Research Article

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Asymmetric Auctions with Discretely Distributed Valuations

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Abstract: We examine a two-bidder auction setting in which the distributions for the bidders' valuations are asymmetric over a support consisting of three elements. For the first price auction, for each parameter values we derive the unique Bayes Nash Equilibrium in closed form. We rely on this result to compare the revenue in the first price auction with the revenue in the second price auction. The latter is often revenue superior to the former, and we determine precisely, given a distribution for the value of a bidder, when a distribution for the value of the other bidder exists such that the first price auction is superior to the second price auction.

Keywords: asymmetric auctions; first price auction; second price auction; revenue ranking

JEL Classification: D44; D82

1 Introduction

This paper is about an auction setting in which bidders have asymmetrically distributed values, but for which it is possible to characterize in closed form the unique Bayes Nash Equilibrium for the first price auction. This allows to derive quite accurate results on the effects of asymmetries on equilibrium bidding, and

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on the revenue comparison between the first price auction and the second price auction.

In the standard auction setting, bidders have private values which are ex ante i.i.d. random variables; this delivers many significant results for the standard setting. Conversely, the important and realistic extension in which bidders have asymmetrically distributed values is more difficult to deal with for a variety of auctions, for instance for the first price auction (FPA), because asymmetric distributions often prevent the existence of a closed form for the equilibrium bidding functions[1] – one exception is the second price auction (SPA), in which bidding the own valuation is a weakly dominant strategy for each bidder. This makes it difficult, in an asymmetric environment, to compare the revenues from different auction formats, or to perform comparative statics analysis about the effect of a change in the distributions of the valuations.

In this paper we examine a setting with two bidders in which the valuation of each bidder has the same support $\{v_L, v_M, v_H\}$, with $v_H - v_M = v_M - v_L > 0$, but the probability distribution for v_1, the value of bidder 1, may be different from the probability distribution for v_2, the value of bidder 2.[2] The only restriction we impose on the distributions, without loss of generality, is $\Pr\{v_1 = v_H\} \geq \Pr\{v_2 = v_H\}$.

We determine in closed form the unique Bayes Nash Equilibrium for the FPA, which involves mixed strategies for both bidders. In particular, the supports for the bids submitted by type 1_H (bidder 1 with value v_H) and type 2_H (bidder 2 with value v_H) share the same maximum bid, which implies that these types have the same utility, and this typically has the consequence that type 1_M (or type 2_M, but not both) puts a probability mass on the bid v_L.[3] This "mass" feature of the equilibrium in the FPA increases the winning probability and the utility for type 1_M or for type 2_M above the winning probability and the utility under the SPA. This is relevant when we compare the FPA and the SPA in terms of revenue, because the SPA allocates the object efficiently – unlike the FPA – and a sufficient condition for R^S, the expected revenue under the SPA, to be higher than R^F, the expected revenue under the FPA, is that the bidders' rents in the FPA are greater than in the SPA. We prove that this is often the case because of the mass feature of the equilibrium for the FPA.[4] More

1 Plum (1992), Cheng (2006), Kaplan and Zamir (2012) derive equilibrium for the FPA in closed form for specific settings.

2 Maskin and Riley (1983), Maskin and Riley (1985), Cheng (2011), Doni and Menicucci (2013) examine settings with discretely distributed values, but restrict to cases in which the value of each bidder has a binary support.

3 In fact, in some cases also type 1_H bids v_L with positive probability.

4 Conversely, the literature has identified several settings in which the opposite result, $R^F > R^S$, holds: see for instance Maskin and Riley (2000a), Li and Riley (2007), Kirkegaard (2012), Kirkegaard (2014), Kirkegaard (2021).

in detail, we show that some probability distributions for v_2 are such that $R^S \geq R^F$ *for each* distribution for v_1 (see set S_2 in Figure 2 in Subsection 4.2), whereas for other distributions for v_2 there is a distribution for v_1 such that $R^F > R^S$. In general, the smaller is $\Pr\{v_2 = v_H\}$, the more likely is that there exists a distribution for v_1 which satisfies $R^F > R^S$, and such distribution induces the strongest bidding in the FPA given the distribution for v_2.

Maskin and Riley (1985) prove that $R^S > R^F$ always holds in a setting in which each bidder's value has a (same) binary support. Conversely, in our setting with ternary support it is possible that R^F is greater than R^S. We explain that this occurs because starting from a symmetric setting, with $R^F = R^S$, a suitable improvement in a bidder's value distribution increases R^F and R^S, which in some cases results in $R^F > R^S$. But when the support is binary, any improvement in the value distribution of a bidder has the effect of increasing R^S, while R^F does not change as neither bidder changes his bid distribution in the FPA.

The rest of the paper is organized as follows: Section 2 introduces the auction environment. Section 3 is about equilibrium bidding in the FPA. Section 4 compares the FPA and the SPA in terms of bidders' rents and in terms of revenue. Section 5 concludes. The Appendix provides the proof of Proposition 1. The proofs of our other results are available in Ceesay, Doni, Menicucci (2024).[5]

2 Model

A (female) seller owns an object to which she attaches no value and faces two (male) bidders interested in buying the object. Bidder 1 (bidder 2) privately observes his own monetary value v_1 (v_2) for the object, which is equal either to v_L, or to v_M, or to v_H, with $v_L \geq 0$ and $v_M = v_L + \Delta$, $v_H = v_M + \Delta$ for a positive Δ. For $i = 1{,}2$, the value v_i of bidder i is viewed by the seller and by the other bidder as a realization of a random variable for which the probabilities of v_L, v_M, v_H are denoted with λ_i,μ_i,η_i:[6]

$$\lambda_i = \Pr\{v_i = v_L\} > 0, \;\; \mu_i = \Pr\{v_i = v_M\} > 0, \;\; \eta_i = \Pr\{v_i = v_H\} > 0$$

5 In addition, Ceesay, Doni, Menicucci (2024) examine two particular classes of asymmetries – shift and stretch – introduced in Maskin and Riley (2000a), and the effects of asymmetry on bidding with respect to a symmetric environment. The latter analysis allows to examine a bidder's incentive to invest ex ante in order to improve the value distribution, an issue we briefly discuss in Section 5, for a procurement setting.

6 This is for mnemonic reasons, as λ (μ) is the Greek letter equivalent for L (M), and η is the Greek letter closest to H.

with $\lambda_i + \mu_i + \eta_i = 1$, and the distributions of v_1 and v_2 are stochastically independent.[7]

Although the two random variables have the same support $\{v_L, v_M, v_H\}$, they are asymmetrically distributed unless $(\lambda_1, \mu_1, \eta_1) = (\lambda_2, \mu_2, \eta_2)$. The expected utility of each bidder is given by his value times his probability to win the object, minus his expected payment. The seller is risk neutral.

3 Equilibrium Bidding

3.1 Equilibrium Bidding in the First Price Auction

In the first price auction, FPA henceforth, each bidder simultaneously submits a sealed bid, the highest bidder wins and pays his bid to the seller. For some tie-breaking rules, no pure-strategy equilibrium exists in this game, but Proposition 2 in Maskin and Riley (2000b) establishes that an equilibrium, possibly in mixed strategies, exists under the "Vickrey tie-breaking rule", according to which each bidder i is required to submit both an "ordinary" bid $b_i \geq 0$ and a "tie-breaker" bid $c_i \geq 0$.[8] The tie-breaking rule (see Maskin and Riley 2000b for a complete description) specifies that c_1, c_2 matter only when $b_1 = b_2$, and implies that for each bidder i it is weakly dominant to choose c_i equal to $v_i - b_i$. Hence, in describing a strategy of bidder i, to each b_i we implicitly associate $c_i = v_i - b_i$. As a result, when $b_1 = b_2$ the bidder with the highest value wins and pays to the seller the other bidder's value. Proposition 1 below identifies, for each parameter values, a unique equilibrium for the FPA under the Vickrey tie-breaking rule.

