (13), (14) and nontrivial boundary conditions (15), the boundary layer should be determined from the corresponding homogeneous equations and homogeneous trivial boundary conditions. For the determination of the boundary layer at $x = 0$, inputting the change of the variable $\gamma = \xi/\varepsilon$ we have a boundary value problem

$$\begin{aligned} &\frac{\partial\sigma_{xxb}}{\partial\gamma} + \frac{\partial\sigma_{xyb}}{\partial\zeta} = 0,\quad \frac{\partial\sigma_{\mathrm{xyb}}}{\partial\gamma} + \frac{\partial\sigma_{yyb}}{\partial\zeta} = 0, \\ &\varepsilon^{-1}\frac{\partial U_b}{\partial\gamma} = \frac{1}{E}\left(\sigma_{xxb} - v\sigma_{yyb}\right), \\ &\varepsilon^{-1}\frac{\partial V_b}{\partial\zeta} = \frac{1}{E}\left(\sigma_{yyb} - v\sigma_{xxb}\right), \\ &\varepsilon^{-1}\frac{\partial U_b}{\partial\zeta} + \varepsilon^{-1}\frac{\partial V_b}{\partial\gamma} = \frac{1}{G}\sigma_{xyb}, \\ &\sigma_{xyb}(\gamma,\pm1) = 0,\quad \sigma_{yyb}(\gamma,\pm1) = 0. \end{aligned} \tag{26}$$

It is necessary to find such a solution of the boundary-value problem (26), which has got damping character when removing from $x = 0$ ($\gamma = 0$) inside the rectangle-strip. It has the form of

$$\begin{aligned} &\sigma_{ijb}(\gamma,\zeta) = \varepsilon^{-1+s}\sigma_{ijb}^{(s)}(\zeta)\exp(-\lambda\gamma),\quad s = \overline{0,N}, \\ &(U_b, V_b) = \varepsilon^s\left(u_b^{(s)}(\zeta), v_b^{(s)}(\zeta)\right)\exp(-\lambda\gamma), \end{aligned} \tag{27}$$

where λ is still the unknown number.

Substituting (27) into (26) all the sought values may be expressed through σ_{yyb} :

$$\begin{aligned} &\sigma_{xxb}^{(s)} = \frac{1}{\lambda^2}\frac{d^2\sigma_{yyb}^{(s)}}{d\zeta^2},\quad \sigma_{xyb}^{(s)} = \frac{1}{\lambda}\frac{d\sigma_{yyb}^{(s)}}{d\zeta} \\ &u_b^{(s)} = -\frac{1}{\lambda^3 E}\left(\frac{d^2\sigma_{yyb}^{(s)}}{d\zeta^2} - v\lambda^2\sigma_{yyb}^{(s)}\right), \\ &v_b^{(s)} = -\frac{1}{\lambda^4 E}\left(\frac{d^3\sigma_{yyb}^{(s)}}{d\zeta^3} + (2+v)\lambda^2\frac{d\sigma_{yyb}^{(s)}}{d\zeta}\right) \end{aligned} \tag{28}$$

which is determined from the equation

$$\frac{d^4\sigma_{yyb}^{(s)}}{d\zeta^4} + 2\lambda^2\frac{d^2\sigma_{yyb}^{(s)}}{d\zeta^2} + \lambda^4\sigma_{yyb}^{(s)} = 0. \tag{29}$$

Having solved Equation (29) and satisfying conditions (26) relative to $\sigma_{xyb}, \sigma_{yyb}$, we obtain

$$\sigma_{yyb}^{(s)} = A_n^{(s)}F_n(\zeta),\quad n = \overline{0,N} \tag{30}$$

where $A_n^{(s)}$ is still arbitrary constant, in the expansion – compression problem (symmetric problem)

$$F_n(\zeta) = \zeta\sin\lambda_n\zeta - \tan\lambda_n\cos\lambda_n\zeta,\quad \sin 2\lambda_n + 2\lambda_n = 0. \tag{31}$$

and in the bend problem (skew-symmetric problem)

$$F_n(\zeta) = \sin\lambda_n\zeta - \zeta\tan\lambda_n\cos\lambda_n\zeta,\quad \sin 2\lambda_n - 2\lambda_n = 0. \tag{32}$$

Transcendental equations $\sin 2\lambda_n \pm 2\lambda_n = 0$ have got complex-conjugate roots, the roots with $\mathrm{Re}\lambda_n > 0$, providing damping character of the solution will interest us. The values of the first ten roots λ_n are brought in [12].

