Research Article

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Disclosure of Product Information After Price Competition

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Abstract: We study under what conditions product information sufficiently unravels in a competitive environment. Information sufficiently unravels if the consumer makes the same purchasing decision as under complete information. The consumer is uncertain about the sellers' product characteristics while she has private information about her preference for differentiated products. In contrast to the prior literature, we focus on the case where the sellers compete to attract the consumer by disclosing product information only after they set prices for their individual products. We provide a necessary and sufficient condition on the consumer's relative comparison of one seller's product to the other's for every outcome to be sufficient unraveling under comparative and non-comparative advertisements, respectively. We show, by example, that competition may enhance information disclosure only if the consumer has limited reasoning capability.

Keywords: competition; persuasion games; information disclosure; comparative advertisements; non-comparative advertisements

JEL Classification: C72; D82; L15

1 Introduction

In many markets, consumers need product information that is hardly obtainable unless sellers disclose it, and sellers compete to attract consumers by providing this otherwise private information in verifiable manners: for instance, free samples,

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informative advertisements, third-party test results, and technical reports by independent laboratories. Laws against fraud may prevent deceptive advertisements. This article analyzes the disclosure of verifiable information in a competitive environment and investigates under what conditions the consumers are sufficiently informed about products. We focus on the case where the sellers inform the consumers after price competition. In the car industry, for instance, the manufacturers set the prices but leave it to the dealers to persuade potential clients with verifiable information such as test drives.

Much literature has dealt with the disclosure of verifiable information by a single seller. Under vertical differentiation, where all consumers agree on the ranking of the valuations for products as if they are differentiated only in terms of quality, the well-known unraveling argument establishes that full disclosure should be the unique outcome (see Grossman 1981; Grossman and Hart 1980; Milgrom 1981; Milgrom and Roberts 1986 for a few seminal works).[1] Recently, several authors have investigated the disclosure of product information by allowing for horizontal differentiation, where the rankings of the valuations for products are not necessarily identical across the consumers as if the products differ only in colors. Sun (2011) considers the model of a multidimensional product space. In one dimension, products differ in quality. In the other dimension, they are horizontally differentiated. She shows that if product quality is known to the consumer, the seller with higher quality is less likely to disclose information about the horizontal attribute. In a spatial model, Celik (2014) shows that full disclosure obtains if and only if the consumer's preference for her ideal variety is sufficiently strong. In a general framework allowing for both vertical and horizontal differentiation, Koessler and Renault (2012) provide pairwise monotonicity as a necessary and sufficient condition for full disclosure to be the unique outcome regardless of the prior. In a model similar to Koessler and Renault (2012) and Woo (2023) considers price-dependent orders over products, called the sales-dominance relations, and characterizes pairwise monotonicity as the completeness of the sales-dominance relations at all prices. He shows that if the sales-dominance relation is complete at every price, the seller of a sales-dominant product should not pool with other sales-dominated products. That is, the generalized unraveling argument runs based on the order over products defined in terms price-dependent sales rather than price-independent quality.

1 Even under vertical differentiation, full disclosure may fail due to costly messages (Jovanovic 1982; Verrecchia 1983), uncertainty on the sender's verifiability (Shin 1994), and unawareness (Heifetz, Meier, and Schipper 2021; Li, Peitz, and Zhao 2014). Lipman and Seppi (1995) and Mathis (2008) specify conditions under which full disclosure emerges in a context of partial verifiability. See Milgrom (2008) for a survey of literature on communication with verifiable messages.

In this paper, two sellers know each other's product characteristics and the consumer privately knows her preference for differentiated products. The sellers simultaneously set prices for their individual products. After observing the chosen prices, each seller sends a verifiable message about his and his opponent's products, that is, the advertisements are comparative. Finally, the consumer makes a purchasing decision. We extend the results of Koessler and Renault (2012) and Woo (2023) in a duopoly model and explore under what conditions product information sufficiently unravels for the consumer to make only the purchasing decision that should be made as if she were to know perfectly about the sellers' products.

Of course, we are not the first to study information disclosure in a competitive environment. Board (2009) considers a duopoly model where products are vertically differentiated and advertising is non-comparative (i.e. the sellers are not allowed to inform about their opponent's product). In Hotz and Xiao (2013), products are characterized by quality and horizontal location, consumers know the locations of the sellers, and the duopolists provide information about the quality attributes of their products. Both works show that price competition between two firms can undermine the full unraveling result. Roughly, in these works, full disclosure fails because it triggers intense price competition, which would result in lower prices and profits for the sellers. Cheong and Kim (2004) consider a model of oligopoly where products are vertically differentiated and information revelation is costly. They show that no matter how small the disclosure cost is, no seller will reveal information if the number of firms is sufficiently large. We want to highlight that all aforementioned works adopt the disclosure-then-price setting. That is, the sellers set prices only after they provide product information. Janssen and Teteryatnikova (2016) consider a spatial model where a particular location of each seller represents his type, and the consumer purchases from the seller that she expects to be closest to her ideal position. They consider the four combinations of disclosure-then-price or price-then-disclosure settings and comparative or non-comparative advertisements. They show that while full disclosure is an equilibrium outcome in all cases, it is the unique outcome only under the price-then-disclosure timing and comparative advertisements. While they focus on comparing the equilibrium outcomes across different circumstances, we explore the conditions under which product information sufficiently unravels under the price-then-disclosure setting.

Several features distinguish our model from previous works. Typically, the literature assumes that sellers compete in prices after providing product information (see also Levin, Peck, and Ye 2009). In contrast, we consider the setting in which sellers make disclosure decision after they set prices. That is, we endogenize prices but prices are known before they provide product information. This timeline would be suitable when adjusting prices is less flexible than disclosing information. For instance, the posted prices are more or less stable in the retail stores.

Moreover, this approach would enable us to focus on some reasons, if any, other than intense price competition that may impede information disclosure in a competitive environment. Secondly, we allow both vertical and horizontal preferences for the consumer, which is essential to argue that, in a competitive environment, whether products are differentiated vertically or horizontally does not really matter to obtain information unraveling at every outcome. Thirdly, we allow types to be correlated. In Janssen and Teteryatnikova (2016), in particular, types are assumed independent. After prices are selected, the sellers compete in a zero-sum game by providing information about each other's products. Namely, an increase in the market share for one seller implies a decrease in the market share for the other. Therefore, if a pooling outcome exists, the market share at this pooling outcome should be the same as the market shares that would result from the full disclosure of the suppressed product information, which is proved to be non-generic in Janssen and Teteryatnikova (2016). When types are correlated, this intuition for unraveling does not apply simply because information disclosure may not be a zero-sum game in market shares. Lastly, we adopt, as a solution concept, prudent rationalizability that is a version of extensive-form rationalizability featuring cautious behaviors and forward induction, and an extensive-form version of iterated admissibility, one of the oldest solution concept in game theory. Heifetz, Meier, and Schipper (2021) show that the unraveling result is obtained by replacing sequential equilibrium and "skepticism" of the uninformed consumer with prudent rationalizability. Schipper and Woo (2019) apply prudent rationalizability to electoral campaigning games in which voters may be unaware of some political issues and uncertain about political positions of the candidates, and the candidates use the sophisticated campaign strategy, microtargeting. Most importantly, Li and Schipper (2020) experiment with persuasion games in which a single seller provides verifiable information to the consumer. They observe that participants play actions that are consistent with prudent rationalizable strategies.

Having described the framework, we now discuss the results and explain the intuition. A product state refers to a pair of product types, one type for each seller. We find that in a competitive environment, how the consumer evaluates one seller relative to the other over the product states matters for information unraveling to be the unique outcome. In other words, whether products are differentiated vertically or horizontally does not really matter to obtain the unraveling result in a competitive environment. For simplicity, suppose that products are vertically differentiated. By fully disclosing the realized product type, a monopolist informs that his product is of the highest quality among all other possible products the consumer may believe. In other words, the consumer's valuations for products matter when the monopolist makes a decision regarding information disclosure. In a competitive setup, by fully disclosing the realized product state, seller a informs that he is the most competitive against seller b among all other possible product states the consumer may believe. Hence, the differences in the consumer's valuations for the sellers' products matter for their decisions about information disclosure. Given a product state, the relative competitiveness of seller a to b refers this difference. For instance, if each seller $i \in \{a, b\}$ could have either a high-quality product H_i or low-quality product L_i, (H_a, L_b) is the best product state to seller a since he is the most competitive against b in this product state. Likewise, (L_a, H_b) is the worst product state for seller a. However, the ranking of the relative competitiveness of seller a for product states (H_a, H_b) and (L_a, L_b) is ambiguous. Intuitively, the "superiority" between these product states may not be well-defined as if products may not be well-ranked under horizontal differentiation.

Given a price pair, sales from seller a in a product state (say, X) refer to a set of consumer types who prefer to purchase from a in X. We say that (given the price pair) product state X sales-dominates product state Y for seller a if sales from a in X are larger than sales from a in Y. Sales from seller b in a product state and the sales-dominance relation for seller b are analogously defined.[2] We show that the consumer is sufficiently informed at every outcome if and only if at every price pair, every two product states are well-ordered in terms of sales. This is reminiscent of the standard persuasion games under vertical differentiation. The single seller with the product of some quality sends a verifiable message that rules out the possibility of the consumer believing products of lower qualities, and full disclosure uniquely obtains. Under the complete sales-dominance relations, unraveling proceeds analogously except that two informed sellers are involved and that product states are well-ordered in terms of price-dependent sales. In a competitive environment, the extended unraveling argument runs as follows. Consider an arbitrary price pair. Each seller in the most sales-dominant product state discloses the realized product state. Each seller in the second-most sales-dominant product state would inform that some product state at least as "good" as the second-most sales-dominant state is realized. Inductive reasoning continues to apply until all relevant product information sufficiently unravels.

To get an intuition of why the complete sales-dominance relations are necessary for information unraveling to be the unique outcome, consider product states X and Y that are incomparable in terms of sales at some price pair.[3] By the definition

2 Indeed, at every price pair, the sales-dominance relation for seller a is dual to the sales-dominance relation for seller b. That is, whenever product state X sales-dominates product Y for a, Y sales-dominates X for b.

3 (H_a, H_b) and (L_a, L_b) in a previous paragraph may be an example of the incomparable two product states.

of the sales-dominance relations, at this price pair, some consumer types must prefer seller a in product state X and b in Y, while the optimal choices of some other consumer types should be seller b in product state X and a in Y, which are opposite to the choices of the first type. In each product state, information suppression can be rationalizable to every seller. For instance, to seller a in product state X, it affects the probability of the consumer purchasing his product. Firstly, it increases the probability since the second consumer type may purchase from a under uncertainty. Secondly, it decreases the probability since the first consumer type may purchase from b under uncertainty. If seller a in X optimistically believes that the gain outweighs the loss, it is rational for him not to reveal the realized product state X.

In the US, Federal Trade Commission has allowed to name competing brands in advertisements since 1970. In 1997, the EU legalized comparative advertising subject to the restriction that it should not be misleading. However, in some Asian markets, comparative advertising has not been profoundly common. For example, in Malaysia and China, comparative advertisements are legal but without directly naming their competitors, and the use of comparative advertisements was sanctioned in India by the Advertising Standards Council of India in the late 1990s. Comparative advertising has been allowed officially in Korea only since 2001 but has not been widely used. We consider an alternative model of non-comparative advertisements. Not surprisingly, the necessary and sufficient condition required to obtain information unraveling at every outcome is stronger under non-comparative advertisements. Consider arbitrary doubleton sets of product types, one set for each seller. As an concrete example, suppose that each seller $i \in \{a, b\}$ could have either a high-quality product H_i or low-quality product L_i. Note that the products of each seller are vertically differentiated (see Proposition 2 for the detailed discussion). Let's assume, for the moment, that the preference of the consumer is known to the sellers. Then, the consumer is sufficiently informed even without any information disclosure, for instance, if the price of a is sufficiently higher than the price of b so that she prefers b regardless of the product state. Otherwise, some seller $i \in \{a, b\}$ improves his payoff through full disclosure of H_i. For instance, seller a should fully reveal product type H_a when the consumer prefers to purchase from a in product state (H_a, H_b) since she prefers a even in (H_a, L_b). Likewise, seller b should fully reveal H_b if she prefers b in (H_a, H_b). Now, we assume that the preference of the consumer is not known to the sellers, as is in our setup. Under the condition for sufficient unraveling, the consumer types can be partitioned in the following way. Between any two groups, the values for the relative competitiveness of seller a to b are "separated". Namely, all these values of the consumer types in one group are higher than those of all consumer types in the other. Moreover, the relative competitiveness of seller a takes the same value in the product state (H_a, H_b)

to all consumer types within a group. Then, either of the followings holds at every price pair, as does in the case with the known consumer type. The consumer is sufficiently informed even without any information disclosure or seller a of type H_a or b of H_b voluntarily disclose their product types.

Section 2 formalizes the baseline model, the one with comparative advertisements. In Section 3, we define the price-dependent sales-dominance relations over product states and show that product information sufficiently unravels at every prudent rationalizable outcome if and only if the sales-dominance relation is complete at every price pair. In Section 4, we consider the model of non-comparative advertisements and provide a necessary and sufficient condition required for every outcome to achieve information unraveling in this alternative circumstance. In Section 5, we provide an example in which competition enhances information disclosure when advertisements are comparative and the consumer has limited reasoning capability. In Section 6, we conclude and discuss some advantages of the solution concept, prudent rationalizability.

2 Baseline Model

Two male sellers a and b (called players a and b, respectively) compete for a single female consumer (called player c) by providing verifiable information about their products. The sellers have complete information about each other's product attributes. The consumer has complete and private information about her preference for differentiated products. Let G_i denote a finite set of product types for seller $i \in \{a, b\}$. A product state refers to an ordered pair $(g_a, g_b) \in G_a \times G_b$ of product types. A set of product states is denoted in bold by $\boldsymbol{G} = G_a \times G_b$ and a typical product state by $\boldsymbol{g} \in \boldsymbol{G}$. Let T be a finite set of the consumer's preference types.[4]

The game proceeds through four stages (see Figure 1). In the information stage, nature chooses the type of each player $k \in \{a, b, c\}$. Each seller observes the realized product state but remains uncertain about the consumer's preference type. The consumer learns only about her preference type. In the price stage, the sellers simultaneously set binding prices for their products. After observing each other's prices, the sellers simultaneously inform the consumer of the product state through costless messages. The information provided in the message stage must be truthful,

4 Alternatively, we can consider the model with the market of unit size. Let T be a partition of the set of consumers such that all consumers in each cell of the partition have an identical preference over the products. Then, we can call a class of this partition a consumer type. Essentially, the specification with a single consumer whose real preference type is among the ones in T is identical to the specification with the market of unit size where consumers can be partitioned into $|T|$ equivalence groups according to their preferences over the products.

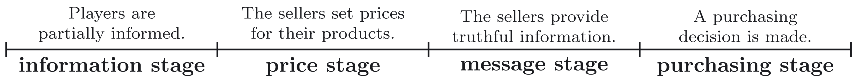

Figure 1: Timeline of the game.

while it may not be totally detailed. In the purchasing stage, the consumer makes a purchasing decision given the offered prices and the received messages. After the purchasing stage, the payoffs of all players realize.

Let P denote a set of all prices a seller can charge the consumer. We assume that set P is finite with an arbitrary grid (e.g. like cents in the U.S.). This assumption is made for reality to retain a finite game. A set of price pairs $(p_a, p_b) \in P \times P$ is denoted in bold by $\boldsymbol{P}$ and a typical price pair by $\boldsymbol{p} \in \boldsymbol{P}$. For every price pair $\boldsymbol{p} = (p_a, p_b)$, Δp denotes the difference of p_a from p_b, that is, $\Delta p = p_a - p_b$.

For simplicity, we assume that the marginal cost of each seller is zero regardless of his product type. The payoff to seller $i \in \{a, b\}$ is given by his profit. It is p_i if the consumer purchases his product at a price $p_i \in P$, while it is zero otherwise. If a consumer type $t \in T$ purchases the product of a type $g \in G_a \cup G_b$ at a price $p \in P$, her payoff is $v + u(g, t) - p$, where $v > 0$ is a type-independent willingness-to-pay of the consumer and $u\colon \left(G_a \cup G_b\right) \times T \to \mathbb{R}_+$. We assume that v is sufficiently large so that the consumer must purchase from either seller.[5] For every product state $\boldsymbol{g} = (g_a, g_b) \in \boldsymbol{G}$ and consumer type $t \in T$, the relative competitiveness of seller a to b is denoted by $\Delta u\left(\boldsymbol{g}, t\right) = u(g_a, t) - u(g_b, t)$. Attention is restricted only to a consumer with generic preferences. Formally, her preference is generic if $\Delta u(\boldsymbol{g}, t) \neq \Delta p$ for every $\boldsymbol{g} \in \boldsymbol{G}$, $t \in T$, and $\boldsymbol{p} \in \boldsymbol{P}$.[6] Under this assumption, the consumer strictly prefers one seller over the other at every price pair if the realized product state fully unravels.

For every $k \in \{a, b, c\}$, H_k denotes a set of player k's information sets. At every information set $h_i \in H_i$ of seller $i \in \{a, b\}$, he has learned the realized product state. Let $g_a(h_i)$ and $g_b(h_i)$ respectively denote the product type of seller a and b realized on the path to h_i and $\boldsymbol{g}(h_i)$ the realized product state $\left(g_a\left(h_i\right), g_b\left(h_i\right)\right)$. At every information set $h_i \in H_i$ of seller $i \in \{a, b\}$ in the message stage, he has observed the prices chosen in the price stage. By $p_a(h_i)$ and $p_b(h_i)$, we respectively denote

5 When the payoff of not purchasing is normalized zero, we can alternatively interpret v as the type-independent penalty or loss of not purchasing the product. In other words, the product of interest is a "must-have" item such as a car insurance to a vehicle owner.

6 We implicitly assume that the sellers' products are hardly homogeneous due to different technologies, customer services, or brand loyalty. Given any price pair, an in-depth inspection over heterogeneous products would make the consumer strictly prefer to purchase from one seller over the other.

the price chosen by seller a and b on the path to h_i and by $\boldsymbol{p}(h_i)$ the price pair $\left(p_a(h_i), p_b(h_i)\right)$. Every information set $h_c \in H_c$ of the consumer is identified with a tuple of her preference type, prices, and messages observed in the previous stages. Let $t(h_c)$ denote the preference type learned on the path to h_c. For every $i \in \{a, b\}$, $p_i(h_c)$ and $m_i(h_c)$ respectively denote the offered price and the message provided by seller i on the path to h_c. By $\boldsymbol{p}(h_c)$, we denote price pair $\left(p_a(h_c), p_b(h_c)\right)$.

