

Research Article

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Games with Unawareness

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Abstract: We provide a tool to model and solve strategic situations where players' perceptions are limited, as well as situations where players realize that other players' perceptions may be limited and so on. We define normal, repeated, incomplete information, and extensive form games with unawareness using a unified methodology. A game with unawareness is defined as a collection of standard games (of the corresponding form). The collection specifies how each player views the game, how she views the other players' perceptions of the game and so on. The modeler's description of perceptions, the players' description of other players' perceptions, etc. are shown to have consistent representations. We extend solution concepts such as rationalizability and Nash equilibrium to these games and study their properties. It is shown that while unawareness in normal form games can be mapped to incomplete information games, the extended Nash equilibrium solution is not mapped to a known solution concept in the equivalent incomplete information games, implying that games with unawareness generate novel types of behavior.

Keywords: games, unawareness

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1 Introduction

As game theoretical modeling becomes more prevalent as a practical modeling tool, a central problem arises due to a multitude of models that can be employed. The reasoning players in a realistic strategic interaction seem likely to come up

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with different games representing the situation at hand. This discrepancy may be the result of players being unaware of some aspects of the situation, or that the mere act of modeling and reasoning about the situation leads the players to incorporate only a selection of aspects they deem most crucial. In either case the players may formulate a restricted perception of the game. The objective of this paper is to provide a model that allows the players to have differing game models for their interaction, recognize and model other players' different perceptions of the game and iteratively higher order perceptions, and analyze reasoning and behavior in this setting.

Players who model the game differently should also be allowed to recognize that other players may well have different models for the strategic interaction. We provide a new game form that allows the representation of players with varying restricted perception of the environment at hand. The game form that we call – games with unawareness – is a collection of standard games describing the perception of each player, their perception of other players' perceptions and so on. The standard games in the collection are related to each other based on properties of higher order reasoning in a manner that guarantees that how each player views all interactive perceptions of the game is itself a game with unawareness – players perceive the situation much like the modeler does as a collection of standard games. We use the same methodology to extend normal (strategic) form games, repeated games, incomplete information games and extensive form games, providing a unified framework for reasoning about limited perceptions, or unawareness. We then turn to solution concepts and extend rationalizability and Nash equilibria to normal form games with unawareness, as well as extend Bayesian Nash equilibria (BNE) for games with incomplete information and unawareness, and Nash equilibria and refinements such as sequential equilibria are extended to extensive form games with unawareness.

Our first result demonstrates the consistency of the representation of games with unawareness for the aforementioned game forms, i.e., that every player in every game form views the game as a game with unawareness, that all players perceive other players' view of the game as a game with unawareness, and so on. We also prove the existence of the extended Nash equilibrium (NE) solution. Finally, we show that while games with unawareness can be naturally mapped to games with incomplete information, the latter cannot replace the explicit modeling of games with limited perceptions since the mapping does preserve this new solution concept. In particular, the extended Nash solution concept generates a tighter (subset) prediction about behavior than the Bayesian NE of the equivalent incomplete information game, whereas the latter corresponds to an extended rationalizable solution.

1.1 Three Examples

We operationalize restricted perception of the game as follows: A player – Alice – can be unaware of some of the actions available to another player – Bob. This will imply that the game she perceives does not contain some of Bob's actions. Alternatively, Alice may not include a third player in the game at all. This again represents a restricted game. It might also be the case that Bob perceives that Alice is unaware of some aspect of the game, in this case his view of her view of the game is restricted but it does not imply that Alice is indeed restricted in this manner. It might well be that Alice is aware of all aspects of the interaction, yet Bob's perception of her perception is limited. The two examples below provide some variations on these type of situations. They also shed some light on how the solutions to these games are constructed. The guiding principle for behavior is that every player chooses a strategy in the game they perceive which reacts (e.g., with a best response) to strategies the player believes others will be playing in the game she thinks they perceive.