From the first two formulae (28) and conditions (26) very important property in the boundary layer follows-self-balance of stresses σ_{xxb} and σ_{xyb} in the arbitrary cross-section $\gamma = \gamma_k$:

$$\int_{-1}^{+1} \sigma_{xxb}(\gamma,\zeta)d\zeta = 0, \quad \int_{-1}^{+1} \zeta\sigma_{xxb}(\gamma,\zeta)d\zeta = 0, \tag{33}$$
$$\int_{-1}^{+1} \sigma_{xyb}(\gamma,\zeta)d\zeta = 0, \quad \forall\gamma = \gamma_k.$$

Based on the general form of solution (12), using character (33), the rest of the above mentioned three constants in the solution of the outer problem are determined, therefore the solution of the outer problem itself as well. The constants $A_n^{(s)}$ in the solution of the boundary layer are determined from the conditions

$$\sigma_{xxb}^{(s)}(\gamma = 0,\zeta) = \phi^{(s-2)} - \sigma_{xx}^{(s)}(0,\zeta), \tag{34}$$
$$\sigma_{xyb}^{(s)}(\gamma = 0,\zeta) = \psi^{(s-2)} - \sigma_{xy}^{(s-1)}(0,\zeta).$$

Note, that the solution of the boundary layer, determined by formulae (27), (28), (30), is a mathematically exact solution and is well-known in elasticity theory as Fuss-Papkovich-Lourier homogeneous solution. Accepting the plane sections hypothesis, this solution as well as the approximations $s \geqslant 1$ in the outer problem are practically neglected.

Asymptotics (18), (19) remain in force for the anisotropic and layered beams-strips and bars [13].

2 The solution of the space first boundary-value problem for an anisotropic thermoelastic plate

The asymptotic method of the solution without essential changes, described for plane problems, remains in force for the solution of space problems of anisotropic plates, as well.

In order to solve 3D problem for plates $G = \{(x,y,z) : 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b \leqslant, -h \leqslant z \leqslant h\}$ possessing general anisotropy (21 constant of elasticity), we input dimensionless coordinates $x = \ell\xi, y = \ell\eta, z = h\zeta$ ($\ell = \min(a,b)$), $h << \ell$ and dimensionless displacements $U = u/\ell, V = v/\ell, W = w/\ell$. The corresponding system of the equations of thermoelasticity

$$\frac{\partial\sigma_{xx}}{\partial\xi} + \frac{\partial\sigma_{xy}}{\partial\eta} + \varepsilon^{-1}\frac{\partial\sigma_{xz}}{\partial\zeta} + \ell F_x(\xi,\eta,\zeta) = 0, (x,y,z;\xi,\eta,\zeta),$$

$$\begin{aligned}\frac{\partial U}{\partial\xi} &= a_{i1}\sigma_{xx} + a_{i2}\sigma_{yy} + a_{i3}\sigma_{zz} + a_{i4}\sigma_{yz} + \\ &+ a_{i5}\sigma_{xz} + a_{i6}\sigma_{xy} + \alpha_{ii}\theta, (U,V;\xi,\eta; i = 1,2),\end{aligned} \tag{35}$$

$$\begin{aligned}\varepsilon^{-1}\frac{\partial W}{\partial\zeta} &= a_{13}\sigma_{xx} + a_{23}\sigma_{yy} + a_{33}\sigma_{zz} + a_{34}\sigma_{yz} + \\ &+ a_{35}\sigma_{xz} + a_{36}\sigma_{xy} + \alpha_{33}\theta,\end{aligned}$$

