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The Algebraic Formulation: Why and How to Use it

Abstract: Finite Element, Boundary Element, Finite Volume, and Finite Difference Analysis are all commonly used in nearly all engineering disciplines to simplify complex problems of geometry and change, but they all tend to oversimplify. This paper shows a more recent computational approach developed initially for problems in solid mechanics and electro-magnetic field analysis. It is an algebraic approach, and it offers a more accurate representation of geometric and topological features.

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1 Introduction

The computational methods currently used in physics are based on the discretisation of the differential formulation, by using one of the many methods of discretisation, such as the finite element method (FEM), the boundary element method (BEM), the finite volume method (FVM), the finite difference method (FDM), and so forth. Infinitesimal analysis [1] has without doubt played a major role in the mathematical treatment of physics in the past, and will continue to do so in the future, but, as discussed in Section 2, we must also be aware that several important aspects of the phenomenon being described, such as its geometrical and topological features, remain hidden, in using the differential formulation [2]. This is a consequence not of performing the limit, in itself, but rather of the numerical technique used for finding the limit [3]. In Section 2, we analyse and compare the two most known techniques, the iterative technique and the application of the Cancelation Rule for limits. It is shown how the first technique, leading to the approximate solution of the algebraic formulation, preserves information on the trend of the function in the neighbourhood of the estimation point, while the second technique, leading to the exact solution of the differential formulation, does not. Under the topological point of view, this means that the algebraic formulation preserves information on the length scales associated with the solution, while the differential formulation does not. On the basis of this observation, it is also proposed to consider that the limit provided by the Cancelation Rule for limits is exact only in the broad sense (i.e., the numerical sense), and not in the narrow sense (involving also topological information). Moreover, applying the limit process introduces some limitations as regularity conditions must be imposed on the field variables. These regularity conditions, in particular those concerning differentiability, are the price we pay for using a formalism that is both very advanced and easy to manipulate.

The Cancelation Rule for limits leads to point-wise field variables, while the iterative procedure leads to global variables (Section 2.2), which, being associated with elements provided with an extent, are set functions. The use of global variables instead of field variables allows us to obtain a purely algebraic approach to physical laws, called the direct algebraic formulation [4] – [50]. The term “direct” emphasises that this formulation is not induced by the differential formulation, as is the case for the so-called discrete formulations (Section 2.3). By performing densities and rates of the global variables, it is then always possible to obtain the differential formulation from the direct algebraic formulation.

Since the algebraic formulation is developed before the differential formulation, and not vice-versa, the direct algebraic formulation cannot use the tools of the differential formulation for describing physical variables and equations. Therefore, the need for new suitable tools arises, which allows us to translate physical notions into mathematical notions through the intermediation of topology and geometry. The most convenient mathematical setting where to formulate a geometrical approach of physics is algebraic topology [51] – [57], the branch of mathematics that develops notions corresponding to those of the differential formulations [58] – [78], but based on global variables instead of field variables. This approach leads us to use algebra [79] – [96] instead of differential calculus. In order to provide a better understanding of what using algebra instead of differential calculus means, Section 3 deals with exterior algebra and geometric algebra [97] – [113], the

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two fundamental settings for the geometric study of spaces not just of geometric vectors, but of other vector-like objects such as vector fields or functions. Algebraic topology and its features are then treated in Section 4.

2 A Comparison Between Algebraic and Differential Formulations Under the Geometrical and Topological Viewpoints

2.1 Relationship Between how to Compute Limits and Numerical Formulations in Computational Physics

2.1.1 Some Basics of Calculus

In order to explain why the algebraic approach of the Cell Method (CM) is a winning strategy, if compared to that of the differential formulation, let us start with a brief excursus on the foundation of the differential formulation, calculus.

As is well known, calculus is the mathematical study of how things change and how quickly they change. Calculus uses the concept of **limit** to consider end behaviour in the infinitely large and to provide the behaviour of the output of a function as the input of that function gets closer and closer to a certain value. The second type of behaviour analysis is similar to looking at the function through a microscope and increasing the power of the magnification so as to zoom in on a very small portion of that function. This principle is known as **local linearity** and guarantees that the graph of any continuous smooth function looks like a line, if you are close enough to any point **P** of the curve. We will call this line the **tangent line** (from the Latin word *tangere*, "to touch"). **P** is the **point of tangency**.

Calculus has two major branches, differential calculus and integral calculus, related to each other by the fundamental theorem of calculus. Differential calculus concerns rates of change and slopes of curves, while integral calculus concerns accumulation of quantities and the areas under curves.

As far as differential calculus is concerned, there are two kinds of rate of change, **average rates of change** and **instantaneous rates of change**.

If a quantity changes from a value of m to a value of n over a certain interval from a to b, then the average rate of change is the change, $n - m$, divided by the length of the interval:

$$\text{Average rate of change} = \frac{n-m}{b-a}. \tag{1}$$

The instantaneous rate of change at a point on a curve describing the change is the slope of the curve at that point, where the slope of a graph at a point is the slope of the tangent line at that point (provided that the slope exists). Unless a function describing the change is continuous and smooth at a point, the instantaneous rate of change does not exist at that point.

By summarizing the differences between the two rates of change, average rates of change

- measure how rapidly (on average) a quantity changes over an interval,
- are a difference of output values,
- can be obtained by calculating the slope of the secant line (from the Latin word *secare*, "to cut") between two points, the line that passes through the two points on the graph,
- require data points or a continuous curve to calculate.

Instantaneous rates of change (or **rates of change** or **slopes of the curve** or **slope of the tangent line** or **derivatives**)

- measure how rapidly a quantity is changing at a point,
- describe how quickly the output is increasing or decreasing at that point,
- can be obtained by calculating the slope of the tangent line at a single point,
- require a continuous, smooth curve to calculate.

The line tangent to a graph at a point **P** can also be thought of as the limiting position of nearby secant lines – that is, secant lines through **P** and nearby points on the graph. The slope function is continuous as long as the original function is continuous and smooth.

Equations involving derivatives are called **differential equations** and a numerical formulation using differential equations is called **differential formulation**.

Limits give us the power to evaluate the behaviour of a continuous function at a point. In particular, limits may be used in order to evaluate the behaviour of the function giving the slope of another function, when this slope is a continuous function.

2.1.2 The ε–δ Definition of a Limit

The $\varepsilon - \delta$ definition of a limit is the formal mathematical definition of a limit.

Let f be a real-valued function defined everywhere on an open interval containing the real number c (except possibly at c) and let L be a real number. The statement:

$$\lim_{x \to c} f(x) = L, \tag{2}$$

means that, for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that, for all real x, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Symbolically:

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall x \, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon) . \tag{3}$$

Equation 2 can be read as "the limit of $f(x)$, as x approaches c, is L".

The above definition of a limit is true even if

$$f(c) \neq L. \tag{4}$$

Indeed, the function f does not need to be defined at c. The value of the limit does not depend on the value of $f(c)$, nor even that c be in the domain of f.

The absolute value $|x - c|$ in 3 means that x is taken sufficiently close to c from either side (but different from c). The limit value of $f(x)$ as x approaches c from the left, $x \to c^-$, is denoted as **left-hand limit**, and the limit value of $f(x)$ as x approaches c from the right, $x \to c^+$, is denoted as **right-hand limit**. Left-handed and right-handed limits are called **one-sided limits**. A limit exists only if the limit from the left and the limit from the right are equal. Consequently, the limit notion requires a smooth function.

The derivative $f'(x)$ of a continuous function $f(x)$ is defined as either of the two limits (if they exist):

$$f'(x) \triangleq \lim_{s \to x} \frac{f(s) - f(x)}{s - x}, \tag{5}$$

and

$$f'(x) \triangleq \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \quad h > 0, \tag{6}$$

where the argument of the limit in 6 is called a difference quotient. Both in 5 and 6, the slope of the tangent line is obtained by finding a formula for the slope of a secant line in terms of the length of the interval, $s - x$ and h, respectively, and then determining the limiting value of the formula as the length of the interval approaches zero.

Because the derivative of a function is the limit of quotients whose numerator comes from values of the function and whose denominator comes from values of the independent variable, the units in which the derivative is measured are the units in which values of the function are measured divided by the units in which the independent variable is measured.

Due to the absolute value $|x - c|$ in the $\varepsilon - \delta$ definition of a limit, the limit can also be evaluated on the backward difference of the function $f(x)$:

$$f'(x) \triangleq \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x - h)}{h}, \quad h > 0. \tag{7}$$

A further way for finding the derivative $f'(x)$ is making use of the central difference of the function $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)}{h}, \quad h > 0. \tag{8}$$

The ratio in 6 is not a continuous function at $h = 0$, because it is not defined there. In fact, the limit 6 has the indeterminate form $(\to 0)/(\to 0)$ as $h \to 0$, since both the numerator and the denominator approach 0 as $h \to 0$.

We can compute the limiting value 6 both in an approximated way, by reducing the error with subsequent iterations, as per the $\varepsilon - \delta$ definition of a limit, or in an exact way, by making use of the Cancellation Rule for limits.

In the perspective of a computational analysis using the differential formulation, it is obvious that the choice falls on the exact, rather than the approximated, computation of limits. In effect, in doing so, one can obtain an exact solution of the physical phenomenon under consideration only in few elementary cases, with simple geometric shapes of the domain and under particular boundary conditions. Anyway, the most important aspect is not that the exact numerical solution is hardly ever attained in real cases, but rather, that the choice itself of the term "exact" for the limit promised by the Cancelation Rule is not entirely appropriate. Actually, in order to provide the solution of the limit directly, the Cancellation Rule for limits reduces the order of zero both in the numerator and the denominator by one. Under the numerical point of view, this reduction is made by cancelling a quantity with the order of a length, both in the numerator and in the denominator. Under the topological point of view, we could say that the reduction degrades the solution, in the sense that, being deprived of one length scale, the solution given by the Cancelation Rule provides us with a lower degree of detail in describing the physical phenomenon under consideration. In other words, we pay the direct solution of the Cancelation Rule by losing some kind of information on the solution itself. This is why we can say that the solution provided by the Cancelation Rule is not exact in a narrow sense, but only in a broad sense.

2.1.3 A Discussion on the Cancelation Rule for Limits

Let us see, in more detail, which is the information we lose by reducing the order of zero of numerator and denominator. Due to the Cancelation Rule for limits, we can factor h out of the numerator in 6:

$$[f(x+h) - f(x)]|_{x=\bar{x}} = h \cdot g(h), \tag{9}$$

and cancel this common factor in the numerator and denominator. Then, we can find the limit by evaluating the new expression at $h = 0$, that is, by plugging in 0 for h, because the new expression is continuous at $h = 0$:

$$\lim_{h\to 0} \frac{h \cdot g(h)}{h} = g(h)|_{h=0}, \tag{10}$$

where the result is a real number.

The equality in 10, established by the Cancelation Rule, is undoubtedly numerically correct, in the sense that the results of the left- and right-hand-side expressions are actually numerically equal, but the way in which these results are achieved is radically different in the two cases [3], [16], [19]. As a matter of fact, the limit on the left side is defined on the open interval of length h, while the function $g(h)$ is evaluated for a given value of the variable, $h = 0$. This difference, negligible from the purely numerical viewpoint, is instead essential from the topological viewpoint. In effect, it is so much essential that the opportunity of using an algebraic rather than a differential formulation could be discussed just on the basis of the equality between the left- and right-hand-side terms in 10.

In order to understand this last statement, we must recall that the $\varepsilon - \delta$ definition of a limit implies choosing an (open) interval, containing the point in which we want to estimate a function, with the aim of making the distance between the points in which we compute the function and the point in which we want to estimate the function as small as we want. In other words, the limit on the left side in 10 is strictly bonded to the idea of interval of a point and cannot be separated from it. The result of the limit is the value to which the function output appears to approach as the computation point approaches the estimation point. For evaluating this result, we must enough carefully choose the computation points, in order to derive the trend of the output to a specific degree of approximation. That is, the result we obtain by choosing increasingly close points is only an estimation of the actual result and the approximation of the estimation is as much better (the degree of approximation is as much low) as the computation point is close to the estimation point. In conclusion, the $\varepsilon - \delta$ definition of a limit also bounds the limit to the notions of approximation and degree of approximation, or accuracy.

Completely different is the discussion on the right-hand-side function of 10. Actually, the new function $g(h)$ is computed at a point, the point $h = 0$, without any need of evaluating its trend on an interval. The consequence is that the result we obtain is exact (in a broad sense), and we do not need to prefix any desired accuracy for the result itself. This is very useful from the numerical point of view, but, from the topological point of view, we lose information on what happens approaching the evaluation point. It is the same type of information we lose in passing from the description of a phenomenon in a space to the description of the same phenomenon in the tangent space at the evaluation point.

Now, the question is if this is an acceptable loss. In the spirit of the principle of the local linearity, factorisation and cancelation of the common factors are usually carried out under the implicit assumption that it is. In particular, as far as numerical modelling is concerned, any possible implication of the Cancellation Rule on the numerical result was so far neglected, leading the researchers to use the differential formulation as if it were the natural formulation for computational physics. Better still, we could say that three centuries of differential formulation accustomed us to think that only a differential equation can provide the exact solution of a problem, in general, and a physical problem, in our peculiar case. This unconditional hope in the exact solution of the differential formulation did not leave space to any other consideration, in particular on what the cancelation of a length scale may involve from the physical and topological viewpoints.

Anyway, the Cancellation Rule can actually be employed only in those cases where the specific phenomenon uniquely depends on what happens at the point under consideration [3]. In effect, this happens in few physical problems, while, in most cases, the physical phenomenon under consideration also depends on what happens in a neighbourhood centred at the point. By extension of 10 to functions of more than one variable, studying the physical phenomenon as if it were a point-wise function means that we are using the right-hand side of 10, while studying the physical phenomenon as a function of all the points contained in a neighbourhood means that we are using the left-hand side, with h approaching zero but never equal to zero. In the first case, we are facing a differential formulation, while, in the second case, we are facing an algebraic formulation.

Operatively, we are using an algebraic formulation whenever we choose increasingly close points (to both the right and the left) of the estimation point, until the outputs remain constant to one decimal place beyond the desired accuracy for two or three calculations. How much the

computation points must be close to the estimation point depends on how fast the result of the limit is approached as we approach the point in which the limit is estimated. Therefore, the dimension of the neighbourhood is fixed by the trend of the phenomenon around the point under consideration, or, in other words, the distance δ for the evaluation of $f'(c)$ depends both on the error ε and on $f''(c)$. The information we lose by using the Cancellation Rule lies just in the trend of the phenomenon, that is, in the curvature, since the curvature cannot be accounted for in passing from a space to its tangent space at the evaluation point.

In the differential formulation, the notion of limit is used not only for defining derivatives, but also densities. In this second case, the denominator that tends to zero has the dimensions of a length raised to the power of 1, 2, or 3. The Cancelation Rule for limits can be employed also in this second case, by factorizing and cancelling length scales in dimension 1, 2, or 3, respectively. This leads to point-wise variables in any cases, the line, surface, and volume densities.

Finally, the Cancelation Rule for limits is used also for finding rates, by factorizing and cancelling time scales in dimension 1. This latter time, the limit, which is a time derivative, provides an instant-wise variable.

In conclusion, with reference to the space of the physical phenomena, the differential formulation provides the numerical solution in the tangent space of degree 0, where we can describe each physical phenomenon in terms of the space elements of degree 0, the points, and the time elements of degree 0, the time instants. Conversely, the algebraic formulation of the CM allows us to take account of, we could say, the curvatures in space and time at a point, where a point of the space of the physical phenomena is a given physical phenomenon, in a given configuration, at a given time instant [3].

In other words, the solution given by the Cancelation Rule for limits is the projection of the actual solution from the multi-dimensional space to the tangent space of degree 0. The cancelation of the common factors in numerator and denominator acts as a projection operator and the equality in 10 should more properly be substituted by a symbol of projection. Consequently, the solution of the differential formulation is the shadow of the actual solution in the tangent space of degree 0. On the contrary, the CM, avoiding the projection process, provides us with a higher degree solution, approximated in any case, which is more adherent to the physical nature of the phenomenon under consideration.

It is worth noting that concerns about the soundness of arguments involving infinitesimals date back to ancient Greek mathematics, with Archimedes replacing such proofs with ones using other techniques, such as the method of exhaustion. This discussion also permeated the history of calculus, which is fraught with philosophical debates about the meaning and logical validity of fluxions, Newton's term for differential calculus (fluents was his term for integral calculus).

By using the language of non-standard analysis, which is a rigorous formalisation of calculations with infinitesimals, the infinite and infinitesimal quantities can be treated by the system of **hyperreal numbers**, or **hyperreals**, or **nonstandard reals**. Denoted by ${}^*\mathbb{R}$, the hyperreal numbers are an extension of the real numbers, $\mathbb{R}$, that contains numbers greater than anything of the form:

$$1 + 1 + \cdots + 1. \tag{11}$$

Such a number is infinite, and its reciprocal is infinitesimal.

The hyperreal numbers satisfy the transfer principle, a rigorous version of Leibniz's heuristic Law of Continuity. The transfer principle states that true first order statements about $\mathbb{R}$ are also valid in ${}^*\mathbb{R}$. Therefore, the hyperreals were logically consistent if and only if the reals were. This put to rest the fear that any proof involving infinitesimals might be unsound, provided that they were manipulated according to the logical rules which Robinson delineated.

Non-standard analysis deals primarily with the **hyperreal line**, which is an extension of the real line, containing infinitesimals, in addition to the reals (Fig. 1). In the hyperreal line every real number has a collection of numbers (called a **monad**, or **halo**) of hyperreals infinitely close to it.

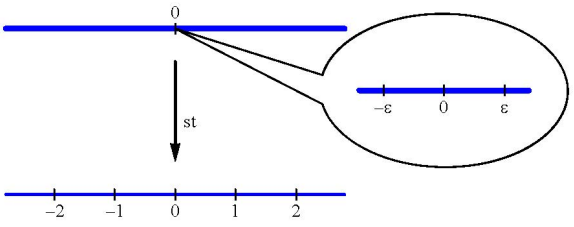


Figure 1: The bottom line represents the "thin" real continuum. The line at top represents the "thick" hyperreal continuum. The "infinitesimal microscope" is used to view an infinitesimal neighbourhood of 0.

The **standard part function** is a function from the limited (finite) hyperreal to the reals. It associates with a finite hyperreal x, the unique standard real number x_0 which is infinitely close to it (Fig. 1):

$$\mathrm{st}(x) = x_0. \tag{12}$$

As such, the standard part function is a mathematical implementation of the historical concept of adequality introduced by Pierre de Fermat. It can also be thought of as a mathematical implementation of Leibniz's Transcendental Law of Homogeneity. The standard part function was first defined by Abraham Robinson as a key ingredient in defining the concepts of the calculus, such as the derivative and the integral, in non-standard analysis.

The standard part of any infinitesimal is 0. Thus, if N is an infinite hypernatural, then $1/N$ is infinitesimal, and

$$\mathrm{st}\left(\frac{1}{N}\right) = 0. \tag{13}$$

The standard part function allows the definition of the basic concepts of analysis, such as derivative and integral, in a direct fashion. The derivative of f at a standard real number x becomes

$$f'(x) = \mathrm{st}\left(\frac{{}^{\star}f(x+\Delta x) - {}^{\star}f(x)}{\Delta x}\right), \tag{14}$$

where Δx is an infinitesimal, smaller than any standard positive real, yet greater than zero, and ${}^{\star}f$ is the natural extension of f to the hyperreals (* is the transfer operator applied to f). Similarly, the integral is defined as the standard part of a suitable infinite sum.

In this approach, $f'(x)$ is the real number infinitely close to the hyperreal argument of st. For example, the non-standard computation of the derivative of the function $f(x) = x^2$ provides

$$f'(x) = \mathrm{st}\left(\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right) = \mathrm{st}(2x + \Delta x) = 2x, \tag{15}$$

since

$$2x + \Delta x \approx 2x, \tag{16}$$

where the symbol "≈" is used for indicating the relation "is infinitely close to". In order to make $f'(x)$ a real-valued function, we must dispense with the final term, Δx, which is the error term. In the standard approach using only real numbers, that is done by taking the limit as Δx tends to zero. In the non-standard approach using hyperreal numbers, the quantity Δx is taken to be an infinitesimal, a nonzero number that is closer to 0 than to any nonzero real, which is discarded by the standard part function.

The notion of limit can easily be recaptured in terms of the standard part function, st, namely:

$$\lim_{x \to c} f(x) = L, \tag{17}$$

if and only if, whenever the difference $|x - c|$ is infinitesimal, the difference $|f(x) - L|$ is infinitesimal, as well. In formulas

$$\mathrm{st}(x) = c \Rightarrow \mathrm{st}(f(x)) = L. \tag{18}$$

The standard part of x is sometimes referred to as its **shadow**. Therefore, the derivative of $f(x)$ is the shadow of the difference quotient [3].

We can thus conclude that the standard part function is a form of projection from hyperreals to reals. As a consequence, using the algebraic formulation is somehow similar to performing non-standard calculus, the modern application of infinitesimals, in the sense of non-standard analysis, to differential and integral calculus. In effect, the extension of the real numbers, $\mathbb{R}$, is equivalent to providing the space of reals with a supplementary structure of infinitesimal lengths. This configures the hyperreal number system as an infinitesimal-enriched continuum, and the algebraic approach can be viewed as the algebraic version of non-standard calculus.

The great advantage of the infinitesimal-enrichment is that of successfully incorporating a large part of the technical difficulties at the foundational level of non-standard calculus. Similarly, in the algebraic formulation many numerical problems, mainly instability or convergence problems, are avoided by the presence of a supplementary structure of (finite) lengths both in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

The use of an algebraic formulation instead of a differential one also has a justification based on the microstructure of matter. As previously said, when performing densities and rates, the intention is to formulate the field laws in an exact form. Nevertheless, the density finding process is carried out without considering whether a physical significance exists for the limit one is performing. In fact, since matter is discrete on a molecular scale, performing the limit process of the mean densities with the extent of the geometrical object going to zero makes no physical sense.

2.2 Field and Global Variables

We can classify the physical variables according to their nature, global or local.

Broadly speaking, the global variables are variables that are neither densities nor rates of other variables. In particular, we will call:

- **Global variable in space**, or space global variable, a variable that is not the line, surface or volume density of another variable.
- **Global variable in time**, or time global variable, a variable that is not the rate or time derivative of another variable.

The field variables are obtained from the global variables as densities of space global variables and rates of

time global variables. Due to their point-wise nature, they are local variables.

The variables obtained by line, surface, or volume integration of field variables, the integral variables, are global variables, but there also exist global variables that are not integral variables.

One of the main consequences of using the left- rather than the right-hand side in 10 is that the nature of physical variables is different in the algebraic rather than the differential formulation, global in the first case and local in the second case.