A pure strategy s_i of seller $i \in \{a, b\}$ prescribes what price to charge at every information set in the price stage and what message to send about the product state at every information set in the message stage.[7] A seller at an information set in the price stage chooses a price in P. A message $s_i(h_i)$ chosen at an information set h_i in the message stage satisfies $\boldsymbol{g}(h_i) \in s_i(h_i)$ and $s_i(h_i) \subseteq \boldsymbol{G}$. The latter implies that advertisements are comparative and the provided information may be vague. The former requires that each seller provide only truthful information. Note that the real product state must be among the states mentioned in the message. A pure strategy s_c of the consumer specifies from which seller to purchase at every information set in the purchasing stage, that is, $s_c\colon H_c \to \{a, b\}$. For every $k \in \{a, b, c\}$, S_k denotes the set of player k's pure strategies.

The above mentioned are common knowledge among the players. As a solution concept, we use prudent rationalizability, which is an extensive-form version of iterated admissibility. To define prudent rationalizability, we introduce belief systems. At every information set of the consumer, she forms a belief about product states and strategies of the sellers. A belief system b_c of the consumer is a tuple

$$\left(b_c\left(h_c\right)\right)_{h_c \in H_c} \in \prod_{h_c \in H_c} \Delta\left(\boldsymbol{G} \times S_a \times S_b\right)$$

such that for every $h_c \in H_c$, belief $b_c(h_c)$ assigns probability 1 to the set of profiles $(\boldsymbol{g}, s_a, s_c) \in \boldsymbol{G} \times S_a \times S_b$ that reach h_c. From now on, we reserve the notation i and j for one seller and his opponent seller, respectively. At every information set of a seller, he forms a belief about the consumer's preference types and strategies of the other players. For every $i \in \{a, b\}$, a belief system b_i of seller i is a tuple

$$\left(b_i\left(h_i\right)\right)_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta\left(T \times S_j \times S_c\right)$$

such that for every information set $h_i \in H_i$, belief $b_i(h_i)$ assigns probability 1 to the set of profiles $(t, s_j, s_c) \in T \times S_j \times S_c$ that reach h_i. Moreover, Bayesian updating is applied whenever possible. Consider information sets h_i^p and h_i^m of seller i in the price and message stage such that $\boldsymbol{g}\left(h_i^p\right) = \boldsymbol{g}\left(h_i^m\right)$. Then, h_i^m follows h_i^p. For

7 The model of non-comparative advertisements will be discussed in Section 4. In the alternative model, each seller provides information only about his product.

every profiles (t, s_j, s_c) and $\left(t', s'_j, s'_c\right)$ that reach h_i^m, $\frac{b_i(h_i^p)(t',s'_j,s'_c)}{b_i(h_i^p)(t,s_j,s_c)} = \frac{b_i(h_i^m)(t',s'_j,s'_c)}{b_i(h_i^m)(t,s_j,s_c)}$ whenever $b_i\left(h_i^p\right)(t, s_j, s_c) > 0$. For every $k \in \{a, b, c\}$, B_k denotes a set of player k's belief systems.

For every $k \in \{a, b, c\}$, we say that a strategy $s_k \in S_k$ of player k is rational at an information set $h_k \in H_k$ with a belief system if there does not exist an action a_k such that only replacing action $s_k(h_k)$ with a_k yields player k a strictly higher expected payoff. We say that s_k is rationalizable at h_k if there exists a belief system with which s_k is rational at h_k.

Prudent rationalizability is an iterated elimination process. Roughly, in each round of elimination, player $k \in \{a, b, c\}$ at an information set forms a full-support belief by taking into account the move of nature and the strategies of the other players that have survived all previous rounds. Moreover, a strategy of player k survives this round of elimination if it is rational at every information set of player k with some belief system formed in the aforementioned manner. Indeed, prudent rationalizability features cautious behaviors of the players (see Heifetz, Meier, and Schipper (2021) for more details on prudent rationalizability). In each round, the cautiousness of the players enters through the full-support beliefs over the unknown types and the surviving strategies of the other players. Namely, a player does not completely exclude any of the other players' unknown types and not-yet-eliminated strategies. This feature would play an essential role for the results.

Definition 1 . (Prudent rationalizability) For every player $k \in \{a, b, c\}$, $S_k^0 = S_k$. We define inductively for $r \geq 1$. For every seller $i \in \{a, b\}$,

$$B_i^r = \left\{ b_i \in B_i \colon \begin{array}{l} \text{For every information set } h_i \in H_i \text{ of seller } i \text{ that is reached by some} \\ \text{profile in } T \times S_j^{r-1} \times S_c^{r-1}\text{, the support of } b_i(h_i) \text{ is the set of all profiles} \\ \text{in } T \times S_j^{r-1} \times S_c^{r-1} \text{ that reach } h_i. \end{array} \right\},$$

and for the consumer c,

$$B_c^r = \left\{ b_c \in B_c \colon \begin{array}{l} \text{For every information set } h_c \in H_c \text{ of the consumer that is reached by} \\ \text{some profile in } \boldsymbol{G} \times S_a^{r-1} \times S_b^{r-1}\text{, the support of } b_c(h_c) \text{ is the set of all} \\ \text{profilesin } \boldsymbol{G} \times S_a^{r-1} \times S_b^{r-1} \text{ that reach } h_c. \end{array} \right\},$$

and for every $k \in \{a, b, c\}$

$$S_k^r = \left\{ s_k \in S_k^{r-1} \colon \begin{array}{l} \text{There exists } b_k \in B_k^r \text{ with which strategy } s_k \text{ of player } k \text{ is rational} \\ \text{at every } h_k \in H_k. \end{array} \right\}.$$

The set of prudent rationalizable strategies of player $k \in \{a, b, c\}$ is

$$S_k^\infty = \bigcap_{r=1}^{\infty} S_k^r.$$

A prudent rationalizable outcome refers to a profile $(s_a, s_b, s_c) \in S_a^\infty \times S_b^\infty \times S_c^\infty$ of prudent rationalizable strategies. The existence of prudent rationalizable outcomes follows from a result in Heifetz, Meier, and Schipper (2021). We say that product information sufficiently unravels at a prudent rationalizable outcome (s_a, s_b, s_c) if for every product state $\boldsymbol{g} \in \boldsymbol{G}$ and information set $h_c \in H_c$ of the consumer reached by $(\boldsymbol{g}, s_a, s_b)$, $s_c(h_c) = a$ if and only if $\Delta u(\boldsymbol{g}, t(h_c)) > \Delta p_i(h_c)$. Note that at a sufficiently unraveling prudent rationalizable outcome, the consumer makes exactly the purchasing decision that she would make under perfect information of the product state. Note further that in contrast to the consumer at a full-disclosure outcome, she may not learn the realized product state precisely at a sufficiently unraveling outcome. However, she is more or less well-informed of the product state in that any further revelation of product information will not change her purchasing decision, that is, her choice must be ex post optimal regardless of the product state she believes as possible.

3 Sales-Dominance Relations and Sufficient Unraveling

For every price pair $\boldsymbol{p} \in \boldsymbol{P}$ and non-empty set $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states, we define $T_i(\boldsymbol{p}, \boldsymbol{G}')$ as the set of consumer types who prefer to purchase from seller $i \in \{a, b\}$ in some product state $\boldsymbol{g} \in \boldsymbol{G}'$, i.e.

$$\begin{aligned} T_i(\boldsymbol{p}, \boldsymbol{G}') &= \left\{ t \in T\colon u(g_i, t) - p_i > u(g_j, t) - p_j \text{ for some } (g_i, g_j) \in \boldsymbol{G}' \right\} \\ &= \bigcup_{\boldsymbol{g} \in \boldsymbol{G}'} T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) \end{aligned}$$

Suppose that the consumer is offered a price pair $\boldsymbol{p}$ and learns that all and only product states in $\boldsymbol{G}'$ are possible.[8] If $\boldsymbol{G}'$ is singleton (say, $\boldsymbol{G}' = \{\boldsymbol{g}\}$), the consumer precisely learns the realized product state $\boldsymbol{g}$ and all and only consumer types in $T_i(\boldsymbol{p}, \{\boldsymbol{g}\})$ purchase from seller i. We call $T_i(\boldsymbol{p}, \{\boldsymbol{g}\})$ sales from seller i at price

8 During the elimination process of prudent rationalizability, the consumer might form this belief at information set h_c with $\boldsymbol{p}(h_c) = \boldsymbol{p}$. For instance, all surviving strategies of seller $i \in \{a, b\}$ prescribe not to send $m_i(h_c)$ at any information set h_i in the message stage such that $\boldsymbol{g}(h_i) \notin \boldsymbol{G}'$ and $\boldsymbol{p}(h_i) = \boldsymbol{p}$. Moreover, for every product state $\boldsymbol{g} \in \boldsymbol{G}'$, some not-yet-eliminated strategy of seller i prescribes to set $p_i(h_c)$ at h_i' on the price state such that $\boldsymbol{g}(h_i') = \boldsymbol{g}$ and to send $m_i(h_c)$ at h_i'' in the message stage such that $\boldsymbol{g}(h_i'') = \boldsymbol{g}$ and $\boldsymbol{p}(h_i'') = \boldsymbol{p}$.

pair $\boldsymbol{p}$ in product state $\boldsymbol{g}$.[9] Otherwise, if $\boldsymbol{G}'$ is non-singleton, then the consumer remains uncertain about the product state and purchasing from seller i is rationalizable to all and only consumer types in $T_i(\boldsymbol{p}, \boldsymbol{G}')$.[10] Firstly, every consumer type not in $T_i(\boldsymbol{p}, \boldsymbol{G}')$ must purchase from the opponent seller j with whatever full-support belief she forms over $\boldsymbol{G}'$. Secondly, purchasing from i is rational to a consumer type $t \in T_i(\boldsymbol{p}, \boldsymbol{G}')$ with a full-support belief on $\boldsymbol{G}'$ assigning a sufficiently high probability to some product state $(g_i, g_j) \in \boldsymbol{G}'$ such that $u(g_i, t) - p_i > u(g_j, t) - p_j$. We interpret $T_i(\boldsymbol{p}, \boldsymbol{G}')$ as the highest possible sales from seller i when the consumer is offered a price pair $\boldsymbol{p}$ and believes that the true product state is in $\boldsymbol{G}'$.[11]

Definition 2 . (Weak sales-dominance relations over product states) For price pair $\boldsymbol{p} \in \boldsymbol{P}$ and product states $\boldsymbol{g}$ and $\boldsymbol{g}'$ in $\boldsymbol{G}$, we say that $\boldsymbol{g}$ weakly sales-dominates $\boldsymbol{g}'$ at $\boldsymbol{p}$ for seller $i \in \{a, b\}$ (denoted by $\boldsymbol{g} \succsim^i_{\boldsymbol{p}} \boldsymbol{g}'$) if $T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) \supseteq T_i(\boldsymbol{p}, \{\boldsymbol{g}'\})$. Over a non-empty set $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states, $\bar{\boldsymbol{g}} \in \boldsymbol{G}'$ is sales-dominant at $\boldsymbol{p}$ for seller i if $\bar{\boldsymbol{g}} \succsim^i_{\boldsymbol{p}} \boldsymbol{g}$ for every $\boldsymbol{g} \in \boldsymbol{G}'$.

For every price pair $\boldsymbol{p} \in \boldsymbol{P}$, the strict sales-dominance relation and the sales-equivalence relation are defined as usual. I.e. $\boldsymbol{g}$ strictly sales-dominates $\boldsymbol{g}'$ at $\boldsymbol{p}$ for seller $i \in \{a, b\}$ (denoted by $\boldsymbol{g} \succ^i_{\boldsymbol{p}} \boldsymbol{g}'$) if $T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) \supsetneq T_i(\boldsymbol{p}, \{\boldsymbol{g}'\})$. Whenever we refer to sales-dominance without qualifying as strict or weak, the default interpretation will be weak sales-dominance. Product states $\boldsymbol{g}$ and $\boldsymbol{g}'$ are sales-equivalent for seller $i \in \{a, b\}$ at $\boldsymbol{p}$ (denoted by $\boldsymbol{g} \sim^i_{\boldsymbol{p}} \boldsymbol{g}'$) if $T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) = T_i(\boldsymbol{p}, \{\boldsymbol{g}'\})$.

We say that the sales-dominance relation $\succsim^i_{\boldsymbol{p}}$ for seller $i \in \{a, b\}$ is complete at price pair $\boldsymbol{p} \in \boldsymbol{P}$ if $\boldsymbol{g} \succsim^i_{\boldsymbol{p}} \boldsymbol{g}'$ or $\boldsymbol{g}' \succsim^i_{\boldsymbol{p}} \boldsymbol{g}$ for every product states $\boldsymbol{g}$ and $\boldsymbol{g}'$ in $\boldsymbol{G}$. Under the assumption of generic preference, if the sales-dominance relation is complete at $\boldsymbol{p}$ for one seller, so should be for the other because they are dual to each other in that $\boldsymbol{g} \succsim^i_{\boldsymbol{p}} \boldsymbol{g}'$ if and only if $\boldsymbol{g}' \succsim^j_{\boldsymbol{p}} \boldsymbol{g}$. We write that the sales-dominance relation is complete at a price pair if it is the case for each seller. We say that the sales-dominance relations are complete if it is complete at every price pair. Note that the completeness of the weak sales-dominance relations is the condition on the primitives of the model, namely the preferences of the consumer.

9 Since there is a single consumer with unit demand, the ex post sales from a seller is binary, 0 or 1. Rigorously, $T_i(\boldsymbol{p}, \{\boldsymbol{g}\})$ is the expected sales from i at $\boldsymbol{p}$ in $\boldsymbol{g}$ when the consumer is offered $\boldsymbol{p}$ and precisely learns $\boldsymbol{g}$. However, for simplicity in writing, we refer $T_i(\boldsymbol{p}, \{\boldsymbol{g}\})$ to sales rather than the expected sales. See also Footnote 4.

10 In this article, a non-singleton set refers to a non-empty set containing more than one element.

11 Since sales refer to a set of consumer types, the adjectives "large" and "small" may be appropriate usages of language. However, we use "high" and "low" as they are normally used to describe the sales volume.

Δu	t_1	t_2
g^1	6	1
g^2	3	5

	$T_a(p, \{g^1\})$	$T_a(p, \{g^2\})$	$T_b(p, \{g^1\})$	$T_b(p, \{g^2\})$
$\Delta p_1 = 4$	$\{t_1\}$	$\{t_2\}$	$\{t_2\}$	$\{t_1\}$
$\Delta p_2 = 2$	$\{t_1\}$	T	$\{t_2\}$	$\emptyset$

Example 1. Not surprisingly, the sales-dominance relations are not necessarily complete. That is, given the preference of the consumer, a price pair may exist at which the sales-dominance relation is complete for neither seller. Let $\boldsymbol{G} = \{\boldsymbol{g}^1, \boldsymbol{g}^2\}$ and $T = \{t_1, t_2\}$. In the upper table, the relative competitiveness Δu of seller a is provided for every $(\boldsymbol{g}, t) \in \boldsymbol{G} \times T$ and the bottom table shows sales from a seller at price pairs $\boldsymbol{p}_1$ with $\Delta p_1 = 4$ and $\boldsymbol{p}_2$ with $\Delta p_2 = 2$. At $\boldsymbol{p}_1$, no product state sales-dominates the other for any seller. However, at $\boldsymbol{p}_2$, $\boldsymbol{g}^2$ strictly sales-dominates $\boldsymbol{g}^1$ for seller a, while $\boldsymbol{g}^2$ is strictly sales-dominated for b. $\square$

The sales-dominance relation is trivially reflexive and transitive at every price pair since it is defined using set inclusion. If the sales-dominance relation is complete at price pair $\boldsymbol{p} \in \boldsymbol{P}$, $\boldsymbol{G}$ is a completely-preordered set by the sales-dominance relation $\succsim^i_{\boldsymbol{p}}$.[12] It is well-known that a completely-preordered finite set has the greatest element. For example, the most preferred alternative exists over a finite choice set if the preference relation is complete, reflexive, and transitive. Formally, the sales-dominance relation is complete at price pair $\boldsymbol{p}$ if and only if a sales-dominant product state for seller $i \in \{a, b\}$ exists at $\boldsymbol{p}$ over every non-empty set $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states.

Lemma 1. *Consider an arbitrary price pair $\boldsymbol{p} \in \boldsymbol{P}$ and non-empty set $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states. For seller $i \in \{a, b\}$, a product state $\bar{\boldsymbol{g}} \in \boldsymbol{G}'$ is sales-dominant at $\boldsymbol{p}$ over $\boldsymbol{G}'$ if and only if $T_i(\boldsymbol{p}, \{\bar{\boldsymbol{g}}\}) = T_i(\boldsymbol{p}, \boldsymbol{G}')$.*

Proof. Consider an arbitrary price pair $\boldsymbol{p} \in \boldsymbol{P}$ and non-empty set $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states. $T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) \subseteq T_i(\boldsymbol{p}, \boldsymbol{G}')$ for every $\boldsymbol{g} \in \boldsymbol{G}'$ by definition. For seller $i \in \{a, b\}$, product state $\bar{\boldsymbol{g}}$ is sales-dominant at $\boldsymbol{p}$ over $\boldsymbol{G}'$ if and only if $T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) \subseteq T_i(\boldsymbol{p}, \{\bar{\boldsymbol{g}}\})$ for every $\boldsymbol{g} \in \boldsymbol{G}'$, which is equivalent to $T_i(\boldsymbol{p}, \boldsymbol{G}') = \bigcup_{\boldsymbol{g} \in \boldsymbol{G}'} T_i(\boldsymbol{p}, \{\boldsymbol{g}\}) \subseteq T_i(\boldsymbol{p}, \{\bar{\boldsymbol{g}}\})$. $\square$

12 A binary relation is a complete preorder if it is complete, reflexive, and transitive.

Consider an arbitrary price pair $\boldsymbol{p}$ at which $\bar{\boldsymbol{g}}$ is sales-dominant for seller $i \in \{a, b\}$ over message $m_i \subseteq \boldsymbol{G}$. Lemma 1 implies that sufficient-disclosure of $\bar{\boldsymbol{g}}$ is at least as profitable as m_i at information set h_i in the message stage such that $\boldsymbol{g}(h_i) = \bar{\boldsymbol{g}}$ and $\boldsymbol{p}(h_i) = \boldsymbol{p}$. That is, the for-sure sales $T_i(\boldsymbol{p}, \{\bar{\boldsymbol{g}}\})$ achieved by sending a message $\bar{m}_i$ such that $\bar{\boldsymbol{g}} \sim^i_{\boldsymbol{p}} \boldsymbol{g}$ for every $\boldsymbol{g} \in \bar{m}_i$ are never lower than the highest-possible sales achieved by sending m_i. Lemma 1 further implies that in any second-round prudent rationalizable outcome, if $\bar{\boldsymbol{g}} \succ^i_{\boldsymbol{p}} \boldsymbol{g}$ for some $\boldsymbol{g} \in m_i$, seller i should not send message m_i at the above mentioned information set h_i.[13] Suppose to the contrary that seller i sends such message m_i at h_i. Firstly, seller i believes with a positive probability that seller j sends message m_j with $\boldsymbol{g} \in m_j$ at information set h_j in the message stage such that $\boldsymbol{g}(h_j) = \bar{\boldsymbol{g}}$ and $\boldsymbol{p}(h_j) = \boldsymbol{p}$, which would cause the consumer to believe both $\bar{\boldsymbol{g}}$ and $\boldsymbol{g}$ as possible. Secondly, it is rational for every consumer type $t \in T_i(\boldsymbol{p}, \{\bar{\boldsymbol{g}}\}) \cap T_j(\boldsymbol{p}, \{\boldsymbol{g}\})$ to purchase from seller j at $\boldsymbol{p}$ after receiving messages (m_i, m_j) with a belief assigning a sufficient high probability to $\boldsymbol{g}$. Thus, the resulting sales from seller i can be strictly lower than under sufficient-unraveling of $\bar{\boldsymbol{g}}$.