$$\begin{aligned}\varepsilon^{-1}\frac{\partial V}{\partial\zeta} + \frac{\partial W}{\partial\eta} &= a_{1j}\sigma_{xx} + a_{2j}\sigma_{yy} + a_{3j}\sigma_{zz} + a_{4j}\sigma_{yz} + \\ &+ a_{5j}\sigma_{xz} + a_{6j}\sigma_{xy} + \alpha_{23}\theta,\end{aligned}$$

$$(V,U;\xi,\eta;\alpha_{23},\alpha_{13}; j = 4,5)\,,$$

$$\begin{aligned}\frac{\partial U}{\partial\eta} + \frac{\partial V}{\partial\xi} &= a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{36}\sigma_{zz} + a_{46}\sigma_{yz} + \\ &+ a_{56}\sigma_{xz} + a_{66}\sigma_{xy} + \alpha_{12}\theta\end{aligned}$$

is again singularly-perturbed by small parametre $\varepsilon = h/\ell$. The solution of the system (35) has the form of (12). For the outer problem we have the form of (18), where

$$q = -2 \text{ for } \sigma_{xx}, \sigma_{yy}, \sigma_{xy}, U, V; \quad q = -1 \text{ for } \sigma_{xz}, \sigma_{yz}; \tag{36}$$
$$q = 0 \text{ for } \sigma_{zz}; \quad q = -3 \text{ for } W.$$

Substituting (18), (36) into (35) we get a system which permits integration along ζ and all the sought parameters, after satisfying the conditions of the first boundary-value problem at $\zeta = \pm 1$, are expressed through $U^{(s)}(\xi,\eta), V^{(s)}(\xi,\eta), W^{(s)}(\xi,\eta)$, which are determined from the equations

$$\ell_{11}U^{(s)} + \ell_{12}V^{(s)} = P_1^{(s)}, \quad \ell_{12}U^{(s)} + \ell_{22}V^{(s)} = P_2^{(s)}, \tag{37}$$

$$\begin{aligned}B_{11}\frac{\partial^4 W^{(s)}}{\partial\xi^4} + 4B_{16}\frac{\partial^4 W^{(s)}}{\partial\xi^3\partial\eta} + 2\left(B_{12} + 2B_{66}\right)\frac{\partial^4 W^{(s)}}{\partial\xi^2\partial\eta^2} + \\ +4B_{26}\frac{\partial^4 W^{(s)}}{\partial\xi\partial\eta^3} + B_{22}\frac{\partial^4 W^{(s)}}{\partial\eta^4} = q^{(s)},\end{aligned} \tag{38}$$

where

$$\begin{aligned}\ell_{11} &= B_{11}\frac{\partial^2}{\partial\xi^2} + B_{66}\frac{\partial^2}{\partial\eta^2} + \\ &+ 2B_{16}\frac{\partial^2}{\partial\xi\partial\eta}, (1,2;\xi,\eta)\end{aligned} \tag{39}$$

$$\ell_{12} = B_{16}\frac{\partial^2}{\partial\xi^2} + (B_{12}+B_{66})\frac{\partial^2}{\partial\xi\partial\eta} + B_{26}\frac{\partial^2}{\partial\eta^2},$$
$$B_{11} = \left(a_{22}a_{66} - a_{26}^2\right)/\triangle,\ \ B_{22} = \left(a_{11}a_{66} - a_{16}^2\right)/\triangle,$$
$$B_{12} = (a_{16}a_{26} - a_{12}a_{66})/\triangle,\ \ B_{66} = \left(a_{11}a_{22} - a_{12}^2\right)/\triangle,$$
$$B_{16} = (a_{12}a_{26} - a_{22}a_{16})/\triangle,\ \ B_{26} = (a_{12}a_{16} - a_{11}a_{26})/\triangle,$$
$$\triangle = \left(a_{11}a_{22} - a_{12}^2\right)a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{26}^2 - a_{22}a_{16}^2.$$

Equations (37), written in dimensional coordinates, at $s = 0$ coincide with the classic equations of generalized plane problem, and Equation (38) coincides with the classic equation of the plate bend which has elastic symmetry plane. At $s > 0$ the right parts of these equations change, i.e. loading members, where syllables, characterizing general anisotropy, enter, too. Classic theory of plates does not take into account the boundary layer, which consists of antiplane and plane boundary layers, the parameters of which, when removing from the bank surface inwards the plate, diminish exponentially, but with essentially different indexes of the exponent. Asymptotic theory of anisotropic plates and shells is built in [12].