Our first example is of a normal form game with unawareness. We begin with the game depicted in (1) below. This game represents all the actions available to the players – Alice and Bob – and the payoffs associated with each action profile. Assume that in the situation we are modeling Alice and Bob are both aware of all the actions available in the game, so when *they* write a description of the actions and payoffs it corresponds to (1) below. However, Alice is unaware that Bob is aware of *all* her actions. In particular, She thinks that he is only aware of the actions $\{a_1, a_2, b_1, b_2, b_3\}$ – She does not realize that her third action is included in the game as Bob perceives it. This situation may arise if Alice perceives that action a_3 is secret, or if, say, Bob is new to the environment in which this interaction occurs and there is no reason to think he would model action a_3 .

This situation may arise if Bob actually learns about Alice's secret action, or talks to an "old-timer" who reveals the existence of the not so obvious third action available to Alice. We note, that the strategy profile (a_2, b_1) is the unique NE of the game depicted in (1), hence if such games have been played in the past (even if fully perceived by all players) the action a_3 may not have been observed. Hence, even if Alice believes an inexperienced Bob will study similar past situations, she may well conclude that he will not model this action. In this example we further assume that Bob realizes all this. Not only does he consider a_3 he also deduces that Alice does not realize that he is considering it, e.g., if he learned it was a secret action. Hence, Bob is aware that Alice perceives Bob's perception of the game to consist only of the actions $\{a_1, a_2, b_1, b_2, b_3\}$. We also assume that Bob is aware that Alice is aware of the whole action set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$.

Turning to higher levels of interactive views of the game; since Alice is only aware of Bob being aware of $\{a_1, a_2, b_1, b_2, b_3\}$, we assume she cannot be aware that he is aware that she is aware of anything beyond this set, otherwise, she would be aware that he is able to reason about her reasoning about the additional action a_3 , so he must be able to reason about a_3 as far as Alice can deduce, which would contradict our assumption that Alice is unaware that Bob is aware of a_3 . In particular, any higher order iteration of awareness of Alice and Bob which is not considered above is assumed to be associated with the set $\{a_1, a_2, b_1, b_2, b_3\}$.

		Bob			
		b_1	b_2	b_3	
Alice	a_1	0, 2	3, 3	0, 2	(1)
	a_2	2, 2	2, 1	2, 1	
	a_3	1, 0	4, 0	0, 1	

As noted above, the initial game we started out with in (1) has a unique NE (a_2, b_1) obtained by iteratively eliminating strictly dominated strategies. However, while both players are aware that this is the game being played, we assumed that Alice perceives that Bob is aware of only two of her actions. In other words, Alice is unaware that Bob is aware of a_3 . While Alice and Bob both view the game as in (1) Alice perceives that Bob finds the game being played as depicted in (2).

		Bob			
		b_1	b_2	b_3	
Alice	a_1	0, 2	3, 3	0, 2	(2)
	a_2	2, 2	2, 1	2, 1	

Hence, Alice also finds that Bob finds that *she* perceives the game as in (2), and so on for every higher order awareness. We obtain that Alice perceives that Bob views the game as a standard normal form game with complete awareness – a game where all participating players are aware of all aspects of the game, they are aware that all other players are aware of the same game, and so on. Taking Nash equilibria as the solution concept for normal form games, Alice may deduce that Bob plays according to the NE (a_1, b_2) of the game in (2) which is also the Pareto dominant outcome of this normal form game. Alice, who sees herself as being more aware than Bob, will be inclined to choose her best response to b_2 which is a_3 in the game as she perceives it. Bob can make the exact same deduction that we, as modelers, just made, since he is aware of all the actions and he fully realizes how Alice perceives *his* awareness. Hence, Bob may deduce that Alice, being unaware of his full awareness, will assume he plays b_2 and she will play a_3 as her best response. Bob being aware of all that can deduce this behavior based on the iterated best response principle. This would lead Bob to play his best response to a_3 which is b_3 . We will have that Alice chooses a_3 and Bob chooses b_3 as a result of this

higher order unawareness. Starting with a NE and considering best responses, we end up with the worst possible payoff for Alice and a payoff below all Nash equilibria of either games for both players, even though both are aware of the full extent of the game, both are commonly aware of the action profile of the unique NE (a_2, b_1) in that game and both act rationally given their perceived view of the game and assume the other player does as well.