Example 2. Consider the preference of the consumer given below. Let P be a finite set of positive prices. The sales-dominance relation is complete at every price pair. We would like to build an intuition about why sufficient disclosure is the unique prudent rationalizable outcome under the complete sales-dominance relations. Moreover, we illustrate how prudent rationalizability applies before we state and prove the main results.

Δu	t_1	t_2
g^1	3	−4
g^2	1	−2

First round: Every strategy $s_i \in S_i$ of seller $i \in \{a, b\}$ is rational with a belief that the consumer purchases from i at all her information sets reachable by s_i, while she purchases from j at all other information sets. Table 1 summarizes first-round prudent rationalizable strategies of the consumer. Each information set of the consumer is mapped to a cell in this table. At information sets h^1_c of consumer type t_1 such that $\Delta p\left(h^1_c\right) \in (1, 3)$ and $m_a\left(h^1_c\right) \cap m_b\left(h^1_c\right) = \boldsymbol{G}$, she cautiously believes that any product state is possible by forming a full-support belief on $\boldsymbol{G}$. If she believes

13 However, m_i is rational to i at h'_i in the message stage such that $\boldsymbol{g}\left(h'_i\right) = \boldsymbol{g}$ and $\boldsymbol{p}\left(h'_i\right) = \boldsymbol{p}$. He may believe with a sufficiently high probability that j sends m'_j with $\bar{\boldsymbol{g}} \in m'_j$ at h'_j in the message stage such that $\boldsymbol{g}\left(h'_j\right) = \boldsymbol{g}$ and $\boldsymbol{p}\left(h'_j\right) = \boldsymbol{p}$, and that every consumer type $t \in T_i(\boldsymbol{p}, \{\bar{\boldsymbol{g}}\})$ purchases from i at $\boldsymbol{p}$ after receiving $\left(m'_i, m'_j\right)$ with a belief assigning sufficient weight to $\bar{\boldsymbol{g}}$.

Table 1: First round prudent rationalizable strategies of the consumer.

		$m_a(h_c) \cap m_b(h_c)$		
		$\{g^1\}$	$\{g^2\}$	G
$t(h_c) = t_1$	$\Delta p(h_c) < 1$	a	a	a
	$\Delta p(h_c) \in (1, 3)$	a	b	a or b
	$\Delta p(h_c) > 3$	b	b	b
$t(h_c) = t_2$	$\Delta p(h_c) < -4$	a	a	a
	$\Delta p(h_c) \in (-4, -2)$	b	a	a or b
	$\Delta p(h_c) > -2$	b	b	b

$\boldsymbol{g}^1$ with a sufficiently high probability, she prefers a. With an alternative belief assigning a sufficiently high probability to $\boldsymbol{g}^2$, b is the optimal choice. Hence, purchasing from seller $i \in \{a, b\}$ is first-round prudent rationalizable at such a h_c^1. Likewise, at information sets h_c^2 of consumer type t_2 such that $\Delta p\left(h_c^2\right) \in (-4, -2)$ and $m_a\left(h_c^2\right) \cap m_b\left(h_c^2\right) = \boldsymbol{G}$, a full-support belief on $\boldsymbol{G}$ exists with which purchasing from seller $i \in \{a, b\}$ is rational. Whatever the messages from the sellers, t_1 is sufficiently informed at h_c with $\Delta p(h_c) \notin (1, 3)$, and so is t_2 at h_c' with $\Delta p\left(h_c'\right) \notin (-4, -2)$.

Second round: In the second round, every seller $i \in \{a, b\}$ forms a full-support belief by taking into account all consumer types and only the first-round prudent rationalizable strategies of the other players. In the second round, every positive price is prudent rationalizable at every information set of seller $i \in \{a, b\}$ in the price stage (see Appendix B for more detailed discussion). We focus on information sets in the message stage. Firstly, consider information sets reached after a price pair $\boldsymbol{p} \in \boldsymbol{P}$ with $\Delta p \in (-4, -2)$ is chosen in the price stage. From the first-round prudent rationalizable strategies, the sellers know that t_1 would purchase from a regardless of the message received and they are concerned about how t_2 would respond to the provided messages. Suppose that the realized product state is $\boldsymbol{g}^1$. From the first-round prudent rationalizable strategies, each seller believes that the opponent seller may send message $\boldsymbol{G}$ and that t_2 purchases from b with certainty if and only if $\boldsymbol{g}^1$ is fully revealed from either seller. Therefore, seller b must disclose $\boldsymbol{g}^1$ to persuade t_2 for sure and seller a must suppress information not to lose a chance of persuading t_2. Now suppose that product state $\boldsymbol{g}^2$ is realized. At price pair $\boldsymbol{p}$, $\boldsymbol{g}^2$ strictly sales-dominates $\boldsymbol{g}^1$ for a. Following Lemma 1, only full disclosure is rationalizable for seller a. In contrast, seller b should provide trivial information $\boldsymbol{G}$. Analogously, we derive the second-round prudent rationalizable actions of the sellers at information sets in the message stage reached after a price pair $\boldsymbol{p}$ with $\Delta p \in (1, 3)$ is chosen in the price stage. Table 2 summarizes the second-round

Table 2: Second-round prudent rationalizable actions of the sellers in the message stage.

	Seller *a*		**Seller *b***	
	$g(h_a) = g^1$	$g(h_a) = g^2$	$g(h_b) = g^1$	$g(h_b) = g^2$
$\Delta p(h_i) < -4$	$\{g^1\}$ and $\boldsymbol{G}$	$\{g^2\}$ and $\boldsymbol{G}$	$\{g^1\}$ and $\boldsymbol{G}$	$\{g^2\}$ and $\boldsymbol{G}$
$\Delta p(h_i) \in (-4, -2)$	$\boldsymbol{G}$	$\{g^2\}$	$\{g^1\}$	$\boldsymbol{G}$
$\Delta p(h_i) \in (-2, 1)$	$\{g^1\}$ and $\boldsymbol{G}$	$\{g^2\}$ and $\boldsymbol{G}$	$\{g^1\}$ and $\boldsymbol{G}$	$\{g^2\}$ and $\boldsymbol{G}$
$\Delta p(h_i) \in (1, 3)$	$\{g^1\}$	$\boldsymbol{G}$	$\boldsymbol{G}$	$\{g^2\}$
$\Delta p(h_i) > 3$	$\{g^1\}$ and $\boldsymbol{G}$	$\{g^2\}$ and $\boldsymbol{G}$	$\{g^1\}$ and $\boldsymbol{G}$	$\{g^2\}$ and $\boldsymbol{G}$

prudent rationalizable actions of the sellers in the message stage. Each information set of a seller in the message stage is mapped to a cell in the table.

The elimination process ends in the second round. Recall, from Table 1, that the consumer is insufficiently informed only if some price pair $\boldsymbol{p}$ with $\Delta p \in (-4, -2) \cup (1, 3)$ is offered along the path of a prudent rationalizable outcome. From the prudent rationalizable strategies of the sellers, one seller fully discloses for every product state and such price pair $\boldsymbol{p}$. Thus, the consumer is sufficiently informed at every prudent rationalizable outcome. □

In Theorem 1, we claim that product information unravels sufficiently at every prudent rationalizable outcome if the sales-dominance relation is complete at every price pair. For a simple illustration of the proof, attention is restricted to a price pair $\boldsymbol{p} \in \boldsymbol{P}$ at which no two product states are sales-equivalent. Then, we can order the product states in $\boldsymbol{G}$ according to the strict sales-dominance relation $\succ^i_{\boldsymbol{p}}$ for seller $i \in \{a, b\}$ at $\boldsymbol{p}$ and call the product state in the nth place according to $\succ^i_{\boldsymbol{p}}$ the nth most sales-dominant product state for i at $\boldsymbol{p}$. For every $n \geq 1$, $\boldsymbol{g}^n_i$ denotes the nth most sales-dominant product state for seller i at $\boldsymbol{p}$ and h^n_i denotes his information set in the message stage such that $\boldsymbol{g}\left(h^n_i\right) = \boldsymbol{g}^n_i$ and $\boldsymbol{p}\left(h^n_i\right) = \boldsymbol{p}$. Roughly, we show that every first-round prudent rationalizable strategy of the consumer prescribes to purchase from the most preferred seller at $\boldsymbol{p}$ in $\boldsymbol{g}^1_i$ if the received messages fully unravel $\boldsymbol{g}^1_i$ and that every second-round prudent rationalizable strategy of seller i prescribes to disclose $\boldsymbol{g}^1_i$ at h^1_i. For $r > 1$, every $(2r-1)$th-round prudent rationalizable strategy of the consumer prescribes to purchase from the most preferred seller at $\boldsymbol{p}$ in $\boldsymbol{g}^r_i$ if the provided messages inform that the worst product state according to $\succsim^i_{\boldsymbol{p}}$ is $\boldsymbol{g}^r_i$ (for instance, if messages (m_a, m_b) are received such that $\boldsymbol{g}^r_i \in m_a \cap m_b$ and $\boldsymbol{g} \succsim^i_{\boldsymbol{p}} \boldsymbol{g}^r_i$ for every $\boldsymbol{g} \in m_a \cap m_b$). Moreover, every $2r$th-round of prudent rationalizable strategy of seller i prescribes to send at h^r_i a message including only product states $\boldsymbol{g}$ that are at least as good as $\boldsymbol{g}^r_i$, i.e. $\boldsymbol{g} \succsim^i_{\boldsymbol{p}} \boldsymbol{g}^r_i$. After a finite number of rounds, all "undisclosed" product states are sales-equivalent at $\boldsymbol{p}$ and the elimination process stops.

One may misleading believe that the number of rounds of prudent rationalizability required for sufficient unraveling is determined by the number of product states in $\boldsymbol{G}$. Instead, it depends on the way how $\boldsymbol{G}$ is partitioned according to the sales-dominance relation. Pick an arbitrary price pair $\boldsymbol{p} \in \boldsymbol{P}$ that can be chosen at a prudent rationalizable outcome. If $\boldsymbol{G}$ is partitioned into one equivalence class according to $\succsim^a_{\boldsymbol{p}}$, all product states are sales-equivalent at $\boldsymbol{p}$ and the consumer is sufficiently informed without any information disclosure. Now, assume that $\boldsymbol{G}$ is partitioned into $\bar{n}$ equivalence classes according to $\succsim^a_{\boldsymbol{p}}$, where $\bar{n} = 2n$ or $2n + 1$ for some $n \geq 1$. Then, given $\boldsymbol{p}$, product information unravels sufficiently after $2n$ rounds. In Example 2, the two products states are different in sales only at $\boldsymbol{p}$ with $\Delta p \in (-4, -2) \cup (1, 3)$ and some strategies of the sellers are eliminated until the second round.

Theorem 1. *If the sales-dominance relation is complete at every price pair, then product information sufficiently unravels at every prudent rationalizable outcome.*

What restriction would the complete sales-dominance relations impose on the consumer's ranking of Δu in the pairwise comparison of product states? Consider t$_{\text{wo}}$ product states $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ in $\boldsymbol{G}$. Completeness of the sales-dominance relation at $\boldsymbol{p}$ requires that $T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\}) \setminus T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\})$ or $T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\}) \setminus T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\})$ are empty. Now, we consider consumer types t_1 and t_2 such that $\Delta u\left(\boldsymbol{g}^2, t_1\right) < \Delta u\left(\boldsymbol{g}^1, t_1\right)$ and $\Delta u\left(\boldsymbol{g}^1, t_2\right) \leq \Delta u\left(\boldsymbol{g}^2, t_2\right)$. For any price pair $\boldsymbol{p}$ with $\Delta p \in \left(\Delta u\left(\boldsymbol{g}^2, t_1\right), \Delta u\left(\boldsymbol{g}^1, t_1\right)\right]$, $t_1 \in T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\}) \setminus T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\})$, implying $T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\}) \setminus T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\}) = \emptyset$. Therefore, Δu should be monotone with respect for consumer types in that either $\Delta u\left(\boldsymbol{g}^2, t_1\right) < u\left(\boldsymbol{g}^1, t_1\right) \leq u\left(\boldsymbol{g}^1, t_2\right) \leq \Delta u\left(\boldsymbol{g}^2, t_2\right)$ or $\Delta u\left(\boldsymbol{g}^1, t_2\right) \leq u\left(\boldsymbol{g}^2, t_2\right) \leq u\left(\boldsymbol{g}^2, t_1\right) < \Delta u\left(\boldsymbol{g}^1, t_1\right)$.

For every non-empty sets $\boldsymbol{G}' \subseteq \boldsymbol{G}$ and $T' \subseteq T$, a saddle point is defined as a pair $\left(\boldsymbol{g}^*, t^*\right) \in \boldsymbol{G}' \times T'$ such that $\Delta u(\boldsymbol{g}^*, t^*) = \max_{\boldsymbol{G}'} \min_{T'} \Delta u(\boldsymbol{g}, t) = \min_{T'} \max_{\boldsymbol{G}'} \Delta u(\boldsymbol{g}, t)$. According to Gurvich and Libkin (1990), a saddle point exists for every non-empty sets $\boldsymbol{G}'$ and T' if and only if every doubleton subsets $\boldsymbol{G}'' \subseteq \boldsymbol{G}$ and $T'' \subseteq T$ has a saddle point. Therefore, if no saddle point exists for some non-empty sets $\boldsymbol{G}'$ and T', some doubleton sets $\boldsymbol{G}'' = \{\boldsymbol{g}^1, \boldsymbol{g}^2\}$ and $T'' = \{t_1, t_2\}$ exist for which either case in Table 3 holds. We say that the set $\boldsymbol{P}$ of prices is sufficiently fine if for every different pairs $(\boldsymbol{g}, t)$ and $(\boldsymbol{g}', t')$ in $\boldsymbol{G} \times T$, there exist a price pair $\boldsymbol{p} \in \boldsymbol{P}$ such that $\Delta u(\boldsymbol{g}, t) < \Delta p < \Delta u(\boldsymbol{g}', t')$ or $\Delta u(\boldsymbol{g}', t') < \Delta p < \Delta u(\boldsymbol{g}, t)$. We believe that this assumption would be appropriate in the model with a single consumer as prices would be more "densely" populated on the real line than the values of relative competitiveness.

Table 3: Non-existence of a saddle point for sets $\{g^1, g^2\}$ and $\{t_1, t_2\}$.

	(a) Case 1	
$\Delta u(g^1, t_1)$	$>$	$\Delta u(g^1, t_2)$
$\vee$		$\wedge$
$\Delta u(g^2, t_1)$	$<$	$\Delta u(g^2, t_2)$
	(b) Case 2	
$\Delta u(g^1, t_1)$	$<$	$\Delta u(g^1, t_2)$
$\wedge$		$\vee$
$\Delta u(g^2, t_1)$	$>$	$\Delta u(g^2, t_2)$

Proposition 1. *Suppose that the set P of prices is sufficiently fine. The following statements are equivalent.*

(i) *For every price pair $\boldsymbol{p} \in \boldsymbol{P}$ and seller $i \in \{a, b\}$, the sales-dominance relation $\succsim_{\boldsymbol{p}}^{i}$ is complete.*

(ii) *A saddle point exists for every pair of non-empty subsets $\boldsymbol{G}' \subseteq \boldsymbol{G}$ and $T' \subseteq T$.*

(iii) *For every $\boldsymbol{g} \neq \boldsymbol{g}'$ in $\boldsymbol{G}$ and $t \neq t'$ in T, $\Delta u(\boldsymbol{g}, t) > \Delta u(\boldsymbol{g}', t)$ implies $\Delta u(\boldsymbol{g}, t') \geq \Delta u(\boldsymbol{g}', t')$ or $\Delta u(\boldsymbol{g}, t) > \Delta u(\boldsymbol{g}, t')$ implies $\Delta u(\boldsymbol{g}', t) \geq \Delta u(\boldsymbol{g}', t')$.*[14]

Proof. The equivalence of statements (*ii*) and (*iii*) follows from Gurvich and Libkin (1990).

(*i*) ⇒ (*ii*): Suppose to the contrary that (*ii*) fails. Without loss of generality, assume Case 1 in Table 3. Since $\boldsymbol{P}$ is sufficiently fine, there exists a price pair $\boldsymbol{p} \in \boldsymbol{P}$ such that $\max\{\Delta u(\boldsymbol{g}^1, t_2), \Delta u(\boldsymbol{g}^2, t_1)\} < \Delta p < \min\{\Delta u(\boldsymbol{g}^1, t_1), \Delta u(\boldsymbol{g}^2, t_2)\}$. Because $T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\}) = T_b(\boldsymbol{p}, \{\boldsymbol{g}^2\}) = \{t_1\}$ and $T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\}) = T_b(\boldsymbol{p}, \{\boldsymbol{g}^1\}) = \{t_2\}$, the sales-dominance relation is complete at $\boldsymbol{p}$ for neither seller, a contradiction.