3 Nonclassical boundary-value problems of plates and shells

Classic theory of plates and shells considers only one class of boundary-value problems – it is considered that on the facial surfaces of the plate or shell the signs of the corresponding components of the stresses tensor are given. Meanwhile, a lot of important problems of construction, technics and seismology bring to other boundary value problems, particularly, when on the facial surfaces the values of the displacement vector components or mixed conditions are given. By direct check it is possible to be convinced, that for the solution of these classes of problems hypotheses of classical theory are not applicable. The asymptotic method permitted us to solve these classes of problems. Principally new asymptotics for stresses tensor and displacement vector was found.
The procedure of the solution finding remains unchangeable.
State the problem: to find in the domain G occupied by the plate of the equations system solutions (35), satisfying at $y = -h(\zeta = -1)$ conditions

$$u(\xi,\eta,-1) = u^-(\xi,\eta),\ (u,v,w) \tag{40}$$

and at $y = h(\zeta = 1)$ conditions

$$u(\xi,\eta,1) = u^+(\xi,\eta),\quad (u,v,w) \tag{41}$$

or

$$\sigma_{jz}(\xi,\eta,1) = \sigma_{jz}^+(\xi,\eta,1),\ \ j = x,y,z. \tag{42}$$

The solutions of the formulated problems again have the form of (12), and the solution of the outer problem – (18), yet, after the substitution of (18) into system (35) we only get noncontradictory system at

$$q_{\sigma_{ij}} = -1,\ \ q_{u,v,w} = 0, \tag{43}$$

as we see, asymptotics (43) principally differs from asymptotics (36) of the first boundary-value problem. Substituting (18), (43) into system (35), after the well-known procedure and satisfaction of conditions (40)-(42), we determine the final solutions of these problems. It is notable that the final solution of the outer problem here is fully expressed through the boundary functions $u^\pm$, $v^\pm$, $w^\pm$, $\sigma_{jz}^\pm$. If these functions are polynomials from ξ,η , the iterative process cuts off on the definite approximation, as a result we obtain mathematically exact solution of the outer problem. For the illustration of the above said we bring solutions of some space problems for orthotropic plates.
Let

$$\text{a) } u^- = v^- = w^- = 0,\ u^+ = const,\ v^+ = const,\ w^+ = const. \tag{44}$$

The iteration cuts off on the initial approximation, we have

$$\sigma_{xz} = G_{13}\frac{u^+}{2h},\ \sigma_{yz} = G_{23}\frac{v^+}{2h},\ \sigma_{zz} = \frac{1}{A_{33}}\frac{w^+}{2h} \tag{45}$$
$$\sigma_{xz} = \frac{A_{13}}{A_{33}}\frac{w^+}{2h},\ \sigma_{yy} = \frac{A_{23}}{A_{33}}\frac{w^+}{2h},\ \sigma_{xy} = 0$$
$$u = \frac{u^+}{2h}(h+z),\ v = \frac{v^+}{2h}(h+z),\ w = \frac{w^+}{2h}(h+z)$$
$$A_{13} = (a_{12}a_{23} - a_{13}a_{22})/\triangle_1,$$
$$A_{23} = (a_{12}a_{13} - a_{23}a_{11})/\triangle_1,$$
$$A_{33} = a_{13}A_{13} + a_{23}A_{23} + a_{33},$$
$$\triangle_1 = a_{11}a_{22} - a_{12}^2$$

$$\text{b) } u^- = v^- = w^- = 0,\ \ \sigma_{xz}^+ = \sigma_{yz}^+ = 0,\ \ \sigma_{zz}^+ = -(b\xi + c\eta) \tag{46}$$