This example illustrates the general formulation of a game with unawareness as a collection of standard normal form games: A game describing how each player views the game, a game describing how each player views another player's view of the game, games describing how they view how others view others' view of the game and so on. We also introduced one of the properties linking these viewpoints when we assumed that if Alice is aware that Bob is aware of X she is also aware of X . This corresponds to the interpretation that if Alice thinks Bob thinks X is an important part of the game, she can definitely reason about X and would model and consider it as well. We also introduced how solutions can be extended by assuming players use reasoning principles based on solutions in the games as they perceive them while taking into account other players will apply those principles in the games *they* perceive. Here we have Alice, in the game as she perceives it, plays a best response to a strategy of Bob in the game Alice think that Bob is considering and Bob using this reasoning w.r.t. the viewpoints he perceives to devise his best response.

Our second example applies the same principles of modeling and reasoning to a repeated game setting. In this example a simple twist in the players unawareness leads to a cooperative result based on best response reasoning. Consider the following prisoner's dilemma repeated, say ten times:

$$\begin{array}{cc}
 & \text{Bob} \\
 & \begin{array}{cc} b_1 & b_2 \end{array} \\
 \text{Alice} & \begin{array}{cc} a_1 & \begin{array}{cc} 2, 2 & -1, 3 \end{array} \\ a_2 & \begin{array}{cc} 3, -1 & 0, 0 \end{array} \end{array}
 \end{array} \quad (3)$$

We denote this stage game by G_1 . Assume both Alice and Bob are fully aware and perceive the game being played is G_1 . However, both players initially perceive that the other player is unaware of the non-cooperative action, i.e., each player perceives the other player as perceiving the game as follows:

$$\begin{array}{cc}
 & \text{Bob} \\
 & \begin{array}{cc} b_1 & \end{array} \\
 \text{Alice} & \begin{array}{cc} a_1 & 2, 2 \end{array}
 \end{array} \quad (4)$$

Let us denote the game with the single cooperative action by G_2 . As the game is played repeatedly these perceptions remain unchanged as long as the players both cooperate, however if any one of the players defects the players immediately

become commonly aware of the full game, in other words they attribute the same full perception of the game as G_1 to each other iteratively to any high order.

We can capture this situation with the following representation. For a given history of play h we can denote by $G_{(h,i)}$ how player i views the game after observing the history h . Similarly, we denote by $G_{(h,i) \wedge (h',j)}$ how i after history h perceives that $j \neq i$ will perceive the game after j observed history h' . In our example, $G_{(\phi,i)} = G_1$, $G_{(\phi,i) \wedge (\phi,j)} = G_2$, $G_{(\phi,i) \wedge (\phi,j) \wedge (\phi,i)} = G_2$ and so on. Similarly, as long as the histories h, h', h'' being considered only include the cooperative actions (a_1, b_1) we have $G_{(h,i)} = G_1$, $G_{(h,i) \wedge (h',j)} = G_2$, $G_{(h,i) \wedge (h',j) \wedge (h'',i)} = G_2$ and so on. However, if defection occurs the perceptions change. Hence if either (a_2, b_1) , (a_1, b_2) or (a_2, b_2) has occurred in h' we have that $G_{(h,i) \wedge (h',j)} = G_1$. In particular, player i realizes that if defection is played then player j 's perception will include the complete game.