We use i_j to denote type j of bidder i, for $j = L, M, H$ and $i = 1, 2$, and as a notation for mixed strategies we let G_{ij} denote the c.d.f. of the (ordinary) bid submitted by type i_j; u^F_{ij} is type i_j's equilibrium expected utility. In order to fix the ideas, without loss of generality we assume

$$\lambda_1 + \mu_1 \leq \lambda_2 + \mu_2, \text{that is } \ \eta_1 \geq \eta_2 \tag{1}$$

This means that bidder 1 is ex ante weakly stronger than bidder 2 in the sense that $\Pr\{v_1 = v_H\}$ is no less than $\Pr\{v_2 = v_H\}$, a condition weaker than first order stochastic dominance.

7 Although we require here $\lambda_i > 0$, $\mu_i > 0, \eta_i > 0$ for $i = 1, 2$, in the following we consider sometimes cases in which some of the above probabilities are zero. In such cases the equilibrium can be obtained by applying a limit argument to the equilibrium obtained when $\lambda_i > 0$, $\mu_i > 0, \eta_i > 0$ for $i = 1, 2$.

8 A very similar idea appears in Lebrun (2002), in the auction denoted with $F\bar{P}A$.

Arguing as in Maskin and Riley (1985) and in Riley (1989), we deduce that each Bayes Nash Equilibrium is such that for $i = 1, 2$, type i_L bids v_L with probability 1 (a pure strategy), the set of possible realizations of G_{iM} is an interval $[v_L, \bar{b}_{iM}]$ in which $\bar{b}_{iM}$ may be equal to v_L, the set of possible realizations of G_{iH} is an interval $[\bar{b}_{iM}, \bar{b}_{iH}]$ in which $\bar{b}_{iM} < \bar{b}_{iH}$ (with no probability mass on a single bid different from v_L), and $\bar{b}_{1H} = \bar{b}_{2H}$. In the following, $\bar{b}_H$ denotes both $\bar{b}_{1H}$ and $\bar{b}_{2H}$.

In order to determine the mixed strategy G_{ij} for each type i_j, we define $G_i(b)$ as $\lambda_i G_{iL}(b) + \mu_i G_{iM}(b) + \eta_i G_{iH}(b)$, that is G_i is the c.d.f. of the bids submitted by bidder i. In equilibrium, type i_j is indifferent among all the bids in the set of the possible realizations of G_{ij}. In particular, for type 1_M the equality $(v_M - b)G_2(b) = u^F_{1M}$ holds for each $b \in (v_L, \bar{b}_{1M}]$, and u^F_{1M} coincides with $\lim_{b \downarrow v_L}(v_M - b)G_2(b)$, that is with $G_2(v_L)\Delta$. We know that $G_2(v_L) \geq \lambda_2$ since type 2_L bids v_L, but we cannot rule out that also type 2_M bids v_L with positive probability. Hence $G_2(v_L)$ may be greater than λ_2, and we use ρ_2 to denote $G_2(v_L)$; likewise, we set $\rho_1 = G_1(v_L)$. Therefore[9]

$$u^F_{1M} = \rho_2\Delta \quad \text{with} \quad \rho_2 \geq \lambda_2 \qquad \text{and} \qquad u^F_{2M} = \rho_1\Delta \quad \text{with} \quad \rho_1 \geq \lambda_1 \tag{2}$$

The indifference conditions for types $1_M, 1_H, 2_M, 2_H$ yield the following equalities, in which $v_H - \bar{b}_H = u^F_{1H} = u^F_{2H}$:

$$(v_M - b)G_2(b) = \rho_2\Delta \text{ for each } b \in (v_L, \bar{b}_{1M}] \tag{3}$$

$$(v_H - b)G_2(b) = v_H - \bar{b}_H \text{ for each } b \in [\bar{b}_{1M}, \bar{b}_H] \tag{4}$$

$$(v_M - b)G_1(b) = \rho_1\Delta \text{ for each } b \in (v_L, \bar{b}_{2M}] \tag{5}$$

$$(v_H - b)G_1(b) = v_H - \bar{b}_H \text{ for each } b \in [\bar{b}_{2M}, \bar{b}_H] \tag{6}$$

Example: The case of binary support As an example, we illustrate here the role played by ρ_1, ρ_2 when for each bidder there are just two possible values, v_L and v_H (with $v_H - v_L = 2\Delta$), with probabilities λ_1 and $1 - \lambda_1$ for bidder 1, λ_2 and $1 - \lambda_2$ for bidder 2 (that is, $\mu_1 = \mu_2 = 0$) and $\lambda_1 \leq \lambda_2$.[10] If $\lambda_1 = \lambda_2$, then we obtain $\bar{b}_H = v_H - 2\lambda_2\Delta$ and $G_1(b) = G_2(b) = \frac{v_H - \bar{b}_H}{v_H - b}$ for each $b \in [v_L, \bar{b}_H]$, with $\rho_1 = \lambda_1 = \rho_2 = \lambda_2$. When $\lambda_1 < \lambda_2$, we find that $\bar{b}_H$ is unchanged because type 1_H's equilibrium utility is still $2\lambda_2\Delta$, his utility from bidding v_L. Hence also type 2_H's equilibrium utility is

9 Although $\rho_1 \geq \lambda_1, \rho_2 \geq \lambda_2$, at least one of these weak inequalities is an equality because if $\rho_1 > \lambda_1, \rho_2 > \lambda_2$, then types $1_M, 2_M$ both bid v_L with positive probability and either type wants to raise the own bid a bit above v_L to increase his probability to win by a discrete amount while increasing his payment in case of victory just a bit.

10 This setting has already been examined in Maskin and Riley (1985), which establish the result about the revenue comparison mentioned just before Subsection 4.1. Here we use this setting as a sort of benchmark.

$2\lambda_2\Delta$, and type 2_H needs to earn the same utility $2\lambda_2\Delta$ from any bid in $(v_L, \bar{b}_H]$. However, if type 1_H puts no probability mass on v_L, then 2_H's utility from a bid $b \in (v_L, \bar{b}_H)$ tends to $2\lambda_1\Delta$ as b tends to v_L, rather than to $2\lambda_2\Delta$. Hence in equilibrium ρ_1 needs to be equal to λ_2, that is $\rho_1 > \lambda_1$, which requires that type 1_H bids v_L with probability $\frac{\lambda_2-\lambda_1}{1-\lambda_1} > 0$. As a result, when λ_1 smaller than λ_2 the c.d.f.s G_1, G_2 and the revenue do not depend on λ_1. □

We prove below that in our setting with three types for each bidder, sometimes it is bidder 1 who bids v_L with probability ρ_1 greater than λ_1, sometimes it is bidder 2 who bids v_L with probability $\rho_2 > \lambda_2$.