The iteration cuts off after the first two step, we have

$$\sigma_{xx} = -A_{13}(b\xi + c\eta),\ \sigma_{yy} = -A_{23}(b\xi + c\eta),\ \sigma_{xy} = 0 \tag{47}$$
$$\sigma_{zz} = -(b\xi + c\eta),\ \sigma_{yz} = -A_{23}c(h-z),$$
$$\sigma_{xz} = -A_{13}b(h-z)$$
$$u = \frac{1}{2}A_{33}b(z+h)^2 - \frac{1}{2}A_{13}a_{55}b\left(3h^2 + 2zh - z^2\right)$$
$$v = \frac{1}{2}A_{33}c(z+h)^2 - \frac{1}{2}A_{23}a_{44}c\left(3h^2 + 2zh - z^2\right)$$
$$w = -A_{33}(z+h)(b\xi + c\eta)$$

In the monographs [12, 14] the solution of nonclassical boundary-value problems set for one-layered and layered anisotropic plates and shells are brought.

4 Space dynamic problems of thin bodies

The asymptotic method turned to be very effective for the solution of dynamic problems of beams, plates and shells. We shall show it for the orthotropic plates. We consider the following classes of forced vibrations, representing greatest interest for the supplements:
a) the vibrations of plates $G = \{(x, y, z) : 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, -h \leqslant z \leqslant h, \min(a, b) = \ell, h << \ell\}$ on the absolutely rigid base:

$$u(x, y, -h) = 0,\ v(x, y, -h) = 0,\ w(x, y, -h) = 0 \qquad (48)$$

when at $z = h$ the conditions

$$\sigma_{jz}(x, y, h) = \sigma_{jz}^{(+)}(x, y) \exp(i\Omega t), j = x, y, z \qquad (49)$$

or

$$u(x, y, h) = u^{(+)}(x, y) \exp(i\Omega t),\ (u, v, w)$$

are given;
b) vibrations, caused by the displacement vector, applied to the facial surface $z = -h$

$$u(x, y, -h) = u^{(-)}(x, y) \exp(i\Omega t),\ (u, v, w) \qquad (50)$$

and the surface $z = h$ is free or rigidly fastened

$$\sigma_{jz}(x, y, h) = 0,\ j = x, y, z \text{ or } u(x, y, h) = 0,\ (u, v, w) \quad (51)$$

c) the vibrations caused by supplements to the facial surfaces by harmonically changing in time loadings

$$\sigma_{jz}(x, y, \pm h) = \pm\sigma_{jz}^{\pm}(x, y) \exp(i\Omega t). \qquad (52)$$

It is required to find a solution of dynamic equations of elasticity theory of an orthotropic body, satisfying one of the groups of conditions (48)-(52).
Finding out the solution in the form of

$$\begin{gathered} \sigma_{\alpha\beta}(x, y, z, t) = \sigma_{jk}(x, y, z) \exp(i\Omega t), \qquad (53)\\ \alpha, \beta = x, y, z;\ \ j, k = 1, 2, 3,\\ (u, v, w) = \left(u_x(x, y, z), u_y, u_z\right) \exp(i\Omega t) \end{gathered}$$

Then passing to dimensionless coordinates $\xi = x/\ell, \eta = y/\ell, \zeta = z/h$ and displacements $U = u_x/\ell$, $V = u_y/\ell$, $W = u_z/\ell$ a singularly perturbed system will be obtained