The description above does not paint a full picture of how the players perception of the game might evolve. For example, after seeing defection by Alice it could be the case that Bob realizes that Alice was aware of the full game all along, or Bob may reason that she just recently became aware of the possibility of defection. These are different scenarios and can be captured by correspondingly completing the definition of various higher order perceptions in this game. For example, we can assume that $G_{(h,i) \wedge (h',j) \wedge (h'',i)} = G_1$ when defection has occurred in h' for all h'' including say $h'' = \phi$. This amounts to assuming that i believes that once defection occurs then j will realize that i was aware of the full game all along. Or, we can model a situation where i thinks that j thinks that i learned about the defection action, rather than perceived it all along. Similarly, further variations could represent a difference in how higher order perceptions evolve based on the particular player who defected. For example, if in the history h' player i was the first to defect then player j may reason that player i knew about defection all along, while if it was only j that defected first, i may think that j only learned about defection at that point in time from her. Clearly this can be further complicated as one considers various higher level bifurcations of how players interpret each others' past perceptions based on learning something unexpected.

No matter how we complete some these higher order perceptions, it is important to note that the only perceptions relevant are those that the players think possible. Hence, since $G_{(\phi,1) \wedge (\phi,2)} = G_2$ we do not consider the perception of $(\phi, 1)$ – Alice before actions are taken – of the perception of $((a_1, b_2), 2)$, since Alice at the beginning does not think that Bob is reasoning about action b_2 . In other words we do not consider the following iterated view of the game $G_{(\phi,1) \wedge ((a_1,b_2),2)}$ as relevant. Hence, the higher order perceptions that are considered respect the restrictions of reasoning that the players attribute to one another for each perception associated with a player and a relevant history of play.

We can now turn to a quick intuitive analysis of reasoning in this example. Each player reasons that if they choose to defect at any point it will cause the common perception that the game being played is G_1 . Hence, each player would anticipate that as long as they play cooperatively, they will receive a payoff of 2 since they assume the other player will play cooperatively as they perceive them to be unaware of any other action. Each player will reason that playing cooperatively until the final round and defecting in round ten would lead to a payoff of $2 \cdot 9 + 3 = 21$ from their perspective. Any deviation earlier would lead to a payoff of 3 instead of 2 in that early round of assumed unilateral defection, followed by defection by both players until the end if we assume any of the standard solution concepts to the remaining commonly perceived finite prisoner's dilemma. Hence the maximum payoff attained by defecting earlier is strictly lower. We have that both players will therefor play cooperatively in all nine rounds and defect in the last one. Needless to say that both players would be surprised to see the other player defect in the last round. The best response reasoning we described is based on assuming that once the game unravels to be G_1 both players will deviate throughout which is supported in the extension of standard solution concepts to our game forms.

We note that one can consider higher order variations of this example, such as a case where each player perceives the full game G_1 and perceives that the other player also perceives the game as G_1 but that the other player perceives them as perceiving the game as G_2 , in other words $G_{(\phi,i) \wedge (\phi,j)} = G_1$ and $G_{(\phi,i) \wedge (\phi,j) \wedge (\phi,i)} = G_2$. In this case, under the assumption that any defection unravels any misperception and that reasoning in the unraveled game leads to defection throughout, there is a cooperative solution where the players will cooperate up to round eight, and then defect in the last two rounds expecting the other player to defect only in round ten.

Our third example applies the same principles of modeling and reasoning to a dynamic setting which allow us to also capture how the players' perceptions might change or be changed strategically. We begin with a story in which Alice – the baker – has contracted with Bob – the coffee shop owner – to produce and deliver baked goods from her bakery to his coffee shop. Alice and Bob have contracted on the baked goods based on their expected costs and revenues from the transaction. Alice's cost for producing and delivering the baked goods on Monday is 4. The value to Bob if the goods arrive on Monday is 7 and we assume they contracted on a price of 5 for the baking and delivery of the goods on Monday. Hence, their payoffs will be 1, 2 respectively. However, Alice can also bake the goods and deliver them on Tuesday. Given how busy Monday is, her cost of baking and delivering the goods on Tuesday is lower and stands at 2, the

value to Bob for receiving the goods on Tuesday is much lower and stands at 3 since many of his customers prefer to purchase the goods on Monday. Anticipating that, Alice and Bob have stipulated in the contract that if Alice delivers the goods on Tuesday Bob will only need to pay 2 instead of 5. Leading to a final payoff 0, 1 to Alice and Bob respectively. This situation so far is depicted in Figure 1.