In effect, in the differential formulation, some variables arise directly as functions of points and time instants, while the remaining variables are reduced to points and time instants functions by performing densities and rates and making use of the Cancelation Rule for limits. Thus, the physical variables of the differential formulation are point-wise and/or instant-wise field functions. Moreover, since we are ignoring approximations and accuracies, we must consider any infinitesimal region around a given point as a uniform region.

On the other hand, avoiding factorisation and cancelation, the algebraic formulation uses global variables. Moreover, since, in doing so, the algebraic formulation preserves the length and time scales of the global physical variables, the physical variables, in spatial description, turn out to be naturally associated with one of the four space elements (point, line, surface, and volume, which are denoted with their initial capital letters in bold, **P**, **L**, **S**, and **V**, respectively, as shown in Fig. 2) and/or with one of the two time elements (time instant and time interval, which are denoted with **I** and **T**, respectively, as shown in Fig. 3). Due to their association with elements provided with an extent, the global variables are set functions.

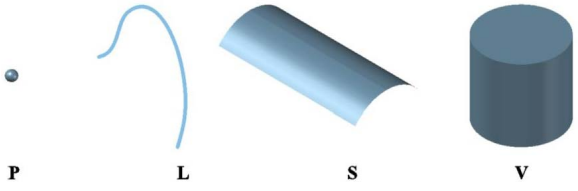


Figure 2: The four space elements and their notations.

Figure 3: The two time elements and their notations.

Physical theory	**Global variable**	**Referent geometrical object**
Thermal conduction	**Temperature**	[P]
Thermal conduction	**Electrical potential**	[P]
Solid mechanics	**Displacement**	[P]
Fluid mechanics	**Velocity**	[P]
Electromagnetism	**Voltage**	[L]
Solid mechanics	**Stretching**	[L]
Acoustics	**Velocity circulation**	[L]
Electromagnetism	**Charge flow**	[S]
Fluid dynamics	**Discharge**	[S]
Thermal conduction	**Heat**	[S]
Solid mechanics	**Surface force**	[S]
Mechanics	**Mass content**	[V]
Mechanics	**Momentum content**	[V]
....		

Figure 4: Association between physical variables and points (**P**), lines (**L**), surfaces (**S**), and volumes (**V**) for several physical theories.

The association between physical variables and space elements was discovered by Franklin Branin [114], an IBM engineer, in the late 1960s. Branin also pointed out the intimate relation between algebraic topology, network theory, and the vector calculus.

As an example of association between physical variables and space elements, the flux and the flow are associated with a surface (Fig. 4). The voltage, the magnetomotive force, the line integral of the fluid velocity, and the work of a force along a line in a force field are associated with a line. The mass content, the energy content, the entropy content, and the momentum content are associated with a volume. Moreover, displacements in solid mechanics, the kinetic potential in flow mechanics, the gauge function of electromagnetism, the iconal function in optics, and temperature are examples of variables associated with points in space (and time), without being densities or rates.

In the differential material description of physics, the physical laws are ordinary differential equations, where the field variables are time derivatives and rates, while the algebraic material description makes use of global variables in space, which change with the two time elements (in material formulations, all the variations are time variations, while the system of which we are following the motion is prefixed).

In the differential spatial description of physics, the physical laws are partial differential equations, where the field variables are densities, while the algebraic spatial description makes use of global variables in time, which change with the four space elements (in spatial formulations, all the variations are geometrical variations, while the instant in which to compute the variations is prefixed).

In some physical theories, one may use both the material and the spatial descriptions. This is the case of fluid dynamics, electromagnetism, and quantum mechanics [115].

Note that there is a remarkable difference between global and field variables, when the domain of the physical problem is composed of more than one medium. In fact, while global variables are continuous through the interface of two different media, their variations can be discontinuous. Consequently, even field variables, which are densities and rates, are generally discontinuous. The same can be said for any other kind of discontinuities of the domain or the sources of the physical problem (some examples of continuity of the global variables are collected in Fig. 5).

Global variables are essential to the philosophy of the Cell Method, since, by using these variables, it is possible to obtain an algebraic formulation directly and, what is most important, the global variables involved in obtaining the formulation do not have to be differentiable functions.

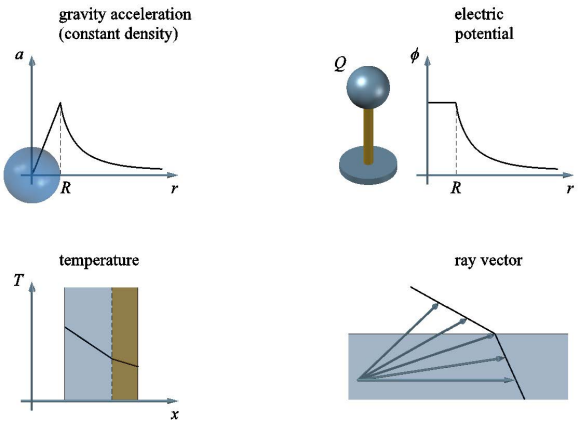


Figure 5: Continuity of the global variables associated with points in domains made of more than one medium.

Moreover, by using the limit process on the mean densities and rates of the global variables, we can obtain the traditional field functions of the differential formulation.

The main difference between the two formulations – algebraic and differential – lies precisely in the fact that the limit process is used in the latter. In effect, since calculating the densities and rates of the domain variables is based on the assumption that global variables are continuous and differentiable, the range of applicability of differential formulation is restricted to regions without material discontinuities or concentrated sources, while that of the algebraic formulation is not restricted to such regions.

The purpose of this paper is not simply to classify the physical variables according to the geometrical and time elements on which they depend. The question takes on a much more deep meaning, since the ability of the CM to solve some of the problems affected by spurious solutions in the differential formulation lies, in part, just on the association between variables and space and/or time elements.

The comparison between global and field variables deserves a final comment, which will be essential for the comprehension of the difference between computational methods based on an algebraic rather than a differential formulation. Even in this last case, the comment arises from the association between physical variables and space elements in dimensions 0, 1, 2 and 3, which is ignored in the differential formulation, while it is emphasised in the algebraic formulation. In particular, the space distribution of the point-wise field functions of the differential formulation requires the introduction of coordinate systems, whose purpose is to create a correspondence between the points of the space and the numbers, that is, their coordinates. This allows us to describe geometry through mathematics. Now, in a differential formulation, the coordinate systems, together with a time axis, are the most appropriate frameworks to treat variables, since variables are point- and instant-wise functions, but we cannot state the same in an algebraic formulation, where variables also depend on geometrical and time objects of dimension greater than 0. Consequently, we must introduce some kind of generalisation of the coordinate systems and time axes, in order to describe global variables in the algebraic formulation. In particular, we need some suitable reference structures, whose elements are endowed with spatial or time extents.

The generalisation of the coordinate systems is achieved by introducing cell complexes and by associating the global variables with the related space elements of the cell complexes, that is, the four space elements, **P**, **L**, **S**, and **V**, of the cell complexes. This allows us to describe global variables directly.

The branch of the mathematics that develops notions corresponding to those of the differential formulations, but based on global variables instead of field variables, is the algebraic topology. This is why the algebraic formulation uses notations of algebraic topology, in order to describe the four space elements of the cell complexes. Physical notions are therefore translated into mathematical notions through the intermediation of topology and geometry, and we are led to use algebra instead of differential calculus. Many properties of cell complexes were developed in algebraic topology, among them the notions of **orientation** [116], **duality**, and **incidence numbers** (Section 4).

In this context, the role played by coordinate systems is substituted by a suitable labelling of the four space elements, which also takes into account some of the topological features of the four space elements, as their inner and outer orientations and duality (Section 3.2). We can thus

conclude that the cell complexes and their labelling are the algebraic version of the points and their continuous mapping, established by the coordinate systems of the differential setting, respectively.

Also the generalisation of the time axis is made by means of cell complexes, which, in this second case, are associated with the two time elements, that is, time instant, **I**, and time intervals, **T**. Algebraic topology is then employed even for the time cell complexes, in order to describe the time global variables.

In conclusion, the CM cell complexes do not have the same role that cell complexes have in the differential formulation. In fact, the CM cell complexes are not the result of a domain discretisation, a process needed in differential formulations for managing the working region, as in the case of the FEM. They are required in the algebraic formulation in order to provide a suitable structure for describing global variables, since global variables are associated not only with points and time instants, as for the differential formulation, but also with lines, surfaces, volumes, and time intervals.

2.3 A Comparison Between the Cell Method and the Discrete Methods

The Cell Method has often been compared to the direct or physical approach, initially used in the finite element method, or to the finite volume method and the finite difference method. In particular, the Cell Method may seem very similar to the vertex-based scheme of the FVM. However, on deeper analysis of the similarities and differences between the CM and the discrete methods ([3], [117]), the CM is shown to be, for the moment, the only truly algebraic method.

In effect, the key point to bear in mind in building a truly algebraic formulation is that all operators must be discrete and use of the limit process must be avoided at each level of the formulation. The direct or physical approach is not suited to this, since it starts from the pointwise conservation equations of differential formulation (Fig. 6) and, for differential formulation, there is the need for field functions, which depend on the point position and the instant value.

If the field functions are not described directly in terms of point position and instant values, they can be obtained by calculating the densities and rates of the global variables, as shown in Fig. 6.

The algebraic formulation is then derived from the differential formulation through an integration process (Fig. 6) that is needed because, while in differential formulation geometry must be eliminated from the physical laws, in the numerical solution geometry is essential.

The finite volume method and the finite difference method are also based on a differential formulation (Fig. 6). The Cell Method, on the contrary, uses global variables and formulates the governing equations in algebraic form directly.

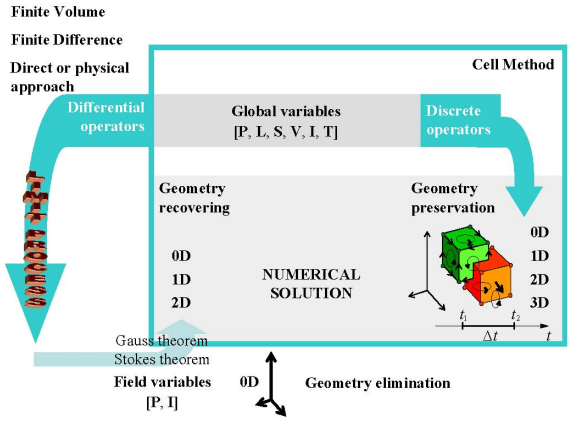


Figure 6: Building an algebraic formulation through the direct or physical approach, the finite volume method, the finite difference method, and the Cell Method.

3 Algebra and the geometric interpretation of vector spaces

The exterior algebra provides an algebraic setting in which to answer geometric questions and makes sense for spaces not just of geometric vectors, but of other vector-like objects such as vector fields or functions. The exterior algebra, or Grassmann algebra after Hermann Grassmann, is the algebraic system whose product is the **exterior product**, or **wedge product**. It is the largest algebra that supports an alternating product on vectors.

Vector algebra and geometric algebra (GA) are alternative approaches to providing additional algebraic structures on vector spaces, with geometric interpretations, particularly vector fields in multivariable calculus and applications in mathematical physics. Vector algebra is specific to Euclidean three-space, while geometric algebra uses multilinear algebra and applies in all dimensions and signatures, notably 3+1 space-time as well as 2 dimensions. They are mathematically equivalent in 3 dimensions, though the approaches differ. Vector algebra is more widely used in elementary multivariable calculus,

while geometric algebra is used in some more advanced treatments and is proposed for elementary use as well. In advanced mathematics, particularly differential geometry [118], [119], neither is widely used, with differential forms [51], [63], [72] being far more widely used.

Geometric algebra gives emphasis on geometric interpretations and physical applications. The distinguishing multiplication operation that defines the geometric algebra as a unital ring [120] is the **geometric product**.

3.1 Inner and Outer Products Originated by the Geometric Product

Taking the geometric product among vectors can yield bivectors, trivectors, or general p-vectors. The addition operation combines these into general multivectors, which are the elements of the ring. This includes, among other possibilities, a well-defined sum of a scalar and a vector, an operation that is impossible by the traditional vector addition.

Given a finite-dimensional real quadratic space $V = \mathbb{R}^n$ with quadratic form:

$$Q = V \to \mathbb{R}; \tag{19}$$

the geometric algebra for this quadratic space is the Clifford algebra $\mathcal{C}\ell(V, Q)$. If a, b, and c are vectors, then the geometric product is associative:

$$a(bc) = (ab)c; \tag{20}$$

and distributive over addition:

$$a(b + c) = ab + ac. \tag{21}$$

Moreover, the geometric product has the following property:

$$a^2 \in \mathbb{R}; \tag{22}$$

where the square is not necessarily positive. A further important property of the geometric product is the existence of elements with multiplicative inverse, also known as units. If $a^2 \neq 0$ for some vector a, then a^{-1} exists and is equal to

$$a^{-1} = \frac{a}{a^2}. \tag{23}$$

Not all the elements of the algebra are necessarily units. For example, if u is a vector in V such that $u^2 = 1$, the elements $1 \pm u$ have no inverse since they are zero divisors:

$$(1 - u)(1 + u) = 1 - uu = 1 - 1 = 0. \tag{24}$$

From the axioms above, we find that we may write the geometric product of any two vectors a and b as the sum of a symmetric product and an antisymmetric product:

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba). \tag{25}$$

The symmetric product defines the inner product [101] of vectors a and b:

$$a \cdot b := \frac{1}{2}(ab + ba) = \frac{1}{2}\left((a + b)^2 - a^2 - b^2\right); \tag{26}$$

which is a real number, because it is a sum of squares, and is not required to be positive definite. It is not specifically the inner product on a normed vector space.

The antisymmetric product in 25 is equal to the exterior product of the contained exterior algebra and defines the outer product of vectors a and b:

$$a \wedge b := \frac{1}{2}(ab - ba). \tag{27}$$

Note that the outer product of GA is not the outer product of linear algebra. Geometrically, the outer product $a \wedge b$ can be viewed by placing the tail of the arrow b at the head of the arrow a and extending vector a along vector b (Fig. 7).

The resulting entity is a two-dimensional subspace, and we call it a bivector. It has an area equal to the size of the parallelogram spanned by a and b. The senses of a and b orientate the sides of the parallelogram and define a sense of traversal of its boundary. In the case of Fig. 7, the traversal sense is a clockwise sense, which can be depicted by a clockwise arc.

Note also that a bivector has no shape. Using a parallelogram to visualise the area provides an intuitive way of understanding, but a bivector is just an oriented area, in the same way a vector is just an oriented length (Fig. 8).

Being the extension of a vector along another vector, the outer product can be considered, in some ways, the opposite of the dot product, which projects a vector onto another vector.

The geometric interpretation of the outer product $b \wedge a$ is achieved by placing the tail of the arrow a at the head of the arrow b and extending vector b along vector a. This reverses the circulation of the boundary (Fig. 9), while it does not change the area of the parallelogram spanned by a and b.

In conclusion, the geometric product in 25 can be written as the sum between a scalar and a bivector:

$$ab = a \cdot b + a \wedge b. \tag{28}$$

The scalar and the bivector are added by keeping the two entities separated, in the same way in which, in com-

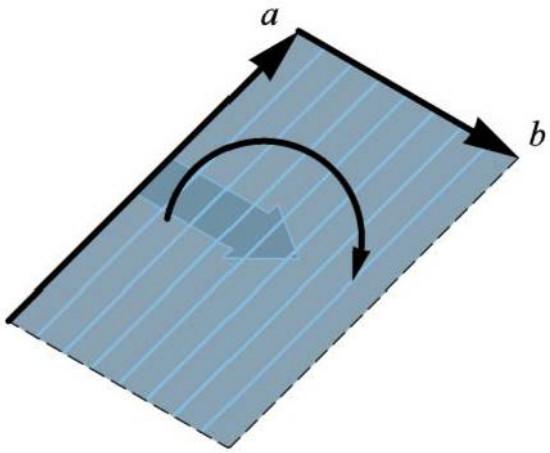


Figure 7: The extension of vector a along vector b provides the geometric interpretation of the outer product $a \wedge b$.

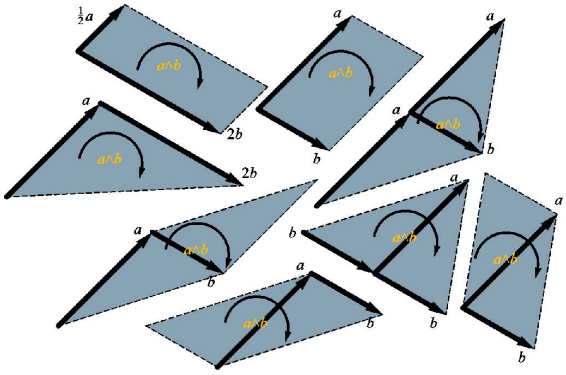


Figure 8: Different shapes for the same bivector, $a \wedge b$.

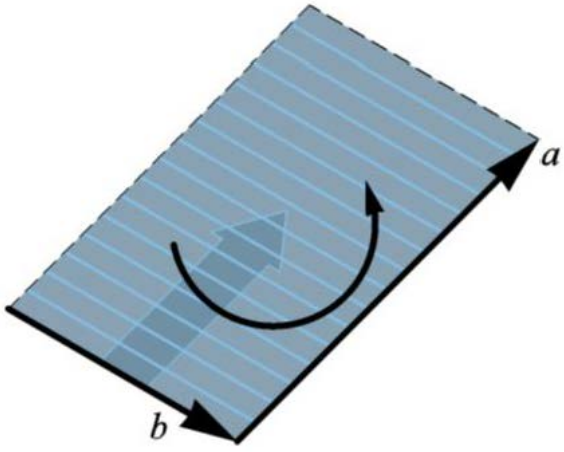


Figure 9: The extension of vector b along vector a provides the geometric interpretation of the outer product $b \wedge a$.

plex numbers, we keep the real and imaginary parts separated. Being the sum of a scalar and a bivector, the geometric product is a particular combination of blades, that is, is a multivector.

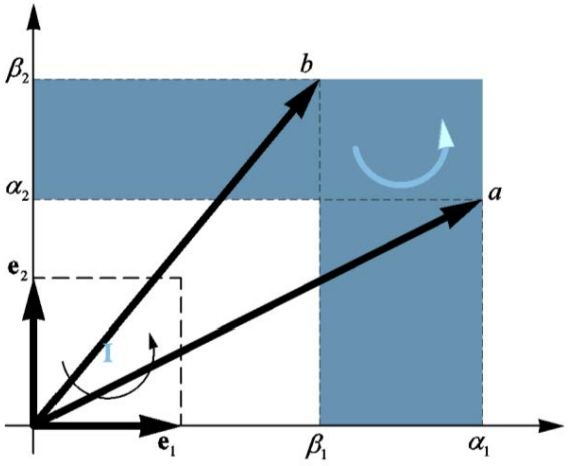


Figure 10: Vector decomposition in two-dimensional basis.

In a geometric algebra for which the square of any nonzero vector is positive, the inner product of two vectors can be identified with the dot product of standard vector algebra.

As we can express an n-dimensional vector as a linear combination of the basis vectors, e_i, writing it as a n-tuple of real numbers, likewise bivectors can be expressed as linear combinations of basis bivectors. By way of example, consider the real number decomposition $\mathbf{a} = (\alpha_1, \alpha_2)$ and $\mathbf{b} = (\beta_1, \beta_2)$, of the two vectors $\mathbf{a}$ and $\mathbf{b}$ of the Euclidian Plane $\mathbb{R}^2$, onto the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ (Fig. 10):

$$\mathbf{a} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2; \tag{29}$$

$$\mathbf{b} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2. \tag{30}$$

The outer product of $\mathbf{a}$ and $\mathbf{b}$ provides

$$\mathbf{a} \wedge \mathbf{b} = (\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2) \wedge (\beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2) = \\ = \alpha_1\mathbf{e}_1 \wedge \beta_1\mathbf{e}_1 + \alpha_1\mathbf{e}_1 \wedge \beta_2\mathbf{e}_2 + \alpha_2\mathbf{e}_2 \wedge \beta_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \wedge \beta_2\mathbf{e}_2. \tag{31}$$

By reordering the scalar multiplications, we obtain

$$\mathbf{a}\wedge\mathbf{b} = \alpha_1\beta_1\mathbf{e}_1\wedge\mathbf{e}_1 + \alpha_1\beta_2\mathbf{e}_1\wedge\mathbf{e}_2 + \alpha_2\beta_1\mathbf{e}_2\wedge\mathbf{e}_1 + \alpha_2\beta_2\mathbf{e}_2\wedge\mathbf{e}_2. \tag{32}$$

Now, since the outer product of a vector by itself equals zero, the outer product $\mathbf{a} \wedge \mathbf{b}$ can be written as

$$\mathbf{a} \wedge \mathbf{b} = \alpha_1\beta_2\mathbf{e}_1 \wedge \mathbf{e}_2 + \alpha_2\beta_1\mathbf{e}_2 \wedge \mathbf{e}_1; \tag{33}$$

where, since (Fig. 10):

$$\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{I}; \tag{34}$$

$$\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{I}; \tag{35}$$

we can finally express the bivector $\mathbf{a} \wedge \mathbf{b}$ in terms of the basis bivector, $\mathbf{I}$, which, in Euclidean plane represents $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$:

$$\mathbf{a} \wedge \mathbf{b} = (\alpha_1\beta_2 - \alpha_2\beta_1)\,\mathbf{I}. \tag{36}$$

Since $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$ denote the vector components of $\mathbf{a}$ and $\mathbf{b}$, the round bracketed term can be geometrically interpreted as the norm of a sum of bivectors, that is, an algebraic sum of areas:

$$\mathbf{a} \wedge \mathbf{b} = \begin{bmatrix} +1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1\beta_2 \\ \alpha_2\beta_1 \end{bmatrix} \mathbf{I} = \boldsymbol{\alpha}_1 \wedge \boldsymbol{\beta}_2 + \boldsymbol{\alpha}_2 \wedge \boldsymbol{\beta}_1; \tag{37}$$

where $\alpha_1\beta_2$ is multiplied by +1 because $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\beta}_2$ has the same sense as $\mathbf{I}$, while $\alpha_2\beta_1$ is multiplied by −1 because $\boldsymbol{\alpha}_2 \wedge \boldsymbol{\beta}_1$ has the opposite sense.

The difference between $\alpha_1\beta_2$ and $\alpha_2\beta_1$ equals the size of the coloured area in Fig. 10.

Since all spaces $\mathbb{R}^n$ generate a set of basis blades that make up a geometric algebra of subspaces, denoted by $\mathcal{C}\ell_n$, a possible basis for $\mathcal{C}\ell_2$, generated by $\mathbb{R}^2$, is

$$\{\ \underbrace{\mathbf{1}}_{\substack{basis\\ scalar}}\ ,\ \underbrace{\mathbf{e}_1, \mathbf{e}_2}_{\substack{basis\\ vectors}}\ ,\ \underbrace{\mathbf{I}}_{\substack{basis\\ bivector}}\ \},\quad , \tag{38}$$

where $\mathbf{1}$ is used to denote the basis 0-blade, or scalar-basis.