In order to derive G_1, G_2 from (3) to (6), we need to determine $\bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H, \rho_1, \rho_2$.[11] Next lemma shows that (1) implies $\bar{b}_{1M} \le \bar{b}_{2M}$, hence type 1_H (type 1_M) is less aggressive in terms of the set of possible bids than type 2_H (than type 2_M).[12]

Lemma 1. *If $\lambda_1 + \mu_1 < \lambda_2 + \mu_2$, then $v_L \le \bar{b}_{1M} < \bar{b}_{2M}$; if $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$, then $v_L < \bar{b}_{1M} = \bar{b}_{2M}$. Moreover, $G_1(\bar{b}_{2M}) = \lambda_2 + \mu_2$.*

It turns out that $\bar{b}_{1M} > v_L$ for some parameter values, but $\bar{b}_{1M} = v_L$ for other parameter values. In the first case, from (3) to (6) it follows that

$$\bar{b}_{1M} = v_M - \frac{\rho_1}{\lambda_1 + \mu_1}\Delta, \quad \bar{b}_{2M} = v_M - \frac{\rho_1}{\lambda_2 + \mu_2}\Delta,$$
$$\bar{b}_H = v_H - (\rho_1 + \lambda_2 + \mu_2)\Delta \tag{7}$$

In the second case, (7) still applies to $\bar{b}_{2M}, \bar{b}_H$, but the first equality in (7) is replaced by $\bar{b}_{1M} = v_L$.

For each parameter values, Proposition 1 determines ρ_1, ρ_2 uniquely, thus identifies a unique equilibrium, which is one of the following three strategy profiles:

$$P_{2M}: \begin{cases} \text{the distributions of bids are given by } G_1, G_2 \text{ satisfying } (3)-(6), \text{with} \\ \bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H \text{ in (7) and } \rho_1 = \lambda_1,\ \rho_2 = \lambda_1 \dfrac{\lambda_1 + \lambda_2 + \mu_2}{2\lambda_1 + \mu_1} \end{cases} \tag{8}$$

11 From G_1 it is possible to derive G_{1M}, G_{1H} using $G_1(b) = \lambda_1 + \mu_1 G_{1M}(b) + \eta_1 G_{1H}(b)$ for $b > v_L$ and the equalities $G_{1M}(b) = 1$ for $b \ge \bar{b}_{1M}$, $G_{1H}(b) = 0$ for $b < \bar{b}_{1M}$. Similarly, from G_2 it is possible to derive G_{2M}, G_{2H}.

12 Ceesay, Doni, Menicucci (2024) prove that actually (1) implies that the bid distribution of type 1_H is weaker than the bid distribution of type 2_H in the sense of first order stochastic dominance [a result analogous to that in Proposition 1 of Fibich, Gavious, Sela (2002)], but no analogous result holds when comparing the bids of type 1_M and the bids of type 2_M.

$$P_{1M}\colon \begin{cases} \text{the distributions of bids are given by } G_1, G_2 \text{ satisfying (3)} - \text{(6), with} \\ \bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H \text{ in (7) and } \rho_1 = \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2(\lambda_1 + \mu_1)} - \frac{1}{2}\mu_2,\ \rho_2 = \lambda_2 \end{cases} \tag{9}$$

$$P_{1MH}\colon \begin{cases} \text{type } 1_M \text{ bids } v_L \ \left(\text{that is, } \bar{b}_{1M} = v_L\right)\text{; the distributions of bids are given} \\ \text{by } G_1, G_2 \text{ satisfying (4)} - \text{(6), with } \bar{b}_{2M}, \bar{b}_H \text{ in (7) and } \rho_1 = \lambda_2 - \mu_2,\ \rho_2 = \lambda_2 \end{cases} \tag{10}$$

In each of these profiles, types 1_L and 2_L both bid v_L. The profiles mainly differ because of the additional bidder types who bid v_L with positive probability: in P_{2M} it is only type 2_M; in P_{1M} it is only type 1_M; in P_{1MH}, both type 1_M (with probability 1) and type 1_H bid v_L with positive probability. Notice from (10) that in P_{1MH} bidding is not affected by $\lambda_1,\ \mu_1$.

Proposition 1. *Suppose that* (1) *is satisfied. Then the unique equilibrium in the FPA is P_{2M} if*

$$\lambda_2(\lambda_1 + \mu_1) < \lambda_1(\lambda_1 + \mu_2) \tag{11}$$

The unique equilibrium is P_{1M} if (11) *is violated and*

$$\lambda_2 - \mu_2 < \lambda_1 + \mu_1 \tag{12}$$

The unique equilibrium is P_{1MH} if (12) *is violated.*

By Proposition 1, there exist three different equilibrium regimes, (8)–(10), and (11), (12) determine the regime which applies. Figures 1a and 1b below provide a graphical illustration of Proposition 1 by fixing $\lambda_2,\ \mu_2$ and representing the space of $(\lambda_1,\ \mu_1)$ which satisfy (1), that is the triangle with bold edges and vertices (0,0), $(\lambda_2 + \mu_2, 0)$, $(0, \lambda_2 + \mu_2)$; the point $(\lambda_1,\ \mu_1) = (\lambda_2,\ \mu_2)$ is on the hypothenuse of this triangle.

Figure 1a refers to the case with $\lambda_2 \leq \mu_2$, which makes (12) satisfied for each $(\lambda_1,\ \mu_1)$, hence P_{1M} or P_{2M} is the equilibrium: Region R_{2M} is the set of $(\lambda_1,\ \mu_1)$ for which (11) holds and P_{2M} is the equilibrium; R_{1M} is the region in which (11) is violated and P_{1M} is the equilibrium. The curve C, connecting point (0,0) to $(\lambda_2,\ \mu_2)$, is the set of $(\lambda_1,\ \mu_1)$ such that (11) is an equality, which implies $(\rho_1, \rho_2) = (\lambda_1, \lambda_2)$ and only types $1_L, 2_L$ bid v_L.

Figure 1b is about the case of $\lambda_2 > \mu_2$. Then there exist $(\lambda_1,\ \mu_1)$ close to (0,0) which violate (12) and R_{1MH} is the region consisting of such $(\lambda_1,\ \mu_1)$; P_{1MH} is the equilibrium for each $(\lambda_1,\ \mu_1) \in R_{1MH}$. In P_{1MH}, type 1_H bids v_L with positive probability and we notice that this may occur only if $\lambda_2 > \mu_2$ because 1_H's utility from

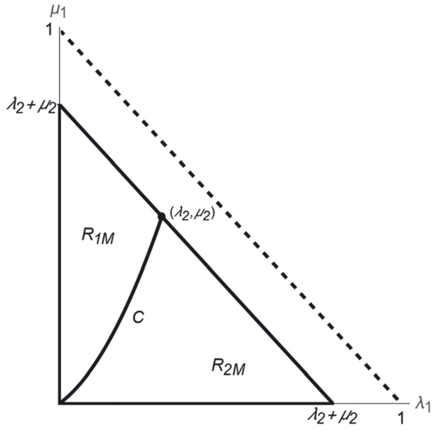

Figure 1a: The regions R_{1M}, R_{2M} when $\lambda_2 \leq \mu_2$.

bidding v_L is $2\lambda_2\Delta$, but by bidding v_M, type 1_H wins with probability greater than $\lambda_2 + \mu_2$, earning utility greater than $(\lambda_2 + \mu_2)\Delta$. Thus $\lambda_2 \leq \mu_2$ makes the bid v_L less profitable than the bid v_M, and rules out that 1_H bids v_L. This is why P_{1MH} is never an equilibrium when $\lambda_2 \leq \mu_2$.