(*ii*) ⇒ (*i*): Suppose to the contrary that the sales-dominance relation is incomplete at some price pair $\boldsymbol{p} \in \boldsymbol{P}$ for some seller. Without loss of generality, we assume that it is not complete at $\boldsymbol{p}$ for seller a. Product states $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ in $\boldsymbol{G}$ exist such that neither $\boldsymbol{g}^1 \succsim_{\boldsymbol{p}}^{a} \boldsymbol{g}^2$ nor $\boldsymbol{g}^2 \succsim_{\boldsymbol{p}}^{a} \boldsymbol{g}^1$ holds. Namely, for some consumer types t_1 and t_2,

14 Using the terms in Koessler and Renault (2012), the relative competitiveness Δu of seller a to b can be said statewise monotone with respect to the product state if $\Delta u(\boldsymbol{g}, t) > \Delta u(\boldsymbol{g}', t)$ implies $\Delta u(\boldsymbol{g}, t') \geq \Delta u(\boldsymbol{g}', t')$ for every $\boldsymbol{g} \neq \boldsymbol{g}'$ in $\boldsymbol{G}$ and $t \neq t'$ in T, and statewise monotone with respect to the consumer type if $\Delta u(\boldsymbol{g}, t) > \Delta u(\boldsymbol{g}, t')$ implies $\Delta u(\boldsymbol{g}', t) \geq \Delta u(\boldsymbol{g}', t')$ for every $\boldsymbol{g} \neq \boldsymbol{g}'$ in $\boldsymbol{G}$ and $t \neq t'$ in T. Moreover, the relative competitiveness Δu of seller a to b can be said pairwise monotonic if it is statewise monotone with respect to the product state or with respect to the consumer type for every $\boldsymbol{g} \neq \boldsymbol{g}'$ in $\boldsymbol{G}$ and $t \neq t'$ in T. Therefore, statement (*iii*) can be alternatively expressed that the relative competitiveness of seller a to b is pairwise monotonic.

we have $t_1 \in T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\}) \setminus T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\})$ and $t_2 \in T_a(\boldsymbol{p}, \{\boldsymbol{g}^2\}) \setminus T_a(\boldsymbol{p}, \{\boldsymbol{g}^1\})$. Case 1 of Table 3 holds, a contradiction. □

The assumption that P is sufficiently fine is crucial for Proposition 1. Even without this assumption, statements (*ii*) and (*iii*) are equivalent and they imply statement (*i*). However, they are not necessarily implied by (*i*). Assume that $P = \{0, 3, 6\}$ and that $\Delta u(\boldsymbol{g}^1, t_1) = 2$, $\Delta u(\boldsymbol{g}^2, t_2) = 1$, and $\Delta u(\boldsymbol{g}^1, t_2) = \Delta u(\boldsymbol{g}^2, t_1) \in (0, 1)$ for $\boldsymbol{G} = \{\boldsymbol{g}^1, \boldsymbol{g}^2\}$ and $T = \{t_1, t_2\}$. While no saddle point exists for $\boldsymbol{G}$ and T, the sales-dominance relation is complete at every price pair.

According to Proposition 1, the completeness of the sales-dominance relations depends on how the consumer evaluates one seller relative to the other over the type space. Namely, sufficient disclosure of product information is not necessarily implied by vertical differentiation nor by horizontal differentiation. The valuation of t_2 for product type g_a^2 is denoted by x, i.e. $x = u(g_a^2, t_2)$. Products are vertically differentiated if value x of t_2 for g_a^2 lies in the range (5, 9). If x takes some low value (for example, $x = 3$ or $x = 6$), the sales-dominance relation is complete at every price pair since $\Delta u(\boldsymbol{g}, t_1) \geq \Delta u(\boldsymbol{g}, t_2)$ for every $\boldsymbol{g}$. Otherwise, if x takes some high value (for instance, $x = 8$ or $x = 11$), the sales-dominance relation is incomplete at price pair $\boldsymbol{p}$ with $\Delta p \in (5, 6)$ since $\boldsymbol{g}^1$ and $\boldsymbol{g}^4$ are not well-ranked in terms of sales.

u	t_1	t_2
g_a^1	10	9
g_a^2	7	x
g_b^1	4	5
g_b^2	2	1

Δu	Complete				Incomplete			
	$x = 3$		$x = 6$		$x = 8$		$x = 11$	
	t_1	t_2	t_1	t_2	t_1	t_2	t_1	t_2
$\boldsymbol{g}^1 = (g_a^1, g_b^1)$	6	4	6	4	6	4	6	4
$\boldsymbol{g}^2 = (g_a^1, g_b^2)$	8	8	8	8	8	8	8	8
$\boldsymbol{g}^3 = (g_a^2, g_b^1)$	3	−2	3	1	3	3	3	6
$\boldsymbol{g}^4 = (g_a^2, g_b^2)$	5	2	5	5	5	7	5	10

3.1 Product Information at Insufficient Unraveling Outcome

We explore what product information can be provided if the sales-dominance relations are incomplete and show that the complete sales-dominance relations are necessary for the consumer to be sufficiently informed at every prudent rationalizable outcome.

Recall Example 1. For any seller, no product state is sales-dominant at price pair $\bar{\boldsymbol{p}} = (\bar{p}_a, \bar{p}_b)$ with $\Delta\bar{p} = 4$. We assume for the moment that the sellers are restricted to choose price pair $\bar{\boldsymbol{p}}$ in the price stage. Then, a prudent rationalizable outcome emerges at which the consumer is insufficiently informed. Note that to each seller in every product state, suppressing product information causes a tradeoff between a gain and a loss in the probability of his product being purchased. For instance, seller a in $\boldsymbol{g}^1$ may lose consumer type t_1 by sending message $\boldsymbol{G}$ if she gets to purchase from b under uncertainty about the product state with a belief assigning a sufficiently high probability to $\boldsymbol{g}^2$. However, by sending message $\boldsymbol{G}$, seller a may successfully persuade consumer type t_2 to purchases from a if she gets to form a belief assigning a sufficiently high probability to $\boldsymbol{g}^2$. Roughly, at every information set in the message stage, sending message $\boldsymbol{G}$ can be rational to seller $i \in \{a, b\}$ with an "optimistic" belief that the "gain" outweighs the "loss". For example, seller a in product state $\boldsymbol{g}^1$ could believe with a sufficient high probability that the consumer is of t_2 type and seller b would send message $\boldsymbol{G}$. Further, he could believe that t_2 would purchase from a after receiving messages $\boldsymbol{G}$ from the sellers, while she should purchase from b with knowledge of $\boldsymbol{g}^1$.

Now, we assume that the sellers are free to choose any price in the price stage. Roughly, at every information set in the price stage, price $\bar{p}_i$ can be rational to seller $i \in \{a, b\}$ with a belief that any price higher than $\bar{p}_i$ would result in a substantial loss in the probability of his product being purchased. For example, in product state $\boldsymbol{g}^1$, seller a in the price stage could believe with a sufficient high probability that the consumer is of t_2 type, and seller b would set the price at $\bar{p}_b$ in the price stage and send message $\boldsymbol{G}$ in the message stage whenever price pair $\bar{\boldsymbol{p}}$ is selected in the price stage. Further, he could believe that consumer type t_2 would purchase from a after observing price pair $\bar{\boldsymbol{p}}$ and receiving messages $\boldsymbol{G}$ from both sellers, while she would purchase from b if price pair $(p_a, \bar{p}_b)$ is charged, where $p_a > \bar{p}_a$, with a belief assigning a sufficiently high probability to $\boldsymbol{g}^1$.

Theorem 2 shows that if a price pair $\boldsymbol{p} \in \boldsymbol{P}$ exists at which for every seller no product state is sales-dominant over $\boldsymbol{G}' \subseteq \boldsymbol{G}$, some prudent rationalizable outcome exists at which the consumer faces price pair $\boldsymbol{p}$ and believes that true product state is in $\boldsymbol{G}'$. Since every $\boldsymbol{g} \in \boldsymbol{G}'$ is not sales-dominant at $\boldsymbol{p}$ over $\boldsymbol{G}'$ for seller $i \in \{a, b\}$, $\boldsymbol{g}$ does not sales-dominate some product state $\boldsymbol{g}' \in \boldsymbol{G}'$, implying that $t \in T_j(\boldsymbol{p}, \{\boldsymbol{g}\}) \cap T_i(\boldsymbol{p}, \{\boldsymbol{g}'\})$ for some $t \in T$. Therefore, for every product state $\boldsymbol{g} \in \boldsymbol{G}'$ and seller $i \in \{a, b\}$, $T_j(\boldsymbol{p}, \{\boldsymbol{g}\}) \cap T_i(\boldsymbol{p}, \boldsymbol{G}')$ is non-empty. This observation suggests that given price pair $\boldsymbol{p}$, sending message $\boldsymbol{G}'$ is rational to seller i in product state $\boldsymbol{g}$ with an "optimistic" belief. Note that consumer type t is insufficiently informed because any purchasing decision that could be made along the path of this prudent rationalizable outcome is not ex post optimal at some product state in $\boldsymbol{G}'$.

Theorem 2. *Suppose that there exist a price pair $\boldsymbol{p} \in \boldsymbol{P}$ of positive prices and a non-empty set $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states such that no product state is sales-dominant at $\boldsymbol{p}$ for seller $i \in \{a, b\}$ over $\boldsymbol{G}'$. Then the consumer faces $\boldsymbol{p}$ and believes that true product state is in $\boldsymbol{G}'$ along the path of an insufficiently unraveling prudent rationalizable outcome.*

If the sales-dominance relations are incomplete, some price pair and non-singleton set of product states exist for which no seller has a sales-dominant product state (see Table 3). Corollary 1 is implied by Theorem 2.

Corollary 1. *If product information sufficiently unravels at every prudent rationalizable outcome, then the sales-dominance relation is complete at every price pair.*

4 Non-Comparative Advertisements

Now, we consider the model with non-comparative advertisements. Advertisements could be non-comparative either because sellers have private information about their own products, respectively, or because comparative advertisement is forbidden by law. For direct comparison to the results in the previous section, we focus on the latter case and keep the assumption that all sellers have complete information about each other's product attributes.

The alternative game proceeds analogously to the baseline model. In the information stage, nature chooses the type of each player $k \in \{a, b, c\}$. Each seller observes the realized product state. The consumer privately learns her preference for differentiated products. In the price stage, the sellers simultaneously set binding prices for their individual products. In the message stage, each seller informs the consumer through a costless message about his product but never about his opponent's. In the purchasing stage, the consumer purchases from one and only one seller given the offered prices and the received messages. As for the preference of the consumer, we additionally assume that $u(g, t) \neq u(g', t)$ for every consumer type $t \in T$ and different product types $\{g, g'\} \subseteq G_i$ of seller $i \in \{a, b\}$, implying that between any two products of seller i, the consumer strictly prefers one over the other.

A pure strategy s_i of seller $i \in \{a, b\}$ prescribes what price to charge at every information set in the price stage and what message to send about his realized product type at every information set in the message stage. A message $s_i(h_i)$ chosen at information set h_i in the message stage satisfies $g_i(h_i) \in s_i(h_i) \subseteq G_i$. Namely, the provided message may be vague but must be truthful in that the realized product type must be among the types mentioned in the message. The players' belief systems and prudent rationalizability are defined according to these modified strategies.

Example 3. We show by example that even under the complete sales-dominance relations, product information may unravel insufficiently in the model with non-comparative advertisements. The tables in (a) provide the consumer's type-dependent preference. We name the product states as in the bottom table. The sales-dominance relation is complete at every price pair since the ranking of Δu for product states is identical to all consumer types. For simplicity, attention is restricted to a price pair $\boldsymbol{p} \in \boldsymbol{P}$ with $\Delta p = 0$. In Table (b), the first three rows of each column are matched to an information set of the consumer reached after observing $\boldsymbol{p}$. Let h_i^k denote an information set of seller $i \in \{a, b\}$ in the message stage such that $\boldsymbol{g}\left(h_i^k\right) = \boldsymbol{g}^k$ and $\boldsymbol{p}\left(h_i^k\right) = \boldsymbol{p}$. We claim that for each information set h_i^k, a prudent rationalizable strategy s_i of seller i exists such that $s_i\left(h_i^k\right) = G_i$.

(a) Preference of the consumer

u	t_1	t_2	t_3
g_a^1	5	3	3
g_a^2	3	2	1
g_b^1	1	1	2
g_b^2	4	4	5

Δu	t_1	t_2	t_3
$\boldsymbol{g}^1 = \left(g_a^1, g_b^1\right)$	4	2	1
$\boldsymbol{g}^2 = \left(g_a^1, g_b^2\right)$	1	−1	−2
$\boldsymbol{g}^3 = \left(g_a^2, g_b^1\right)$	2	1	−1
$\boldsymbol{g}^4 = \left(g_a^2, g_b^2\right)$	−1	−2	−4

(b) Prudent rationalizable actions of the consumer when $\Delta p(h_c) = 0$

$t(h_c)$	t_1	t_1	t_2	t_2	t_2	t_3	t_3
$m_a(h_c)$	G_a	G_a	$\left\{g_a^1\right\}$	G_a	G_a	$\left\{g_a^2\right\}$	G_a
$m_b(h_c)$	$\left\{g_b^2\right\}$	G_b	G_b	$\left\{g_b^1\right\}$	G_b	G_b	G_b
	a and *b*	*a* and *b*	*a* and *b*	*a*	*a* and *b*	*b*	*a* and *b*

Firstly, products of seller a are vertically differentiated. However, suppressing information is rationalizable at information set h_a^k, where $k \in \{1, 2\}$. Seller a may believe with a sufficiently high probability that seller b sends message G_b at h_b^k, which makes the consumer believe $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ as possible whatever message sent by a at h_a^k. Further, he may believe with a sufficiently high probability that the consumer is of type t_2 and she purchases from b after receiving messages $\left(\left\{g_a^1\right\}, G_b\right)$, while purchases from a after receiving (G_a, G_b) (see the first and last columns for t_2).

Secondly, each of product states $\boldsymbol{g}^2$ and $\boldsymbol{g}^4$ (associated with g_b^2) strictly sales-dominates both $\boldsymbol{g}^1$ and $\boldsymbol{g}^3$ (associated with g_b^1) at $\boldsymbol{p}$ for seller b. However, for every $k \in \{2, 4\}$, message G_b is rational at information set h_b^k with a belief assigning a sufficiently high probability to consumer type t_1. Further, he believes that seller a sends message G_a at h_a^k, which makes the consumer believe $\boldsymbol{g}^2$ and $\boldsymbol{g}^4$ as possible whatever message sent by b at h_b^k, and thereafter t_1 purchases from seller a when she receives message $\left\{g_b^2\right\}$ from b, while purchases from b when receiving G_b.

Lastly, we argue that message G_b of seller b is rational at information set h_b^k, where $k \in \{1, 3\}$. Seller b may believe with a sufficiently high probability that seller a sends message G_a at h_a^k, the consumer is of type t_2, and she purchases from a for sure after receiving message pair $\left(G_a, \left\{g_b^1\right\}\right)$, while from b after (G_a, G_b) is received (see the last two columns for t_2). Likewise, message G_a of seller a is rational at information set h_a^k, where $k \in \{3, 4\}$ with an analogous belief assigning a sufficiently high probability to consumer type t_3. $\square$

It is not surprising that a stronger condition is required for sufficient disclosure to be unique outcome under non-comparative advertisements as each seller is allowed to inform only about one "side" of product states. For instance, even when a seller is in the sales-dominant product state, he cannot reveal the product type of the opponent seller and the consumer may believe in some other product states that are not sales-dominant. In Example 3, that is the underlying reason why suppressing information is rationalizable to seller a at h_a^1 and to seller b at h_b^4.

Definition 3 . (Absolute sales-dominant product types) For every price pair $\boldsymbol{p} \in \boldsymbol{P}$ and non-empty set $G_i' \times G_j' \subseteq \boldsymbol{G}$ of product states, we say that product type $\bar{g}_i \in G_i'$ of seller $i \in \{a, b\}$ is absolute sales-dominant for i at $\boldsymbol{p}$ over $G_i' \times G_j'$ if every product state in $\{\bar{g}_i\} \times G_j'$ is sales-dominant at $\boldsymbol{p}$ for i over $G_i' \times G_j'$.

By definition, if product type $\bar{g}_i$ of some seller i is absolute sales-dominant at $\boldsymbol{p}$ over $G_a' \times G_b'$, all product states in $\{\bar{g}_i\} \times G_j'$ are sales-equivalent at $\boldsymbol{p}$. If each seller has an absolute sales-dominant product type at $\boldsymbol{p}$ over $G_a' \times G_b'$, it can be easily shown that all product states in $G_a' \times G_b'$ are sales-equivalent at $\boldsymbol{p}$.

In Theorem 3, we provide a necessary and sufficient condition for sufficient unraveling to be the unique prudent rationalizable outcome in the model with non-comparative advertisements. The proof for Theorem 3 proceeds analogous to Theorems 1 and 2. For a simple illustration of the proof for sufficiency, consider an arbitrary price pair. We essentially show that in every odd round of prudent rationalizability, the consumer updates her belief and makes a purchasing decision accordingly and that in every subsequent even round, at least one seller sufficiently discloses his product type in the message stage. The elimination process continues until all "undisclosed" product states are sales-equivalent.

Theorem 3. *Product information sufficiently unravels at every prudent rationalizable outcome of the model with non-comparative advertisements if and only if for every price pair* $\boldsymbol{p} \in \boldsymbol{P}$ *and non-empty set* $G_i' \times G_j' \subseteq \boldsymbol{G}$ *of product states, a product type* $\bar{g}_i \in G_i'$ *of some seller* $i \in \{a, b\}$ *is absolute sales-dominant for i at* $\boldsymbol{p}$ *over* $G_i' \times G_j'$.

With slight modification of the proof for Theorem 3, we can show that the condition described in Theorem 3 is necessary and sufficient for sufficient unraveling even when the sellers have private information about their own products.[15] Suppose that the consumer faces a price pair $\boldsymbol{p}$ and learns that all and only product states in $\boldsymbol{G}' = G_a' \times G_b'$ are possible. By definition, if $\bar{g}_a \in G_a'$ is absolute sales-dominant for seller a at $\boldsymbol{p}$ over $\boldsymbol{G}'$, then product state $(\bar{g}_a, g_b)$ is sales-dominant for a for any $g_b \in G_b'$. Since it is rational for seller a to disclose his type $\bar{g}_a$ regardless of the actual type $g_b \in G_b'$, his rational choice would not vary even if he remains uncertain about b's product type.

Proposition 2. *Suppose that set P of prices is sufficiently fine. The followings are equivalent.*

(i) *For every non-empty sets* $G_a' \times G_b' \subseteq \boldsymbol{G}$ *of product states and price pair* $\boldsymbol{p} \in \boldsymbol{P}$, *a product type* $g_i \in G_i'$ *of some seller* $i \in \{a, b\}$ *is absolute sales-dominant for* $\boldsymbol{p}$ *over* $G_a' \times G_b'$.

(ii) *For every non-singleton sets* $G_a' \subseteq G_a$ *and* $G_b' \subseteq G_b$ *of product types and set* $\{t_1, t_2\} \subseteq T$ *of consumer types, (a) or (b) holds.*

  (a) $\min_{g \in G_a' \times G_b'} \Delta u(\boldsymbol{g}, t_1) \geq \max_{g \in G_a' \times G_b'} \Delta u(\boldsymbol{g}, t_2)$ *or* $\min_{g \in G_a' \times G_b'} \Delta u(\boldsymbol{g}, t_2) \geq \max_{g \in G_a' \times G_b'} \Delta u(\boldsymbol{g}, t_1)$.

  (b) *A product state* $(g_a, g_b) \in G_a' \times G_b'$ *exists such that* $g_i = \arg\max_{g \in G_i'} u(g, t)$ *for every* $t \in \{t_1, t_2\}$ *and seller* $i \in \{a, b\}$, *and* $\Delta u(g_a, g_b, t_1) = \Delta u(g_a, g_b, t_2)$.