Analogously, in three-dimensional space $\mathbb{R}^3$, there are three basis vectors, three basis bivectors, and one basis trivector, besides than the scalar-basis (Fig. 11). Therefore, a possible basis for $\mathcal{C}\ell_3$ is:

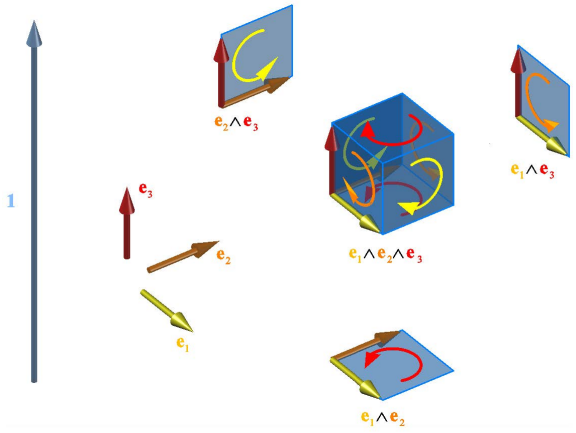


Figure 11: Standard *n-vector* basis in 3D: unit scalar 1 (represented by a black number line), unit vectors, unit bivectors, and a unit trivector.

$$\left\{\ \underbrace{\mathbf{1}}_{\substack{basis\\ scalar}}\ ,\ \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\substack{basis\\ vectors}}\ ,\ \underbrace{\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}}_{\substack{basis\\ bivectors}}\ ,\ \underbrace{\mathbf{e}_{123}}_{\substack{basis\\ trivector}}\ \right\}. \tag{39}$$

The total number of basis blades for an algebra can be calculated by adding the numbers required for all basis k-blades:

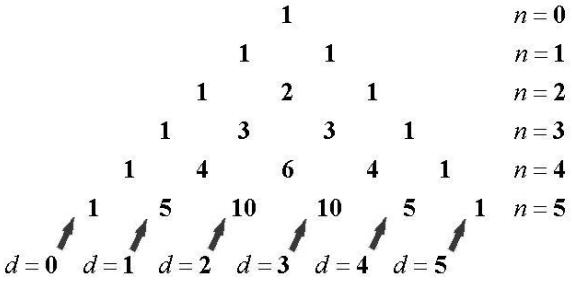


Figure 12: The first six rows of Pascal's triangle, from row 0 to row 5.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n; \tag{40}$$

where the number of basis k-blades needed in an n-dimensional space to represent arbitrary k-blades is provided by the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{41}$$

This is because a basis k-blade is uniquely determined by the $k$basis vectors from which it is constructed. There are $n$different basis vectors in total and $\binom{n}{k}$ is the number of ways to choose $k$elements from a set of $n$elements.

Therefore, the number of basis k-blades in an n-dimensional space is the k-th entry in row n of Pascal's triangle (Fig. 12).

3.2 The Features of *p*-vectors and the Orientations of Space Elements

The various objects of geometric algebra, the p-vectors, are charged with three attributes, or features: **attitude**, **orientation**, and **magnitude**. The second feature, taken singularly and combined with the first feature, gives rise to the two kinds of orientation in space, inner and outer orientations.

3.2.1 Inner Orientation of Space Elements

The second feature of p-vectors, the orientation, is, more properly, an **inner orientation**, because it does not depend on the embedding space. For example, a vector $\mathbf{u}$ in three dimensions has an orientation given by the sense of a straight line parallel to it (often indicated by an arrowhead, as in Fig. 13).

Similarly, a bivector $\mathbf{u} \wedge \mathbf{v}$ in three dimensions, generated by the exterior product between $\mathbf{u}$ and $\mathbf{v}$, has an orientation (sometimes denoted by a curved arrow in the plane containing $\mathbf{u}$ and $\mathbf{v}$, as in Fig. 13) indicating a choice of sense of traversal of its boundary (its **circulation**). Finally, a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ has an orientation (sometimes denoted by curved arrows in the planes of its 2-faces – see Section 4.3 – as in Fig. 13) indicating a choice of sense of circulation of the faces on its boundary. All the former orientations are inner orientations.

The term "inner" refers to the fact that the circulations are defined for the boundaries of the elements, by choosing an order for the vertexes. Therefore, we move and stay on the boundaries of the elements, without going out from the elements themselves.

In geometric algebra, the inner orientation is the geometric interpretation of the exterior geometric product among vectors. In particular, the inner orientation of a plane surface can be viewed as the orientation of the exterior product between two vectors $\mathbf{u}$ and $\mathbf{v}$ (the bivector $\mathbf{u} \wedge \mathbf{v}$) of the plane on which the surface lies, that is, the sense of the rotation that would align the first vector, $\mathbf{u}$, with the second vector, $\mathbf{v}$ (Fig. 13). Analogously, the inner orientation of a volume can be viewed as the orientation of the exterior product between three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$) of the three-dimensional space containing the volume (Fig. 13).

It is worth noting that, contrarily to the definition of an inner orientation in a narrow sense, the definition of a positive or a negative inner orientation implicitly implies the notion of external observer. In fact, as far as a plane surface is concerned, the result of the exterior product $\mathbf{u} \wedge \mathbf{v}$ is positive if and only if the sense of the rotation that would align $\mathbf{u}$ with $\mathbf{v}$ is anticlockwise for the observer that watches the plane in which $\mathbf{u}$ with $\mathbf{v}$ lie. In other words, since speaking of clockwise or anticlockwise sense of the rotation makes no sense in a two-dimensional space, an observer that lives in dimension 2 cannot discriminate whether the clockwise or anticlockwise is the positive sense of rotation. A sense of rotation in a plane is clockwise or anticlockwise only if it is related to the oriented normal direction.

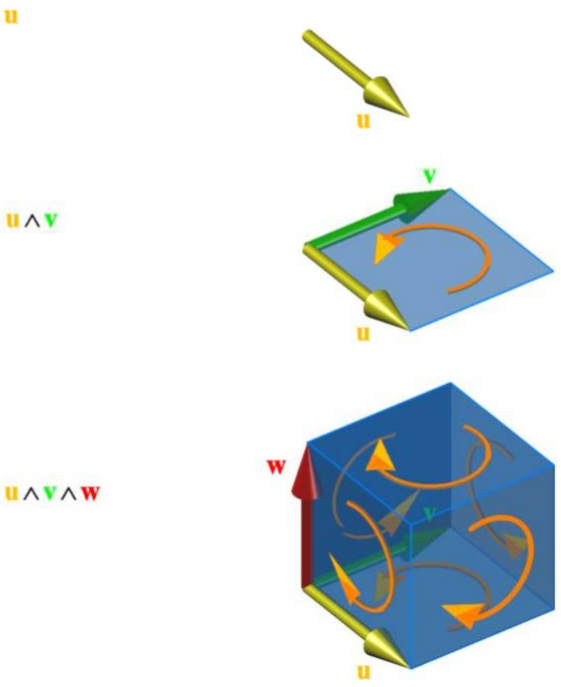


Figure 13: Geometric interpretation for the exterior product of p vectors to obtain an p-vector (parallelotope elements), where $p = 1, 2, 3$. The "circulations" show the inner orientation.

By choosing the anticlockwise as the positive sense of rotation, the observer can be represented as a vector $\mathbf{w}$ that forms an ordered right-handed triple of vectors $(\mathbf{u}, \mathbf{v}, \mathbf{w})$, together with $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{w}$ is normal to the plane in which $\mathbf{u}$ and $\mathbf{v}$ lie. If the magnitude of $\mathbf{w}$ is the scalar given by the exterior product $\mathbf{u} \wedge \mathbf{v}$, $\mathbf{w}$ is the result of the vector product $\mathbf{u} \times \mathbf{v}$. Consequently, the exterior product is provided with a sign only if "watched" by an observer living in the three-dimensional space, thus giving rise to a vector product. In other words, the exterior product lives in a geometrical vector space of dimension 2 [121], while the vector product that provides its sign can be defined only in three-dimensional (or more than three-dimensional) spaces.

In conclusion, the inner orientation of a surface is not positive or negative in itself. Neither choosing the sign of the inner orientation can be considered an arbitrary convention. Providing the inner orientation of a surface with a sign makes sense only when the surface is "watched" by an external observer, that is, only when the surface is studied in an embedding space of dimension greater than 2, the dimension of the surface. If this is the case, a plane surface has a positive or negative inner orientation when the associated (in the three-dimensional space) vector product is positive or negative, respectively. For example, the surface originated by the bivector $\mathbf{u} \wedge \mathbf{v}$ in Fig. 13 has a positive inner orientation (the surface is watched form above).

The same surface would have a negative inner orientation if watched from below.

As can be seen in Fig. 13, all the six faces of the positive trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ have a negative inner orientation when they are watched by an external observer, while they have a positive inner orientation when they are watched by a local observer that is inside the volume. This happens since the inner volume of the trivector is the intersection of the six positive half-spaces, that is, the half-spaces of the six observers that watch the positive surfaces originated by the trivector. By relating the sign of the inner orientation to the external observer also in this second case, the positive inner orientation of a volume is the one watched by the external observer. As a consequence, the inner orientation of a volume is positive when the inner orientations of all its faces are negative. In the case of Fig. 13, therefore, the volume generated by the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ has a positive inner orientation.

The concept of inner orientation defined above did not apply to zero-dimensional vector spaces (points). However, it is useful to be able to assign different inner orientations to a point. Since every zero-dimensional vector space is naturally identified with $\mathbb{R}^0$, the operation on one point that is equivalent to an exterior product produces elements in $\mathbb{R}^1$, which is a vector space provided with an orientation (given by the standard basis). These one-dimensional elements have a positive inner orientation if they are oriented as the basis of $\mathbb{R}^1$, a negative inner orientation otherwise. In other words, also this latter time we fix the sign of the inner orientation of the element by choosing an order of traversal for the elements of its boundary, that is, the points that compose it. The extension of the outer product to zero-dimensional vectors (points) provides:

$$\mathbf{P} \wedge \mathbf{Q} \triangleq \mathbf{u}; \tag{42}$$

which has the geometrical meaning of point **P** extended toward point **Q**. The extension of the outer product preserves the antisymmetric property of the product, since $\mathbf{Q} \wedge \mathbf{P}$ (point **Q** extended toward point **P**) is the negation of $\mathbf{P} \wedge \mathbf{Q}$:

$$\mathbf{Q} \wedge \mathbf{P} = -\mathbf{u}. \tag{43}$$

What if we extend a zero-dimensional subspace along a one-dimensional one? A point extended by a vector results in an oriented length, which can be represented by the vector itself (Fig. 14). Consequently, a bound vector with origin in **P**, $(\mathbf{P}, \mathbf{u})$, can be seen as the outer product between **P** and the free vector **u**:

$$\mathbf{P} \wedge \mathbf{u} \triangleq (\mathbf{P}, \mathbf{u}). \tag{44}$$

Then, since the bound vector $(\mathbf{P}, \mathbf{u})$ is often denoted by simply **u**, as its free vector, we can also write

$$\mathbf{P} \wedge \mathbf{u} = \mathbf{u}. \tag{45}$$

For consistency, we must therefore define the outer product between the vector **u** and the point **P** as the negation of **u**:

$$\mathbf{u} \wedge \mathbf{P} \triangleq -\mathbf{u}. \tag{46}$$

Note that, as for the bivector, whether or not the one-dimensional element is in the same sense of the basis can be evaluated only by an observer that does not lie on the one-dimensional space. Since there always exists a plane that contains a given line (the one-dimensional space) and a given point that does not belong to the line (the point in which the observer is), the observer belongs to a two-dimensional space, at least. Thus, the evaluation of the sign of the one-dimensional element is possible only in dimension 2.

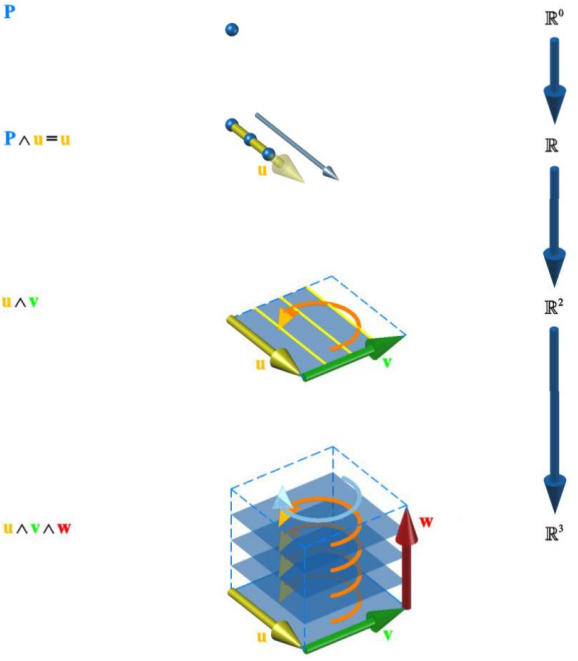


Figure 14: The inner orientation of a p-space element is induced by the $(p-1)$-space elements on its boundary.

Moreover, in analogy to the direction of the vector product $\mathbf{u} \times \mathbf{v}$, which is orthogonal both to **u** and **v**, the result of the operation $\mathbf{P} \times \mathbf{Q}$, defined in $\mathbb{R}^1$ on elements of $\mathbb{R}^0$, has the direction of a line that is orthogonal both to **P** and **Q**. In three-dimensional space, where we can define infinite sub-spaces of dimension 1, each provided with its own basis, this operation produces elements in the direction of any line of the three-dimensional space. Being orthogonal

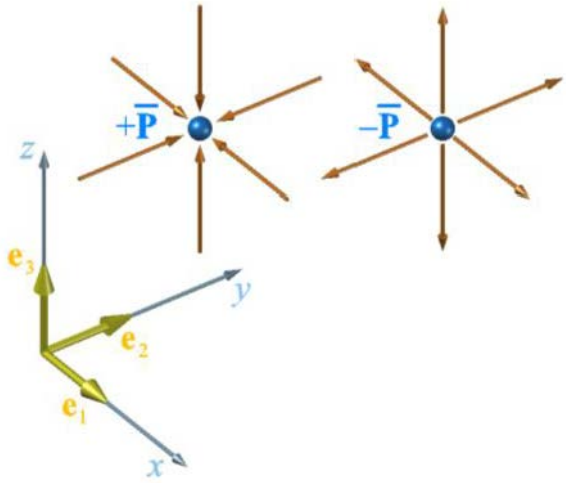


Figure 15: Positive and negative inner orientations of a point.

to each direction of the three-dimensional space, the point is orthogonal to the three-dimensional space itself and to each volume of the space.

In conclusion, we can define two inner orientations of a point, the outward and the inward orientations. The first orientation is given by the outgoing lines, while the second orientation is given by the incoming lines (Fig. 15). In the first case, the point is called a **source**, while, in the second case, is called a **sink**.

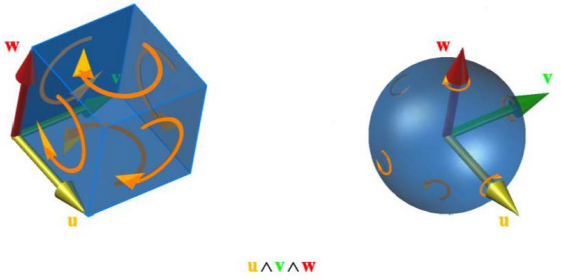


Figure 16: The geometric interpretation of the exterior product of three vectors **u**, **v**, **w** as an oriented volume. Two possible shapes are shown: a parallelepiped and a sphere. The actual shape is irrelevant to the exterior product.

Figure 17: The two inner orientations of a curve.

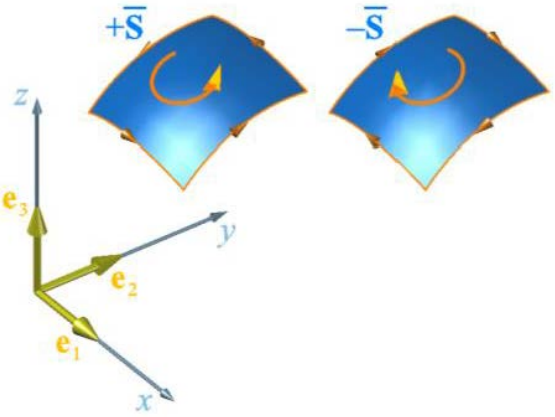


Figure 18: Positive and negative inner orientations of a surface.

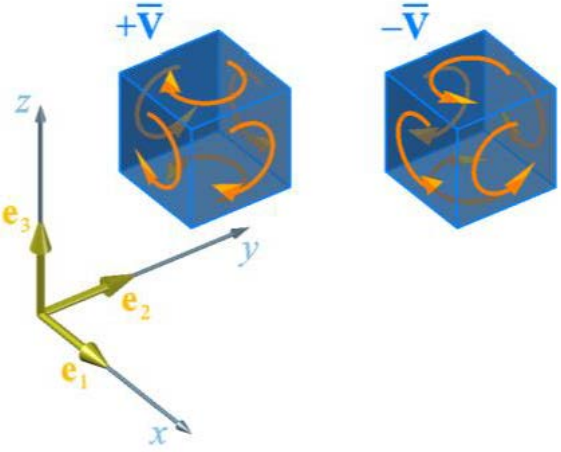


Figure 19: Positive and negative inner orientations of a volume.

As far as the sign of the inner orientation of the point is concerned, by making use of the notion of observer in this latter case too, each incoming line can be viewed as the sense along which the external observer watches the point. In this sense, a sink is a point with a positive inner orientation, while a source is a point with a negative inner orientation (Fig. 15).

Note also that even the trivector can be viewed as an extension. In fact, it is originated by a bivector extended by a third vector (Fig. 14). Like bivectors, a trivector has no shape, but only volume and sign. Even though a box helps to understand the nature of trivectors intuitively, it could have been any shape (Fig. 16).

In conclusion, since the positive or negative inner orientation of a p-space element is induced by the positive or negative inner orientation of the $(p-1)$-space elements on its boundary, we derive the inner orientations and their signs inductively (Fig. 14). This allows us to extend the pro-

cedure for finding the inner orientation of the space elements to spaces of any dimension.

In the following, a point, a line, a surface, and a volume endowed with inner orientations will be denoted by putting bars over their symbols (Figs. 15, 18, 19).

The inner orientations of a curve, a surface (not necessarily plane), and a volume are given by the inner orientations of the p-vectors of greater dimension in their tangent spaces (Figs. 17 – 19).

Electric potential and velocity potential are examples of physical variables associated with the inner orientations of points.

Strains, in continuum mechanics, are examples of physical variables associated with the inner orientations of lines.

The work involved in a thermodynamic cycle is an example of a physical variable whose sign depends on the inner orientation of the area that the cycle includes.

3.2.2 Outer Orientation of space elements

As far as the first feature of p-vectors is concerned, the attitude is part of the description of how p-vectors are placed in the space they are in. Thus, the notion of attitude is related to the notion of embedding of a p-vector in its space, or space immersion.

In particular, a vector in three dimensions has an attitude given by the family of straight lines parallel to it (possibly specified by an unoriented ring around the vector), a bivector in three dimensions has an attitude given by the family of planes associated with it (possibly specified by one of the normal lines common to these planes), and a trivector in three dimensions has an attitude that depends on the arbitrary choice of which ordered bases are positively oriented and which are negatively oriented.

Between a p-vector and its attitude there exists the same kind of relationship that exists between an element a of a set X and the equivalence class of a in the quotient set of X by a given equivalence relationship. In the special case of the attitude of a vector in the three-dimensional space, the set is that of the straight lines and the equivalence relationship is that of parallelism between lines. In this sense, the family of lines mentioned in the definition of an attitude of a vector is one equivalence class of the quotient set of straight lines by the equivalence relation of parallelism. One of the invariants of the equivalence relation of parallelism is the family of planes that are normal to the lines in a given equivalence class. Since we can choose any of the parallel planes for representing the invariant, we can speak both in terms of family of parallel planes and in terms of one single plane.

Similar considerations may also be applied to the relationship between bivectors and their attitudes, or trivectors and their attitudes. Thus, we can describe the attitude of a p-vector either in terms of its equivalence class (the family of parallel lines, when the p-vector is a vector), or in terms of its class invariant (the family of normal planes, when the p-vector is a vector), that is, the equivalence class of its orthogonal complement. In particular, as far as the description in terms of class invariants is concerned, the attitude of a vector $\mathbf{u}$ can be viewed as a family of normal planes (Fig. 20), each one originated by the translation of a plane normal to $\mathbf{u}$, along the direction of $\mathbf{u}$ (the planes span the direction of $\mathbf{u}$). Equivalently, the attitude of $\mathbf{u}$ can be represented by an arbitrary plane of the family of normal planes.

Analogously, the attitude of a bivector $\mathbf{u} \wedge \mathbf{v}$ can be viewed as two families of parallel planes (Fig. 20), the first family normal to $\mathbf{u}$ and the second family normal to $\mathbf{v}$ (the planes span both the directions of $\mathbf{u}$ and $\mathbf{v}$). Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent in their common plane, the planes that span both the directions of $\mathbf{u}$ and $\mathbf{v}$ originate all the planes normal to $\mathbf{u} \wedge \mathbf{v}$, that is, all the planes parallel to $\mathbf{u} \times \mathbf{v}$. These planes can be represented by the line of intersection between an arbitrary plane of the first family and an arbitrary plane of the second family. The intersection line is parallel to all planes of the two families and to vector $\mathbf{u} \times \mathbf{v}$.

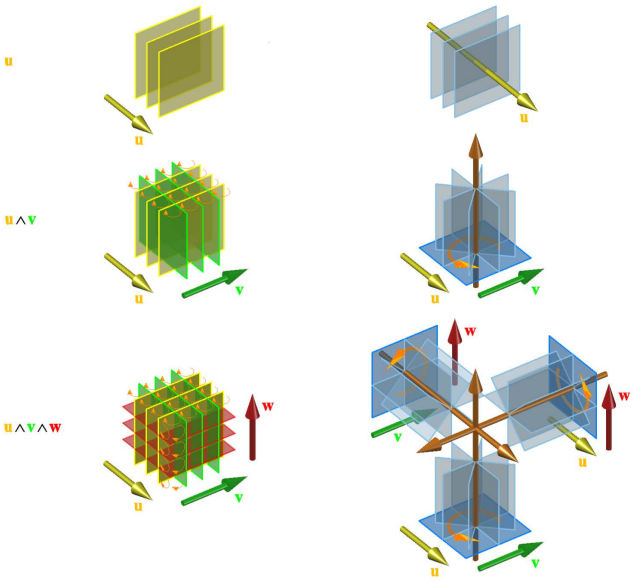


Figure 20: Geometric interpretation of the attitude of a p-vector in terms of class invariants.