$$\begin{gathered} \frac{\partial\sigma_{11}}{\partial\xi} + \frac{\partial\sigma_{12}}{\partial\eta} + \varepsilon^{-1}\frac{\partial\sigma_{13}}{\partial\zeta} + \varepsilon^{-2}\Omega_*^2 U = 0, \qquad (54)\\ (1, 2, 3; U, V, W),\\ \frac{\partial U}{\partial\xi} = a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33}, (1, 2; \xi, \eta; U, V),\\ \varepsilon^{-1}\frac{\partial W}{\partial\zeta} = a_{13}\sigma_{11} + a_{23}\sigma_{22} + a_{33}\sigma_{33},\\ \frac{\partial U}{\partial\eta} + \frac{\partial V}{\partial\xi} = a_{66}\sigma_{12},\ \frac{\partial W}{\partial\xi} + \varepsilon^{-1}\frac{\partial U}{\partial\zeta} = a_{55}\sigma_{13},\\ \frac{\partial W}{\partial\eta} + \varepsilon^{-1}\frac{\partial V}{\partial\zeta} = a_{44}\sigma_{23},\ \Omega_*^2 = \rho h^2\Omega^2,\ \varepsilon = h/\ell \end{gathered}$$

The solution of system (54) again has the form of (12), and for the outer problem asymptotics (18), (43) is just. Substituting it into system (54), all the stresses will be expressed through the displacements by formulae

$$\begin{gathered} \sigma_{11}^{(s)} = -A_{23}\frac{\partial W^{(s)}}{\partial\zeta} + A_{22}\frac{\partial U^{(s-1)}}{\partial\xi} - A_{12}\frac{\partial V^{(s-1)}}{\partial\eta}, \qquad (55)\\ \sigma_{22}^{(s)} = -A_{13}\frac{\partial W^{(s)}}{\partial\zeta} - A_{12}\frac{\partial U^{(s-1)}}{\partial\xi} + A_{33}\frac{\partial V^{(s-1)}}{\partial\eta},\\ \sigma_{33}^{(s)} = A_{11}\frac{\partial W^{(s)}}{\partial\zeta} - A_{23}\frac{\partial U^{(s-1)}}{\partial\xi} - A_{13}\frac{\partial V^{(s-1)}}{\partial\eta},\\ \sigma_{12}^{(s)} = \frac{1}{a_{66}}\left(\frac{\partial U^{(s-1)}}{\partial\eta} + \frac{\partial V^{(s-1)}}{\partial\xi}\right),\\ \sigma_{13}^{(s)} = \frac{1}{a_{55}}\left(\frac{\partial U^{(s)}}{\partial\zeta} + \frac{\partial W^{(s-1)}}{\partial\xi}\right),\\ \sigma_{23}^{(s)} = \frac{1}{a_{44}}\left(\frac{\partial V^{(s)}}{\partial\zeta} + \frac{\partial W^{(s-1)}}{\partial\eta}\right),\\ Q^{(m)} \equiv 0,\ m < 0,\\ A_{11} = \left(a_{11}a_{22} - a_{12}^2\right)/\triangle,\ A_{22} = \left(a_{22}a_{33} - a_{23}^2\right)/\triangle,\\ A_{33} = \left(a_{11}a_{33} - a_{13}^2\right)/\triangle,\ A_{12} = (a_{33}a_{12} - a_{13}a_{23})/\triangle,\\ A_{13} = (a_{11}a_{23} - a_{12}a_{13})/\triangle,\ A_{23} = (a_{22}a_{13} - a_{12}a_{23})/\triangle,\\ \triangle = a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 \end{gathered}$$

The displacements are determined from the equations

$$\begin{gathered} \frac{\partial^2 U^{(s)}}{\partial\zeta^2} + a_{55}\Omega_*^2 U^{(s)} = R_u^{(s)}\ (u, V; a_{55}, a_{44}; R_u, R_v), \qquad (56)\\ A_{11}\frac{\partial^2 W^{(s)}}{\partial\zeta^2} + \Omega_*^2 W^{(s)} = R_w^{(s)} \end{gathered}$$

where

$$\begin{gathered} R_u^{(s)} = -\frac{\partial^2 W^{(s-1)}}{\partial\xi\partial\zeta} - a_{55}\left(\frac{\partial\sigma_{11}^{(s-1)}}{\partial\xi} + \frac{\partial\sigma_{12}^{(s-1)}}{\partial\eta}\right), \qquad (57)\\ (u, v; \xi, \eta; a_{55}, a_{44}; 1, 2),\\ R_w^{(s)} = A_{23}\frac{\partial^2 U^{(s-1)}}{\partial\xi\partial\zeta} + A_{13}\frac{\partial^2 V^{(s-1)}}{\partial\eta\partial\zeta} - \left(\frac{\partial\sigma_{13}^{(s-1)}}{\partial\xi} + \frac{\partial\sigma_{23}^{(s-1)}}{\partial\eta}\right). \end{gathered}$$