Consider arbitrary non-singleton sets $G_a' \subseteq G_a$ and $G_b' \subseteq G_b$ and a consumer type $\bar{t} \in T$. Let $\boldsymbol{G}' = G_a' \times G_b'$. To illustrate an intuition for the equivalence result

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15 The proof of this claim proceeds analogously to the proof of Theorem 3. The primary distinction lies in a seller's inability to specify the exact information set of the other seller. Consider the actual product state (g_a, g_b). With perfect information of this state, seller a at the information set reached after the selection of price pair $\boldsymbol{p}$ knows that b is at information set h_b satisfying $\boldsymbol{p}(h_b) = \boldsymbol{p}$ and $g_b(h_b) = g_b$. With only private information about the product type, seller a knows that b is at one of several information sets h_b satisfying $\boldsymbol{p}(h_b) = \boldsymbol{p}$.

in Proposition 2, we partition T into three groups: $T^+ = \left\{t \in T\colon \min_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t) \geq \max_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, \bar{t})\right\}$, $T^- = \left\{t \in T\colon \max_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t) \leq \min_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, \bar{t})\right\}$, and $\bar{T} = T \backslash (T^+ \cup T^-)$. By construction, $\bar{t} \in \bar{T}$.

Firstly, condition (*b*) of (*ii*) requires that an identical product type (say $\bar{g}_i$) in G_i' of seller $i \in \{a, b\}$ be the most preferred to all consumer types in $\bar{T}$. Moreover, in product state $(\bar{g}_a, \bar{g}_b)$, the relative competitiveness Δu takes the same value (say, $\bar{\Delta}$) to all consumer types in $\bar{T}$ (i.e. $\Delta u\big((\bar{g}_a, \bar{g}_b), t\big) = \bar{\Delta}$ for every $t \in \bar{T}$). Statement (*ii*) further implies that the ranges of the values for the relative competitiveness are "separated" between any two groups. Namely, $\max_{\boldsymbol{G}'\times T^-} \Delta u(\boldsymbol{g}, t) \leq \min_{\boldsymbol{G}'\times \bar{T}} \Delta u(\boldsymbol{g}, t)$ and $\max_{\boldsymbol{G}'\times \bar{T}} \Delta u(\boldsymbol{g}, t) \leq \min_{\boldsymbol{G}'\times T^+} \Delta u(\boldsymbol{g}, t)$. Otherwise, if there exist consumer types $t' \neq \bar{t}$ in $\bar{T}$ and $t^+ \in T^+$ such that $\min_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t') < \max_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, \bar{t}) \leq \min_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t^+) < \max_{\boldsymbol{g}\in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t')$, $\Delta u\big((\bar{g}_a, \bar{g}_b), t\big) = \bar{\Delta}$ for every $t \in \{\bar{t}, t^+, t'\}$ due to condition (*b*), a contradiction.

Consider a price pair $\boldsymbol{p}$ such that Δp lies between the ranges of Δu for T^- and $\bar{T}$ or T^+ and $\bar{T}$. For instance, let Δp be such that $\max_{\boldsymbol{G}'\times T^-} \Delta u(\boldsymbol{g}, t) < \Delta p < \min_{\boldsymbol{G}'\times \bar{T}} \Delta u(\boldsymbol{g}, t)$. All consumer types in T^+ and $\bar{T}$ prefer a regardless of the product state in $\boldsymbol{G}'$ and all consumer types in T^- prefer b regardless of the product state in $\boldsymbol{G}'$. Since all product states in $\boldsymbol{G}'$ are sales-equivalent at $\boldsymbol{p}$, all product types of each seller are absolute sales-dominant at p over $\boldsymbol{G}'$. Now consider a price pair $\boldsymbol{p}$ such that Δp lies in the range of Δu for $\bar{T}$, i.e. $\min_{\boldsymbol{G}'\times \bar{T}} \Delta u(\boldsymbol{g}, t) < \Delta p < \max_{\boldsymbol{G}'\times \bar{T}} \Delta u(\boldsymbol{g}, t)$. Further, if $\Delta p < \bar{\Delta}$, $\bar{g}_a$ is absolute sales-dominant for a at $\boldsymbol{p}$ over $\boldsymbol{G}'$ since $\Delta u\big((\bar{g}_a, g_b), t\big) > \bar{\Delta}$ for every $t \in \bar{T}$ and $g_b \in G_b'$. Likewise, if $\Delta p > \bar{\Delta}$, $\bar{g}_b$ is absolute sales-dominant for b at $\boldsymbol{p}$ over $\boldsymbol{G}'$.

As is for Proposition 1, the assumption that P is sufficiently fine is crucial for Proposition 2. Even without this assumption, statement (*ii*) in Proposition 2 implies statement (*i*). However, it is not necessarily implied by (*i*).

It is noteworthy that the completeness of the sales-dominance relations is implied by the necessary and sufficient condition for unraveling under non-comparative advertisements, but not vice versa as seen in Example 3. Let $\boldsymbol{g}^1 = \left(g_a^1, g_b^1\right)$ and $\boldsymbol{g}^2 = \left(g_a^2, g_b^2\right)$ be different product states. If g_i^1 of some seller i is absolute sales-dominant at $\boldsymbol{p}$ over $\left\{g_a^1, g_a^2\right\} \times \left\{g_b^1, g_b^2\right\}$, $\boldsymbol{g}^1$ and $\left(g_i^1, g_j^2\right)$ are sales-dominant at $\boldsymbol{p}$ for i over $\left\{g_a^1, g_a^2\right\} \times \left\{g_b^1, g_b^2\right\}$. Otherwise, if g_i^2 of some seller i is absolute sales-dominant at $\boldsymbol{p}$ over $\left\{g_a^1, g_a^2\right\} \times \left\{g_b^1, g_b^2\right\}$, $\boldsymbol{g}^2$ sales-dominates $\boldsymbol{g}^1$ at $\boldsymbol{p}$ for i.[16]

16 In Appendix A.4, we show that the sales-dominance relation is complete at every price pair if statement (*ii*) of Proposition 2 is satisfied (see Lemma A.1).

5 Discussions

5.1 Effects of Competition on Information Unraveling

Board (2009), Hotz and Xiao (2013), and Cheong and Kim (2004) report that competition can undermine the unraveling result in a competitive environment. In their models, the sellers set prices after they provide product information. Roughly, full disclosure fails because it triggers intense price competition. It may be interesting to study whether competition enhances information unraveling in the alternative price-then-disclosure timing. We now show, by example, that competition is not the main driving force for sufficient disclosure under comparative advertisements. However, it may enhance information disclosure if the consumer lacks sophisticated reasoning capability.

To see the effect of competition on information unraveling, we consider an example in which seller a is able to provide information about product states but seller b is unable to provide any information. Alternatively, we may assume that the only available message to seller b is $\boldsymbol{G}$. Hence, they do not compete through information disclosure. $G_i = \left\{g_i^1, g_i^2\right\}$ for every $i \in \{a, b\}$ and $T = \{t_1, t_2\}$. The consumer's preference is given below. At every price pair, the sales-dominance relation is complete. For simplicity, consider a price pair $\boldsymbol{p} \in \boldsymbol{P}$ such that $\Delta p = 4$. For seller a, $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ are sales-dominant at $\boldsymbol{p}$, while $\boldsymbol{g}^3$ is sales-dominated by all others. Let h_i^k be the information set of seller i in the message stage such that $\boldsymbol{g}\left(h_i^k\right) = \boldsymbol{g}^k$ and $\boldsymbol{p}\left(h_i^k\right) = \boldsymbol{p}$.

$\Delta u(\boldsymbol{g}, t)$	$\boldsymbol{g}^1 = (g_a^1, g_b^1)$	$\boldsymbol{g}^2 = (g_a^1, g_b^2)$	$\boldsymbol{g}^3 = (g_a^2, g_b^1)$	$\boldsymbol{g}^4 = (g_a^2, g_b^2)$
t_1	5	8	3	6
t_2	5	7	1	3

Every second round prudent rationalizable strategy of seller a prescribes to sufficiently reveal at h_a^1 and h_a^2 (i.e. message $\{\boldsymbol{g}^1\}$ or $\{\boldsymbol{g}^1, \boldsymbol{g}^2\}$ at h_a^1, and $\{\boldsymbol{g}^2\}$ or $\{\boldsymbol{g}^1, \boldsymbol{g}^2\}$ at h_a^2). In the third round, the consumer believes neither $\boldsymbol{g}^1$ nor $\boldsymbol{g}^2$ as possible if she receives from a any message that includes $\boldsymbol{g}^3$ or $\boldsymbol{g}^4$ (for instance, message $\boldsymbol{G}$). Every forth round prudent rationalizable strategy of a prescribes not to include $\boldsymbol{g}^3$ in the message sent at h_a^4. Finally, in the fifth round, the consumer precisely learns product state $\boldsymbol{g}^3$ if she receives a message including $\boldsymbol{g}^3$. The elimination procedure ends in the fifth round and the consumer is sufficiently informed at every prudent rationalizable outcome even though seller b does not provide any product information. Note that this unraveling result still holds at every other price pair. In this example, competition neither enhances nor impedes information unraveling.

Prudent rationalizable entails forward induction. Namely, the consumer asks herself why a seller provides this or that information. For instance, in the third round, the consumer receiving message $\boldsymbol{G}$ from seller a deduces that neither $\boldsymbol{g}^1$ nor $\boldsymbol{g}^2$ is real since message $\boldsymbol{G}$ should not be sent at h_a^1 and h_a^2 as long as seller a is rational.

Rather surprisingly, competition may enhance information unraveling if the consumer has limited reasoning capability. Suppose that the consumer believes that the sellers believe in the consumer's rationality but she cannot form any higher-order belief of rationality. Then, the consumer believes in the second-round prudent rationalizable strategies of the sellers and the elimination process stops in the third round. Therefore, the consumer remains insufficiently informed at some prudent rationalizable outcomes. For instance, she remains uncertain over $\boldsymbol{g}^3$ and $\boldsymbol{g}^4$ when she receives message $\boldsymbol{G}$ from seller a.

Now, we assume that both sellers provide product information, while the consumer still has limited reasoning capability. Then, every second round prudent rationalizable strategy of seller a prescribes to sufficiently disclose at h_a^1 and h_a^2 and every second round prudent rationalizable strategy of b prescribes to fully disclose at h_b^3. In the third round, the consumer learns product state $\boldsymbol{g}^4$ if she receives any messages including $\boldsymbol{g}^4$ from both sellers. Recall that prudent rationalizability entails forward induction. With competition, the consumer deduces about the true product state not only from the message sent by seller a but also the message sent by b. That is, the consumer further learns that the true state is not $\boldsymbol{g}^3$ when receiving messages $\boldsymbol{G}$ from both sellers. Moreover, she knows that every message sent at h_i^4 of seller $i \in \{a, b\}$ must be truthful. The elimination procedure ends in the third round and the consumer is sufficiently informed at every prudent rationalizable outcome.

Remark. With a slight modification, the complete sales-dominance relations work as a necessary and sufficient condition for the consumer to be sufficiently informed at every prudent rationalizable outcome in the model with a single seller. Suppose that only seller a serves for the consumer and provides product information. Assume that v is not so large that there exists a sufficiently high price $p \in P$ such that $p > u(g, t)$ for every $(g, t) \in \left(G_a \cup G_b\right) \times T$. Then, if the sellers price themselves out of the market, the consumer would decide not to purchase. In other words, the purchase is no longer mandatory. The consumer's choice of b would be interpreted as the best out-of-market option available to the consumer, which is assumed to give the payoff of zero. For every price $p_a \in P$ and product type $g_a \in G_a$ of seller a, sales from a is given by $T_a(p_a, \{g_a\}) = \{t \in T\colon u(g_a, t) > p_a\}$. By defining the sales-dominance relation over seller a's product types, we can conclude that the sales-dominance relation for seller a is complete at every price $p_a \in P$ if and only if the consumer is sufficiently informed of seller a's product at every prudent rationalizable outcome. Indeed, Woo (2023) shows that in a model with a single seller, pairwise monotonicity, a necessary and sufficient condition provided in Koessler and Renault (2012) for full-disclosure to be the unique outcome regardless of the prior, is equivalent to the completeness of the sales-dominance relations at all prices.

5.2 When Purchase is Not Mandatory

We have considered the case in which the product of interest is a must-have item. For instance, car insurance is mandatory to every vehicle owner in some countries, and some professors require students to purchase the textbook. We show, by example, that sufficient unraveling may fail even under the complete sales-dominance relations if purchase is not mandatory. The extension of our results to the case when purchase is not mandatory remains as a future study.

We redefine strategies of the consumer. A pure strategy s_c of the consumer specifies for every information set in the purchasing stage whether to purchase or from whom to purchase, that is, $s_c\colon H_c \to \{a, b, 0\}$, where the consumer's action 0 indicates that she decides not to purchase. Belief systems and prudent rationalizability are redefined with the modified strategies.

Let $G_a = \left\{g_a^1, g_a^2\right\}$, $G_b = \left\{g_b^1, g_b^2\right\}$, and $T = \{t_1, t_2, t_3\}$. Consider the preference of the consumer given in Table (a) below. We assume that the purchase is not mandatory and every consumer type receives utility zero if she does not purchase. For every product state $\boldsymbol{g} \in \boldsymbol{G}$, Δu gets smaller as the index of the consumer gets larger. Hence, the sales-dominance relation is complete at every price pair. Let $\bar{\boldsymbol{p}} = (\bar{p}_a, \bar{p}_b) = (12, 11)$. At some prudent rationalizable outcome, the consumer faces price pair $\bar{\boldsymbol{p}}$ and believes all product states in $\boldsymbol{G}$. For lack of space, we skip the arguments for why price $\bar{p}_i$ of seller $i \in \{a, b\}$ is prudent rationalizable at every information set in the price stage. Roughly, this price is rational to seller i in the price stage if he believes with a sufficiently high probability that any price higher than $\bar{p}_i$ results in a substantial loss in the probability of his product being purchased.

(a) Preference of the consumer

$u(g, t)$	t_1	t_2	t_3
g_a^1	10	13	2
g_a^2	16	10	4
g_b^1	2	8	16
g_b^2	1	15	10

$\Delta u(\boldsymbol{g}, t)$	t_1	t_2	t_3
$\boldsymbol{g}^1 = (g_a^1, g_b^1)$	8	5	−14
$\boldsymbol{g}^2 = (g_a^1, g_b^2)$	9	−2	−8
$\boldsymbol{g}^3 = (g_a^2, g_b^1)$	14	2	−12
$\boldsymbol{g}^4 = (g_a^2, g_b^2)$	15	−5	−6

(b) Prudent rationalizable actions of the consumer with $\Delta p = 1$

	t_1	t_2	t_3
$\boldsymbol{g}^1 = (g_a^1, g_b^1)$	0	*a*	*b*
$\boldsymbol{g}^2 = (g_a^1, g_b^2)$	0	*b*	0
$\boldsymbol{g}^3 = (g_a^2, g_b^1)$	*a*	0	*b*
$\boldsymbol{g}^4 = (g_a^2, g_b^2)$	*a*	*b*	0
$\boldsymbol{G}$	*a* and 0	*a*, *b* and 0	*b* and 0

In Table (b), we provide a prudent rationalizable action of the consumer at some information sets reached after price pair $\bar{\boldsymbol{p}}$ is offered. The first four rows show the consumer's prudent rationalizable actions when she is fully informed of the realized product state, and the last row is for the case when she receives messages $\boldsymbol{G}$ from the sellers. For every $\boldsymbol{g} \in \boldsymbol{G}$ and $i \in \{a, b\}$, a consumer type exists who should not purchase from seller i at $\bar{\boldsymbol{p}}$ with complete information of product state $\boldsymbol{g}$, while she may purchase from i with a full-support belief over $\boldsymbol{G}$. Each seller could optimistically believe that the probability of his product being purchased would rise by providing trivial information $\boldsymbol{G}$ at every information set in the message stage reached after $\bar{\boldsymbol{p}}$ is chosen. For instance, message $\boldsymbol{G}$ is rational to seller a in product state $\boldsymbol{g}^1$ with a belief assigning a sufficient high probability to the event that seller b sends message $\boldsymbol{G}$, the consumer is of t_1 type and she purchases from a at $\bar{\boldsymbol{p}}$ only when she receives messages $\boldsymbol{G}$ from both sellers. Analogously, message $\boldsymbol{G}$ is rational to seller b in $\boldsymbol{g}^1$ with a belief assigning a sufficient high probability to the event that the consumer is of t_2 type and she purchases from b at $\bar{\boldsymbol{p}}$ only when she receives messages $\boldsymbol{G}$ from both sellers.

6 Conclusions

We study the disclosure of verifiable information in a competitive environment. The consumer is uncertain about the sellers' product characteristics, but she has complete and private information about her preference. Two sellers provide verifiable information about their products to attract the consumer. In particular, they compete by providing product information only after they set prices for their individual product, which is the main feature that distinguishes our model from previous works.

For each advertisements features, we provide the necessary and sufficient condition for every prudent rationalizable outcome to achieve sufficient unraveling of product information. Under comparative advertisements, the necessary and sufficient condition is satisfied whenever the sales-dominance relations is complete at every price pair. Under non-comparative advertisements, a stronger condition is required. Based on these conditions, we conclude that how the consumer evaluates one seller relative to the other over the product states matters to obtain sufficient unraveling at every prudent rationalizable outcome. In contrast to the results of some previous literature, we find that competition does not impede information disclosure in the price-then-disclosure timing. Moreover, if the consumer has limited reasoning capability, competition helps to overcome asymmetry in product information.

We conclude by discussing some advantages of the solution concept, prudent rationalizability over an equilibrium concept. Firstly, prudent rationalizability features the cautious behaviors of the players and embodies forward induction. Namely, the consumer would ask herself why the sellers provide this or that product information by cautiously reasoning about the rationality of the sellers and their cautious reasoning about her cautious reasoning, etc. This interactive reasoning introduces some degree of "skepticism" akin to Milgrom and Roberts (1986) since the consumer's cautious beliefs put some weight on unfavorable (from the perspective of the sellers) product states. Secondly, prudent rationalizability is an extensive-form of iterated admissibility. Hence, we can figure out how the rationality of the players applies in every round of elimination, which is useful to provide an iterative unraveling argument that extends the standard unraveling argument in a competitive environment. Prudent rationalizability measures the reasoning capabilities by the numbers of rounds. Thus, it yields a prediction for every level-k of reasoning, which is useful to show that with comparative advertisements, competition can enhance information unraveling if the consumer has limited reasoning capability. Lastly, prudent rationalizability is a prior-free solution concept. We do not need any auxiliary assumptions on probability distributions like common prior and independent types, thus adding robustness to the results.