Finally, the attitude of a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ can be viewed as three families of parallel planes (Fig. 20), provided that the three families are normal to **u**, **v**, and **w**, respectively (the planes span the three directions of **u**, **v**, and **w**). If **u**, **v**, and **w** are linearly independent, then the three families originate all the plane of the three-dimensional space. A possible representation of all the planes of the three-dimensional space, under the equivalence relation of parallelism, is achieved by choosing a point of the space and considering the set of all the planes that contain the point. Being common to all the planes, the point can be used for representing the whole set of planes, which, in turn, represents all the planes of the three-dimensional space.

In conclusion, as for the inner orientation of the p-vectors, also the attitude of the p-vectors is defined inductively, starting from the 1-vector. This allows us to define the attitude of the p-vectors even in dimension greater than 3.

It follows that the same family of parallel planes represents both the set of planes that are normal to **u** and the set of hyperplanes of $\mathbf{u}^*$, the dual vector of **u**. Consequently, the attitude of the class invariant of a vector **u** equals the attitude of the covector $\mathbf{u}^*$. This is ultimately a consequence of the Riesz representation theorem, which allows us to represent a covector by its related vector. Said in terms of equivalence relation of parallelism, the class invariant of a vector equals the equivalence class of its covector. This establishes a bijective correspondence between the attitude of the orthogonal complement of a vector **u** and the attitude of the covector, $\mathbf{u}^*$, of **u**.

Due to the Riesz representation theorem, the bijective correspondence extends also to the second feature, that is, the orientations of a vector and its covector, since the order of the hyperplanes is determined by the sense of **u**. This allows us to define a second type of orientation for the covector $\mathbf{u}^*$, which we call the **outer orientation** since it is induced by the (inner) orientation of **u** and has the geometrical meaning of sense of traversal of the hyperplanes of $\mathbf{u}^*$. In doing so, we have established a bijective correspondence between the inner orientation of a vector and the outer orientation of its covector. On the other hand, since it is always possible to define an inner orientation for $\mathbf{u}^*$ (by choosing a basis bivector for $\mathbf{u}^*$), the duality between vectors and covectors will result in an outer orientation for **u**, induced by the inner orientation of $\mathbf{u}^*$. Therefore, the inner orientation of a covector induces an outer orientation on its vector.

Moreover, since the equivalence classes of $\mathbf{u}^*$ are in bijective correspondence with the attitude of **u**, to fix the inner orientation of $\mathbf{u}^*$ is also equivalent to fixing an orientation, which is an inner orientation, for the attitude of **u**. In doing so, the attitude of **u** becomes an attitude vector, and its inner orientation equals the outer orientation of **u**. Therefore, by providing the attitude with an orientation, we establish an isomorphism between the orthogonal complement and the dual vector space of any subset of vectors. There is a natural analog of this relationship in general Banach spaces. In this case one defines $W^\perp$, the orthogonal complement of W, to be a subspace, of the dual of V, defined similarly as the annihilator:

$$W^\perp = \left\{ x \in V^* : \forall y \in W, x(y) = 0 \right\}; \tag{47}$$

where W is a closed linear subspace in V.

This means that the pairing between the geometric algebra and its dual can be described by the invariants of the equivalence relation of parallelism.

In conclusion, we can define the orientation of a vector by providing either its inner orientation or the inner orientation of its attitude vector (which is also the outer orientation of the vector). The latter, in turn, is equal to the inner orientation of the covector.

It remains to be determined how to relate the inner orientation of a vector with its outer orientation. In other words, it remains to be put in bijective correspondence the inner orientation of **u** with the inner orientation of the hyperplanes of $\mathbf{u}^*$ or, which is the same, with the inner orientation of the attitude vector of **u**. We can do this in two ways, by establishing two different criteria. The starting point is, once again, the equivalence class of **u** by the equivalence relation of parallelism. We have said that each equivalence class is a family of (unoriented) parallel straight lines. Therefore, we can represent an equivalence class by ideally holding together all the straight lines, or a representative set of straight lines, of the class. In other words, holding together a family of lines is equivalent to partitioning the straight lines of the three-dimensional space in subsets of lines that are parallel to a given vector.

Since a curved arrow is often used, both in mathematics and in physics, for indicating the sense of fingers curling when closing a hand around something, we can use a curved arrow for denoting the closed hand that holds the straight lines, thus considering the entire family (the equivalence class) of straight lines. This means that we can use a curved arrow for denoting the attitude vector of a vector. In particular, we can establish the desired duality relation by extending the thumb in the sense of the vector and using the sense of closure of the hand for providing the attitude vector with an inner orientation. Since we have two hands, the use of the right hand rather than the left hand gives rise to a criterion, the right-hand grip rule, rather than another criterion, the left-hand grip rule. It is

usual to use right hands in right-handed coordinate systems (Fig. 21) and left hands in left-handed coordinate systems.

This bijective correspondence allows us to represent the orientation of **u** as the inner orientation of $\mathbf{u}^*$, rather than the inner orientation of **u** (Fig. 22). As a consequence, the orientation of **u** can be described either in terms of inner or outer orientations, being the two descriptions absolutely interchangeable.

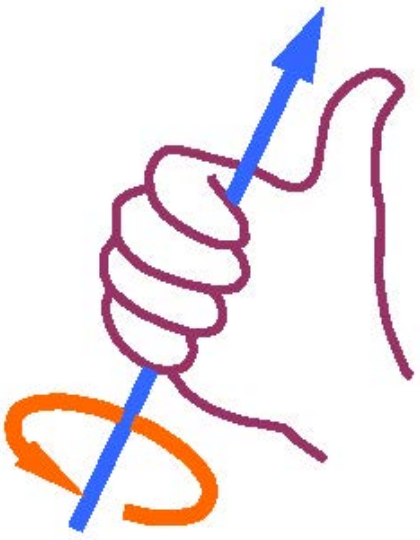


Figure 21: Relationship between the right-hand grip rule and the two orientations of a surface: inner orientation (on the plane on which the right fist is) and outer orientation (along the direction of the thumb). The figure also provides the two orientations of a vector: inner orientation (along the direction of the thumb) and outer orientation (on the plane on which the right fist is).

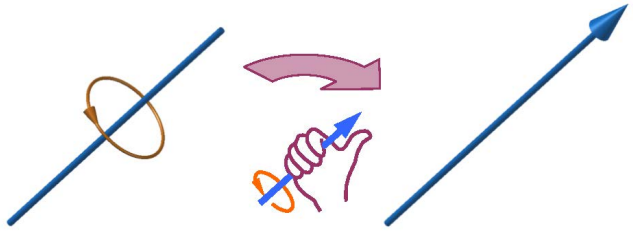


Figure 22: Equivalence between outer and inner orientations of a vector, in right-handed coordinates.

As discussed in Section 3.2.1, in the three-dimensional space a point is orthogonal to any trivector and a trivector is orthogonal to any point. This allows us to extend the bijective correspondence between orthogonal complements and dual vectors to the relationship between points and volumes (including the relationship between their inner and outer orientations). As a consequence, the outer orientation of a point is defined by the inner orientation of a surrounding volume of arbitrary shape. This, in turn, is determined by the inner orientations of the planes that are tangent to the volume surface (Fig. 16), that is, by the outer orientations of the normal straight lines that cross the volume surface. Since it is always possible to choose a spherical volume around the point, the outer orientation of the point is ultimately given by the half-lines that have the given point in common (Fig. 23).

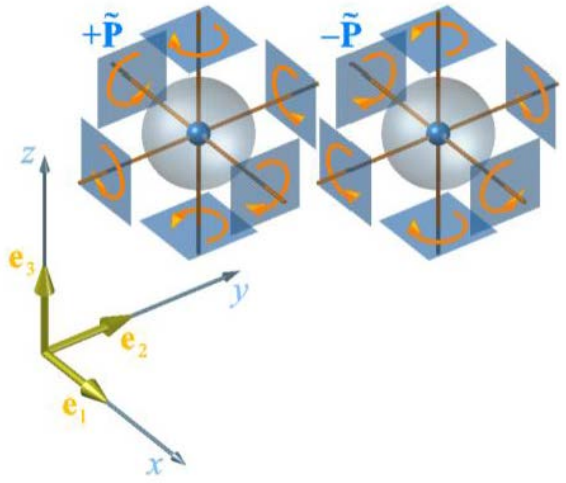


Figure 23: Relationship between outer orientation of a point and inner orientation of a volume.

In conclusion, the correspondence between space elements provided with outer orientation and their orthogonal complements, which turn out to be provided with inner orientation, leads us to define a relation of duality between space elements of different dimensions. In the three-dimensional space, the **dual space element**, or just **dual** for short, of a space element of dimension p is the normal space element of dimension $3 - p$. That is, the sum of the dimensions of a space element and its dual element always equals three, the dimension of the space they are in. In other words, if we regard any space element equipped with an orientation as the result of an exterior product on vectors, we can define the dual of the p-dimensional space element as the orthogonal complement of the p-dimensional space element.

In particular, the dual element of a point is a volume, the dual element of a straight line is a surface, the dual element of a surface is a straight line, and the dual element of a volume is a point. The outer orientation of a space el-

ement is given by the inner orientation of its dual space element (Fig. 24).

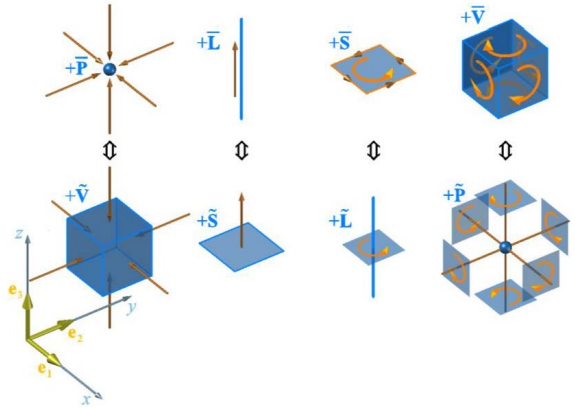


Figure 24: Relationship between the inner orientation of a p-space element and its dual element, of dimension $p - 1$.

Note that the relationship between space elements and their duals extends to the signs of inner and outer orientations. Consequently, a positive or negative inner orientation of a p-space element induces a positive or negative outer orientation in its dual element, respectively. In particular, since the point with a positive inner orientation is a sink, a volume has a positive outer orientation when the normal lines enter the volume, as in the old scientific literature. For accuracy, according to what observed on the relationship between the sign of the orientation and the observer, the positive outer orientation of a volume is indicated by inward normals and the point with a positive inner orientation is a sink whenever the observer is external, while the positive outer orientation of a volume is indicated by outward normals and the point with a positive inner orientation is a source whenever the observer is a local observer, ideally positioned inside the volume. Therefore, the signs of the outer orientations of points and volumes depend on the kind of description, material rather than spatial. Since different physical disciplines use different descriptions, this explains why the physical disciplines use different notations for the positive orientation of volumes.

Note also that the choice of the inward orientation as the positive outer orientation of the volume is compatible with the inner orientation of its faces. In fact, the positive inner orientation of the faces of the volumes is clockwise for an external observer, thus fixing in the inward direction the positive direction for crossing the faces themselves.

Finally, a surface has a positive outer orientation when the line crossing it has a positive inner orientation, represented by a vector that points toward the external observer. A line has a positive outer orientation when the normal plane has a positive inner orientation for the observer that is watching the plane.

A point, a line, a surface, and a volume endowed with outer orientations will be denoted by putting tildes over their symbols (Figs. 23 – 27).

The outer orientations of a curve, a (not necessarily plane) surface and a volume are given by the outer orientations of the p-vectors of greater dimension in their tangent spaces (Figs. 25 – 27).

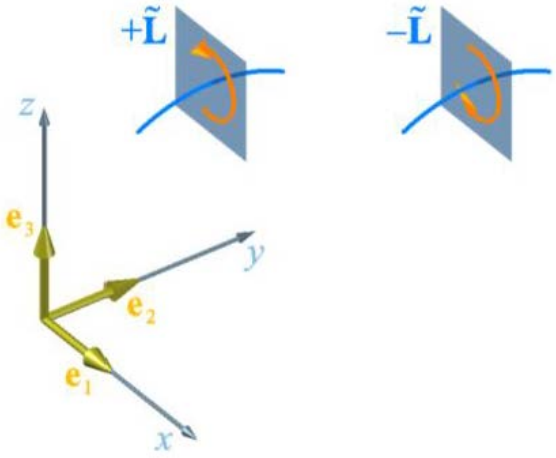


Figure 25: Positive and negative outer orientations of a curve.

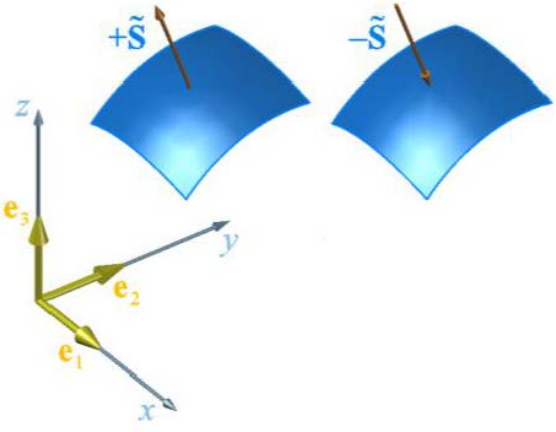


Figure 26: Positive and negative outer orientations of a surface.

Scalar magnetic potential and stream function in fluid dynamics are examples of physical variables associated with points with outer orientations.

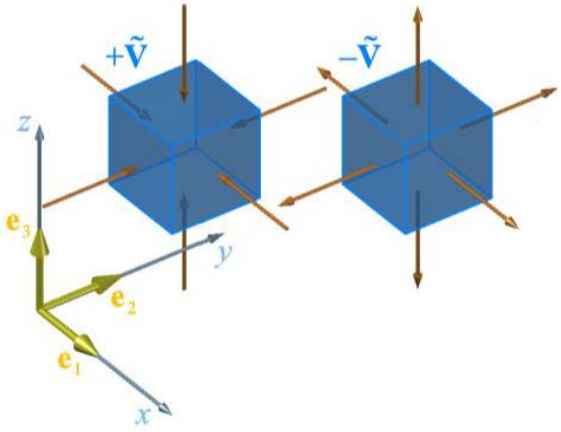


Figure 27: Positive and negative outer orientations of a volume.

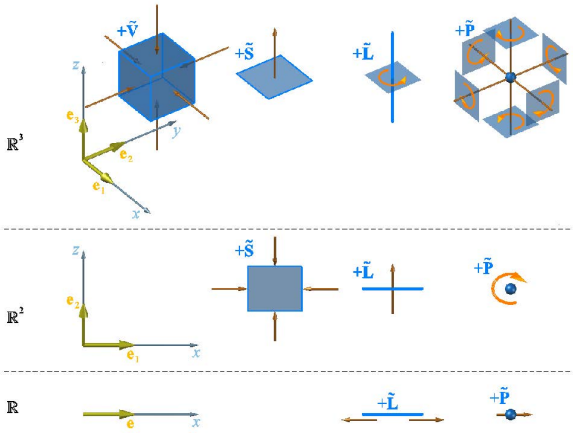


Figure 29: How the positive outer orientation depends on the dimension n of the embedding space.

The outer orientation of a line is useful for describing, for example, the rotatory polarisation of a light beam.

The heat that crosses a surface, the surface forces in continuum mechanics, the mass current, the energy current, and the charge current are examples of surface variables whose sign depends on the crossing direction of the surface, that is, on the outer orientation of the surface to which they are related.

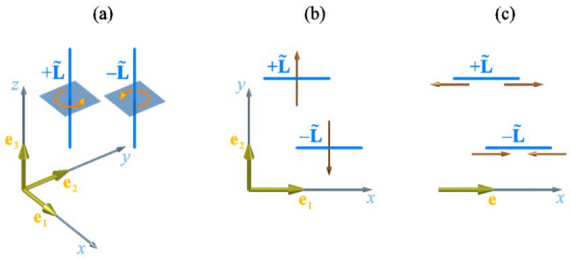


Figure 28: The two outer orientations (positive and negative orientations) of a line in three-dimensional, two-dimensional, and one-dimensional spaces.

The notion of divergence is associated with the idea that outward normals indicate a positive volume (observer inside the volume).

Being an orthogonal complement, the dual of a p-dimensional space element has dimension $n - p$, in the n-dimensional space. This means that the outer orientation depends on the dimension of the embedding space, while the inner orientation does not. As an example, the outer orientation of a line in the three-dimensional space is the orientation of a curved arrow on a normal plane (Fig. 28a), while it is the direction of a crossing arrow in the two-dimensional space (Fig. 28b), and two outgoing or incoming arrows, denoting expansion or contraction of the line, respectively, when it is embedded in the one-dimensional space (Fig. 28c). In Fig. 29, we have collected the positive outer orientations of the space elements in three-dimensional, two-dimensional, and one-dimensional spaces.

The dependence of the outer orientation on the dimension of the embedding space also allows us to define the class invariant by the equivalence relation of parallelism between p-vectors in the n-dimensional space.

In conclusion, the notions of inner and outer orientations are implicit in geometric algebra, despite they are not formalised there.

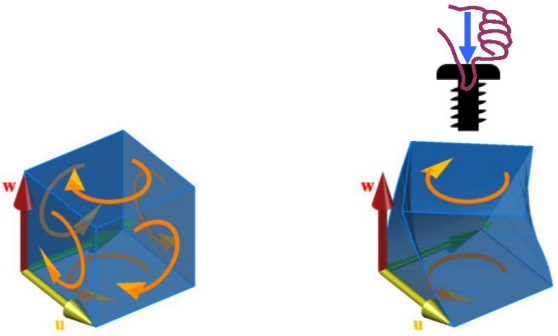


Figure 30: Relationship between the inner orientation of a cube and the orientation of a screw.

Even the relationship between the inner and outer orientations and the related notion of orthogonal complement (or dual element) are implicit, both in mathematics and physics. They are given by the right-hand rule, which is equivalent to the right-hand grip rule and the right-handed screw rule. As far as the right-hand grip rule is concerned, in fact, we have already seen that the sense of fin-

gers curling gives both the inner orientation of the plane on which the right fist is and the outer orientation of its dual, that is, the line that is directed as the thumb (Fig. 21). Analogously, the direction of the thumb gives both the inner orientation of the line that is directed as the thumb and the outer orientation of its dual, that is, the plane on which the right fist is. Thus, broadly speaking, the thumb (with its sense) and the fist (with the curling sense of its fingers) are each the orthogonal complement of the other and their orientations stand in relation of duality.

Moreover, the idea that the line in the direction of the thumb is one arbitrary element of the set of lines that are normal to the plane of the fist is implicit in the definition of cross product, which is a free vector. This makes the position of the cross product indeterminate, allowing us to draw the vector everywhere in the space. Analogously, the four fingers of the fist are associated with the four parallel planes, that is, with the common attitude of the four planes.

Since a rotation of a right-handed screw in the curling sense of the right hand fingers will cause the right-handed screw to move in the direction of the right thumb, we can identify the right fist with a right-handed screw. This allows us to extend the relationships established above to the right-handed screw rule. This second time, however, we can also establish relations of dual orientation between the volume and its orthogonal complement, the point. In fact, in a right-handed coordinate system the three pairs of opposite faces of a positive cube rotate as three right-handed screws (Fig. 30), which are screwed on the faces of the cube. Therefore, the clockwise rotation of the right-handed screw on the cube faces gives the positive inner orientation of a volume in a right-handed coordinate system. On the other hand, the clockwise rotations of the three right-handed screws define the outer orientations of the normals to the faces of the cube, that is, the positive outer orientation of the point that is the dual of the volume. Thus, the rotation of the right-handed screw gives both the inner orientation of a volume and the outer orientation of its dual, the point.

Finally, the motion directions of the right-handed screws define the outer orientation of the volume originated by the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are directed as the three right-handed coordinate axes. This latter time, in fact, the clockwise rotation of the right-handed screws on the cube faces will cause the screws to move inward, thus providing also the positive inner orientation of the point that is the dual of the volume. Therefore, the motion direction of the right-handed screw gives both the inner orientation of a point and the outer orientation of its dual, the volume. This means that the right-handed screw rule implicitly establishes also a relation of duality between the rotation sense and the motion direction of a right-handed screw. Thus, broadly speaking, rotation sense and motion direction are each the orthogonal complement of the other.

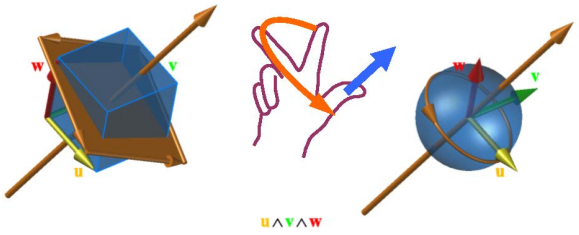


Figure 31: The geometric interpretation of the exterior product of three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ as an oriented volume compared with the right-hand rule. The actual shape is irrelevant.

As far as the third rule is concerned, the right-hand rule, the finger disposition in the right hand naturally provides both the inner orientation of the point that is the origin of a right-handed coordinate system, with axes in the directions of the fingers, and the outer orientation of the volume that surrounds the hand.

At the same time, by rotating one finger at a time toward the other, we obtain both the (equatorial) inner orientation of the volume that surrounds the hand and the outer orientation of the "origin" of the fingers, given by the rotation sense around the third finger, which represents one of the half-lines outgoing the "origin". This is the same relationship provided by the spinning-top analogy of the exterior product $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ between three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ directed as three right-handed axes, where the orientation on a solid ball representing the trivector can be thought of as giving the ball a counterclockwise spin around any given axis (Fig. 31).

Finally, by rotating the first finger toward the second finger, the rotation sense gives both the inner orientation of the plane in which the two fingers lie and the outer orientation of the line in the direction of the third finger, while the third finger gives both the inner orientation of the line along it and the outer orientation of the plane in which first and second fingers lie.

In conclusion, we could say that the right-hand grip rule, the right-hand rule, and the right-handed screw rule (or the equivalent corkscrew-rule) establish the pairing between the orientations of the p-vectors and their duals, the p-covectors, for $p = 1, 2, 3$. This allows us to describe space elements in dimension $p = 1, 2, 3$ either in terms of inner or outer orientations.

Inner and outer orientations of space elements assume a great importance in the description of physics, because the signs of global physical variables (Section 2.2) depend on the orientation. This is why we formalise their definitions in this paper, making their notions and their relationships explicit.