The contract also stipulates that in the case of unforeseen contingencies beyond her control that prevent Alice from delivering the baked goods on Monday, she is expected to deliver them on Tuesday but will be paid the full 5 payment at that point and not the reduced payment. The payoffs may depend on the particular contingency. For example, if Alice produces the goods but her delivery truck meets an unforeseen contingency she may need to produce and deliver the baked goods again on Tuesday which will cost her a total cost of $6 = 4 + 2$. In such a case, even though she is paid 5, she ends up with -1 . While, if there is a severe snow storm on Monday and roads are closed, she could reasonably decide not to produce on Monday and produce and deliver on Tuesday and incur only a cost of 2 and receive 5 from Bob to end up with 3.

Driving to his coffee shop early Monday morning Bob notices that unanticipated major road work is scheduled for that day (and that day only). He estimates that if Alice attempts to deliver the baked goods on Monday then there is a 50% chance it will not arrive on time and in that case the baked goods will be ruined. Bob is confident that Alice is not aware of the pending construction since the bakery is in the other side of town and he knows Alice lives close to the bakery. Bob realizes that the strategic situation is no longer as he envisioned it and was depicted in Figure 1. Bob realizes that there was a nature move that resulted in major road delays. Moreover, Bob realizes that he now has new

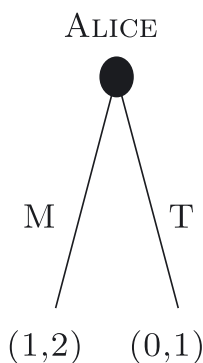


Figure 1: Alice's shipping decision.

possible actions; he can either call Alice and let her know about the road work or not.

Let's consider the payoffs given this new situation. If Bob does not call Alice then Alice will not know about the possibility of road work, she will be unaware of it and assume the game is as depicted in Figure 1. However, the payoffs will be different. If Alice attempts the delivery on Monday then with 50% chance it will go through and Bob's payoff will be 2, at the same time Alice will receive 1 if it gets through on time. With 50% chance the delivery will not go through and in this case Alice will need to bake and deliver the goods again on Tuesday. In that case unforeseen contingencies will be invoked and Alice will still receive the full payment on Tuesday. Bob's payoff will be $-2 = 3 - 5$, at the same time Alice will receive $-1 = 5 - 6$ since her cost will include the production and delivery costs for both days. Note that these are the eventual payoffs and are also the payoffs as perceived by Bob's viewpoint. But when Alice makes her decision whether to make and deliver the goods on Monday or Tuesday, she is unaware of these payoffs and believes the game is as depicted in Figure 1.

If Bob decides to tell Alice about the impending road work, the expected payoffs for an attempted delivery on Monday will not change for either player. But now Alice will be aware of the unforeseen contingency and realize that road work is possible and actually occurred. At that decision point she can invoke it as a justifiable reason to produce and deliver the baked goods on Tuesday without penalty based on the unforeseen contingency clause. Hence, her payoff will be $3 = 5 - 2$, while Bob will have to pay the full price and end up with $-2 = 3 - 5$.