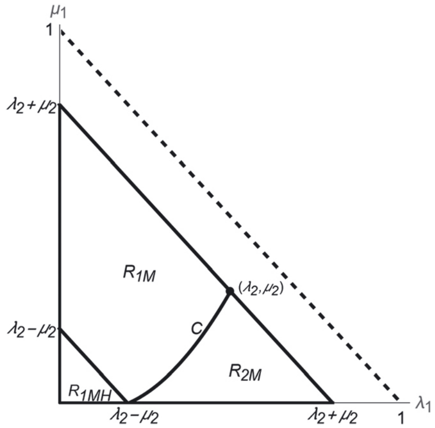

Figure 1b: The regions R_{1M}, R_{2M}, R_{1MH} when $\lambda_2 > \mu_2$.

The expected revenue R^F in the FPA is the expectation of the highest bid, which is equal to v_L with probability $\rho_1\rho_2$ and has c.d.f. $G_1(b)G_2(b)$ for $b \in (v_L, \bar{b}_H]$, with $G_1(b), G_2(b)$ determined by (3)–(6):

$$R^F = \rho_1\rho_2 v_L + \int_{v_L}^{\bar{b}_H} bd\big(G_1(b)G_2(b)\big)$$

On the Effects of Asymmetry on Bidding in the FPA The literature on asymmetric auctions mainly focuses on settings with two bidders and provides sufficient conditions on the c.d.f.s for the bidders' values to draw conclusions about the comparison between the bidders' equilibrium bidding, for instance one bidder's bid distribution is stronger than the other bidder's. In our setting, Ceesay, Doni, Menicucci (2024) use Proposition 1 to show that the comparison results can be proved under conditions which are typically weaker than the literature's conditions, and to examine the effects of asymmetry on bidding.

3.2 Equilibrium Bidding in the Second Price Auction

In the second price auction, SPA henceforth, for each bidder it is weakly dominant to bid the own valuation. We use u_{ij}^S to denote the expected utility of type i_j, for $j = L, M, H$, $i = 1, 2$. Hence $u_{1L}^S = u_{2L}^S = 0$ and

$$u_{1M}^S = \lambda_2\Delta, \quad u_{1H}^S = (2\lambda_2 + \mu_2)\Delta, \qquad u_{2M}^S = \lambda_1\Delta, \quad u_{2H}^S = (2\lambda_1 + \mu_1)\Delta \tag{13}$$

The expected revenue R^S in the SPA is the expectation of the second highest valuation, that is $R^S = v_L + (\mu_1\mu_2 + \mu_1\eta_2 + \eta_1\mu_2)\Delta + 2\eta_1\eta_2\Delta$, and after simple manipulations it can be written as

$$R^S = v_L + \big((2 - 2\lambda_2 - \mu_2)(1 - \lambda_1) - \big(1 - \lambda_2 - \mu_2\big)\mu_1\big)\Delta \tag{14}$$

4 Comparison Between the FPA and the SPA

In this section we compare the expected revenue R^F in the FPA with the expected revenue R^S in the SPA. To this purpose, it is useful to define

$$U^F = \mu_1 u_{1M}^F + \eta_1 u_{1H}^F + \mu_2 u_{2M}^F + \eta_2 u_{2H}^F,$$
$$U^S = \mu_1 u_{1M}^S + \eta_1 u_{1H}^S + \mu_2 u_{2M}^S + \eta_2 u_{2H}^S$$

as the total bidders' expected utility under the FPA and under the SPA, respectively.

The SPA always allocates the object to a bidder with the highest value, whereas the FPA implements an inefficient allocation with positive probability when (1) holds strictly because then $\bar{b}_{1M} < \bar{b}_{2M}$ (by Lemma 1) and type 2_M wins with positive probability when facing type 1_H.[13] Therefore social welfare is greater in the SPA than in the FPA, and whenever $U^F \geq U^S$ holds, we can conclude that $R^S > R^F$.

Example: Revenue ranking for the case of binary support The comparison between U^F and U^S yields an immediate conclusion in the setting with binary

13 Conversely, $\bar{b}_{1M} = \bar{b}_{2M}$ when (1) holds with equality, and then the FPA allocates the object efficiently.

support with $\mu_1 = 0$, $\mu_2 = 0$ and $\lambda_1 < \lambda_2$. The equilibrium in the FPA described just after (6) coincides with P_{1MH} in (10) with $\rho_1 = \lambda_2$, $\rho_2 = \lambda_2$. As a result, types $1_L, 2_L, 1_H$ earn the same utility in the FPA as in the SPA, but type 2_H's utility is higher in the FPA than in the SPA, $2\lambda_2\Delta$ rather than $2\lambda_1\Delta$. Therefore $U^F > U^S$ and $R^S > R^F$. □

In the following we show that significantly different results hold for the setting with three types: we prove that $U^F < U^S$ and $R^F > R^S$ in some cases, we illustrate when this holds, and we determine the source of the difference with respect to the binary setting.

4.1 Comparison of Rents

It is immediate from (2) and (13) that both type 1_M and 2_M weakly prefer the FPA, that is $u^F_{1M} \geq u^S_{1M}$ and $u^F_{2M} \geq u^S_{2M}$, since $\rho_2 \geq \lambda_2$ and $\rho_1 \geq \lambda_1$ (with one strict inequality unless (11) is an equality). This occurs because in the FPA type 1_M or type 2_M bids v_L with positive probability, which makes type 2_M or type 1_M better off than in the SPA.

The same preference holds for type 2_H, that is $u^F_{2H} \geq u^S_{2H}$, because in the FPA type 2_H benefits from $\bar{b}_{1M} < \bar{b}_{2M}$, that is type 1_H bids below $\bar{b}_{2M}$ with positive probability. Hence type 1_H loses with positive probability against $\bar{b}_{2M}$, the highest bid submitted by type 2_M, and when bidding $\bar{b}_{2M}$, type 2_H beats types $1_L, 1_M$ for sure, and also type 1_H with positive probability. Thus 2_H wins with probability greater than $\lambda_1 + \mu_1$. Conversely, under the SPA type 2_H wins and earns a positive utility only when facing type 1_L or 1_M, that is with probability $\lambda_1 + \mu_1$.

Matters are different for type 1_H, because u^F_{1H} is equal to u^F_{2H}, but $u^S_{1H} = (2\lambda_2 + \mu_2)\Delta$ may be higher than $u^S_{2H} = (2\lambda_1 + \mu_1)\Delta$; next lemma identifies precisely when $u^F_{1H} < u^S_{1H}$ holds.

Lemma 2. *Types $1_M, 2_M, 2_H$ all weakly prefer the FPA to the SPA. Type 1_H prefers the FPA if $\lambda_1 > \lambda_2$; type 1_H is indifferent between the two auctions if $\lambda_1 = \lambda_2$, or if $\lambda_1 < \lambda_2$ and* (1) *holds with equality; type 1_H prefers the SPA if $\lambda_1 < \lambda_2$ and* (1) *holds strictly.*

By Lemma 2, only type 1_H may prefer the SPA to the FPA, hence it is intuitive that $U^F \geq U^S$, and therefore $R^F < R^S$, hold frequently. In particular, when $\lambda_1 \geq \lambda_2$ each bidder type weakly prefers the FPA to the SPA, hence $U^F > U^S$ if $\lambda_1 \geq \lambda_2$. We state this result in Proposition 2 below, jointly with another sufficient condition for $U^F > U^S$. We remark that this goes in the opposite direction with respect to most of the literature, which finds that the revenue under the FPA is higher than under the SPA.