Appendix A: Proofs

A.1 Proof of Theorem 1

We prove by induction. Suppose that the sales-dominance relation is complete at every price pair. For every price pair $\boldsymbol{p} \in \boldsymbol{P}$ and seller $i \in \{a, b\}$, let $\boldsymbol{G}_{\boldsymbol{p}}^{0,i} = \emptyset$ and $\boldsymbol{\Gamma}_{\boldsymbol{p}}^{0,i} = \boldsymbol{G}$. We define inductively for $r \geq 1$, $\boldsymbol{\Gamma}_{\boldsymbol{p}}^{r,i} = \boldsymbol{\Gamma}_{\boldsymbol{p}}^{r-1,i} \backslash \boldsymbol{G}_{\boldsymbol{p}}^{r-1,i}$ and

$$G_{\boldsymbol{p}}^{r,i} = \left\{ \boldsymbol{g} \in \boldsymbol{\Gamma}_{\boldsymbol{p}}^{r,i} : \boldsymbol{g} \text{ is sales} - \text{dominant at } \boldsymbol{p} \text{ for seller } i \text{ over } \boldsymbol{\Gamma}_{\boldsymbol{p}}^{r,i} \right\}.$$

That is, $\boldsymbol{\Gamma}_{\boldsymbol{p}}^{1,i} = \boldsymbol{G}$ and $\boldsymbol{G}_{\boldsymbol{p}}^{1,i}$ is the set of all sales-dominant product states at $\boldsymbol{p}$ for seller i over $\boldsymbol{G}$. Inductively, for $r > 1$, $\boldsymbol{\Gamma}_{\boldsymbol{p}}^{r,i}$ includes all product states that are weakly sales-dominated by the rth most sales-dominant product states for i at $\boldsymbol{p}$, and $\boldsymbol{G}_{\boldsymbol{p}}^{r,i}$ is the set of all rth most sales-dominant product states at $\boldsymbol{p}$ for i. For every $r \geq 1$, $\boldsymbol{G}_{\boldsymbol{p}}^{r,i}$ is non-empty as long as $\boldsymbol{\Gamma}_{\boldsymbol{p}}^{r,i}$ is non-empty. Since $\boldsymbol{G}$ is finite, all product states in $\boldsymbol{\Gamma}_{\boldsymbol{p}}^{\bar{r},i}$ are sales-equivalent at $\boldsymbol{p}$ and $\boldsymbol{G}_{\boldsymbol{p}}^{\bar{r},i} = \bar{\Gamma}_{\boldsymbol{p}}^{\bar{r},i}$ for some finite number $\bar{r} \geq 1$. For $r > 1$, we say that information set $h_k \in H_k$ of player $k \in \{a, b, c\}$ is reachable in the rth round if some move of nature and $(r-1)$th-round prudent rationalizable strategies of the other players reach h_k.

First round: For every belief system and information set $h_i \in H_i$ of seller $i \in \{a, b\}$, he forms a full-support belief about all consumer types and strategies of the other players reaching h_i. With any belief system, the price of zero cannot be not rational at every information set in the price stage since seller i believes with a positive probability that the consumer purchases from his product at whatever positive price he charges. A strategy $s_i \in S_i$ of seller i that maps every information set in the price stage to a positive price is rational with a belief system $b_i \in B_i^1$ such that for every information set $h_i \in H_i$, the full-support belief $b_i(h_i)$ assigns a sufficiently high probability to strategies of the consumer that prescribes to purchases from seller i at all information sets reached by $s_i(h_i)$, while she chooses the opponent seller at all other information sets.

At every information set $h_c \in H_c$ of the consumer, she believes all and only product states in $m_i(h_c) \cap m_j(h_c)$. If $t(h_c) \in T_i\left(\boldsymbol{p}(h_c), m_i(h_c) \cap m_j(h_c)\right)$ for some seller $i \in \{a, b\}$, purchasing from i is first-round prudent rationalizable at h_c with a belief system b_c such that belief $b_c(h_c)$ at h_c assigns a sufficient high probability to a product state $\boldsymbol{g} \in m_i(h_c) \cap m_j(h_c)$ with $t(h_c) \in T_i(\boldsymbol{p}(h_c), \{\boldsymbol{g}\})$. In particular, every first-round prudent rationalizable strategy of the consumer prescribes to purchase from the seller whose product is the most preferred at price pair $\boldsymbol{p}\left(h_c'\right)$ in a product state in $m_a\left(h_c'\right) \cap m_b\left(h_c'\right)$ if h_c' is such that $m_a\left(h_c'\right) \subseteq \boldsymbol{G}_{\boldsymbol{p}(h_c')}^{1,a}$ or $m_b\left(h_c'\right) \subseteq \boldsymbol{G}_{\boldsymbol{p}(h_c')}^{1,b}$

Second round: For every belief system and information set $h_i' \in H_i$ of seller $i \in \{a, b\}$ that is reachable in the second round, he forms a full-support belief over all consumer types and first-round prudent rationalizable strategies of the other players reaching h_i'. We claim that if strategy s_i of seller $i \in \{a, b\}$ is second-round prudent rationalizable, $s_i(h_i) \subseteq \boldsymbol{G}_{\boldsymbol{p}(h_i)}^{1,i}$ at every information set h_i in the message stage reachable in the second round such that $\boldsymbol{g}(h_i) \in \boldsymbol{G}_{\boldsymbol{p}(h_i)}^{1,i}$ and $\boldsymbol{g}(h_i) \notin \boldsymbol{G}_{\boldsymbol{p}(h_i)}^{1,j}$. Suppose to the contrary that a second-round prudent rationalizable strategy s_i exists such that $s_i(h_i) \nsubseteq \boldsymbol{G}_{\boldsymbol{p}(h_i)}^{1,i}$ at some information set h_i in the message stage reachable in the second round satisfying $\boldsymbol{g}(h_i) \in \boldsymbol{G}^{1,i}_{\boldsymbol{p}(h_i)}$ and $\boldsymbol{g}(h_i) \notin \boldsymbol{G}^{1,j}_{\boldsymbol{p}(h_i)}$. With every belief system $b_i \in B^2_i$, belief $b_i(h_i)$ at h_i must assign a positive probability to the first-round prudent rationalizable strategies of seller j that prescribe to send a message m_j such that $\boldsymbol{g}(h_i) \succ^i_{\boldsymbol{p}(h_i)} \boldsymbol{g}$ for some $\boldsymbol{g} \in s_i(h_i) \cap m_j$ at information set h_j with $\boldsymbol{g}(h_j) = \boldsymbol{g}(h_i)$ and $\boldsymbol{p}(h_j) = \boldsymbol{p}(h_i)$. For simplicity in notation, let $\boldsymbol{G}' = s_i(h_i) \cap m_j$. $T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \cap T_j(\boldsymbol{p}(h_i), \boldsymbol{G}')$ is trivially non-empty. We can partition T into $\{T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \cap T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), T_j(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_i(\boldsymbol{p}(h_i), \boldsymbol{G}')\}$. Belief $b_i(h_i)$ at h_i must assign a positive probability to first-round prudent rationalizable strategies of the consumer such that

$$\text{if } t \in \begin{cases} T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), & \text{the consumer purchases from seller } i \\ T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \cap T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), & \text{the consumer purchases from seller } j \\ T_j(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_i(\boldsymbol{p}(h_i), \boldsymbol{G}'), & \text{the consumer from purchases seller } j. \end{cases}$$

Yet, given belief $b_i(h_i)$ at h_i, seller i can be strictly better off by replacing $s_i(h_i)$ with message $\{\boldsymbol{g}(h_i)\}$, because $T_i(\boldsymbol{p}(h_i), \{\boldsymbol{g}(h_i)\}) = T_i(\boldsymbol{p}(h_i), \boldsymbol{G}')$ according to Lemma 1 and every first-round prudent rationalizable strategies of the consumer prescribes consumer type $t \in T_i(\boldsymbol{p}(h_i), \boldsymbol{G}')$ to purchase from i at the information set reached by the modified strategy, a contradiction.

For the consumer, $S^2_c = S^1_c$ since the set of product states believed at every information set reachable in the second round is the same as the one in the first round.

Induction hypothesis: For $r \geq 1$, we say that strategy $s_c \in S_c$ of the consumer satisfies condition r if at every information set $h_c \in H_c$ reachable in the $(2r-1)$th round such that $s_c(h_c) = i$, we have either

(c1) $t(h_c) \in T_i\left(\boldsymbol{p}(h_c), \left[m_i(h_c) \cap \boldsymbol{\Gamma}^{r,i}_{\boldsymbol{p}(h_c)}\right] \cap \left[m_j(h_c)) \cap \boldsymbol{\Gamma}^{r,j}_{\boldsymbol{p}(h_c)}\right]\right)$ or

(c2) $t(h_c) \in T_i\left(\boldsymbol{p}(h_c), \boldsymbol{G}^{x,i}_{\boldsymbol{p}(h_c)}\right)$ for some $1 \leq x \leq r$ satisfying $m_i(h_c) \cap \boldsymbol{\Gamma}^{x,i}_{\boldsymbol{p}(h_c)} \subseteq \boldsymbol{G}^{x,i}_{\boldsymbol{p}(h_c)}$ and $m_j(h_c) \cap \boldsymbol{\Gamma}^{x+1,j}_{\boldsymbol{p}(h_c)} \neq \emptyset$.

For $r \geq 1$, we say that strategy $s_i \in S_i$ of seller $i \in \{a, b\}$ satisfies condition r if $s_i(h_i) \cap \left[\boldsymbol{\Gamma}^{y,i}_{\boldsymbol{p}(h_i)} \cap \boldsymbol{\Gamma}^{y,j}_{\boldsymbol{p}(h_i)}\right] \subseteq \boldsymbol{G}^{y,i}_{\boldsymbol{p}(h_i)}$ at every information set $h_i \in H_i$ of seller i in the message stage reachable in the $2r$th round such that

(s) for some y with $1 \leq y \leq r$, $\boldsymbol{g}(h_i) \in \boldsymbol{G}^{y,i}_{\boldsymbol{p}(h_i)}$ and $\boldsymbol{g}(h_i) \in \boldsymbol{\Gamma}^{y+1,j}_{\boldsymbol{p}(h_i)}$.

Assume now that we have proved that $(2r-1)$th-round prudent rationalizable strategies of the consumer satisfy condition r and that $2r$th round prudent rationalizable strategies of seller $i \in \{a, b\}$ satisfy condition r. We claim that $(2r+1)$th-round prudent rationalizable strategies of the consumer satisfy condition $r+1$ and that $2(r+1)$th round prudent rationalizable strategies of seller $i \in \{a, b\}$ satisfy condition $r+1$.

$(2r+1)$th round: Consider information set $h_c \in H_c$ of the consumer reachable in the $(2r+1)$th round for which (c1) or (c2) of condition $(r+1)$ is satisfied. It is sufficient to show that the consumer believes all and only product states in $\left[m_i(h_c) \cap \boldsymbol{\Gamma}^{r+1,i}_{\boldsymbol{p}(h_c)}\right] \cap \left[m_j(h_c) \cap \boldsymbol{\Gamma}^{r+1,j}_{\boldsymbol{p}(h_c)}\right]$ at information set h_c for which this intersection is non-empty. With every belief system $b_c \in B_c^{2r+1}$ of the consumer, belief $b_c(h_c)$ at h_c assigns a positive probability to all product states and $2r$th-round prudent rationalizable strategies of the sellers reaching h_c. Since $2r$th-round prudent rationalizable strategies of seller $k \in \{a, b\}$ satisfy condition r by induction hypothesis, belief $b_c(h_c)$ at h_c must not assign a positive probability to product states in $m_k(h_c) \cap \boldsymbol{G}^{r',k}_{\boldsymbol{p}(h_c)}$, where $r' \leq r$. That is, belief $b_c(h_c)$ at h_c must assign a positive probability to only product states in $m_k(h_c) \cap \boldsymbol{\Gamma}^{r+1,k}_{\boldsymbol{p}(h_c)}$.

$2(r+1)$th round: Consider an information set h_i of seller $i \in \{a, b\}$ in the message stage reachable in the $2(r+1)$th round for which condition $(r+1)$ is satisfied. It is sufficient to show that with every $2(r+1)$th-round prudent rationalizable strategy, seller i sends a message $s_i(h_i)$ satisfying $s_i(h_i) \cap \left[\boldsymbol{\Gamma}^{r+1,i}_{\boldsymbol{p}(h_i)} \cap \boldsymbol{\Gamma}^{r+1,j}_{\boldsymbol{p}(h_i)}\right] \subseteq \boldsymbol{G}^{r+1,i}_{\boldsymbol{p}(h_i)}$ if h_i is such that $\boldsymbol{g}(h_i) \in \boldsymbol{G}^{r+1,i}_{\boldsymbol{p}(h_i)}$ and $\boldsymbol{g}(h_i) \in \boldsymbol{\Gamma}^{r+2,j}_{\boldsymbol{p}(h_i)}$. Suppose to the contrary that a $2(r+1)$th-round prudent rationalizable strategy s_i exists such that $s_i(h_i) \cap \left[\boldsymbol{\Gamma}^{r+1,i}_{\boldsymbol{p}(h_i)} \cap \boldsymbol{\Gamma}^{r+1,j}_{\boldsymbol{p}(h_i)}\right] \nsubseteq \boldsymbol{G}^{r+1,i}_{\boldsymbol{p}(h_i)}$ at some information set h_i in the message stage reachable in $2(r+1)$th round such that $\boldsymbol{g}(h_i) \in \boldsymbol{G}^{r+1,i}_{\boldsymbol{p}(h_i)}$ and $\boldsymbol{g}(h_i) \in \boldsymbol{\Gamma}^{r+2,j}_{\boldsymbol{p}(h_i)}$. With every belief system $b_i \in B_i^{2(r+1)}$, belief $b_i(h_i)$ at h_i must assign a positive probability to $(2r+1)$th-round prudent rationalizable strategies of seller j that prescribe to send a message m_j such that $\boldsymbol{g}(h_i) \succ^i_{\boldsymbol{p}(h_i)} \boldsymbol{g}$ for some $\boldsymbol{g} \in \left[s_i(h_i) \cap \boldsymbol{\Gamma}^{r+1,i}_{\boldsymbol{p}(h_i)}\right] \cap \left[m_j \cap \boldsymbol{\Gamma}^{r+1,j}_{\boldsymbol{p}(h_i)}\right]$ at information set h_j with $\boldsymbol{g}(h_j) = \boldsymbol{g}(h_i)$ and $\boldsymbol{p}(h_j) = \boldsymbol{p}(h_i)$. For simplicity in notation, let $\boldsymbol{G}' = \left[s_i(h_i) \cap \boldsymbol{\Gamma}^{r+1,i}_{\boldsymbol{p}(h_i)}\right] \cap \left[m_j \cap \boldsymbol{\Gamma}^{r+1,j}_{\boldsymbol{p}(h_i)}\right]$. $T_a(\boldsymbol{p}(h_i), \boldsymbol{G}') \cap T_b(\boldsymbol{p}(h_i), \boldsymbol{G}')$ is trivially non-empty. We can partition T into $\{T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \cap T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), T_j(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_i(\boldsymbol{p}(h_i), \boldsymbol{G}')\}$. Belief $b_i(h_i)$ at h_i must assign a positive probability to $(2r+1)$th-round prudent rationalizable strategies of the consumer such that

$$\text{if } t \in \begin{cases} T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), & \text{the consumer purchases from seller } i \\ T_i(\boldsymbol{p}(h_i), \boldsymbol{G}') \cap T_j(\boldsymbol{p}(h_i), \boldsymbol{G}'), & \text{the consumer purchases from seller } j \\ T_j(\boldsymbol{p}(h_i), \boldsymbol{G}') \backslash T_i(\boldsymbol{p}(h_i), \boldsymbol{G}'), & \text{the consumer purchases from seller } j. \end{cases}$$

Yet, given belief $b_i(h_i)$ at h_i, seller i can be strictly better off by replacing $s_i(h_i)$ with message $\{\boldsymbol{g}(h_i)\}$, because $T_i(\boldsymbol{p}(h_i), \{\boldsymbol{g}(h_i)\}) = T_i(\boldsymbol{p}(h_i), \boldsymbol{G}')$ according to

Lemma 1 and every $(2r+1)$th-round prudent rationalizable strategy of the consumer prescribes consumer type $t \in T_i(\boldsymbol{p}(h_i), \boldsymbol{G}')$ to purchase from i at information set reached by the modified strategy, a contradiction.

At every prudent rationalizable price pair $\boldsymbol{p} \in \boldsymbol{P}$, $\boldsymbol{G}$ is partitioned into finite subsets of $\boldsymbol{G}$. After some finite rounds of prudent rationalizability, no more strategy of the sellers is eliminated and the consumer is sufficiently informed at every prudent rationalizable outcome.

A.2 Proof of Theorem 2

Consider a non-empty sets $\boldsymbol{G}' \subseteq \boldsymbol{G}$ of product states and a pair $\bar{\boldsymbol{p}} = (\bar{p}_a, \bar{p}_b) \in \boldsymbol{P}$ of positive prices for which no seller has a sales-dominant product at $\bar{\boldsymbol{p}}$ over $\boldsymbol{G}'$. In every product state $\boldsymbol{g} \in \boldsymbol{G}'$, seller $i \in \{a, b\}$ has a consumer type $t \in T$ such that $t \in T_j(\bar{\boldsymbol{p}}, \{\boldsymbol{g}\}) \cap T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$. Otherwise, if some product state $\boldsymbol{g}$ and seller i exist such that $t \notin T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$ for every consumer type $t \in T_j(\bar{\boldsymbol{p}}, \{\boldsymbol{g}\})$, $\boldsymbol{g}$ is sales-dominant at $\bar{\boldsymbol{p}}$ over $\boldsymbol{G}'$ for i. For every $i \in \{a, b\}$ and $\boldsymbol{g} \in \boldsymbol{G}'$, let $x_i^{\boldsymbol{g}}$ be the information set of seller i in the price stage such that $\boldsymbol{g}\left(x_i^{\boldsymbol{g}}\right) = \boldsymbol{g}$ and $y_i^{\boldsymbol{g}}$ be the information set of seller i in the message stage such that $\boldsymbol{g}\left(y_i^{\boldsymbol{g}}\right) = \boldsymbol{g}$ and $\boldsymbol{p}\left(y_i^{\boldsymbol{g}}\right) = \bar{\boldsymbol{p}}$. By induction, we prove that for every product state $\boldsymbol{g} \in \boldsymbol{G}'$, a prudent rationalizable strategy $s_i \in S_i^\infty$ of seller $i \in \{a, b\}$ exists such that $s_i\left(x_i^{\boldsymbol{g}}\right) = \bar{p}_i$ and $s_i\left(y_i^{\boldsymbol{g}}\right) = \boldsymbol{G}'$.

First round: As is in the proof of Theorem 1, a strategy $s_i \in S_i$ of seller $i \in \{a, b\}$ such that $s_i(h_i) > 0$ at every information set h_i in the price stage is the first-round prudent rationalizable.