Once the space elements and their orientations have been viewed in these terms, the elements of the cell complexes used in computational physics are elements of a topological vector space [52, 53], [55] – [57] and can be studied in the context of the geometric algebra. Being a p-vector, each cell has a dual cell and the set of dual cells can be viewed as the set of multilinear forms on the topological vector space. In particular, if X is a normed space equipped with the weak topology, then the dual space X^* is itself a normed vector space and its norm gives rise to a topology, called the strong topology, on X^*. Consequently, the cell complexes are, more properly, CW cell complexes. Moreover, since, different from the weak topology, the strong topology is not a coarsest topology, having a strong topology on the dual space means that there is not a unique way for defining the dual space elements. Thus, different definitions for the dual space elements may lead them to overlap.

4 Algebraic topology as a tool for treating global variables

4.1 Some Notions of Algebraic Topology

Topology is the mathematical study of shapes and spaces. It is a major area of mathematics, concerned with the most basic properties of space, such as connectedness, continuity and boundary. Topology studies the properties of geometric figures that are preserved under continuous deformations, including stretching and bending, but not tearing or overlapping (gluing). Since these kind of deformations are homeomorphisms, that is, continuous, invertible, and with continuous inverses transformations, topology studies the properties of geometric figures that are invariant under homeomorphisms.

Topology has three subfields:

- **Algebraic topology** uses tools from abstract algebra to study topological spaces.
- **Point-set topology** establishes the foundational aspects of topology and investigates concepts inherent to topological spaces. Examples include compactness and connectedness.
- **Geometric topology** primarily studies manifolds and their embeddings (placements) in other manifolds. A particularly active area is **low-dimensional topology**, which studies manifolds of four or fewer dimensions. This includes **knot theory**, the study of mathematical knots.

The basic goal of algebraic topology is that to measure degrees of connectivity, using algebraic constructs such as homology and homotopy groups. This is achieved by looking for invariants, which are used in order to classify the topological spaces.

An n-dimensional closed cell is the image of an n-dimensional closed ball, or closed n-ball, under an attaching map, where

- an **image** is the subset of a function's codomain, which is the output of the function on a subset of its domain;

in three-dimensional Euclidean space, a **ball** is the space inside a sphere. It may be a **closed ball** (including the boundary points) or an **open ball** (excluding them). The definition extends to lower and higher-dimensional Euclidean spaces and to metric spaces in general. In particular, the ball is a bounded interval for $n = 1$ and the interior of a circle (a **disk**) for $n = 2$. In a topological space, X, balls are not necessarily induced by a metric. An (open or closed) ***n*-dimensional topological ball** of X is any subset of X which is homeomorphic to an (open or closed) Euclidean n-ball.

- Let X and Y be topological spaces with A a subspace of Y. The **attaching map** is any continuous map, f, from A to X:

$$f : A \to X. \tag{48}$$

In algebraic topology, it is usual to introduce cell complexes, mainly in the restricted form of simplicial complexes (Section 4.2), and to consider the vertices, edges, surfaces, and volumes of a cell complex as p-cells, that is, cells of different dimensions.

The ***n*-skeleton** (pl. n-skeleta) of a cell complex is the union of the cells whose dimension is at most n. If the union of a set of cells is closed, then this union is itself a cell complex, called a **subcomplex**. Thus the n-skeleton is the largest subcomplex of dimension n or less.

A cell complex is often constructed by defining its skeleta inductively. Begin by taking the 0-skeleton to be a discrete space. Next, attach 1-cells to the 0-skeleton. Here, the 1-cells are attached to points of the 0-skeleton via some continuous map from unit 0-sphere, that is, S_0. Define the 1-skeleton to be the identification space obtained from the

union of the 0-skeleton, 1-cells, and the identification of points of boundary (Section 4.5) of 1-cells by assigning an identification mapping from the boundary of the 1-cells into the 1-cells. In general, given the $(n-1)$-skeleton and a collection of (abstract) closed n-cells, as above, the n-cells are attached to the $(n-1)$-skeleton by some continuous mapping from S_{n-1}, and making an identification (equivalence relation) by specifying maps from the boundary of each n-cell into the $(n-1)$-skeleton. The n-skeleton is the identification space obtained from the union of the $(n-1)$-skeleton and the closed n-cells by identifying each point in the boundary of an n-cell with its image.

Following the terminology of algebraic topology, we will denote the vertices of a cell complex as 0-cells, the edges as 1-cells, the surfaces as 2-cells and the volumes as 3-cells (Fig. 32).

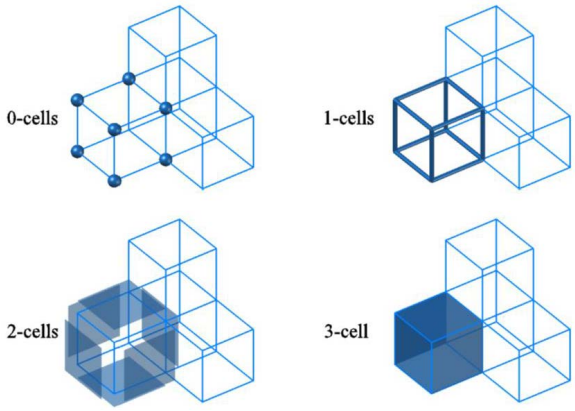


Figure 32: The four space elements in algebraic topology.

Note the difference between the terminologies of algebraic topology and geometry, where the points are 1-cells, the edges are 2-cells, the surfaces are 3-cells and the three-dimensional volumes are 4-cells.

4.2 Simplices and Simplicial Complexes

A **simplex** (plural **simplexes** or **simplices**) is the smallest convex set containing the given vertices.

The formal definition of simplex makes use of the notion of **polytope** (Fig. 33, [122–127]), a geometric object with flat sides that exists in any general number of dimensions. Specifically, a ***k*-simplex** is a k-dimensional polytope that is the convex hull of its $k+1$ vertices, where the **convex hull**, or **convex envelope**, of a set X of points in the Euclidean plane or Euclidean space may be defined as (Fig. 34):

1. The (unique) minimal convex set containing X.
2. The intersection of all convex sets containing X.
3. The set of all convex combinations of points in X.
4. The union of all simplices with vertices in X.

A 2-simplex is a triangle, a 3-simplex is a tetrahedron, a 4-simplex is a 4-cell, and so on, where the **4-cell** in algebraic topology (5-cell in geometry) is a four-dimensional object bounded by five tetrahedral cells. It is also known as the **pentachoron**, or **pentatope**, or **tetrahedral hyperpyramid**, or **hypercell**. It is a 4-simplex, the simplest possible convex regular 4-polytope (four-dimensional analogue of a polyhedron) and is analogous to the tetrahedron in three dimensions and the triangle in two dimensions. Its maximal intersection with three-dimensional space is the triangular prism. The pentachoron can be constructed from a tetrahedron, by adding a 5th vertex (Fig. 35) such that it is equidistant from all the other vertices of the tetrahedron (it is essentially a four-dimensional pyramid with a tetrahedral base).

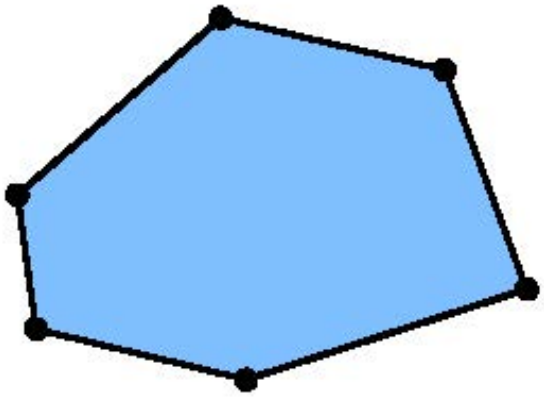


Figure 33: A two-dimensional polytope.

A single point may be considered a 0-simplex, and a line segment may be considered a 1-simplex.

In algebraic topology, simplices are used as building blocks to construct an interesting class of topological spaces, called **simplicial complexes**. These spaces are built from simplices glued together in a combinatorial fashion.

A simplicial complex, K, satisfies the following two conditions:

1. Any face (see Section 4.3) of a simplex from K is also in K.
2. The intersection of any two simplices $\sigma_1, \sigma_2 \in \mathrm{K}$ is a face of both σ_1 and σ_2.

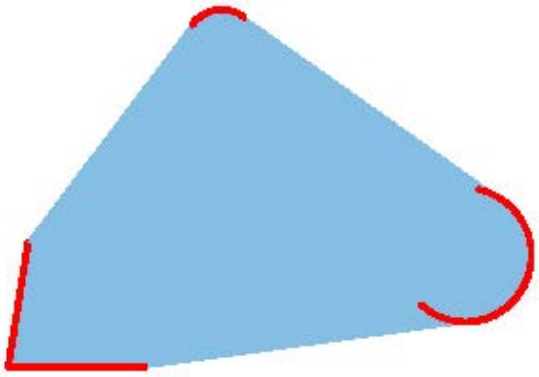


Figure 34: The convex hull of the **the red** set is the **blue** convex.

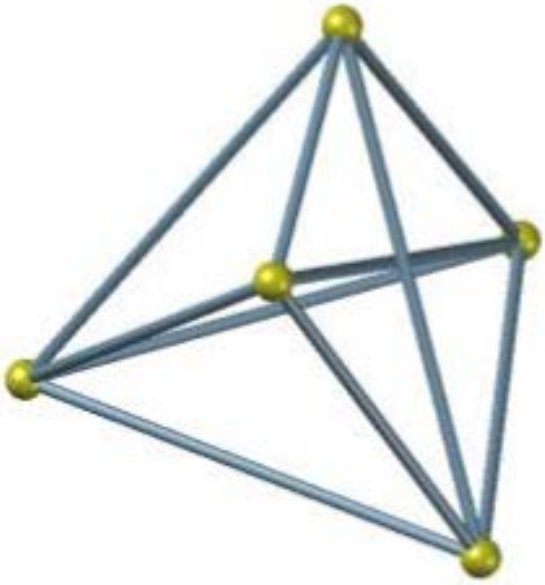


Figure 35: The pentachoron.

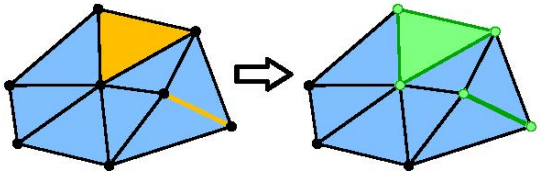


Figure 36: Two simplices (**yellow**) and their closure (**green**).

A **simplicial *k*-complex** is a simplicial complex where the largest dimension of any simplex equals k.

Let K be a simplicial complex and let S be a collection of simplices in K.

The **closure** of S (denoted by $\mathrm{Cl}\, S$) is the smallest simplicial subcomplex of K that contains each simplex in S

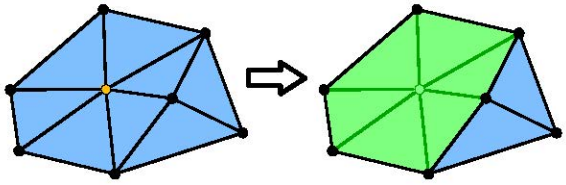


Figure 37: A simplex (**yellow**) and its star (**green**).

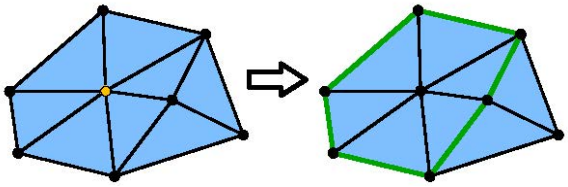


Figure 38: A simplex (**yellow**) and its link (**green**).

(Fig. 36). $\mathrm{Cl}\, S$ is obtained by repeatedly adding to S each face (Section 4.3) of every simplex in S.

The **star** of S (denoted by $\mathrm{St}\, S$) is the set of all simplices in K that have any faces in S (Fig. 37). The star is generally not a simplicial complex itself.

The **link** of S (denoted by $\mathrm{Lk}\, S$) is the closed star of S minus the stars of all faces of S (Fig. 38):

$$\mathrm{Lk}\ S = \mathrm{Cl}\ \mathrm{St}\ S - \mathrm{St}\ \mathrm{Cl}\ S. \tag{49}$$

The space distribution of the point-wise field variables in the differential formulation, when discretised, gives rise to a discrete distribution of points that can be viewed as the 0-simplices of a simplicial cell-complex. On the contrary, the Cell Method makes use of 3-simplices, or 4-simplices when also time is involved, allowing us to associate global variables with all the dimensions of the 4-cell complex. Using 4-cell complexes instead of 1-cell complex is the topological equivalent of avoiding to apply the Cancelation Rule for limits, and is the main reason why the CM is able to take into account the length scales in computational Physics—until the third dimension—while the differential formulation does not.

4.3 Faces and cofaces

In higher-dimensional geometry, the convex hull of any nonempty subset of the points that define a k-dimensional polytope, or k-polytope, is called a **face** of the k-polytope. Moreover, a polytope A is a **coface** of a polytope B if B is a face of A.

The set of faces of a polytope includes the polytope itself and the empty set, which for consistency may be defined as having dimension –1.

The $(k-1)$-faces of a k-dimensional polytope are called the **facets of the *k*-polytope**. For example:

- The facets of a line segment are its 0-faces, or vertices.
- The facets of a polygon are its 1-faces, or edges.
- The facets of a polyhedron [128], or plane tiling, are its 2-faces.
- The facets of a 4-polytope, or 3-honeycomb, are its 3-faces.
- The facets of a 5-polytope, or 4-honeycomb, are its 4-faces.

The term "facet" is also used in reference to simplicial complexes. In this second case, the **facet of a simplicial complex** is any simplex in the complex that is not a face of any larger simplex.

A $(k-2)$-face of a k-dimensional polytope is called a **ridge**. A $(k-3)$-face of a k-dimensional polytope is called a **peak**. The same definitions of ridge and peak apply to polytopes that are also simplices.

When the polytopes are simplices, the faces are simplices themselves. In particular, the convex hull of a subset of size $m+1$ (of the $k+1$ defining points) is an m-simplex, called an ***m*-face** of the k-simplex.

Excluding from the list of the m-faces the empty set, the total number of faces of a k-simplex, $F(k)$, is always a power of two minus one:

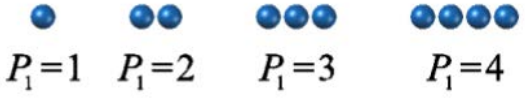


Figure 39: The first four linear numbers.

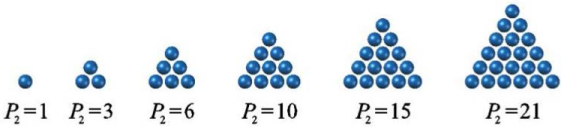


Figure 40: The first six triangular numbers

$$F(k) = 2^{k+1} - 1. \tag{50}$$

In this case, the number of m-faces of a k-simplex is a figurate number, where a **figurate number** is the number of stacking spheres forming some regular geometrical figure on the plane (**polygonal numbers**, or ***m*-gonal** numbers), or in the space (**polyhedral numbers**, or ***m*-hedral** numbers). Thus, a figurate number represents a regular, discrete geometric pattern. In particular:

- The number of 0-faces (vertices) of the k-simplex is the $(k+1)$-th linear number, $P_1(k+1)$, where a **linear number** is a figurate number that represents a line segment. It counts the aligned spheres in a row of spheres (Fig. 39). The n-th linear number is the sum of the first n elements of the sequence of units:

$$P_0 = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \tag{51}$$

The linear numbers are collected along the first diagonal from left (entry 1, second position in each row), arranged in order from top to bottom (Fig. 12). Due to the symmetry of the elements in Pascal's triangle, the linear numbers also form the first diagonal from right.

- The number of 1-faces (edges) of the k-simplex is the k-th triangle number, $P_2(k)$, where a **triangle number**, or **triangular number**, is a figurate number that represents an equilateral triangle. It counts the juxtaposed spheres that are arranged on a plane in the shape of an equilateral triangle (Fig. 40). The n-th triangle number is the number of spheres composing a triangle with n spheres on a side and is equal to the sum of the first n linear numbers. The triangle numbers are in third position (entry two, second diagonal from left), arranged in order from top to bottom (Fig. 12). A second set of ordered triangle numbers forms the second diagonal from right.
- The number of 2-faces (surfaces) of the k-simplex is the $(k-1)$-th tetrahedron number, $P_3(k-1)$, where a **tetrahedron number**, or **tetrahedral number**, or **triangular pyramidal number**, is a figurate number that represents a tetrahedron. It counts the stacking spheres that form a tetrahedron. The n-th tetrahedral number is the sum of the first n triangular numbers. The tetrahedral numbers are in fourth position (entry three, third diagonal from left), arranged in order from top to bottom (Fig. 12). A second set of ordered tetrahedral numbers forms the third diagonal from right.
- The number of 3-faces (volumes) of the k-simplex is the $(k-2)$-th pentatope number, $P_4(k-2)$, where a **pentatope number**, or **pentatopic number**, or **pentachoron number**, or **4-simplex number**, or **4-topic number**, counts the number of intersections created when the corners of a polygon of size n are all connected to one another. For example, a triangle has zero intersections, a square has one intersection, a pentagon has five intersections, a hexagon

has 15 intersections, and a heptagon has 35 intersections. The n-th pentatope number is the sum of the first n tetrahedral numbers. Pentatope numbers are arranged in order along the fourth diagonal (Fig. 12), either from left or right, of Pascal's triangle.

– The number of m-faces of the k-simplex is the $(k-m+1)$-th $(m+1)$-simplex number, $P_{m+1}(k-m+1)$, where the n-th **$(m+1)$-simplex number**, or n-th **$(m+1)$-topic number**, $P_{m+1}(n)$, is the sum of the first n m-simplex numbers.

Since all the figurate numbers are binomial coefficients, the number of m-faces of the k-simplex is also given by the binomial coefficient $P_{m+1}(k-m+1)$:

$$P_{m+1}(k-m+1) = \begin{pmatrix} k+1 \\ m+1 \end{pmatrix}. \tag{52}$$

Consequently, the numbers of the m-faces of a k-simplex may be found in row $k+1$ and diagonal $m+1$, either from left or right, of Pascal's triangle (Fig. 12).

In conclusion, the numbers of the m-faces of a k-simplex are ordered along the $(k+1)$-th row of Pascal's triangle, deprived of the 0-entry (belonging to the left diagonal, diagonal 0). They are arranged in Table 1, for $0 \le k \le 10$ and $0 \le m \le 10$.

The total number of faces of a k-simplex is the sum of the elements of the $(k+1)$-th row in Pascal's triangle, minus 1 (the first element of the row). It is also given by the sum of the elements of the $(k+1)$-th row in Table 1.

If we introduce the local systems of coordinate with origins in the 0-cells of the k-simplex, the triangular array of Pascal's triangle takes on a further meaning. By way of example, consider the 3-simplex in Fig. 41, where a local reference system is given by the three edges outgoing from the same vertex. These three axes define a system of affine coordinates.

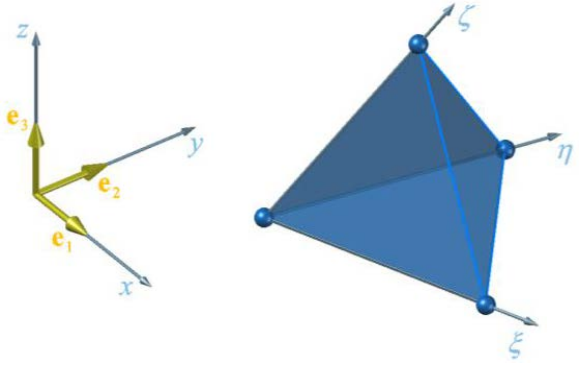


Figure 41: Local reference system of the affine coordinates for a 3-symplex (tetrahedron).

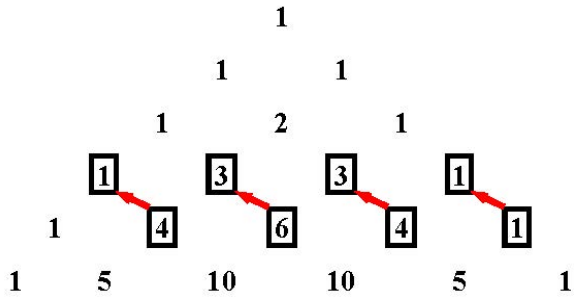


Figure 42: How to move from the element providing the total number of m-faces of the 3-simplex to the element providing the number of those m-faces of the 3-simplex that obey to conditions of belonging, parallelism, or inclusion.

Whichever be the 0-cell where the origin has been fixed, there are always three edges that are parallel to the axes of the local reference system, the ξ, η, and ζ axes. There are also three surfaces that are parallel to the coordinate planes, the ξ/η, η/ζ, and ζ/ξ planes. Finally, there is one volume included inside the local axes and one vertex on the origin of the reference system. In other words, only one of the four vertices is on the origin, only three of the six edges are along the coordinate axes, only three of the four surfaces are on the coordinate planes and the volume of the tetrahedron is also the volume that is included inside the coordinate axes.

Now, both the ordered sequence of the m-faces of the tetrahedron (4, 6, 4, 1) and the ordered sequence of those m-faces that obey to conditions of belonging, parallelism, or inclusion (1, 3, 3, 1) are sequences of binomial coefficients. The first sequence may be found in the $(k+1)$ th row of Pascal's triangle, deprived of the first entry, while the second sequence is given by the kth row of Pascal's triangle (see Fig. 42 for $k = 3$).

By extending the former analysis to the k-simplices, with k not necessarily equal to three, we can define the following general rule:

Given a reference system with the origin in one vertex of the k-simplex and the axes along those edges of the k-simplex that are connected with the chosen vertex, for finding how many m-faces of the k-simplex obey to conditions of belonging or inclusion in the reference axes, start from the element in row $k+1$ and diagonal $m+1$ of Pascal's triangle (which provides the total number of m-faces of the k-simplex) and move diagonally, on the number above and to the left.

As can be easily checked in Table 1, this means that the number of those m-faces of the k-simplex that obey to conditions of belonging or inclusion is equal to the total number of $(m-1)$-faces of the $(k-1)$-simplex. In terms

Table 1: m-faces of a k-simplex for $0 \le k \le 10$ and $0 \le m \le 10$: in the $(k+1)$ th row, the sum of the elements is equal to $2^{k+1} - 1$.