However, the opposite inequalities $U^F < U^S$ and $R^F > R^S$ hold for some parameter values, for instance if $\lambda_1 = \frac{1}{5}$, $\mu_1 = \frac{1}{10}$ and $\lambda_2 = \mu_2 = \frac{2}{5}$. Then $U^F = 1.02 < U^S = 1.06$ and $R^F = 0.6214 > R^S = 0.62$.[14] In next subsection we investigate more deeply the comparison between R^F and R^S.

4.2 Comparison of Revenues

In this subsection we focus on the direct comparison between R^F and R^S. Since Lemma 2 suggests that $R^F < R^S$ often holds, we perform the comparison by fixing $(\lambda_2,\ \mu_2)$ and inquiring whether there exist $(\lambda_1,\ \mu_1)$ such that $R^F > R^S$, or if instead $R^F \leq R^S$ for each $(\lambda_1,\ \mu_1)$ which satisfies (1). We denote with F_2 the set of $(\lambda_2,\ \mu_2)$ such that $R^F > R^S$ for some $(\lambda_1,\ \mu_1)$, and with S_2 the (complementary) set of $(\lambda_2,\ \mu_2)$ such that $R^F \leq R^S$ for each $(\lambda_1,\ \mu_1)$. In determining F_2 and S_2, Proposition 2 below distinguishes the case of $\lambda_2 \leq \mu_2$ from the case of $\lambda_2 > \mu_2$ because in the first case two equilibrium regimes exist, P_{1M} and P_{2M}, whereas in the second case a third equilibrium regime exists, P_{1MH}.

Proposition 2(i). *Suppose that $\lambda_2 \leq \mu_2$. If*

$$(\lambda_2 + \mu_2)^2 - \mu_2 > 0 \tag{15}$$

then $R^F > R^S$ for $(\lambda_1,\ \mu_1)$ close to (0,0), but $R^F < R^S$ when $\lambda_1 \geq \lambda_2$, or $\lambda_1 < \lambda_2$ and μ_1 is large – that is (1) *holds with equality or with approximate equality. If instead* (15) *is violated, then $R^F \leq R^S$ for each $(\lambda_1,\ \mu_1)$ which satisfies* (1).

(ii) Suppose that $\lambda_2 > \mu_2$. If

$$3\lambda_2 + \mu_2 - 1 - 2\frac{\lambda_2}{\mu_2}(\lambda_2 - \mu_2)\ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right) > 0 \tag{16}$$

then $R^F > R^S$ for $(\lambda_1,\ \mu_1)$ close to $(\lambda_2 - \mu_2, 0)$, but $R^F < R^S$ when $\lambda_1 \geq \lambda_2$, or $\lambda_1 < \lambda_2$ and μ_1 is large – that is (1) *holds with equality or with approximate equality. If instead* (16) *is violated, then $R^F \leq R^S$ for each $(\lambda_1,\ \mu_1)$ satisfying* (1).

Corollary 1. *If $\lambda_2 + \mu_2 \leq \frac{1}{2}$, then $R^F \leq R^S$ holds for each $(\lambda_1,\ \mu_1)$.*

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14 From (13), (14) it is possible to derive U^S, R^S. From the proof of Proposition 1 it is possible to derive U^F, R^F.

Proposition 2 establishes that given $(\lambda_2,\ \mu_2)$, in order to determine whether there exist $(\lambda_1,\ \mu_1)$ such that $R^F > R^S$ it suffices to evaluate $R^F - R^S$ at a specific $(\lambda_1,\ \mu_1)$ – which we argue below induces the most aggressive bidding in the FPA. As a result, the sets F_2 and S_2 are identified: F_2 consists of all $(\lambda_2,\ \mu_2)$ which satisfy $\lambda_2 \le \mu_2$ and (15), or $\lambda_2 > \mu_2$ and (16), and is the grey set in Figure 2; S_2 is the white set in Figure 2, and by Corollary 1 it includes each $(\lambda_2,\ \mu_2)$ such that $\lambda_2 + \mu_2 \le \frac{1}{2}$.[15]

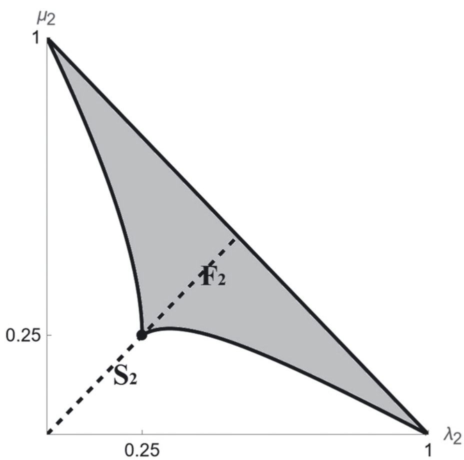

Figure 2: The set of $(\lambda_2,\ \mu_2)$ which satisfy (15) when $\lambda_2 \le \mu_2$, or (16) when $\lambda_2 > \mu_2$.

The case of $\lambda_2 \le \mu_2$ In the following, for ease of language, instead of $(\lambda_1,\ \mu_1)$ close to (0,0) we write $(\lambda_1,\ \mu_1) = (0, 0)$. When $\lambda_2 \le \mu_2$, Proposition 2(i) establishes that there exist $(\lambda_1,\ \mu_1)$ such that $R^F > R^S$ if and only if $R^F > R^S$ holds when $(\lambda_1,\ \mu_1) = (0, 0)$, and such condition is equivalent to (15).

We remark that $(\lambda_1,\ \mu_1) = (0, 0)$ induces the most aggressive bidding in the FPA, given $\lambda_2 \le \mu_2$, because (3)–(6), (7) reveal that G_1, G_2 are more aggressive the lower are ρ_1, ρ_2. Precisely, $\rho_1 \ge \lambda_1, \rho_2 \ge \lambda_2$ by (2) and $(\lambda_1,\ \mu_1) = (0, 0)$ implies $\rho_1 = 0, \rho_2 = \lambda_2$, which are the lowest possible values for ρ_1, ρ_2 given $\lambda_2,\ \mu_2$. But we stress that $(\lambda_1,\ \mu_1) = (0, 0)$ alone is not sufficient for $R^F > R^S$ to hold: $(\lambda_1,\ \mu_1) = (0, 0)$ induces the most aggressive bidding also in the SPA, and the sign of $R^F - R^S$ when $(\lambda_1,\ \mu_1) = (0, 0)$ is determined by whether (15) is satisfied, which occurs if and only if $\mu_2 \ge \frac{1}{4}$ and λ_2 is large enough given $\lambda_2 \le \mu_2$. We explain below why $R^F > R^S$ when these conditions hold.

First notice that when $\lambda_1,\ \mu_1$ are about 0, bidder 1 almost certainly has value v_H and there are three relevant states of the world: (v_1, v_2) equal to (v_H, v_L), or

15 Proposition 2 implies Corollary 1 since $\lambda_2 + \mu_2 \le \frac{1}{2}$ makes (15) violated if $\lambda_2 \le \mu_2$, and makes (16) violated if $\lambda_2 > \mu_2$.

equal to (v_H, v_M), or equal to (v_H, v_H). From (7) we see that $\bar{b}_{2M}$ is close to v_M, and (5) implies that $G_1(b)$ is close to 0 for each $b < v_M$, that is in the limit as $(\lambda_1,\ \mu_1) \to (0,0)$, G_1 puts all the probability on bids no less than v_M. In other words, bidder 1 (i.e. type 1_H) bids at least v_M;[16] hence the revenue under the FPA is greater than v_M. Conversely, under the SPA the revenue coincides with v_2 in each state mentioned above as $\min\{v_1, v_2\} = \min\{v_H, v_2\} = v_2$. Hence the revenue is greater in the FPA when $(v_1, v_2) = (v_H, v_L)$ or $(v_1, v_2) = (v_H, v_M)$, but is greater in the SPA when $(v_1, v_2) = (v_H, v_H)$. This makes it intuitive that $R^F > R^S$ if λ_2 is large, as the greater is λ_2, the greater is the probability of state (v_H, v_L), in which the FPA has a higher revenue, and the lower is the probability of state (v_H, v_H), in which the SPA has a greater revenue. In particular, $R^F > R^S$ if λ_2 is close to $1 - \mu_2$ because then (v_H, v_H), the only state in which the SPA is superior to the FPA, has probability about zero.