Now, we claim that a strategy s_c of the consumer with $s_c(h_c) = i$ is first-round prudent rationalizable at information set $h_c \in H_c$ such that $t(h_c) \in T_i\left(\bar{\boldsymbol{p}}, \boldsymbol{G}'\right)$ for some seller $i \in \{a, b\}$, $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, and $m_a(h_c) = m_b(h_c) = \boldsymbol{G}'$. With every belief system of the consumer, her belief at h_c assigns a positive probability to all product states and strategies of the sellers reaching h_c. In particular, a product state $\boldsymbol{g} \in \boldsymbol{G}'$ with $t(h_c) \in T_i(\bar{\boldsymbol{p}}, \{\boldsymbol{g}\})$ should be believed with a positive probability. Strategy s_c is rational at h_c with a belief system $b_c \in B_c^1$ such that belief $b_c(h_c)$ assigns a sufficient high probability to product state $\boldsymbol{g}$.

Induction step: For $r \geq 1$, we say that condition r is satisfied if

(c) A strategy s_c of the consumer such that $s_c(h_c) = i$ is rth-round prudent rationalizable at information set $h_c \in H_c$ such that $t(h_c) \in T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$, $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, and $m_a(h_c) = m_b(h_c) = \boldsymbol{G}'$, and

(s) For every product state $\boldsymbol{g} \in \boldsymbol{G}'$, a rth-round prudent rationalizable strategy $s_i \in S_i^r$ of seller $i \in \{a, b\}$ exists such that $s_i\left(x_i^{\boldsymbol{g}}\right) = \bar{p}_i$ and $s_i\left(y_i^{\boldsymbol{g}}\right) = \boldsymbol{G}'$.

Assuming that we have proved that condition r is satisfied in the rth-round of prudent rationalizability, we claim that condition $(r+1)$ holds in the $(r+1)$th-round of prudent rationalizability.

Firstly, we claim that part (c) of condition $(r+1)$ is satisfied in the $(r+1)$th-round. From the induction hypothesis, information set h_c such that $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, and $m_a(h_c) = m_b(h_c) = \boldsymbol{G}'$ is reached in $(r+1)$th round and the consumer believes all and only product states in $\boldsymbol{G}'$ at h_c. In particular, if $t(h_c) \in T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$ for some seller $i \in \{a, b\}$, some $\boldsymbol{g} \in \boldsymbol{G}'$ with $t(h_c) \in T_i(\bar{\boldsymbol{p}}, \{\boldsymbol{g}\})$ should be believed with a positive probability. A strategy s_c with $s_c(h_c) = i$ is rational with a belief system $b_c \in B_c^{r+1}$ such that $b_c(h_c)$ assigns a sufficiently high probability to $\boldsymbol{g}$.

Now, we claim that part (s) of condition $(r+1)$ is satisfied in the $(r+1)$th-round. Consider an arbitrary product state $\boldsymbol{g} \in \boldsymbol{G}'$. By $t_i^{\boldsymbol{g}}$, we denote a consumer type with $t_i^{\boldsymbol{g}} \in T_j(\bar{\boldsymbol{p}}, \{\boldsymbol{g}\}) \cap T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$. Consider a $(r+1)$th-round belief system $b_i \in B_i^{r+1}$ of seller i such that belief $b_i\left(x_i^{\boldsymbol{g}}\right)$ at $x_i^{\boldsymbol{g}}$ assigns a sufficiently high probability to consumer type $t_i^{\boldsymbol{g}}$ and the following rth-round prudent rationalizable strategies of the others reaching $x_i^{\boldsymbol{g}}$.

- Seller j charges price $\bar{p}_j$ at information set $x_j^{\boldsymbol{g}}$ and sends message $\boldsymbol{G}'$ at information set $y_j^{\boldsymbol{g}}$.
- Consumer type $t_i^{\boldsymbol{g}}$ purchases from seller i at information set h_c such that $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$ and $m_a(h_c) = m_b(h_c) = \boldsymbol{G}'$, and purchases from j at h_c' such that $\boldsymbol{p}\left(h_c'\right) = \bar{\boldsymbol{p}}$, $\boldsymbol{g} \in m_i\left(h_c'\right) \neq \boldsymbol{G}'$ and $m_j\left(h_c'\right) = \boldsymbol{G}'$ and at h_c'' such that $p_i(h_c'') > \bar{p}_i$ and $p_j(h_c'') = \bar{p}_j$ and $\boldsymbol{g} \in m_i(h_c'') \cap m_j\left(h_c'\right)$.

According to the induction hypothesis, $s_j\left(x_j^{\boldsymbol{g}}\right) = \bar{p}_j$ and $s_j\left(y_j^{\boldsymbol{g}}\right) = \boldsymbol{G}'$ for some rth-round prudent rationalizable strategy $s_j \in S_j^r$ of seller j. For some rth-round prudent rationalizable strategy $s_c \in S_c^r$ of the consumer, $s_c(h_c) = i$ and $s_c\left(h_c'\right) = s_c(h_c'') = j$. According to the induction hypothesis, s_c is rth-round prudent rationalizable at h_c. Moreover, strategy s_c is rational at h_c' with a belief system such that belief at h_c' assigns a sufficiently high probability to some product state $\boldsymbol{g}'$ with $t_i^{\boldsymbol{g}} \in T_j(\bar{\boldsymbol{p}}, \{\boldsymbol{g}'\})$ and so is at h_c'' with the same belief system such that belief at h_c'' assigns a sufficiently high probability to some product state $\boldsymbol{g}''$ with $t_i^{\boldsymbol{g}} \in T_j((p_i, \bar{p}_j), \{\boldsymbol{g}''\})$.

A strategy s_i of seller i such that $s_i\left(x_i^{\boldsymbol{g}}\right) = \bar{p}_i$ and $s_i\left(y_i^{\boldsymbol{g}}\right) = \boldsymbol{G}'$ is rational with the above belief system. Firstly, it is rational at $y_i^{\boldsymbol{g}}$ because belief $b_i\left(y_i^{\boldsymbol{g}}\right)$ assigns a sufficiently high probability to the event that $t_i^{\boldsymbol{g}}$ purchases from i at h_c if he sends message $\boldsymbol{G}'$, while she purchases from j at h_c' for every alternative message $m_i \neq \boldsymbol{G}'$. Secondly, it is rational at $x_i^{\boldsymbol{g}}$ because belief $b_i\left(x_i^{\boldsymbol{g}}\right)$ assigns a sufficiently high probability to the event that $t_i^{\boldsymbol{g}}$ purchases from i if he sets $\bar{p}_i$ and subsequently sends message $\boldsymbol{G}'$, while she purchases from j if he sets any price higher than $\bar{p}_i$.

A.3 Proof of Theorem 3

For the sufficiency: We prove by induction. Suppose that for every pair of non-empty sets $G'_a \times G'_b \subseteq \boldsymbol{G}$ of product states and every price pair, a product types $g_i \in G'_i$ of some seller $i \in \{a, b\}$ is absolute sales-dominant over $G'_i \times G'_j$. For every price pair $\boldsymbol{p} \in \boldsymbol{P}$ and seller $i \in \{a, b\}$, let $G^{0,i}_{\boldsymbol{p}} = \emptyset$ and $\Gamma^{0,i}_{\boldsymbol{p}} = G_i$, and we define inductively for $r \geq 1$,

$$G^{r,i}_{\boldsymbol{p}} = \left\{ g_i \in \Gamma^{r,i}_{\boldsymbol{p}}\colon\ g_i \text{ is absolute sales} - \text{dominant for } i \text{ at } \boldsymbol{p} \text{ over } \Gamma^{r,a}_{\boldsymbol{p}} \times \Gamma^{r,b}_{\boldsymbol{p}}. \right\}$$

where $\Gamma^{r,a}_{\boldsymbol{p}} = \Gamma^{r-1,a}_{\boldsymbol{p}} \backslash G^{r-1,a}_{\boldsymbol{p}}$ and $\Gamma^{r,b}_{\boldsymbol{p}} = \Gamma^{r-1,b}_{\boldsymbol{p}} \backslash G^{r-1,b}_{\boldsymbol{p}}$. By assumption, whenever $\Gamma^{r,a}_{\boldsymbol{p}} \times \Gamma^{r,b}_{\boldsymbol{p}}$ is non-empty, either $G^{r,a}_{\boldsymbol{p}}$ or $G^{r,b}_{\boldsymbol{p}}$ is non-empty for every $r \geq 1$. Since G_a and G_b are finite, some finite number $\bar{r} \geq 1$ exists such that all product states in $\Gamma^{\bar{r},a}_{\boldsymbol{p}} \times \Gamma^{\bar{r},b}_{\boldsymbol{p}}$ are sales-equivalent at $\boldsymbol{p}$ and $G^{\bar{r},i}_{\boldsymbol{p}} = \Gamma^{\bar{r},i}_{\boldsymbol{p}}$ for every seller $i \in \{a, b\}$. Conversely, if $G^{r,a}_{\boldsymbol{p}}$ and $G^{r,b}_{\boldsymbol{p}}$ are non-empty for some $r \geq 1$, all product states in $\Gamma^{r,a}_{\boldsymbol{p}} \times \Gamma^{r,b}_{\boldsymbol{p}}$ are sales-equivalent at $\boldsymbol{p}$. If $\bar{g}_a \in G^{r,a}_{\boldsymbol{p}}$ and $\bar{g}_b \in G^{r,b}_{\boldsymbol{p}}$, then $\bar{\boldsymbol{g}} = (\bar{g}_a, \bar{g}_b)$ is sales-dominant for each seller at $\boldsymbol{p}$ over $\Gamma^{r,a}_{\boldsymbol{p}} \times \Gamma^{r,b}_{\boldsymbol{p}}$. For every $\boldsymbol{g} \in \Gamma^{r,a}_{\boldsymbol{p}} \times \Gamma^{r,b}_{\boldsymbol{p}}$, we have $\bar{\boldsymbol{g}} \succsim^a_{\boldsymbol{p}} \boldsymbol{g} \succsim^a_{\boldsymbol{p}} \bar{\boldsymbol{g}}$ since $\bar{\boldsymbol{g}} \succsim^b_{\boldsymbol{p}} \boldsymbol{g}$ implies $\boldsymbol{g} \succsim^a_{\boldsymbol{p}} \bar{\boldsymbol{g}}$.

First round: Analogously to the proof of Theorem 1, a strategy $s_i \in S_i$ of seller i that maps every information set in the price stage to a positive price is first-round prudent rationalizable. At every information set h_c of the consumer, she believes all and only product states in $m_i(h_c) \times m_j(h_c)$. If $t(h_c) \in T_i\big(\boldsymbol{p}(h_c), m_i(h_c) \times m_j(h_c)\big)$ for some seller $i \in \{a, b\}$, purchasing from i is first-round prudent rationalizable. In particular, every first-round prudent rationalizable strategy of the consumer prescribes to purchase from the seller whose product is the most preferred at price pair $\boldsymbol{p}\big(h'_c\big)$ in a product state in $m_a\big(h'_c\big) \times m_b\big(h'_c\big)$ if h'_c is such that $m_a\big(h'_c\big) \subseteq G^{1,a}_{\boldsymbol{p}(h'_c)}$ or $m_b\big(h'_c\big) \subseteq G^{1,b}_{\boldsymbol{p}(h'_c)}$.

Second round: We claim that if strategy s_i of seller $i \in \{a, b\}$ is second-round prudent rationalizable, $s_i(h_i) \subseteq G^{1,i}_{\boldsymbol{p}(h_i)}$ at every information set h_i in the message stage reachable in the second round such that $g_i(h_i) \in G^{1,i}_{\boldsymbol{p}(h_i)}$ and $G^{1,j}_{\boldsymbol{p}(h_i)} = \emptyset$. Suppose to the contrary that a second-round prudent rationalizable strategy s_i exists such that $s_i(h_i) \nsubseteq G^{1,i}_{\boldsymbol{p}(h_i)}$ at some information set h_i in the message stage reachable in the second round satisfying $g_i(h_i) \in G^{1,i}_{\boldsymbol{p}(h_i)}$ and $G^{1,j}_{\boldsymbol{p}(h_i)} = \emptyset$. With every belief system $b_i \in B^2_i$, belief $b_i(h_i)$ at h_i must assign a positive probability to the first-round prudent rationalizable strategies of seller j that prescribe to send a message m_j such that $(g_i(h_i), g_j) \succ^i_{\boldsymbol{p}(h_i)} (g_i, g_j)$ for some $(g_i, g_j) \in s_i(h_i) \times m_j$ at information set h_j with $\boldsymbol{g}(h_j) = \boldsymbol{g}(h_i)$ and $\boldsymbol{p}(h_j) = \boldsymbol{p}(h_i)$. For simplicity in notation, let $\boldsymbol{G}' = s_i(h_i) \times m_j$. Then, analogously to the proof of Theorem 1, we reach to a contradiction.

Induction hypothesis: For $r \geq 1$, we say that strategy $s_c \in S_c$ of the consumer satisfies condition r if at every information set $h_c \in H_c$ reachable in the $(2r-1)$th round such that $s_c(h_i) = i$, we have either

(c1) $t(h_c) \in T_i\Big(\boldsymbol{p}(h_c), \Big[m_i(h_c) \cap \Gamma^{r,i}_{\boldsymbol{p}(h_c)}\Big] \times \Big[m_j(h_c) \cap \Gamma^{r,j}_{\boldsymbol{p}(h_c)}\Big]\Big)$ or

(c2) $t(h_c) \in T_i\Big(\boldsymbol{p}(h_c), G^{x,i}_{\boldsymbol{p}(h_c)} \times \Big[m_j(h_c) \cap \Gamma^{x,j}_{\boldsymbol{p}(h_c)}\Big]\Big)$ for some $1 \leq x \leq r$ satisfying $G^{x,i}_{\boldsymbol{p}(h_c)} \neq \emptyset$ and $m_i(h_c) \cap \Gamma^{x,i}_{\boldsymbol{p}(h_c)} \subseteq G^{x,i}_{\boldsymbol{p}(h_c)}$.

For $r \geq 1$, we say that strategy $s_i \in S_i$ of seller $i \in \{a, b\}$ satisfies condition r if $s_i(h_i) \cap \Gamma^{y,i}_{\boldsymbol{p}(h_i)} \subseteq G^{y,i}_{\boldsymbol{p}(h_i)}$ at every information set $h_i \in H_i$ of seller i in the message stage reachable in the $2r$th round such that

(s) for some y with $1 \leq y \leq r$, $g_i(h_i) \in G^{y,i}_{\boldsymbol{p}(h_i)}$ and $G^{y,j}_{\boldsymbol{p}(h_i)} = \emptyset$.

Assume now that we have proved that $(2r-1)$th-round prudent rationalizable strategies of the consumer satisfy condition r and that $2r$th round prudent rationalizable strategies of seller $i \in \{a, b\}$ satisfy condition r. We claim that $(2r+1)$th-round prudent rationalizable strategies of the consumer satisfy condition $r+1$ and that $2(r+1)$th round prudent rationalizable strategies of seller $i \in \{a, b\}$ satisfy condition $r+1$.

$(2r+1)$th round: Consider information set $h_c \in H_c$ of the consumer reachable in the $(2r+1)$th round for which (c1) or (c2) of condition $(r+1)$ is satisfied. Analogously to the proof of Theorem 1, we can show that the consumer believes all and only product states in $\Big[m_i(h_c) \cap \Gamma^{r+1,i}_{\boldsymbol{p}(h_c)}\Big] \times \Big[m_j(h_c) \cap \Gamma^{r+1,j}_{\boldsymbol{p}(h_c)}\Big]$ at information set h_c for which this intersection is non-empty since $2r$th-round prudent rationalizable strategies of seller $k \in \{a, b\}$ satisfy condition r by induction hypothesis.

$2(r+1)$ round: Consider an information set h_i of seller $i \in \{a, b\}$ in the message stage reachable in the $2(r+1)$th round for which (s) of condition $(r+1)$ is satisfied. It is sufficient to show that with every $2(r+1)$th-round prudent rationalizable strategy, seller i sends a message $s_i(h_i)$ satisfying $s_i(h_i) \cap \Gamma^{r+1,i}_{\boldsymbol{p}(h_i)} \subseteq G^{r+1,i}_{\boldsymbol{p}(h_i)}$ if $g_i(h_i) \in G^{r+1,i}_{\boldsymbol{p}(h_i)}$ and $G^{r+1,j}_{\boldsymbol{p}(h_i)} = \emptyset$. Suppose to the contrary that a $2(r+1)$th-round prudent rationalizable strategy s_i exists such that $s_i(h_i) \cap \Gamma^{r+1,i}_{\boldsymbol{p}(h_i)} \nsubseteq G^{r+1,i}_{\boldsymbol{p}(h_i)}$ at some information set h_i in the message stage reachable in $2(r+1)$th round such that $g_i(h_i) \in G^{r+1,i}_{\boldsymbol{p}(h_i)}$ and $G^{r+1,j}_{\boldsymbol{p}(h_i)} = \emptyset$. Not all product states in $\Gamma^{r+1,i}_{\boldsymbol{p}(h_i)} \times \Gamma^{r+1,j}_{\boldsymbol{p}(h_i)}$ are sales-equivalent at $\boldsymbol{p}(h_i)$, otherwise, $G^{r+1,j}_{\boldsymbol{p}(h_i)} \neq \emptyset$. With every belief system $b_i \in B^{2(r+1)}_i$, belief $b_i(h_i)$ at h_i must assign a positive probability to $(2r+1)$th-round prudent rationalizable strategies of seller j that prescribe to send a message m_j such that $(g_i(h_i), g_j) \succ^i_{\boldsymbol{p}(h_i)} (g_i, g_j)$ for some $(g_i, g_j) \in \Big[s_i(h_i) \cap \Gamma^{r+1,i}_{\boldsymbol{p}(h_i)}\Big] \times \Big[m_j \cap \Gamma^{r+1,j}_{\boldsymbol{p}(h_i)}\Big]$ at information set h_j with $\boldsymbol{g}(h_j) = \boldsymbol{g}(h_i)$ and $\boldsymbol{p}(h_j) = \boldsymbol{p}(h_i)$. For simplicity in notation, let

$\boldsymbol{G}' = \left[s_i(h_i) \cap \Gamma^{r+1,i}_{\boldsymbol{p}(h_i)}\right] \times \left[m_j \cap \Gamma^{r+1,j}_{\boldsymbol{p}(h_i)}\right]$. Than, analogously to the proof of Theorem 1, we reach to a contradiction.