(vertices)	0-faces (edges)	1-faces (surfaces)	2-faces (volumes)	3-faces	4-faces	5-faces	6-faces	7-faces	8-faces	9-faces	10-faces	Sum
0-simplex (point)	1											1
1-simplex (line)	2	1										3
2-simplex (triangle)	3	3	1									7
3-simplex (tetrahedron)	4	6	4	1								15
4-simplex	5	10	10	5	1							31
5-simplex	6	15	20	15	6	1						63
6-simplex	7	21	35	35	21	7	1					127
7-simplex	8	28	56	70	56	28	8	1				255
8-simplex	9	36	84	126	126	84	36	9	1			511
9-simplex	10	45	120	210	252	210	120	45	10	1		1023
10-simplex	11	55	165	330	462	462	330	165	55	11	1	2047

of binomial coefficients, the number of those m-faces of the k-simplex that obey to conditions of belonging or inclusion is given by the binomial coefficient $P_m(k - m + 1)$:

$$P_m(k - m + 1) = \begin{pmatrix} k \\ m \end{pmatrix}. \tag{53}$$

This number may be found in row k and diagonal m, either from left or right, of Pascal's triangle (Fig. 43).

$$\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}1\\0\end{pmatrix}\quad\begin{pmatrix}1\\1\end{pmatrix}$$
$$\begin{pmatrix}2\\0\end{pmatrix}\quad\begin{pmatrix}2\\1\end{pmatrix}\quad\begin{pmatrix}2\\2\end{pmatrix}$$
$$\begin{pmatrix}3\\0\end{pmatrix}\quad\begin{pmatrix}3\\1\end{pmatrix}\quad\begin{pmatrix}3\\2\end{pmatrix}\quad\begin{pmatrix}3\\3\end{pmatrix}$$
$$\begin{pmatrix}4\\0\end{pmatrix}\quad\begin{pmatrix}4\\1\end{pmatrix}\quad\begin{pmatrix}4\\2\end{pmatrix}\quad\begin{pmatrix}4\\3\end{pmatrix}\quad\begin{pmatrix}4\\4\end{pmatrix}$$
$$\begin{pmatrix}5\\0\end{pmatrix}\quad\begin{pmatrix}5\\1\end{pmatrix}\quad\begin{pmatrix}5\\2\end{pmatrix}\quad\begin{pmatrix}5\\3\end{pmatrix}\quad\begin{pmatrix}5\\4\end{pmatrix}\quad\begin{pmatrix}5\\5\end{pmatrix}$$

Figure 43: The first six rows of Pascal's triangle as Binomial Coefficients.

By considering also the empty set as a face of the k-simplex, the numbers of the m-faces of a k-simplex are still ordered along the $(k+1)$-th row of Pascal's triangle, included the 0-entry. In this case, the total number of faces of a k-simplex, $F^{\mathrm{O}}(k)$, is the sum of the elements of the $(k+1)$-th row in Pascal's triangle:

$$F^{\mathrm{O}}(k) = 2^{k+1}. \tag{54}$$

In doing so, the numbers of the m-faces of a k-simplex equal the numbers of basis $(m+1)$-blades needed in a $(k+1)$-dimensional space to represent arbitrary $(m+1)$-blades. This is a consequence of the relationship shown in Fig. 42, because the coordinate system of the $(k+1)$-dimensionalspace can be fixed on one vertex of a $(k+1)$-blade.

Given two disjoint k-simplices of the same simplicial k-complex, the m-faces of the first k-simplex are not parallel to the m-faces of the second k-simplex, in general. The condition of parallelism between m-faces is satisfied only in coordinate cell complexes, which are obtained from coordinate systems. In this special case, all the local 1-cells are parallel to the coordinate axes, and all the local 2-cells are parallel to the coordinate planes of the global coordinate system.

Due to the parallelism between local and global axes and planes, we can extend the findings on the single k-simplex to the whole coordinate cell k-complex. This allows us to speak of **families of *m*-faces**.

By way of example, let us consider a Cartesian cell complex in three-dimensional space, a Cartesian 3-cell complex (Fig. 44). If we fix a local reference system for every 3-cell of the 3-cell complex, the number of m-faces that satisfy the conditions of belonging or inclusion for each 3-cell is still provided by the third row of Pascal's triangle in Fig. 42. This second time, however, all the 1-faces (edges) are parallel to one of the three global coordinate axes.

The subset of 1-faces that are parallel to the first global coordinate axis, x, defines the first family of 1-faces. Analogously, the subset of 1-faces that are parallel to the second global coordinate axis, y, defines the second family of 1-faces, and the subset of 1-faces that are parallel to the third global coordinate axis, z, defines the third family of 1-faces.

As far as the 2-faces are concerned, each of them turns out to be parallel to one of the three global coordinate planes, x/y, y/z, and z/x, defining three families of 2-

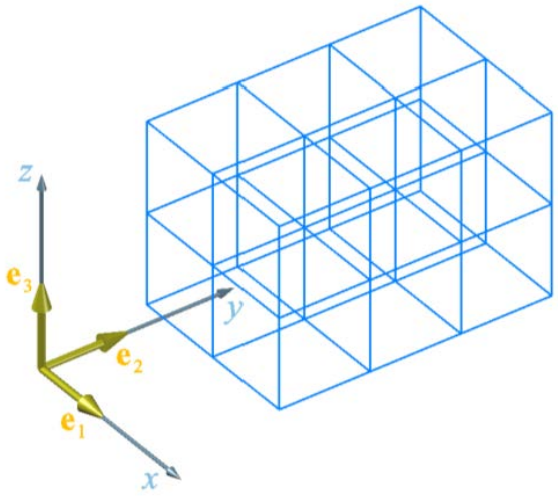


Figure 44: Cartesian cell complex in three dimensions.

faces. Finally, in the 3-cell complex of Fig. 44 there is one family of points and one family of volumes.

In conclusion, the elements of the third row in Pascal's triangle indicate, in this case, the number of families of m-faces in the 3-cell complex, not only the number of m-faces satisfying a given condition for a single 3-cell.

This result can easily be extended to n-dimensional spaces and cell complexes obtained from other kinds of coordinate systems. In any case, the n-th row of Pascal's triangle provides the number of families of m-faces in n-dimensional space (Fig. 45). By comparison between Fig. 45 and Table 1, we can conclude that the number of families of m-faces of a given cell k-complex is equal to the total number of $(m-1)$-faces of the $(k-1)$-simplex.

4.4 Some Notions of the Graph Theory

Graphs are one of the prime objects of study in discrete mathematics [129] – [138]. They are mathematical structures used to model pairwise relations between objects.

A graph is made up of **vertices**, also called **nodes** or **points**, and segments called **edges**, **lines**, or **arcs**, that connect them. Typically, a graph is depicted in a diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges. Normally, the vertices of a graph, by their nature as elements of a set, are distinguishable. This kind of graph may be called **vertex-labelled** (Fig. 46).

Graphs with labelled edges are called **edge-labelled** graphs. Graphs with labels attached to edges or vertices are more generally designated as **labelled**. Consequently, graphs in which vertices are indistinguishable and edges are indistinguishable are called **unlabelled**.

A **loop** is an edge which starts and ends on the same vertex (Fig. 47). Loops may be permitted or not permitted according to the application.

A **multigraph** (or **pseudograph**) is a graph which is permitted to have multiple edges (also called "parallel edges"), that is, edges that have the same end nodes, and sometimes loops (Fig. 47). Thus two vertices may be connected by more than one edge.

Both a graph and a multigraph may be **undirected** (Figs. 46 and 47), meaning that edges have no orientation, or their edges may be **directed** from one vertex to another (Fig. 48).

An undirected graph is an ordered pair $G = (V, E)$ comprising a set V of vertices or nodes together with a set E of edges or lines, which are 2-element subsets of V (that is, an edge is related with two vertices, and the relation is represented as unordered pair of the vertices with respect to the particular edge). The vertices belonging to an edge are called the **ends**, **endpoints**, or **end vertices** of the edge. A vertex may exist in an undirected graph and not belong to an edge.

An undirected graph can be seen as a simplicial complex consisting of 1-simplices (the edges) and 0-simplices (the vertices). As such, cell complexes are generalisations of graphs since they allow for higher-dimensional simplices.

A directed graph, or **digraphs**, is an ordered pair $D = (V, A)$, in which V is a set whose elements are called vertices or nodes, and A is a set of ordered pairs of vertices, called arcs, **directed edges**, or **arrows**. An arc $a = (x, y)$ is considered to be directed from x to y. y is called the **terminal vertex**, or **head**, and x is called the **initial vertex**, or **tail**, of the arc. y is said to be a **direct successor** of x, and x is said to be a **direct predecessor** of y. If a path leads from x to y, then y is said to be a **successor** of x and **reachable** from x, and x is said to be a **predecessor** of y. The arc (y, x) is called the **arc** (x, y) **inverted**. A directed graph D is called **symmetric** if, for every arc in D, the corresponding inverted arc also belongs to D. An **oriented graph** is a directed graph in which at most one of (x, y) and (y, x) may be arcs.

An oriented graph can be seen as a simplicial complex consisting of directed 1-simplices (the edges) and 0-simplices (the vertices). As such, oriented cell complexes are generalisations of oriented graphs to higher-dimensional simplices, which allow us to provide also the vertices with an orientation.

An edge and a vertex on that edge are called **incident**. In set theory, the incidence relations are binary relations, while, in graph theory, an incidence relation can either be a binary or a ternary relation. In particular, if the graph

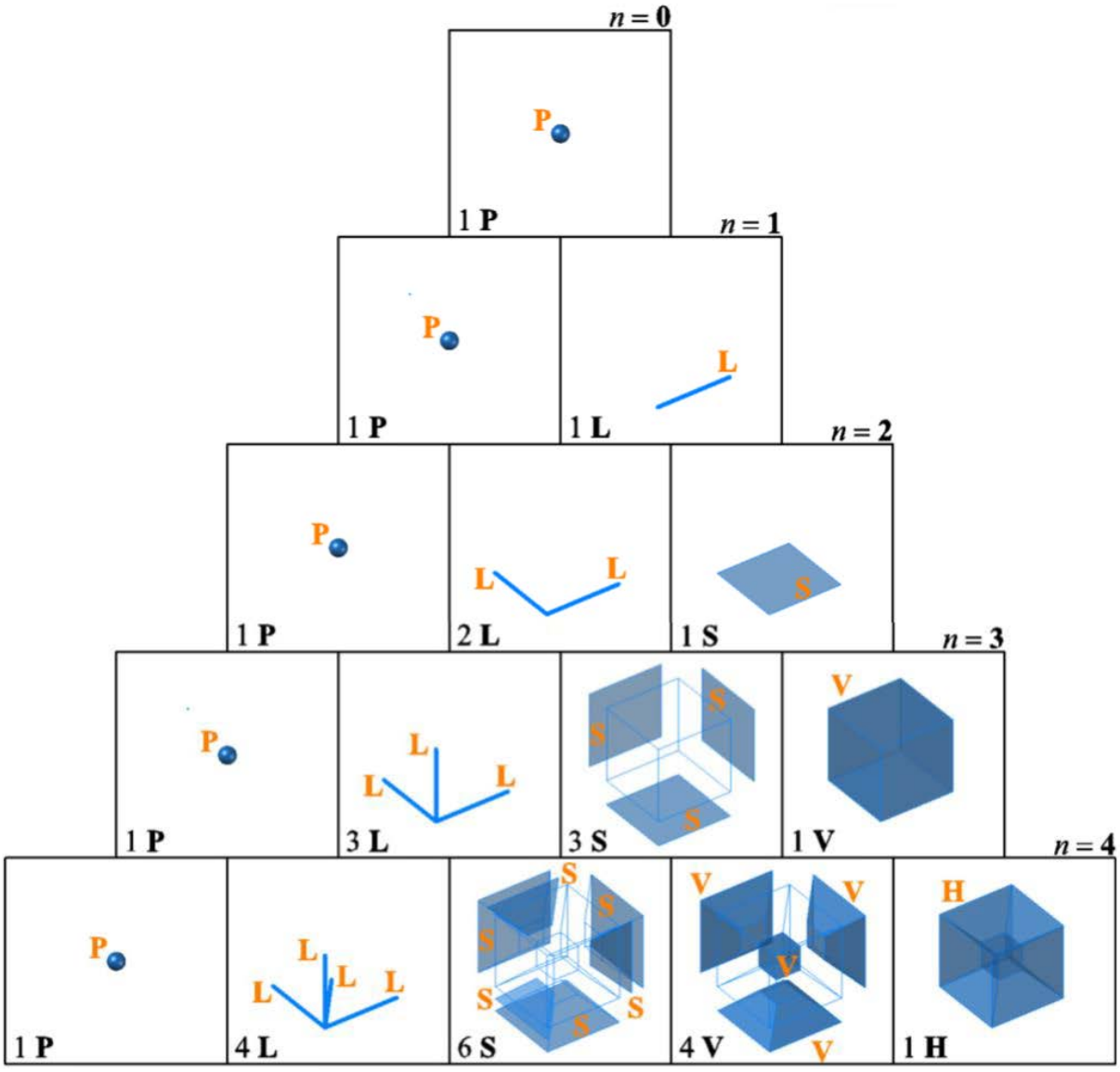


Figure 45: Pascal's triangle and the number of families of m-faces in n-dimensional space.

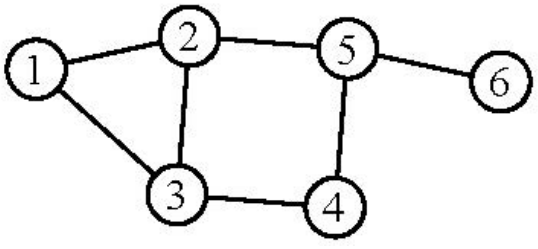


Figure 46: A drawing of a labelled graph on 6 vertices and 7 edges.

is an undirected graph, the incidence relation is a binary relation, while, if the graph is an oriented graph, the incidence relation is a ternary (or triadic) relation.

An **incidence structure** (P, L, I) consists of a set P, whose elements are called the **points**, a disjoint set L, whose elements are called the **lines**, and an incidence relation, I, between them, which is a subset of the Cartesian product $P \times L$. The elements of I are called the **flags**. If we want to take into account also the orientation of elements, such as in oriented graphs, I is a subset of the Cartesian product $P \times L \times O$, where O is the set of the **orientations**. In this second case, the incidence structure is the quadruple (P, L, O, I).

The branch of mathematics that studies incidence structures, mainly in the form of the triples (P, L, I), is called **incidence geometry**.

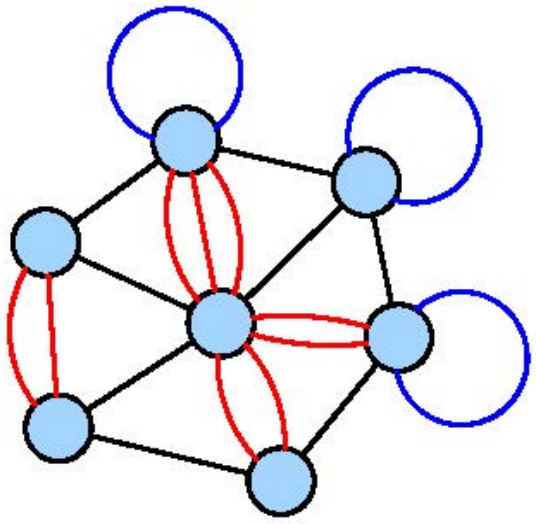


Figure 47: A multigraph with multiple edges (**red**) and several loops (**blue**).

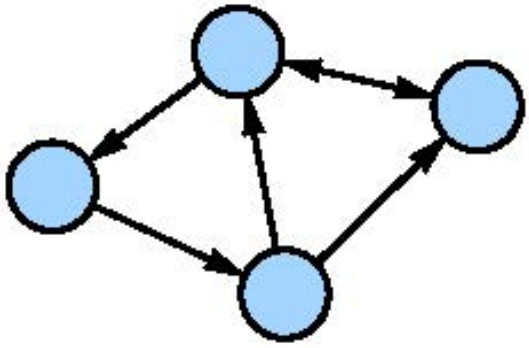


Figure 48: A directed graph.

The matrix that shows all the relationships between two classes of objects in a finite incidence geometry (one with a finite number of points and lines) is called **incidence matrix**. If the first class is X and the second is Y, the matrix has one row for each element of X and one column for each element of Y.The rows of the matrix represent points, while the columns represent lines.

The entries of the incidence matrix are called the **incidence numbers**. An incidence number different from zero in row i and column j means that the point i is incident with the line j. All other incidence numbers are zero. The nonzero incidence numbers are equal to +1 if the incidence relation is a binary relation (undirected graphs), +1 or −1 if the incidence relation is a ternary relation (oriented graphs). In the latter case, the sign of the incidence number b_{ij} tell us whether the edge x_j leaves or enters vertex v_i (there is not a single convention on how to associate the sign with the two cases).

Topological graph theory is a branch of graph theory. It studies the embedding of graphs in surfaces, spatial embeddings of graphs, and graphs as topological spaces. It also studies immersions of graphs.

Embedding a graph in a surface means that we want to draw the graph on a surface, a sphere for example, without two edges intersecting. In particular, an embedding (also spelled imbedding) of a graph G on a surface Σ is a representation of G on Σ in which points of Σ are associated with vertices and simple arcs are associated with edges in such a way that

- the endpoints of the arc associated with an edge e are the points associated with the end vertices of e,
- no arcs include points associated with other vertices,
- two arcs never intersect at a point which is interior to either of the arcs.

A **planar graph** is a graph that can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a **plane graph** or **planar embedding of the graph**.

The **dual graph** of a plane graph G is a graph that has a vertex corresponding to each face of G, and an edge joining two neighbouring faces for each edge in G (Fig. 49). The term "dual" is used because this property is symmetric, meaning that if H is a dual of G, then G is a dual of H (if G is connected). The dual of a plane graph is a plane multigraph.

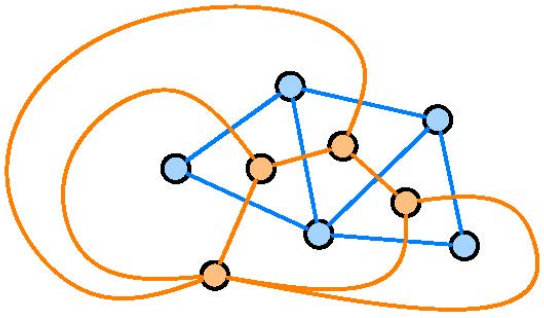


Figure 49: The orange graph is the dual graph of the blue graph.

Since the dual graph depends on a particular embedding, the dual graph of a planar graph is not unique in the sense that the same planar graph can have non-isomorphic dual graphs.

A **graph algebra** is a way of giving a directed graph an algebraic structure. It was introduced in 1983 [54] and has seen many uses in the field of universal algebra since then. Graph algebras have been used, for example, in constructions concerning dualities [129], equational theories [138], flatness [130], groupoid rings [137], topologies [136], varieties [94], finite state automata [132], finite state machines [134], tree languages and tree automata [133].

Let $D = (V, A)$ be a directed graph, and 0 an element not in V. The graph algebra associated with D is the set $V \cup \{0\}$ equipped with multiplication defined by the rules:

$$xy = x \text{ if } x, y \in V, (x, y) \in A; \tag{55}$$

$$xy = 0 \text{ if } x, y \in V \cup \{0\}, (x, y) \notin A. \tag{56}$$

Algebraic graph theory is a branch of mathematics in which algebraic methods are applied to problems about graphs.

4.5 Boundaries, Coboundaries, and the Incidence Matrices

In three-dimensional geometry, only the $(p-1)$-cells that bound a given p-cell, that is, the facets (Section 4.3) of the p-cell, are called the faces of that p-cell. Since the problems of computational physics pertain to three spatial dimension, at most, in the following we will adopt the terminology of three-dimensional geometry. Consequently, if we consider a cell complex made of p-cells of degree 0, 1, 2 and 3, the faces of a 1-cell are its two end nodes (two 0-cells), the faces of a 2-cell are its edges (1-cells), and the faces of a volume are the surfaces (2-cells) that bound the volume (Fig. 50).

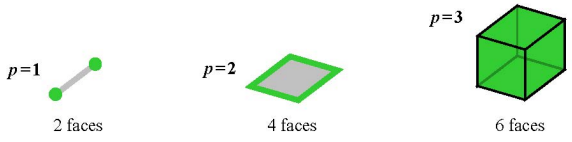


Figure 50: Faces of p-dimensional polytopes, for $1 \le p \le 3$.

The set of faces of a p-cell defines the **boundary** of the p-cell. When the p-cell is a simplex, the boundary of the p-cell is its link (see Section 4.2).

According to the definition of incidence relation (Section 4.4), one can also say that the faces of a p-cell are those $(p-1)$-cells that are incident to the p-cell. In particular, since the four space elements, **P**, **L**, **S** and **V**, are provided with an orientation (Section 3.2) and the cell complexes are, broadly speaking, oriented graphs, the relation of incidence is a ternary relation, which is expressed by the three incidence numbers, 0, +1, and −1.

The incidence numbers can be collected to form the incidence matrices (Section 4.4). In the CM we define as incidence matrix a matrix that has one row for each p-cell and one column for each $(p-1)$-cell. Therefore, the incidence matrices of the CM are not simply the generalisation of the incidence matrices of the graph theory to higher-dimensional simplices, but their transposes.

The incidence number of a p-cell, $\mathbf{e}_p^h$, with a $(p-1)$-cell, $\mathbf{e}_{p-1}^k$, is the relative integer:

$$q_{hk} \triangleq \left[\mathbf{e}_p^h : \mathbf{e}_{p-1}^k\right]; \tag{57}$$

where the first index of q_{hk} refers to the cell of greater dimension. The incidence number is equal to:

- 0, if the $(p-1)$-cell is not on the boundary of the p-cell;
- +1, if the $(p-1)$-cell is on the boundary of the p-cell and the orientations of the p-cell and $(p-1)$-cell are compatible (in the special case when the inner orientation of the p-cell is just the one that would be induced in the p-cell with the inductive procedure described in Section 3.2.1, the inner orientation of the p-cell is called **compatible** to the inner orientations of its faces, as in Fig. 51);
- −1, if the $(p-1)$-cell is on the boundary of the p-cell and the orientations of the p-cell and $(p-1)$-cell are not compatible (Fig. 52).

The $(p+1)$-cells that have a given p-cell as a common face are the cofaces of that p-cell (Fig. 53). One can also say that the cofaces of a p-cell are those $(p+1)$-cells that are incident to the p-cell. The set of cofaces of a p-cell defines the **coboundary** of the p-cell.

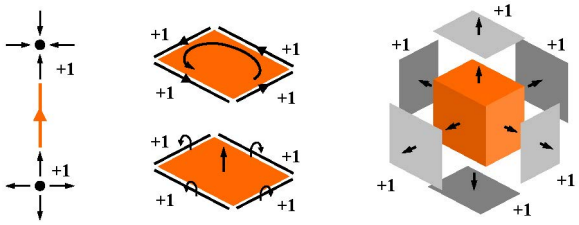


Figure 51: Examples of incidence numbers equal to +1.

In a three-dimensional space, we can define three incidence matrices

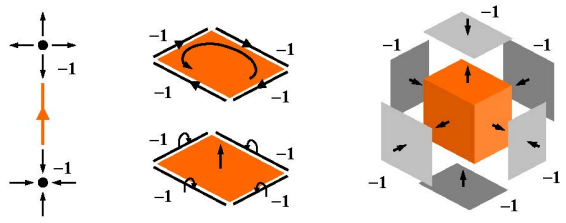


Figure 52: Examples of incidence numbers equal to –1.