The case of $\lambda_2 > \mu_2$ When $\lambda_2 > \mu_2$, Proposition 2(ii) establishes a result analogous to Proposition 2(i), that is $R^F > R^S$ holds for some $(\lambda_1,\ \mu_1)$ if and only if $R^F > R^S$ when $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$, and such condition is equivalent to (16). In a sense, now $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$ plays the role $(\lambda_1,\ \mu_1) = (0,0)$ plays when $\lambda_2 \le \mu_2$. In order to see why, notice that $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$ is a distribution of v_1 which induces the most aggressive bidding in the FPA, given $\lambda_2 > \mu_2$, because when $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$ we have $\rho_2 = \lambda_2$ (this is the minimum value for ρ_2) and $\rho_1 = \lambda_2 - \mu_2$, and for each other $(\lambda_1,\ \mu_1)$, ρ_1 is greater than $\lambda_2 - \mu_2$. Actually, any other $(\lambda_1,\ \mu_1) \in R_{1MH}$ induces the same bidding in the FPA like $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$, as from Subsection 3.1 we know that if $(\lambda_1,\ \mu_1) \in R_{1MH}$, then (12) is violated and R^F is constant with respect to λ_1, μ_1. But R^S in (14) is decreasing in λ_1, μ_1 with $\frac{\partial R^S}{\partial \lambda_1} < \frac{\partial R^S}{\partial \mu_1} < 0$. Hence $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$ is the maximum point for $R^F - R^S$ in R_{1MH}.

However, $R^F > R^S$ at $(\lambda_1,\ \mu_1) = (\lambda_2 - \mu_2, 0)$ if and only if (16) is satisfied, which is equivalent to $\lambda_2 > \frac{1}{4}$ and μ_2 sufficiently large, given $\lambda_2 > \mu_2$. We explain in the following why $R^F > R^S$ if these conditions are satisfied. When $\frac{1}{4} < \lambda_2 \le \frac{1}{2}$ we have that μ_2 large implies μ_2 close to λ_2, which is a setting close to one covered when $\lambda_2 \le \mu_2$, for which we know that $R^F > R^S$. When $\lambda_2 > \frac{1}{2}$, we first describe the effect of an increase in μ_2 on R^F and on R^S. An increase in μ_2 increases R^F as G_2 is unchanged but G_1 improves: see (3)–(6), (7) with $\rho_1 = \lambda_1 = \lambda_2 - \mu_2$, $\rho_2 = \lambda_2$, $\mu_1 = 0$. An increase in μ_2 has ambiguous net effect on R^S as it reduces η_2 (a negative effect) but it also decreases λ_1 and increases η_1 (a positive effect). However, when μ_2 is close to its largest value $1 - \lambda_2$, the event $v_2 = v_H$ has a small probability and then the positive effect on R^S due to the increase in η_1 is small. As a result, the net effect is negative.

16 This is intuitive, as $\rho_1 = 0$ implies $u^F_{2M} = 0$ by (2), which requires that type 2_M cannot win the auction with a bid lower than v_M. This occurs only if bidder 1 bids at least v_M with probability 1.

Hence an increase in μ_2 increases $R^F - R^S$ and makes $R^F - R^S$ positive when μ_2 is close to $1 - \lambda_2$.

Figure 3a represents in grey, for a case in which $\lambda_2 \leq \mu_2$ and (15) holds, the set of (λ_1, μ_1) such that $R^F > R^S$. Figure 3b represents in grey the analogous set for a case in which $\lambda_2 > \mu_2$ and (16) holds. In either case, the white set of (λ_1, μ_1) such that $R^F < R^S$ includes each (λ_1, μ_1) such that $\lambda_1 \geq \lambda_2$ or such that μ_1 is large, consistently with Proposition 2(i–ii). In particular, these conditions imply $R^F < R^S$ when (λ_1, μ_1) is close to (λ_2, μ_2), $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$.[17]

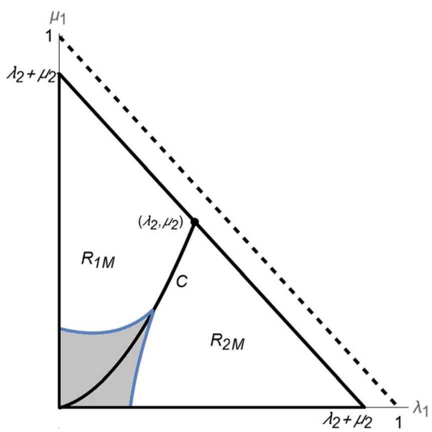

Figure 3a: The set of $(\lambda_1, \text{t}\, \mu_1)$ such that $R^F > R^S$ when $\lambda_2 = 0.4$, $\mu_2 = 0.5$.

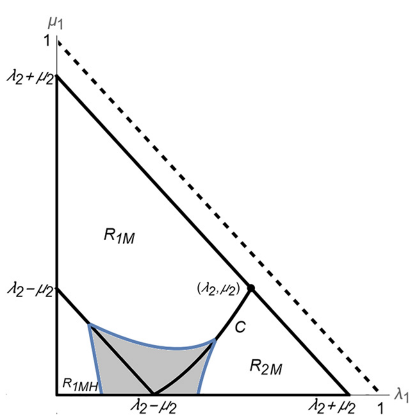

Figure 3b: The set of (λ_1, μ_1) such that $R^F > R^S$ when $\lambda_2 = 0.6$, $\mu_2 = 0.3$.

17 Gavious and Minchuk (2014) prove that no general revenue ranking holds for small asymmetries around the uniform distribution. Conversely, in our setting small asymmetries always favor the SPA.

The Difference with the Setting with Binary Support At the beginning of Section 4 we have remarked that $R^F < R^S$ when the support for each bidder's value is $\{v_L, v_H\}$, that is when $\mu_1 = \mu_2 = 0$, for each $\lambda_1 < \lambda_2$. Conversely, Proposition 2 shows that $R^F > R^S$ in some cases when the support is $\{v_L, v_M, v_H\}$.

In order to explain this difference, start from a symmetric setting with support $\{v_L, v_M, v_H\}$ and $0 < \mu_1 = \mu_2 < \lambda_1 = \lambda_2$; [18] thus $R^F = R^S$. Then consider a change in (λ_1, μ_1) from $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$ to $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$: see Figure 1b. This reduces $\rho_1 = \lambda_1$ and leaves $\rho_2 = \lambda_2$ unchanged, hence improves bidding in the FPA and increases R^F. But it improves bidding also in the SPA and increases R^S. Hence it is uncertain whether $R^F > R^S$ at $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$; this is determined by whether (16) is satisfied. The main point is that the considered improvement in the distribution of v_1 increases both R^F and R^S.