For the necessity: We prove by contradiction. Suppose to the contrary that there exist some positive price pair $\bar{\boldsymbol{p}} = (\bar{p}_a, \bar{p}_b) \in \boldsymbol{P}$ and non-empty set $G'_a \times G'_b \subseteq \boldsymbol{G}$ of product states such that no product type $g_i \in G'_i$ of each seller $i \in \{a, b\}$ is absolute sales-dominant at $\bar{\boldsymbol{p}}$ over $G'_a \times G'_b$. For simplicity in notation, let $\boldsymbol{G}' = G'_i \times G'_j$. For every $i \in \{a, b\}$ and product state $\boldsymbol{g} \in \boldsymbol{G}'$, let $x^{\boldsymbol{g}}_i$ denote the information set of seller i in the price stage such that $\boldsymbol{g}(x^{\boldsymbol{g}}_i) = \boldsymbol{g}$ and $y^{\boldsymbol{g}}_i$ denote the information set of seller i in the message stage such that $\boldsymbol{g}(y^{\boldsymbol{g}}_i) = \boldsymbol{g}$ and $\boldsymbol{p}(y^{\boldsymbol{g}}_i) = \bar{\boldsymbol{p}}$. We show, by induction, that for every $i \in \{a, b\}$ and product state $\boldsymbol{g} \in \boldsymbol{G}'$, a prudent rationalizable strategy $s_i \in S^\infty_i$ of seller i exists such that $s_i(x^{\boldsymbol{g}}_i) = \bar{p}_i$ and $s_i(y^{\boldsymbol{g}}_i) = G'_i$.

First round: Analogously to the proof for Theorem 2, a strategy $s_i \in S_i$ of seller $i \in \{a, b\}$ such that $s_i(h_i) > 0$ at every information set h_i in the price stage is first-round prudent rationalizable. Moreover, a strategy $s_c \in S_c$ of the consumer such that $s_c(h_c) = i$ is first-round prudent rationalizable at information set $h_c \in H_c$ such that $t(h_c) \in T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$ for some $i \in \{a, b\}$, $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, $m_a(h_c) = G'_i$, and $m_j(h_c) = G'_j$.

Induction step: For every $r \geq 1$, we say that condition r is satisfied if

(c) a strategy $s_c \in S^r_c$ of the consumer such that $s_c(h_c) = i$ is rth-round prudent rationalizable at information set $h_c \in H_c$ such that $t(h_c) \in T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$, $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, $m_a(h_c) = G'_i$, and $m_j(h_c) = G'_j$, and

(s) For every seller $i \in \{a, b\}$ and product state $\boldsymbol{g} \in \boldsymbol{G}'$, a rth-round prudent rationalizable strategy $s_i \in S^r_i$ exists such that $s_i(x^{\boldsymbol{g}}_i) = \bar{p}_i$ and $s_i(y^{\boldsymbol{g}}_i) = G'_i$.

Assuming that we have proved that condition r is satisfied in the rth-round of prudent rationalizability, we claim that condition $r + 1$ holds in the $(r + 1)$th-round of prudent rationalizability.

Firstly, we can show that part (c) of condition $(r + 1)$ is satisfied since from the induction hypothesis, every information set h_c such that $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, $m_a(h_c) = G'_a$, and $m_b(h_c) = G'_b$ is reached in $(r + 1)$th round of prudent rationalizability and the consumer believes all and only product states in $\boldsymbol{G}'$ at h_c.

Now, we claim that part (s) of condition $(r + 1)$ is satisfied for every seller $i \in \{a, b\}$ and product state $\boldsymbol{g} \in \boldsymbol{G}'$. We assume that $\boldsymbol{g}$ is not sales-dominant for i at $\boldsymbol{p}$ over $\boldsymbol{G}'$. Then, a consumer type $t^{\boldsymbol{g}} \in T_j(\bar{\boldsymbol{p}}, \{\boldsymbol{g}\}) \cap T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$ exists. Analogous to the proof for Theorem 2, a strategy s_i such that $s_i(x^{\boldsymbol{g}}_i) = \bar{p}_i$ and $s_i(y^{\boldsymbol{g}}_i) = G'_i$ is $(r + 1)$th-round prudent rationalizable. Now, we assume that $\boldsymbol{g}$ is sales-dominant for i at $\boldsymbol{p}$ over $\boldsymbol{G}'$. Then, a consumer type $t^{\boldsymbol{g}}$ exists such that $t^{\boldsymbol{g}} \in T_j\left(\bar{\boldsymbol{p}}, \left\{(g_i(x^{\boldsymbol{g}}_i), g_j)\right\}\right) \cap T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$ for some $g_j \in G'_j$. Note that $t^{\boldsymbol{g}}$ must exist. Otherwise, if $t \in T_i(\bar{\boldsymbol{p}}, \left\{(g_i(x^{\boldsymbol{g}}_i), g_j)\right\})$ for every $g_j \in G'_j$ whenever $t \in T_i(\bar{\boldsymbol{p}}, \boldsymbol{G}')$, then $g_i(x^{\boldsymbol{g}}_i)$ is absolute sales-dominant for i at $\boldsymbol{p}$ over $\boldsymbol{G}'$. Analogous to the proof for

Theorem 2, a strategy s_i such that $s_i(x_i^{\boldsymbol{g}}) = \bar{p}_i$ and $s_i(y_i^{\boldsymbol{g}}) = G_i'$ is $(r+1)$th-round prudent rationalizable with a $(r+1)$th-round belief system $b_i \in B_i^{r+1}$ of i such that the full-support belief $b_i(x_i^{\boldsymbol{g}})$ at $x_i^{\boldsymbol{g}}$ assigns a sufficiently high probability to $t^{\boldsymbol{g}}$ and the following rth-round prudent rationalizable strategies of the others.

1. Seller j charges price $\bar{p}_j$ at information set $x_j^{\boldsymbol{g}}$ and sends message G_j' at information set $y_j^{\boldsymbol{g}}$ and at every other h_j in the message stage such that $\boldsymbol{g}(h_j) = \boldsymbol{g}$ and $\Delta u(\boldsymbol{g}, t^{\boldsymbol{g}}) > \Delta p(h_j)$.
2. Consumer type $t^{\boldsymbol{g}}$ purchases from seller i at h_c such that $\boldsymbol{p}(h_c) = \bar{\boldsymbol{p}}$, $m_i(h_c) = G_i'$, and $m_j(h_c) = G_j'$, and purchases from seller j at h_c' such that $\boldsymbol{p}(h_c') = \bar{\boldsymbol{p}}$, $g_i(x_i^{\boldsymbol{g}}) \in m_i(h_c') \neq G_i'$, and $m_j(h_c') = G_j'$ and at h_c'' such that $p_i(h_c'') > \bar{p}_i$, $p_j(h_c'') = \bar{p}_j$, and either $\Delta u(\boldsymbol{g}, t^{\boldsymbol{g}}) < \Delta p(h_c'')$ or $g_i(x_i^{\boldsymbol{g}}) \in m_i(h_c'')$ and $m_j(h_c'') = G_j'$.

A.4 Proof of Proposition 2

Lemma A.1. *If statement (ii) holds, the sales-dominance relation is complete at every price pair.*

Proof. We prove by contradiction. Pick arbitrary product states $\boldsymbol{g}^1 = (g_a^1, g_b^1)$ and $\boldsymbol{g}^2 = (g_a^2, g_b^2)$. Suppose to the contrary that $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ are not comparable in terms of sales at some price pair $\boldsymbol{p} \in \boldsymbol{P}$. Then, there are consumer types t_1 and t_2 such that $\Delta u(\boldsymbol{g}^2, t_1) < \Delta p < \Delta u(\boldsymbol{g}^1, t_1)$ and $\Delta u(\boldsymbol{g}^1, t_2) < \Delta p < \Delta u(\boldsymbol{g}^2, t_2)$. Firstly, we assume that $g_a^1 = g_a^2$ or $g_b^1 = g_b^2$ but not both hold at the same. If $g_a^1 \neq g_a^2$ and $g_b^1 = g_b^2$, pick arbitrary $g_b \neq g_b^1 \in G_b$. Condition (a) of (ii) is not satisfied for $\{g_a^1, g_a^2\} \times \{g_b^1, g_b\}$. Moreover, $u(g_a^2, t_1) < u(g_a^1, t_1)$ and $u(g_a^1, t_2) < u(g_a^2, t_2)$, a contradiction to condition (b) of (ii). Analogously, we reach a contradiction when $g_a^1 = g_a^2$ and $g_b^1 \neq g_b^2$. Now, we assume that $g_a^1 \neq g_a^2$ and $g_b^1 \neq g_b^2$. Condition (a) of (ii) is not satisfied for $\{g_a^1, g_a^2\} \times \{g_b^1, g_b^2\}$. $\Delta u(\boldsymbol{g}^2, t_1) < \Delta u(\boldsymbol{g}^1, t_1)$ implies that $u(g_a^2, t_1) < u(g_a^1, t_1)$ or $u(g_b^2, t_1) > u(g_b^1, t_1)$. $\Delta u(\boldsymbol{g}^1, t_2) < \Delta u(\boldsymbol{g}^2, t_2)$ implies that $u(g_a^2, t_2) > u(g_a^1, t_2)$ or $u(g_b^2, t_2) < u(g_b^1, t_2)$. If $u(g_a^2, t) < u(g_a^1, t)$ and $u(g_b^2, t) < u(g_b^1, t)$ for every $t \in \{t_1, t_2\}$, $\Delta u(\boldsymbol{g}^1, t_2) < \Delta p < \Delta u(\boldsymbol{g}^1, t_1)$, a contradiction to (b) of (ii). Analogously, if $u(g_a^2, t) > u(g_a^1, t)$ and $u(g_b^2, t) > u(g_b^1, t)$ for every $t \in \{t_1, t_2\}$, $\Delta u(\boldsymbol{g}^2, t_1) < \Delta p < \Delta u(\boldsymbol{g}^2, t_2)$, a contradiction. Finally, if either $u(g_a^2, t_1) < u(g_a^1, t_1)$ and $u(g_a^2, t_2) > u(g_a^1, t_2)$ or $u(g_b^2, t_1) > u(g_b^1, t_1)$ and $u(g_b^2, t_2) < u(g_b^1, t_2)$, a contradiction to condition (b) of (ii). □

$(i) \Rightarrow (ii)$: We prove by contradiction. Consider arbitrary non-singleton sets $G_a' \subseteq G_a$, $G_b' \subseteq G_b$, and $\{t_1, t_2\} \subseteq T$. Let $g_a^1 = \arg\max_{g \in G_a'} u(g, t_1)$, $g_b^1 = \arg\max_{g \in G_b'} u(g, t_1)$, $g_a^2 = \arg\max_{g \in G_a'} u(g, t_2)$, and $g_b^2 = \arg\max_{g \in G_b'} u(g, t_2)$. That is, for every

$i \in \{a, b\}$ and $k \in \{1, 2\}$, g_i^k is the product type that is most preferred by t_k among the ones in G_i'. Note that it may be $g_a^1 = g_a^2$ or $g_b^1 = g_b^2$. For simplicity in notation, let $\boldsymbol{G}' = G_a' \times G_b'$. Suppose to the contrary that condition (*ii*) is not satisfied for $\boldsymbol{G}'$ and $\{t_1, t_2\}$. Then $\min_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_1) < \max_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_2)$ and $\min_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_2) < \max_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_1)$. Moreover, $g_a^1 \neq g_a^2$, $g_b^1 \neq g_b^2$, or $\Delta u((g_a^1, g_b^1), t_1) \neq \Delta u((g_a^2, g_b^2), t_2)$.

Firstly, we consider the case where $\Delta u((g_a^1, g_b^1), t_1) \neq \Delta u((g_a^2, g_b^2), t_2)$. Without loss of generality, assume that $\Delta u((g_a^2, g_b^2), t_2) < \Delta u((g_a^1, g_b^1), t_1)$. Since $\boldsymbol{P}$ is sufficiently fine, a price pair $\boldsymbol{p} \in \boldsymbol{P}$ exists such that

$$\max\left\{ \Delta u((g_a^2, g_b^2), t_2), \min_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_1) \right\} < \Delta p < \min\left\{ \Delta u((g_a^1, g_b^1), t_1), \max_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_2) \right\}.$$

For every $g_a \in G_a'$, $\Delta u((g_a, g_b^2), t_2) < \Delta p < \max_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_2)$ and (g_a, g_b^2) is not sales-dominant at $\boldsymbol{p}$ for a over $\boldsymbol{G}'$. Hence, g_a is not absolute sales-dominant at $\boldsymbol{p}$ for a over $\boldsymbol{G}'$. For every $g_b \in G_b'$, $\min_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t_1) < \Delta p < \Delta u((g_a^1, g_b), t_1)$ and (g_a^1, g_b) is not sales-dominant at $\boldsymbol{p}$ for b over $\boldsymbol{G}'$. Hence, g_b is not absolute sales-dominant at $\boldsymbol{p}$ for b over $\boldsymbol{G}'$, a contradiction to (*i*).

Now, we consider the case where $\Delta u((g_a^1, g_b^1), t_1) = \Delta u((g_a^2, g_b^2), t_2)$ and either $g_a^1 \neq g_a^2$ or $g_b^1 \neq g_b^2$. Without loss of generality, assume that $g_b^1 \neq g_b^2$. Since $\boldsymbol{P}$ is sufficiently fine, a price pair $\boldsymbol{p} \in \boldsymbol{P}$ exists such that

$$\Delta u((g_a^1, g_b^1), t_1) = \Delta u((g_a^2, g_b^2), t_2) < \Delta p < \min\{\Delta u((g_a^1, g_b^2), t_1), \Delta u((g_a^2, g_b^1), t_2)\}.$$

For every $g_a \in G_a'$, $\Delta u((g_a, g_b^2), t_2) < \Delta p < \Delta u((g_a^2, g_b^1), t_2)$ and (g_a, g_b^2) is not sales-dominant at $\boldsymbol{p}$ for a over $\boldsymbol{G}'$. Hence, g_a is not absolute sales-dominant at $\boldsymbol{p}$ for a over $G_a' \times \{g_b^1, g_b^2\}$. Since $\Delta u((g_a^2, g_b^2), t_2) < \Delta p < \Delta u((g_a^2, g_b^1), t_2)$, (g_a^2, g_b^1) is not sales-dominant at $\boldsymbol{p}$ for b over $G_a' \times \{g_b^1, g_b^2\}$. Hence, g_b^1 is not absolute sales-dominant at $\boldsymbol{p}$ for a over $G_a' \times \{g_b^1, g_b^2\}$. Since $\Delta u((g_a^1, g_b^1), t_1) < \Delta p < \Delta u((g_a^1, g_b^2), t_1)$, (g_a^1, g_b^2) is not sales-dominant at $\boldsymbol{p}$ for b over $G_a' \times \{g_b^1, g_b^2\}$. Hence, g_b^2 is not absolute sales-dominant at $\boldsymbol{p}$ for a over $G_a' \times \{g_b^1, g_b^2\}$, a contradiction to statement (*i*).

(*ii*) ⇒ (*i*): Consider arbitrary non-empty sets $G_a' \subseteq G_a$, $G_b' \subseteq G_b$, and price pair $\boldsymbol{p} \in \boldsymbol{P}$. Let $\boldsymbol{G}' = G_a' \times G_b'$. If $T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}') = \emptyset$, all product states in $\boldsymbol{G}'$ are sales-equivalent at $\boldsymbol{p}$, hence for every seller $i \in \{a, b\}$, every $g_i \in G_i'$ is absolute sales-dominant at $\boldsymbol{p}$ for i over $\boldsymbol{G}'$.

Now, suppose that $T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}') \neq \emptyset$. For every consumer type $t \in T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}')$, $\min_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t) < \Delta p < \max_{\boldsymbol{g} \in \boldsymbol{G}'} \Delta u(\boldsymbol{g}, t)$. Firstly, we assume that neither G'_a nor G'_b is singleton. To all consumer types $t \in T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}')$, some product type $\bar{g}_i \in G'_i$ of seller $i \in \{a, b\}$ is the most preferred among all product types in G'_i. Moreover, $\Delta u(\bar{g}_a, \bar{g}_b, t)$ takes the same value to all $t \in T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}')$. Let $\bar{\Delta}$ denote this common value. If $\Delta p < \bar{\Delta}$, $\bar{g}_a$ is absolute sales-dominant for a at $\boldsymbol{p}$ over $\boldsymbol{G}'$ because $\Delta p < \bar{\Delta} = \min_{g \in G'_b} \Delta u(\bar{g}_a, g, t)$ for $t \in T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}')$. Likewise, if $\Delta p > \bar{\Delta}$, $\bar{g}_b$ is absolute sales-dominant for b at $\boldsymbol{p}$ over $\boldsymbol{G}'$ because $\Delta p > \bar{\Delta} = \max_{g \in G'_a} \Delta u(g, \bar{g}_b, t)$ for $t \in T_a(\boldsymbol{p}, \boldsymbol{G}') \cap T_b(\boldsymbol{p}, \boldsymbol{G}')$. Now, we assume that G'_i is non-singleton and G'_j is singleton for seller $i \in \{a, b\}$ and his opponent j. From Lemma A.1, the sales-dominance relation $\succsim^i_{\boldsymbol{p}}$ is complete at $\boldsymbol{p}$. Some product state $(\bar{g}_i, g_j) \in \boldsymbol{G}'$ is sales-dominant at $\boldsymbol{p}$ over $\boldsymbol{G}'$ for i and $\bar{g}_i$ is absolute sales-dominant for i at $\boldsymbol{p}$ over $\boldsymbol{G}'$.

Appendix B: Prudent Rationalizable Prices for Example 2

Recall that P is a set of positive prices. In the first round of prudent rationalizability, every strategy of seller $i \in \{a, b\}$ is rationalizable. We claim that in the second round, every positive price is rationalizable at every information set in the price stage, which implies that prices do not provide any information about the product state. The following notations would be useful. Let $\underline{p} = \min P$ and $\bar{p} = \max P$. We assume that $\bar{p} > \underline{p} + 7$.

Consider seller a at information set h_a in the price stage reached after product state $\boldsymbol{g}^1$ is realized. Intuitively, if he believes with a sufficiently high probability that the consumer is of t_1 type, he would set a more or less “high” price since consumer type t_1 highly values the product of a than the product of seller b. Analogously, seller a would set a more or less “low” price if he believes t_2 with a sufficiently high probability. Firstly, in the second round of prudent rationalizability, every price $p_a \geq \max\{p \in P: p < \underline{p} + 3\}$ is rationalizable to the seller a at h_a. Seller a can believe with a sufficiently high probability that seller b charges a price $p_b \geq \underline{p}$ and that the consumer is of type t_1. Then, charging price $p'_a = \max\{p \in P: p < p_b + 3\}$ yields the highest payoff to seller a. Secondly, in the second round of prudent rationalizability, every price $p_a \leq \max\{p \in P: p < \bar{p} - 4\}$ is rationalizable to seller a at h_a. Seller a can believe with a sufficiently high probability that seller b charges a price $p_b > \underline{p} + 4$ and the consumer is of t_2 type. Moreover, a can believe that t_2 would purchase from seller b at every information set h_c such that $p_a(h_c) - p_b > -4$. Then, charging price $p_a = \max\{p \in P: p < p_b - 4\}$ yields the highest payoff to seller a. With analogous arguments, every positive price is second round prudent rationalizable at every information set of seller $i \in \{a, b\}$ in the price stage. □

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