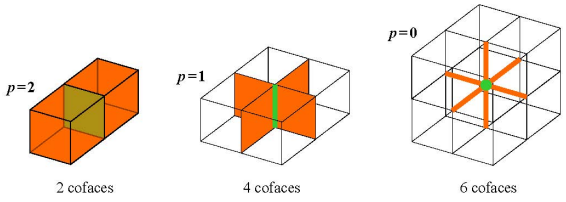
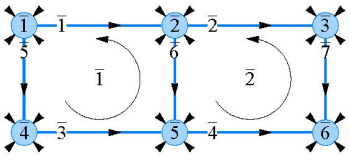


Figure 53: Cofaces of a p-cell of degree 0, 1 and 2.



Incidence matrix $\overline{\mathbf{G}}$

1-cells \ 0-cells	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{1}$	–1	+1	0	0	0	0
$\bar{2}$	0	–1	+1	0	0	0
$\bar{3}$	0	0	0	–1	+1	0
$\bar{4}$	0	0	0	0	–1	+1
$\bar{5}$	–1	0	0	+1	0	0
$\bar{6}$	0	–1	0	0	+1	0
$\bar{7}$	0	0	–1	0	0	+1

Incidence matrix $\overline{\mathbf{C}}$

2-cells \ 1-cells	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{1}$	–1	0	+1	0	+1	–1	0
$\bar{2}$	0	–1	0	+1	0	+1	–1

Figure 54: The incidence matrices in a plane cell complex (the symbol "‾" over the labels of the p-cells means that the$p$-cells are endowed with inner orientation).

- **G** : matrix of the incidence numbers between 1-cells and 0-cells,
- **C** : matrix of the incidence numbers between 2-cells and 1-cells,
- **D** : matrix of the incidence numbers between 3-cells and 2-cells.

In a two-dimensional space, there exist only the two matrices **G** and **C**. An example of matrices **G** and **C** for a two-dimensional domain is shown in Fig. 54 (where all the points are sinks, as usual).

The comparison between Figs. 54 and 10 clarifies in which sense the cell complexes and their labelling are equivalent, in the algebraic setting, to the coordinate systems and their continuous maps in $\mathbb{R}^n$, as stated in Section 2.2. In fact, the inner orientation of the cells, the edges, and the points establish, by the incidence matrices, the same type of relationship given by a basis blades in two-dimensional vector spaces. Note, in particular, the analogy between the incidence matrices and 37, where the +1 and –1 coefficients have the same function as the incidence numbers, since their signs depend on the relationship between the order of the vectors in the outer products and the inner orientation of the basis bivector.

Therefore, we can think each cell of a plane cell complex as a two-dimensional space where the points of the cell, with their labelling and inner orientation, play the role of a basis scalar, the edges of the cell, with their labelling and inner orientation, play the role of basis vectors, and the cell itself, with its inner orientation, plays the role of basis bivector, thus generalizing 38.

In the special case where the cells of the cell complex are simplices, the number of nonzero incidence numbers in each row of the matrices **G**, **C**, and **D** (the latter only for three-dimensional domains) is given by the sequence P_1 of the linear numbers, which form the first diagonal of Pascal's triangle. In particular:

- the number of nonzero incidence numbers in each row of **G** is equal to $P_1(2) = 2$,
- the number of nonzero incidence numbers in each row of **C** is equal to $P_1(3) = 3$,
- The number of nonzero incidence numbers in each row of **D** is equal to $P_1(4) = 4$.

4.6 Chains and Cochains Complexes, Boundary and Coboundary Processes

Chain complex and **cochain complex** are algebraic means of representing the relationships between the cycles and boundaries in various dimensions of a topological space.

A chain complex, $(A_\bullet, d_\bullet)$, is formally defined as a sequence of Abelian groups, or modules, $\ldots, A_{n+2}, A_{n+1}, A_n, A_{n-1}, A_{n-2}, \ldots$ connected by homomorphisms (called **boundary operators**):

$$d_n : A_n \to A_{n-1}; \tag{58}$$

such that the composition of any two consecutive maps is zero for all n:

$$d_n \circ d_{n+1} = 0 \quad \forall n. \tag{59}$$

They are usually written out as

$$\cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \xrightarrow{d_{n-2}} \cdots \tag{60}$$

A chain complex is **bounded above** if all degrees above some fixed degree are 0 and is **bounded below** if all degrees below some fixed degree are 0.

In the special case of a cell complex, a boundary operator is any map from a subset of n p-cells to a subset of m $(p-1)$-cells:

$$d_p : \sum_{i=1}^{n} \mathbf{e}_p^i \to \sum_{j=1}^{m} \mathbf{e}_{p-1}^j; \tag{61}$$

where $\mathbf{e}_p^i$ is the i-th p-cell and $\mathbf{e}_{p-1}^j$ is the j-th $(p-1)$-cell.

When $n = 1$ and m equals the number of faces of the only p-cell, the boundary operator is indicated with the symbol ∂:

$$\partial_p : \mathbf{e}_p \to \sum_{j=1}^{m} \mathbf{e}_{p-1}^j; \tag{62}$$

and ∂_p defines the boundary of $\mathbf{e}_p$.

A closed line and a closed surface are p-cells, $\mathbf{e}_p$, without boundary. They thus belong to bounded below chains:

$$\partial_p : \mathbf{e}_p \to \emptyset. \tag{63}$$

Let h_i be an integer number associated with the i-th p-cell, $\mathbf{e}_p^i$, then h_i can be viewed as a multiplicity, or a weight of $\mathbf{e}_p^i$. This weight induces a weight on the $(p-1)$-cells of the chains of $\mathbf{e}_p^i$, by a process that is called the **boundary process**. In particular, when m is equal to the number of faces of $\mathbf{e}_p^i$, the weight of $\mathbf{e}_{p-1}^j$, h_j, is obtained by adding the weights of the n_j p-cells on the coboundary of $\mathbf{e}_{p-1}^j$, with the +1 or −1 sign according to the mutual incidence numbers:

$$h_j = \sum_{i=1}^{n_j} q_{ij} h_i, \text{ where } n_j = \left| \delta \mathbf{e}_{p-1}^j \right|, \tag{64}$$

where the symbol δ, defined in 73, is used for indicating the coboundary of a cell. Thus, $\delta \mathbf{e}_{p-1}^j$ is the coboundary of the jth $(p-1)$-cell, $\mathbf{e}_{p-1}^j$. Moreover, $\left| \delta \mathbf{e}_{p-1}^j \right|$ denotes the cardinality of the set $\delta \mathbf{e}_{p-1}^j$, that is, the number of elements of the coboundary of $\mathbf{e}_{p-1}^j$, which is equal to the number of cofaces of $\mathbf{e}_{p-1}^j$.

In other words, each $(p-1)$-cell collects the weights that are spread on the $(p-1)$-cell itself by its cofaces, after having multiplied the weights by the mutual incidence numbers (Fig. 55).

The boundary process is defined as the action of the n p-cells, which spread their own weights on their faces, in accordance with the mutual incidence numbers. The pictorial view of the boundary process is provided in Fig. 56 for $n = 1$ and $p = 1, 2, 3$.

By performing the boundary process twice (Fig. 57), we obtain the weight of the k-th $(p-2)$-cell, $\mathbf{e}_{p-2}^k$, which is always equal to zero:

$$h_k = \sum_{j=1}^{n_k} \left(q_{jk} \sum_{i=1}^{n_j} q_{ij} h_i \right) = 0, \text{ where } n_j = \left| \delta \mathbf{e}_{p-1}^j \right|, \quad n_k = \left| \delta \mathbf{e}_{p-2}^k \right|; \tag{65}$$

where n_k is the number of $(p-1)$-cells on the coboundary

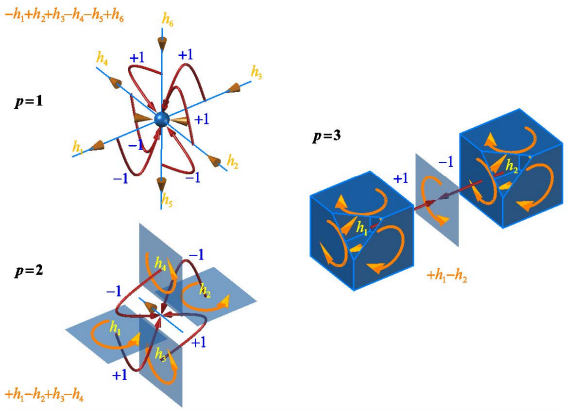


Figure 55: How to find the weight of a $(p-1)$-cell starting from the weights of its cofaces.

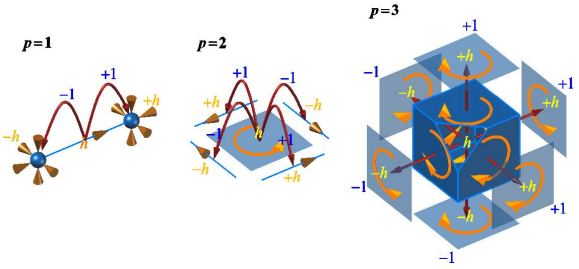


Figure 56: Spreading the weight of one p-cell on its faces.

of $\mathbf{e}_{p-2}^k$, the k-th $(p-2)$-cell.

In other words, as for the composition of the consecutive boundary operators in 59, even the composition of two consecutive boundary processes is zero for all p:

$$\partial (\partial \mathbf{e}_p) = \emptyset. \tag{66}$$

The reason for this is that the double action of a p-cell on its faces, the first time, and on the faces of its faces, the second time, generates some weights that, taken in twos, are equal and opposite. This is shown in Fig. 58 for $n = 1$ and $p = 2$.

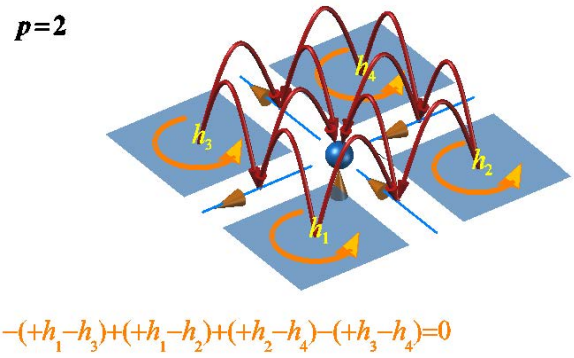


Figure 57: How to find the weight of a $(p-2)$-cell starting from the weights of the cofaces of its cofaces: a two-dimensional example.

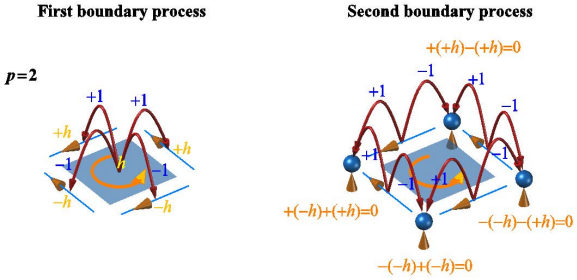


Figure 58: Spreading of the weight of one p-cell on the faces of its faces.

From 65 and the definition of the incidence matrices, **G**, **C**, and **D**, it follows that

$$\mathbf{CG} = \mathbf{0}; \tag{67}$$

$$\mathbf{DC} = \mathbf{0}; \tag{68}$$

where **0** is the null matrix.

The only difference in the definitions of chain and cochain complexes is that, in chain complexes, the boundary operators decrease dimension, whereas in cochain complexes they increase dimension. A cochain complex, $(A^\bullet, d^\bullet)$, is formally defined as a sequence of Abelian groups, or modules, $\ldots, A^{n-2}, A^{n-1}, A^n, A^{n+1}, A^{n+2}, \ldots$ connected by homomorphisms (called **coboundary operators**):

$$d^n : A^n \to A^{n+1}; \tag{69}$$

such that the composition of any two consecutive maps is zero for all n:

$$d^{n+1} \circ d^n = 0 \quad \forall n; \tag{70}$$

$$\cdots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} A^{n+2} \xrightarrow{d^{n+2}} \cdots \tag{71}$$

In the special case of a cell complex, a coboundary operator is any map from a subset of n p-cells to a subset of m $(p+1)$-cell:

$$d^p : \sum_{i=1}^{n} \mathbf{e}_p^i \to \sum_{j=1}^{m} \mathbf{e}_{p+1}^j; \tag{72}$$

where $\mathbf{e}_p^i$ is the i-th p-cell and $\mathbf{e}_{p+1}^j$ is the j-th $(p+1)$-cell.

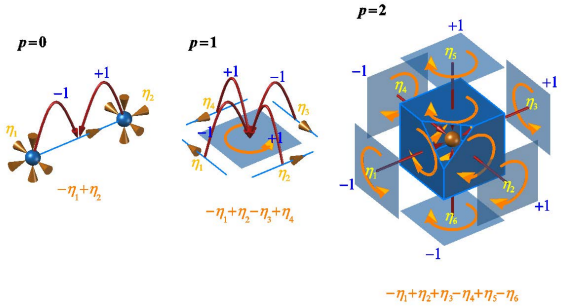


Figure 59: How to find the weight of a $(p+1)$-cell starting from the weights of its faces.

When $m = 1$ and n equals the number of cofaces of the $(p+1)$-cell, the coboundary operator is indicated with the symbol δ:

$$\delta^p : \sum_{i=1}^{n} \mathbf{e}_p^i \to \mathbf{e}_{p+1}; \tag{73}$$

and δ^p defines the coboundary of $\mathbf{e}_{p+1}$.

If $\mathbf{e}_p$ is a closed line or a closed surface, then $\mathbf{e}_p$ belongs to bounded below cochains:

$$\delta^{p-1} : \emptyset \to \mathbf{e}_p. \tag{74}$$

Let η^i be the weight (an integer number) of the i-th p-cell, $\mathbf{e}_p^i$. This weight induces a weight on the $(p+1)$-cells of the cochains of $\mathbf{e}_p^i$, by a process that is called the **coboundary process**. In particular, when n is equal to the number of cofaces of $\mathbf{e}_{p+1}^j$, the weight of $\mathbf{e}_{p+1}^j$, η^j, is obtained by adding the weights of the n_j p-cells on the boundary of $\mathbf{e}_{p+1}^j$, with the +1 or −1 sign according to the mutual incidence numbers:

$$h^j = \sum_{i=1}^{n_j} q_{ji}\eta^i, \text{ where } n_j = \left|\partial \mathbf{e}_{p+1}^j\right|. \tag{75}$$

Note that the two indices of the incidence numbers have been swapped, because the first index refers to the cell of greater dimension.

Each $(p+1)$-cell collects the weights that are spread on the $(p+1)$-cell itself by its faces, after having multiplied the weights by the mutual incidence numbers (Fig. 59).

The coboundary process can be defined as the action of the n p-cells, which spread their own weights on their cofaces, in accordance with the mutual incidence numbers. The pictorial view of the coboundary process is provided in Fig. 60 for $n = 1$ and $p = 1$.

By performing the coboundary process twice, we obtain

$$\delta(\delta \mathbf{e}_p) = \emptyset; \tag{76}$$

which is the algebraic counterpart of the differential identities:

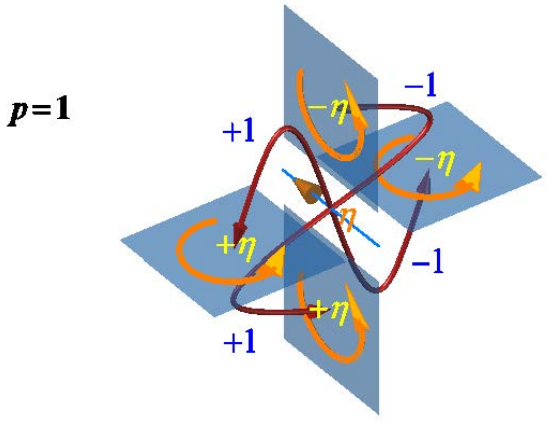


Figure 60: The coboundary process.

$$\text{curl}\,(\text{grad}\ f) \equiv 0; \tag{77}$$

$$\text{div}\,(\text{curl}\ \mathbf{v}) \equiv 0. \tag{78}$$

4.7 Discrete p-forms

A physical variable ϕ associated with one set of p-cells of a cell-complex defines a discrete p-form (or a discrete form of degree p). The potential of a vector field, line integral of a vector, flux and mass content are discrete forms of degree 0, 1, 2, and 3, respectively (Table 2).

The discrete p-forms generalise the notion of field functions, because, in a discrete p-form $\Phi[\mathbf{P}]$, $\Phi[\mathbf{L}]$, $\Phi[\mathbf{S}]$, or $\Phi[\mathbf{V}]$, we associate the value of a physical variable with the space elements of degree p, where $p = 0, 1, 2, 3$, while the field functions, $f(\mathbf{P})$, always associate the value of a physical variable with the points of the domain. As a consequence, $\Phi[\mathbf{P}]$, $\Phi[\mathbf{L}]$, $\Phi[\mathbf{S}]$, and $\Phi[\mathbf{V}]$ are set functions, while $f(\mathbf{P})$ is a point function.

The notion of discrete form is the algebraic version of the exterior differential form, a mathematical formalism that has the great merit of highlighting the geometrical background of physical variables, something ignored by differential calculus, providing a description that is independent of the coordinate system used. Nevertheless, this formalism uses field variables instead of global variables and, for this reason, it must use the notion of derivative.

Let S_p be the set of m p-cells, $\mathbf{e}_p^i$, each one taken with the multiplicity n_i:

$$S_p = \left\{ n_1 \mathbf{e}_p^1, n_2 \mathbf{e}_p^2, \ldots, n_m \mathbf{e}_p^m \right\}; \tag{79}$$

and let $\mathscr{G}$ be an additive and commutative group (a scalar, a vector, a matrix, and so forth), then a discrete p-form, c^p, is a linear function on the set S_p with integer-valued coefficients in the group $\mathscr{G}$.

Denoting by g_i the values assumed by the coefficients when the multiplicities of the p-cells are equal to +1, a discrete p-form can be represented by a row vector, whose elements are the coefficients g_i of the discrete form:

$$c^p = \begin{bmatrix} g_1 & g_2 & \ldots & g_m \end{bmatrix}. \tag{80}$$

The value, g, of the discrete p-form $c^p = \begin{bmatrix} g_1 & g_2 & \ldots & g_m \end{bmatrix}$ on the collection with integer coefficients:

$$\mathbf{c}_p = \sum_{i=1}^{m} n_i \mathbf{e}_p^i; \tag{81}$$

which can also be described by the column vector:

$$\mathbf{c}_p = \begin{bmatrix} n_1 & n_2 & \ldots & n_m \end{bmatrix}^T; \tag{82}$$

is equal to

$$g = \sum_{i=1}^{m} g_i n_i \equiv \begin{bmatrix} g_1 & g_2 & \ldots & g_m \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \ldots \\ n_m \end{bmatrix} = \left\langle c^p, \mathbf{c}_p \right\rangle. \tag{83}$$

Since the p-forms are additive and homogeneous of degree 1, we can re-write g as

$$g = \left\langle c^p, \mathbf{c}_p \right\rangle = \left\langle c^p, \sum_{i=1}^{m} n_i \mathbf{e}_p^i \right\rangle = \sum_{i=1}^{m} n_i \left\langle c^p, \mathbf{e}_p^i \right\rangle = \sum_{i=1}^{m} n_i g_i. \tag{84}$$

Equation 84 means that a p-form (for example, a line integral, a flux, or a mass content) on one of the three space elements with dimension greater than 0 (**L**, **S**, and **V**) is the sum of the values assumed by the p-form on each of the elementary parts in which the space element can be divided.

In particular, remembering again that the discrete p-forms are homogeneous, from 84 it follows the **Oddness Condition** of the p-forms:

$$\left\langle c^p, -\mathbf{e}_p^i \right\rangle = -\left\langle c^p, \mathbf{e}_p^i \right\rangle = -g_i. \tag{85}$$

Table 2: Examples of discrete p-forms

Variable	Potential of a vector field	Line integral of a vector	Flux	Mass content
Evaluated on	0-cells (points)	1-cells (lines)	2-cells (surfaces)	3-cells (volumes)
Discrete p-form	discrete 0-form $\Phi[\mathbf{P}]$	discrete 1-form $\Phi[\mathbf{L}]$	discrete 2-form $\Phi[\mathbf{S}]$	discrete 3-form $\Phi[\mathbf{V}]$

The introduction of the discrete p-forms allows us to find the relation between the boundary operator, ∂, and the coboundary operator, δ

$$\left\langle \delta c^p, \mathbf{c}_{p+1} \right\rangle = \left\langle c^p, \partial \mathbf{c}_{p+1} \right\rangle . \tag{86}$$

This theorem is the algebraic form of the generalised theorem of Stokes, which includes the theorem of Gauss, the proper theorem of Stokes, and the theorem of Leibnitz:

$$\begin{cases} \int_{\mathbf{V}} \nabla \cdot \mathbf{u}\, dV = \int_{\partial \mathbf{V}} \mathbf{u} \cdot dS & \text{Gauss} \\ \int_{\mathbf{S}} (\nabla \times \mathbf{u}) \cdot dS = \int_{\partial \mathbf{S}} \mathbf{v} \cdot dL & \text{Stokes} \\ \int_{\mathbf{V}} \nabla \phi \cdot dL = \phi(\partial \mathbf{L}) = \phi(\partial \mathbf{B}) - \phi(\partial \mathbf{A}) & \text{Leibnitz} \end{cases} \tag{87}$$

For this reason, it is called the **generalised or combinatorial form of Stokes' theorem**. The statement of the theorem is as follows:

The coboundary of the discrete p-form c^p on the $p+1$ collection, $\mathbf{c}_{p+1}$, is equal to the discrete p-form, c^p, on the p-dimensional boundary of the $p+1$ collection, $\mathbf{c}_{p+1}$.

Note that we have derived the theorem of Stokes as a direct consequence of the coboundary process. This means that Stokes' theorem is a purely topological relation, while the various continuity and differentiability conditions usually required in its proof are indeed required by the use of field functions and derivatives. We could say that the most remarkable aspect of the proof of the generalised form of Stokes' theorem lies in highlighting that many differentiability requirements do not belong to physical laws, but are required by the differential apparatus used in their description.

By using 86 and 66, we can proof that the coboundary of a coboundary vanishes, as stated by 76:

$$\begin{aligned} \delta(\delta c^p)(\mathbf{c}_{p+2}) &= (\delta c^p)(\partial \mathbf{c}_{p+2}) = \\ &= c^p(\partial \partial \mathbf{c}_{p+2}) = c^p(\emptyset) = 0. \end{aligned} \tag{88}$$

In other words, when the coboundary process is performed twice in sequence it gives rise to the null element of the group $\mathcal{G}$.