Matters are different when the support is binary because $\mu_2 = 0$ implies that the set of (λ_1, μ_1) which satisfy (1) consists entirely of region R_{1MH} (while R_{1M}, R_{2M} are both empty). Then, when $\mu_1 = \mu_2 = 0$, a reduction in λ_1 below λ_2 keeps (λ_1, μ_1) in region R_{1MH}, in which bidding in the FPA does not depend on (λ_1, μ_1). Hence $R^F < R^S$ for any $\lambda_1 < \lambda_2$ because when the distribution of v_1 becomes stronger, R^S increases but R^F remains constant (as type 1_H puts a probability mass on the bid v_L when $\lambda_1 < \lambda_2$). This feature of the FPA when the support is binary is responsible for the difference between the two settings.

Shift and Stretch Maskin and Riley (2000a) consider a few particular classes of asymmetries, one of which is called shift, another is called stretch. The shift asymmetry is such that the distribution of v_1 is given by the distribution of v_2 shifted to the right by a fixed positive amount. The stretch asymmetry is such that the distribution of v_1 is a rightward stretch of the distribution of v_2. In a setting with continuously distributed values, Maskin and Riley (2000a) prove that the FPA produces a higher revenue than the SPA for any shift and any stretch, under suitable assumptions on the distribution of v_2 which is then shifted or stretched [Kirkegaard (2012) proves these results under slightly weaker assumptions]. Ceesay, Doni, Menicucci (2024) prove that in our context with three types, a significantly more nuanced picture emerges as $R^F < R^S$ in a variety of cases.

5 Conclusions

In this paper we have determined the closed form of the unique equilibrium in the FPA for a two-bidder setting with asymmetric value distributions. Although our

18 This is only to fix the ideas, as a similar argument would apply if $\mu_1 = \mu_2$ were greater than $\lambda_1 = \lambda_2$.

analysis is limited in terms of the set of possible valuations for each bidder, our results do not need restrictions on the distributions over the given set and allow a careful comparison between the FPA and the SPA in terms of revenue and in terms of the effects of an ex ante change in one (or both) value distribution on the resulting equilibrium bidding in FPA.

Arozamena and Cantillon (2004) consider a procurement setting in which the type of each bidder coincides with the bidder's cost to produce the object the auctioneer is interested in, and suppose that (only) one bidder may make an observable investment, before he learns the own type and before the auction takes place, which improves the own ex ante cost distribution. Arozamena and Cantillon (2004) inquire how the bidder's incentive to invest depends on whether the auction is a FPA or a SPA, imposing some restrictions on the effect of the investment on the cost distribution. Our Proposition 1 can be adapted to a procurement setting in which the production cost for each bidder belongs to a set $\{c_L, c_M, c_H\}$ and the cost distributions are asymmetric. This allows to examine the question addressed by Arozamena and Cantillon (2004) without restrictions on the post-investment distribution, and to study more general investment games in which both bidders can invest, possibly starting from asymmetric situations in order to find out whether an initially advantaged bidder has a greater or smaller incentive to invest than a disadvantaged bidder, while comparing the FPA with the SPA.

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Appendix: Proof of Proposition 1

To fix the ideas, we begin with the case in which $v_L < \bar{b}_{1M}$. Then by Lemma 1 the bids $\bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H$ satisfy

$$v_L < \bar{b}_{1M} \leq \bar{b}_{2M} < \bar{b}_H$$

From (5) evaluated at $b = \bar{b}_{1M}$ and at $b = \bar{b}_{2M}$ we derive $\bar{b}_{1M}, \bar{b}_{2M}$ in (7) as a function of ρ_1.[19] Then (3) at $b = \bar{b}_{1M}$ yields $G_2(\bar{b}_{1M}) = \frac{\rho_2}{\rho_1}(\lambda_1 + \mu_1)$, which can be used to evaluate (4) at $\bar{b}_{1M}$ and to derive $\bar{b}_H = v_H - \rho_2(1 + \frac{\lambda_1+\mu_1}{\rho_1})\Delta$.

19 To this purpose we use $G_1(\bar{b}_{1M}) = \lambda_1 + \mu_1$ and, from Lemma 1, $G_1(\bar{b}_{2M}) = \lambda_2 + \mu_2$.

Therefore the equilibrium is fully determined if ρ_1, ρ_2 are identified. This is achieved by evaluating (6) at $\bar{b}_{2M}$, which reduces to $F(\rho_1, \rho_2) = 0$, with F defined as follows:[20]

$$F(\rho_1, \rho_2) = \rho_2\left(1 + \frac{\lambda_1 + \mu_1}{\rho_1}\right) - \rho_1 - \lambda_2 - \mu_2$$

In particular (omitting the common factor Δ), $\rho_2(1 + \frac{\lambda_1+\mu_1}{\rho_1})$ is type 2_H's utility from bidding $\bar{b}_H$ and $\rho_1 + \lambda_2 + \mu_2$ is 2_H's utility from $\bar{b}_{2M}$.[21]

We discuss in the following the three cases which may arise in Proposition 1. Notice that F is strictly decreasing with respect to ρ_1, strictly increasing with respect to ρ_2.

Case of (11) satisfied Inequality (11) is equivalent to $F(\lambda_1, \lambda_2) < 0$, and then $F(\rho_1, \rho_2) = 0$ is satisfied by $\rho_1 = \lambda_1$ and $\rho_2 = \lambda_1 \frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1}$ (the unique solution to $F(\lambda_1, \rho_2) = 0$), which belongs to $(\lambda_2, \lambda_2 + \mu_2)$. The expression of R^F near the end of Subsection 3.1 yields

$$R^F = \lambda_1\rho_2 v_L + \int_{v_L}^{\bar{b}_H} b d\big(G_1(b)G_2(b)\big) = \bar{b}_H - \int_{v_L}^{\bar{b}_H} G_1(b)G_2(b)\mathrm{d}b$$

and G_1, G_2 are obtained from (3) to (6):[22]

$$G_1(b) = \begin{cases} \dfrac{\rho_1\Delta}{v_M - b} & \text{for each } b \in (v_L, \bar{b}_{2M}] \\ \dfrac{v_H - \bar{b}_H}{v_H - b} & \text{for each } b \in [\bar{b}_{2M}, \bar{b}_H] \end{cases}$$

$$G_2(b) = \begin{cases} \dfrac{\rho_2\Delta}{v_M - b} & \text{for each } b \in (v_L, \bar{b}_{1M}] \\ \dfrac{v_H - \bar{b}_H}{v_H - b} & \text{for each } b \in [\bar{b}_{1M}, \bar{b}_H] \end{cases}$$

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20 F is defined for $\rho_1 = \lambda_1$ and $\rho_2 \in [\lambda_2, \lambda_2 + \mu_2]$, or $\rho_1 \in [\lambda_1, \lambda_1 + \mu_1]$ and $\rho_2 = \lambda_2$.

21 Notice that $F(\rho_1, \rho_2) = 0$ implies that $\bar{b}_H$ can be written as $v_H - (\rho_1 + \lambda_2 + \mu_2)\Delta$, which is the expression in (7).