The solution of system (56) is $U^{(s)} = U_0^{(s)} + u_\tau^{(s)}$ (U, V, W), where the first conjugation is the solution of the homogeneous equation, the second is the particular solution of the corresponding inhomogeneous equation. Then by formulae (55) the stresses are calculated and the conditions of each group (48)-(52) are satisfied. For example, the solution

$$\begin{aligned} & U^{(s)} = \\ = & \frac{1}{\cos 2\Omega_* \sqrt{a_{55}}} \left(-u_\tau^{(s)}(\xi, \eta, -1) \cos \Omega_* \sqrt{a_{55}} (1-\zeta) \right) + \\ + & \frac{\sqrt{a_{55}}}{\Omega_*} \left(\sigma_{13}^{+(s)} - \sigma_{13\tau}^{(s)}(\xi, \eta, 1) \right) \sin \Omega_* \sqrt{a_{55}} (1+\zeta) + \\ + & u_\tau^{(s)}(\xi, \eta, \zeta), \quad (U, V; a_{55}, a_{44}; 1, 2) \end{aligned} \tag{58}$$

$$\begin{aligned} & W^{(s)} = \\ = & \frac{1}{\cos \frac{2\Omega_*}{\sqrt{A_{11}}}} \left(-w_\tau^{(s)}(\xi, \eta, -1) \cos \frac{\Omega_*}{\sqrt{A_{11}}} (1-\zeta) \right) + \\ + & \frac{1}{\Omega_* \sqrt{A_{11}}} \left(\sigma_{33}^{+(s)} - \sigma_{33\tau}^{(s)}(\xi, \eta, 1) \right) \sin \frac{\Omega_*}{\sqrt{A_{11}}} (1+\zeta) + \\ + & w_\tau^{(s)}(\xi, \eta, \zeta), \end{aligned}$$

$$\sigma_{j3}^{+(0)} = \varepsilon \sigma_{j3}^{(+)}, \quad \sigma_{j3}^{+(s)} = 0, \quad s \neq 0, \quad j = 1, 2, 3,$$

$$\sigma_{13\tau}^{(s)} = \frac{1}{a_{55}} \left(\frac{\partial u_\tau^{(s)}}{\partial \zeta} + \frac{\partial W^{(s-1)}}{\partial \xi} \right), \ (1, 2; a_{55} a_{44}; \xi, \eta),$$

$$\sigma_{33\tau}^{(s)} = A_{11} \frac{\partial w_\tau^{(s)}}{\partial \zeta} - A_{23} \frac{\partial U^{(s-1)}}{\partial \xi} - A_{13} \frac{\partial V^{(s-1)}}{\partial \eta}$$

corresponding to conditions (48) and the first group (49). The stresses will be calculated by formulae (55).

Solution (58) will be final, if $\cos 2\Omega_* \sqrt{a_{55}} \neq 0$ $(a_{55}, a_{44}, 1/A_{11})$. Otherwise resonance rises, it is not difficult to write out the values of the resonance frequencies. It is easy to write out the solutions, corresponding to the rest of the variants of the boundary conditions (48)-(53).

The asymptotic method is effective for the solution of new classes of the beams, plates and shells problems, particularly problems of interaction of thin bodies with different physical fields [15].

5 Conclusions

The bases of asymptotic theory of beams, plates and shells are described. The connection with classic theory is revealed. Effectiveness of the asymptotic method for the solution of such class of problems, for which the hypotheses of classical theory are not applicable, is shown.

New classes of dynamic problems of elasticity theory for anisotropic plates are solved.

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