Analogously, by using 86 and 76, we can prove 66 (the boundary of a boundary vanishes):

$$\partial\left(\partial c^p\right)(\mathbf{c}_p) = \left(\partial c^p\right)(\delta \mathbf{c}_p) = c^p(\delta \delta \mathbf{c}_p) = c^p(\emptyset) = 0. \tag{89}$$

4.8 Inner and Outer Orientations of Time Elements

Finding the orientations of the time elements could be viewed in the same way that finding the inner and outer orientations of points and lines in a one-dimensional space. In fact, the time axis defines a one-dimensional cell complex, where the time instants, **I**, are points (nodes) and the time intervals, **T**, are the line segments that connect the points (the time instants are the boundary, or the faces, of the time intervals: see Sections 4.3 and 4.5 for details). Moreover, in a one-dimensional space the dual (orthogonal complement) of a point is a line segment and the dual of a line segment is a point. Consequently, the outer orientations could be obtained in accordance with Fig. 29, in the special case where $n = 1$.

As far as the inner orientation is concerned, all the time instants, both those along the positive semi-axis and those along the negative semi-axis, are sinks. Thus, they have an inward inner orientation (Fig. 61).

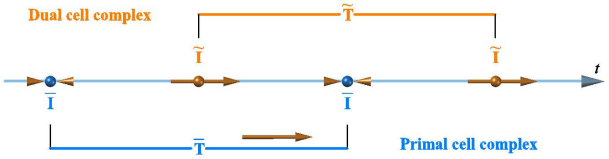


Figure 61: Time elements and their duals.

Finally, we can decide that the inner orientation of the time intervals is the same as the orientation of the time axis.

After a more detailed analysis, however, it is clear that building a cell complex in time makes no sense in itself. In fact, in physics time has not importance in itself. It is just a variable, useful for describing how a physical phenomenon evolves. Now, since any physical phenomenon occurs in space, it follows that the time axis must always be related to one or more axes in space. The perception itself of the time is linked to bodies. Therefore, a cell complex in time must be two-dimensional, at least.

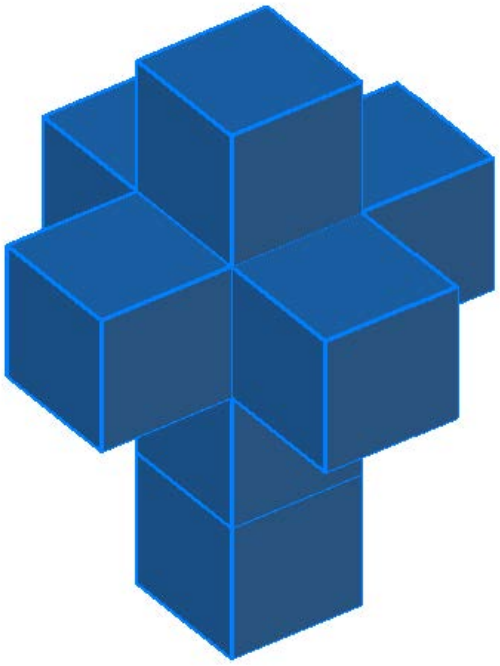


Figure 62: Unfolding of a tesseract in three-dimensional space.

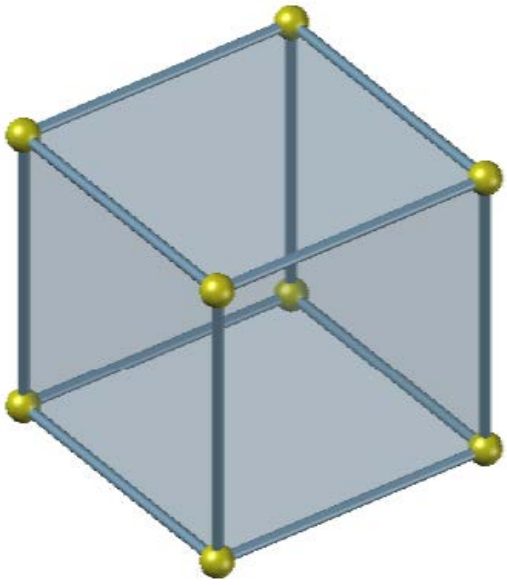


Figure 63: Cell-first perspective projection of the tesseract into 3 dimensions, with hidden surfaces culled.

When a time axis is added to a three-dimensional cell complex where the cell of maximum dimension has been originated by a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$, it gives rise to a four-dimensional space/time cell complex, whose cell of maximum dimension is a **tesseract**, which is one of the six convex regular 4-polytopes (four-dimensional analogue of a polyhedron). In multilinear algebra, a tesseract, also called a **regular octachoron** or **cubic prism**, is a further element of the (graded) exterior algebra on a vector space.

The tesseract is the four-dimensional analog of the cube, in the sense that it is to the cube as the cube is to the square. Each edge of a tesseract is of the same length. Just as the surface of the cube consists of six square faces, the hypersurface of the tesseract consists of eight cubical cells (Fig. 62).

All in all, a tesseract consists of eight cubes, 24 squares, 32 edges, and 16 vertices. There are four cubes, six squares, and four edges meeting at every vertex.

Since a generalisation of the cube to dimensions greater than three is called a "hypercube", or "n-cube", the tesseract is also called the **four-dimensional hypercube**, or **4-cube**.

This structure is not easily imagined, but it is possible to project tesseracts into three- or two-dimensional spaces. One possibility is that of forming the three-dimensional shadows of the tesseract. If the wireframe of a cube is lit from above, the resulting shadow is a square within a square with the corresponding corners connected. Similarly, if the wireframe of a tesseract were lit from "above" (in the fourth direction), its shadow would be that of a three-dimensional cube within another three-dimensional cube (Fig. 63), where five faces are obscured by the visible faces. Similarly, seven cells of the tesseract are not seen in Fig. 63, because they are obscured by the visible cell.

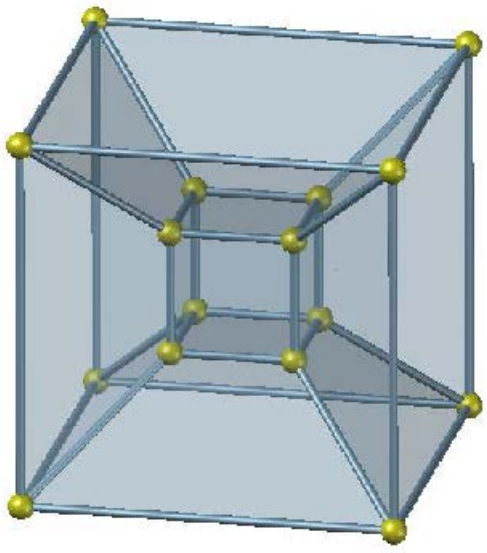


Figure 64: The 8 cubical cells of the tesseract folded, in threes, around the same edge.

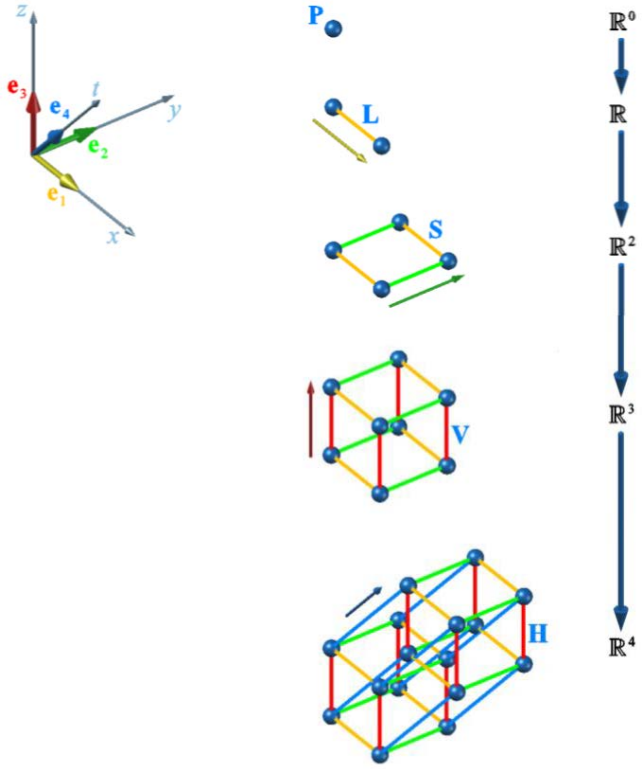


Figure 65: Inductive construction of a 4D hyperprism from dimension 0 to dimension 4, by adding one dimension at a time.

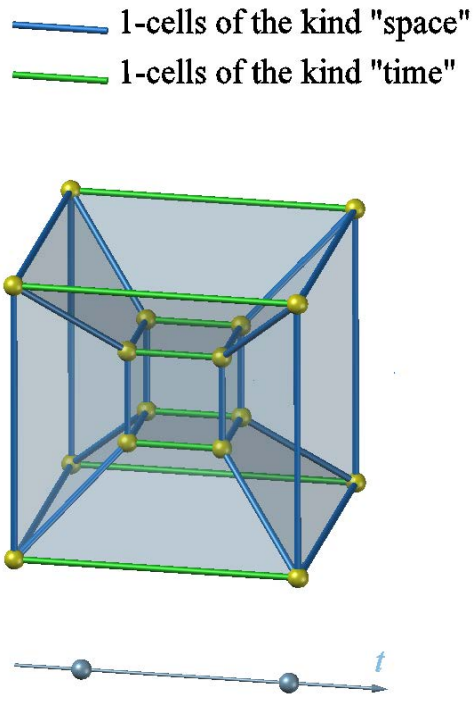


Figure 66: Different kinds of 1-cells in a space/time tesseract.

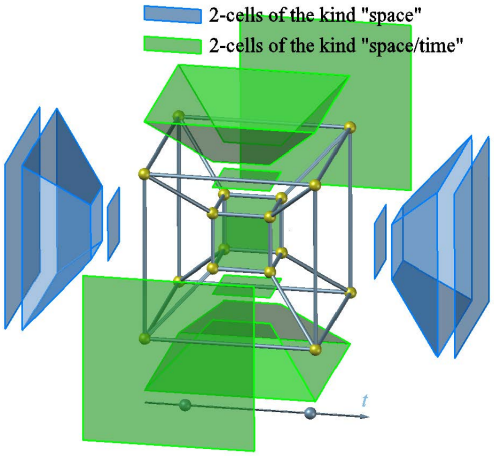


Figure 67: Different kinds of 2-cells in a space/time tesseract.

The tesseract can be constructed in a number of ways:

- As a regular polytope (the generalized analog in any number of dimensions of regular polygons and regular polyhedra) with three cubes folded together around every edge (Fig. 64). In this case, it can be represented by the Schläfli symbol {4,3,3}.
- As a 4D hyperprism. In this case, it is made of two parallel cubes (Fig. 65). The scheme is similar to the construction of a cube from two squares: juxtapose two copies of the lower-dimensional cube and connect the corresponding vertices.
- As a duoprism. In this case, it is represented by the Cartesian product of two squares.
- As an orthotope.
- As the convex hull of the points (±1, ±1, ±1, ±1) in Euclidean 4-space. In this case, it consists of the points:

$$\left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : -1 \le x_i \le +1\right\} \qquad (90)$$

and is bounded by eight hyperplanes ($x_i = \pm 1$).

- As bipartite graphs. In this case, the eight cubes of the tesseract, taken in twos, are independent sets of points.
- As the **4-vector** (multivector of degree four), or Clifford number of degree four, $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \wedge \mathbf{t}$ of the exterior algebra, $\Lambda(V)$, on the vector space, V. This is the construction we will prefer in this paper.

Note that the elements of a CM space/time 4-vector (a tesseract) are of different nature, since some p-cells are associated with a variation of the space variables, some other p-cells are associated with a variation of the time variables, and some other p-cells are associated with a variation of both the space and time variables. In particular,

the points are associated with a variation of both the space and time variables. Therefore, we can say that there exists just one kind of points. As far as the others p-cells are concerned, on the contrary, we can define two different kinds of cells for each $p = 1, 2, 3$. We will denote

- 1-cells of the kind "space": the 1-cells that connect points associated with the same time instant, that is, the edges of the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ at a given instant (Fig. 66);
- 1-cells of the kind "time": the 1-cells that connect points associated with two adjacent time instants, that is, the time intervals (Fig. 66);
- 2-cells of the kind "space": the 2-cells that connect edges associated with the same time instant, that is, the faces of the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ at a given instant (Fig. 67);
- 2-cells of the kind "space/time": the 2-cells that connect edges associated with two adjacent time instants. The area of one of these faces is given by the product between a time interval and an edge of the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ (Fig. 67);
- 3-cells of the kind "space": the 3-cells that connect faces associated with the same time instant, that is, the volume of the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ at a given instant (Fig. 68).
- 3-cells of the kind "space/time": the 3-cells that are enclosed within faces associated with two adjacent time instants. The volume of one of these 3-cells is given by the product between a time interval and two edges of the trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ (Fig. 68).

We can find a formally similar classification in the four-dimensional Minkowski spacetime [139] – [141], where the spacetime interval between two events is either space-like, light-like (null), or time-like.

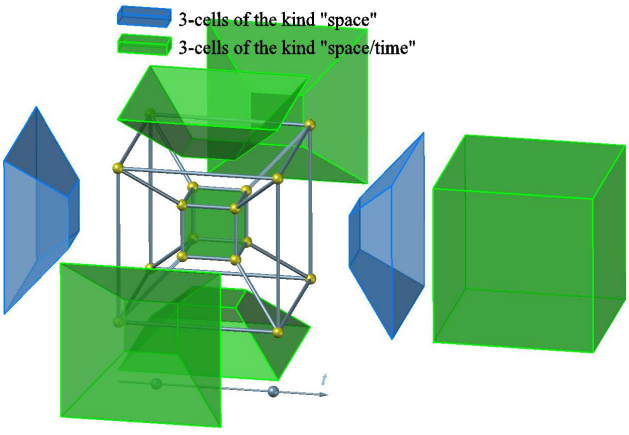


Figure 68: Different kinds of 3-cells in a space/time tesseract.

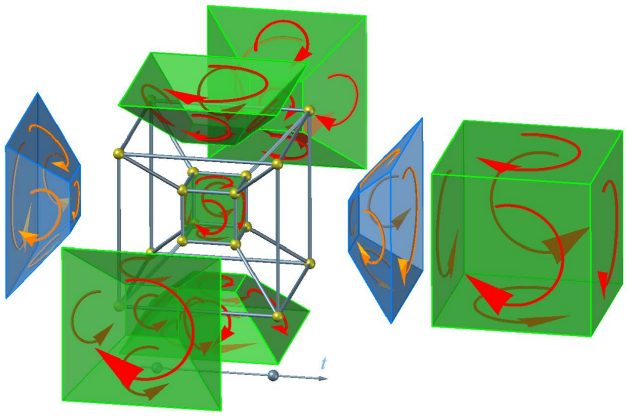


Figure 69: Inner orientations on the 2-cells of the 4-vector.

The inner and outer orientations of the 4-vector are shown in Fig. 69 for the faces of each of the eight cubes. In order to comply with the natural time sequence, from previous time instants to subsequent time instants, we will adopt a different orientation. In particular, by associating the elements of the left cube with the previous instant and the elements of the right cube with the subsequent instant, the eight edges connecting the left to the right cube turn out to have an inner orientation from left to right, that is, the same orientation of the time axis (Fig. 70).

When we associate the global space and time variables with the oriented elements of a 4-vector, we obtain the algebraic version of a four-dimensional Minkowski continuum, called spacetime, whose metric treats the time dimension differently from the three spatial dimensions. Spacetime is thus not a Euclidean space.

In the algebraic formulation, the inner orientation of the 1-cells of the kind "space/time" (time intervals) is the same as the orientation of the time axis (from past to future, in the natural time sequence). Moreover, since the same point of a four-dimensional space denotes both a point in space and a point in time (a time instant), it follows that the time instants have an inward inner orientation, that is, they are sinks.

5 Conclusions

In this paper, the difference between the algebraic and the differential formulation is analysed from the mathematical point of view. Particular attention is devoted to the computation of limits—by highlighting how the Cancelation Rule for limits, used for providing a direct exact solution, may imply a loss of information. Conversely, by computing

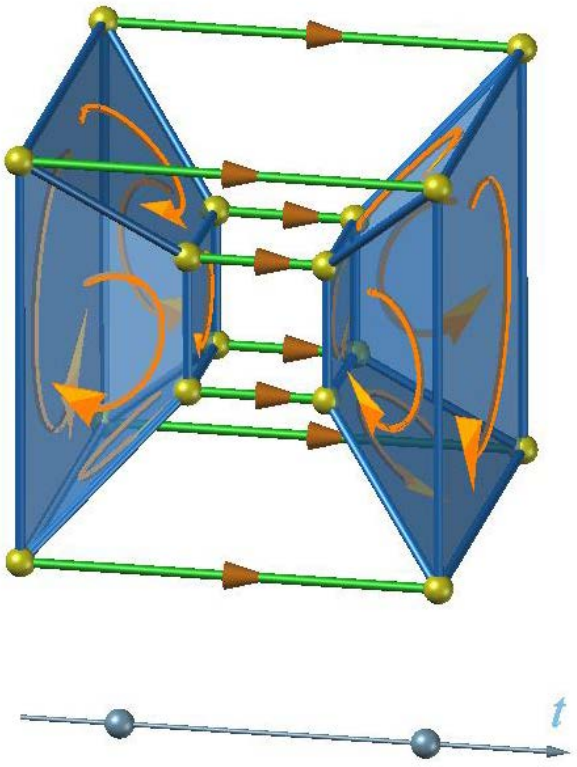


Figure 70: The CM tesseract: inner orientations on the 3-cells of the kind space and the 1-cells of the kind time.

the limit iteratively, with the dimension of the neighbourhood that decreases at each iteration, leading also the error on the solution to decrease, we conserve information on the trend of the function in the neighbourhood of the estimation point. This second way to operate, where the dimension of the neighbourhood approaches zero but is never equal to zero, follows from the $\varepsilon - \delta$ definition of a limit directly and leads to the algebraic formulation. When the Cancelation Rule for limits is used for finding densities and rates, we also lose information on the space and time extent of the geometrical and temporal objects associated with the variables we are computing, obtaining point- and instant-wise variables. By using the algebraic formulation, on the contrary, we preserve both the length and the time scales. Consequently, the physical variables of the algebraic formulation maintain an association with the space and time multi-dimensional elements.

It is also shown how the Cancelation Rule for limits acts on the actual solution of a physical problem as a projection operator. The consequence is that the algebraic formulations is to the differential formulation as the actual solution of a physical problem is to the projection of the actual solution on the tangent space of degree 0, where each physical phenomenon is described in terms of space elements of degree 0, the points, and time elements of degree 0, the time instants. In other words, the differential solution is the shadow of the algebraic solution in the tangent space of degree 0. Moreover, using the algebraic formulation, instead of the differential formulation, is similar to performing non-standard calculus, the modern application of infinitesimals to differential and integral calculus, instead of standard calculus. In this sense, the derivative of a function can be viewed as the standard part, or the shadow, of the difference quotient. It is worth noting that the extension of real numbers, which leads to non-standard calculus, is indeed an attempt to recover the loss of length scales, and the enrichment with a length scale has a regularization effect on the solution.

Of special importance for the philosophy of the Cell Method (CM), the only truly algebraic numerical method, at the moment, are the geometric interpretations of the operations on vectors, provided by both the exterior and geometric algebra, and the notions of extension of a vector by another vector, multivector (or p-vector), dual vector space, bialgebra, and covector. The geometric approach allows us to view the four space elements, **P**, **L**, **S**, and **V**, and the two time elements, **I** and **T**, as p-vectors of a geometric algebra, all inductively generated by the outer product of the geometric algebra. From the attitude and orientation of p-vectors, we have derived the two kinds of orientation for p-vectors, inner and outer orientations, which apply to both the space and the time elements. We have also discussed how the orientation of a p-vector is induced by the orientation of the $(p-1)$-vectors on its boundary, and how the inner orientation of the attitude vector of a vector equals the outer orientation of its covector. This establishes an isomorphism between the orthogonal complement and the dual vector space of any subset of vectors. Finally, we can find a natural analog of this relationship in general Banach spaces. One of the most remarkable consequences of the relationship between inner and outer orientations is that the outer orientation depends on the dimension of the embedding space, while the inner orientation does not.

The notions of inner and outer orientations implicitly permeate the geometric algebra and are equivalent to the rules for the orientation of vectors in right-handed coordinate systems. We have made them explicit in this paper because they are at the basis of the CM description of physics, where the cell complexes for computational analysis are treated algebraically, as topological vector spaces.

The graph theory allows us to introduce the main tools of the incidence geometry, that is, the incidence numbers and the incidence matrices. These tools can be applied even to cell complexes, since an oriented cell complex is

the generalisation of an oriented graph to directed simplices of dimension greater to 1. One of the main consequences of this generalisation is that we can extend to cell complexes the notion of dual graph of a plane graph, attaining a dual cell complex. In particular, since the dual graph depends on a particular embedding, the dual cell complex, as the dual graph, is not unique. Then, the boundary and the coboundary of a p-cell are put in relationship with the notions of incidence relation and incidence matrix. In the special case of the three-dimensional space, the CM uses the incidence matrices for describing the incidence relations between the space elements and their coboundaries, and between the time elements and their coboundaries. We thus obtain three incidence matrices for the four space elements and one incidence matrix for the two time elements. The former three incidence matrices are denoted by **G**, **C**, and **D**. Their incidence numbers are arranged in the matrices in way that **G**, **C**, and **D** turn out to be not simply the generalisation of the incidence matrices of the graph theory, but their transposes. It is also shown how each cell of a plane cell complex can be viewed as a two-dimensional space, where the points of the cell, with their labelling and inner orientation, play the role of a basis scalar, the edges of the cell, with their labelling and inner orientation, play the role of basis vectors, and the cell itself, with its inner orientation, plays the role of basis bivector. This clarifies in which sense the cell complexes and their labelling are equivalent, in the algebraic setting, to the coordinate systems and their continuous maps in $\mathbb{R}^n$.

The chain complex and cochain complex are useful for the definition of the boundary operators, the boundary process, the coboundary operators, and the coboundary process. Coboundary operators and coboundary process, in particular, play an essential role in defining the mathematical structure of the CM governing equations.

The discrete p-forms are further essential tools in the algebraic formulation of the CM. More specifically, the discrete p-forms generalize the notion of field functions and are the algebraic version of the exterior differential forms. The discrete p-forms also allow us to find the relation between the boundary operator and the coboundary operator, leading to the algebraic form of the generalized theorem of Stokes.

Finally, we have discussed the geometrical structure of a four-dimensional cell complex, suitable for a unified description of space and time global variables. The basic cell of this four-dimensional cell complex is the tesseract. Some p-cells of the CM space/time tesseract are associated with a variation of the space variables, some other p-cells are associated with a variation of the time variables, and some other p-cells are associated with a variation of both the space and time variables. This is only one of the many similarities between the CM space/time tesseract and the four-dimensional Minkowski spacetime.

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