22 When $(\lambda_1,\ \mu_1) \in R_{2M}$, no profitable deviation exists for any bidder type. Precisely, for type 1_L the equilibrium utility is 0, and his utility is still 0 if he bids less than v_L, whereas it is negative if he bids more than v_L. For type 1_M, a bid $b \in [\bar{b}_{1M}, \bar{b}_H]$ yields utility $(v_M - b)G_2(b) = (v_H - b - \Delta)G_2(b) = u_{1H}^F - \Delta G_2(b)$, in which the second equality follows from (4). At $b = \bar{b}_{1M}$, $u_{1H}^F - \Delta G_2(b)$ coincides with u_{1M}^F because of (3), and for $b \in (\bar{b}_{1M}, \bar{b}_H]$, $u_{1H}^F - \Delta G_2(b)$ decreases, hence it is smaller than u_{1M}^F. For type 1_H, a bid $b \in [v_L, \bar{b}_{1M}]$ yields utility $(v_H - b)G_2(b) = (v_M - b + \Delta)G_2(b) = u_{1M}^F + \Delta G_2(b)$, in which the second equality follows from (3). At $b = \bar{b}_{1M}$, $u_{1M}^F + \Delta G_2(b)$ coincides with u_{1H}^F because of (4), and for $b \in [v_L, \bar{b}_{1M})$, $u_{1M}^F + \Delta G_2(b)$ increases, hence it is smaller than u_{1H}^F. Similar arguments apply to types $2_L, 2_M, 2_H$, and when $(\lambda_1,\ \mu_1) \in R_{1M} \cup R_{1MH}$.

Hence

$$R^F = \bar{b}_H - \int_{v_L}^{\bar{b}_{1M}} \frac{\lambda_1\rho_2\Delta^2}{(v_M-b)^2}\mathrm{d}b - \int_{\bar{b}_{1M}}^{\bar{b}_{2M}} \frac{\lambda_1\Delta(v_H-\bar{b}_H)}{(v_M-b)(v_H-b)}\mathrm{d}b - \int_{\bar{b}_{2M}}^{\bar{b}_H} \frac{(v_H-\bar{b}_H)^2}{(v_H-b)^2}\mathrm{d}b$$

$$= \bar{b}_H - \lambda_1\rho_2\frac{\bar{b}_{1M}-v_L}{v_M-\bar{b}_{1M}}\Delta - \lambda_1(v_H-\bar{b}_H)$$

$$\times \ln\left(\frac{(v_H-\bar{b}_{2M})(v_M-\bar{b}_{1M})}{(v_M-\bar{b}_{2M})(v_H-\bar{b}_{1M})}\right) - \frac{(v_H-\bar{b}_H)(\bar{b}_H-\bar{b}_{2M})}{v_H-\bar{b}_{2M}}$$

$$= v_L + \Big(2-\rho_2\mu_1-(2-\lambda_2-\mu_2)(\lambda_1+\lambda_2+\mu_2)$$

$$-\lambda_1(\lambda_1+\lambda_2+\mu_2)\ln\left(\frac{\lambda_2+\mu_2+\lambda_1}{2\lambda_1+\mu_1}\right)\Big)\Delta$$

Case of (11) violated, (12) satisfied Inequality (11) is violated and inequality (12) is satisfied if and only if $F(\lambda_1,\lambda_2) \geq 0 > F(\lambda_1+\mu_1,\lambda_2)$. In this case, $F(\rho_1,\rho_2)=0$ is satisfied by $\rho_2=\lambda_2$ and ρ_1 equal to the unique solution to $F(\rho_1,\lambda_2)=0$, that is $\rho_1=\sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)}-\frac{1}{2}\mu_2$, which belongs to $[\lambda_1,\lambda_1+\mu_1)$,[23] and through (7) it determines $\bar{b}_{1M},\bar{b}_{2M},\bar{b}_H$. Hence the expected revenue is

$$R^F = \rho_1\lambda_2 v_L + \int_{v_L}^{\bar{b}_H} b\mathrm{d}\big(G_1(b)G_2(b)\big) = \bar{b}_H - \int_{v_L}^{\bar{b}_H} G_1(b)G_2(b)\mathrm{d}b$$

$$= \bar{b}_H - \int_{v_L}^{\bar{b}_{1M}} \frac{\rho_1\lambda_2\Delta^2}{(v_M-b)^2}\mathrm{d}b - \int_{\bar{b}_{1M}}^{\bar{b}_{2M}} \frac{\rho_1(v_H-\bar{b}_H)}{(v_M-b)(v_H-b)}\Delta\mathrm{d}b - \int_{\bar{b}_{2M}}^{\bar{b}_H} \frac{(v_H-\bar{b}_H)^2}{(v_H-b)^2}\mathrm{d}b$$

$$= \bar{b}_H - \frac{\rho_1\lambda_2(\bar{b}_{1M}-v_L)\Delta}{v_M-\bar{b}_{1M}} - \rho_1(v_H-\bar{b}_H)\ln\left(\frac{(v_H-\bar{b}_{2M})(v_M-\bar{b}_{1M})}{(v_M-\bar{b}_{2M})(v_H-\bar{b}_{1M})}\right)$$

$$- \frac{(v_H-\bar{b}_H)(\bar{b}_H-\bar{b}_{2M})}{v_H-\bar{b}_{2M}} \tag{17}$$

Case of (12) violated Inequality (12) is violated if and only if $F(\lambda_1+\mu_1,\lambda_2)\geq 0$. In this case, no $\rho_1<\lambda_1+\mu_1$ satisfies $F(\rho_1,\lambda_2)=0$. Therefore $\rho_1\geq\lambda_1+\mu_1$, that is type 1_M bids v_L with probability 1 – hence $\bar{b}_{1M}=v_L$ – and if $\rho_1>\lambda_1+\mu_1$ then also type 1_H bids v_L with positive probability. The utility of type 1_H from bidding v_L is $2\lambda_2\Delta$, hence $\bar{b}_H=v_H-2\lambda_2\Delta$, and also the equilibrium utility of type 2_H is $2\lambda_2\Delta$.

23 Precisely, $\rho_1=\lambda_1$ when $F(\lambda_1,\lambda_2)=0$, ρ_1 belongs to $(\lambda_1,\lambda_1+\mu_1)$ if $F(\lambda_1,\lambda_2)>0$.

It is still the case that $\bar{b}_{2M}$ is given by (7), and type 2_H's utility from bidding $\bar{b}_{2M}$ is $\rho_1 + \lambda_2 + \mu_2$. Hence $2\lambda_2\Delta$ needs to be equal to $\rho_1 + \lambda_2 + \mu_2$ and it follows that $\rho_1 = \lambda_2 - \mu_2$, which is indeed greater than $\lambda_1 + \mu_1$.

The expected revenue has the same expression as (17), with $\bar{b}_{1M}$ replaced by v_L:

$$R^F = \bar{b}_H - \rho_1(v_H - \bar{b}_H)\ln\left(\frac{v_H - \bar{b}_{2M}}{2(v_M - \bar{b}_{2M})}\right) - \frac{(v_H - \bar{b}_H)(\bar{b}_H - \bar{b}_{2M})}{v_H - \bar{b}_{2M}}$$
$$= v_L + \left(2 - 2\lambda_2(2 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2 \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right)\right)\Delta$$

Bidders' rents The bidders' rents are given in the following table, in which the common factor Δ is omitted (for P_{1M}, ρ_1 is given by the expression in (9)):

equilibrium \bidder type	1_M	1_H	2_M	2_H
P_{2M}	$\lambda_1 \frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1}$	$\lambda_1 + \lambda_2 + \mu_2$	λ_1	$\lambda_1 + \lambda_2 + \mu_2$
P_{1M}	λ_2	$\rho_1 + \lambda_2 + \mu_2$	ρ_1	$\rho_1 + \lambda_2 + \mu_2$
P_{1MH}	λ_2	$2\lambda_2$	$\lambda_2 - \mu_2$	$2\lambda_2$

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