We depict this game with unawareness using extensive form games as they are perceived by the players in every decision point. The game with unawareness we just described is depicted in Figure 2. The game includes Nature choosing whether there is road work. If there was no road work we have the game depicted in blue which represents how Alice (and Bob) perceive the game when she is unaware of the road work. If road work occurs as in our story (or considered explicitly), Bob can choose whether to tell Alice or not. If he doesn't tell Alice she is still unaware of the road work although it happened, hence she perceives herself to be at a different decision point corresponding to the game depicted in Figure 1 imbedded in the game depicted in Figure 2 after the "no road work" option. We depict this link as a blue arrow indicating where Alice perceives the game is. Hence the game depicted in blue corresponds to the game when there is no awareness of the road work or its possibility. If Bob does let Alice know about the road work she becomes aware of it and realizes the full scope on the game and takes an action at that decision point. The game as perceived by Bob after he learns of the road work and will be perceived by Alice

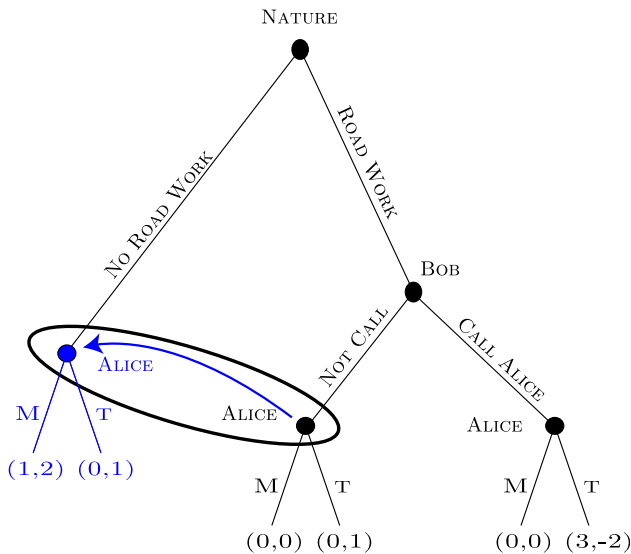


Figure 2: Bob’s Calling Decision. This is the game as seen by Bob, if he does not call Alice he thinks that Alice views the game as in Figure 1 depicted in blue (the arrow indicates that in both decision points of the information set the Alice perceives the same game, in particular she is unaware that there is an information set).

if he tells her is the combined game in black and blue. Whereas aware Bob and Alice also view Alice’s view of the game when she is unaware as denoted by the blue game and the blue arrow.

Given the description of the game we can now consider the players’ behavior. Using a backward induction principle it seems that Alice would choose to produce and deliver on Monday if she believes the game is as depicted in Figure 1. Hence, she is expected to do so also if there is road work but Bob does not call in the game in Figure 2. If Bob does call she is likely to choose Tuesday since she becomes aware of the road work. Hence, Bob can conclude that his payoff will be -2 if he calls Alice and 0 if he doesn’t, hence when he observes the road work he will, by backward induction, not call Alice leading to a payoff of $0, 0$. Hence, we applied a backward induction reasoning where each player is considering how other players in the future will apply backward induction in the game which the player thinks they will perceive.

Consider now another twist of the baked goods delivery scenario. This time, it turns out that Carol who works for Alice at the bakery lives nearby the road construction area and observes the unanticipated road work as well. Carol has two actions, she can text Alice about the potential road work or not. Her payoff for

reporting it is 1 while her payoff if she does not report it is 0. Had there been no road work Carol's payoff is assumed to be 0 as well. In this case Bob is unaware of Carol's actions, in fact he may be unaware of Carol's existence and may have not modeled the possibility of an employee living next to the road work. Hence, Bob's view of the game as is we described in Figure 2. However, the game now has a third player, Carol and her action can impact the awareness of Alice. We depict the game in Figure 3. Here we have the game as originally modeled by Alice in blue, the games in green and blue describe how Bob views the game and how he views Alice will view the game if he does not call her to let her know about the road work (in blue) as well as if he does call her (in green). The game now also has Carol's decision points and payoffs (in black) as well as how will Alice perceive the game if Carol texts her about the road work (in black). Hence, if Carol texts Alice then Alice will view the game as in Figure 3, while viewing Bob's view of the game if he observes road work as depicted in the game with unawareness in Figure 2 which includes how Bob views Alice's view of the game when she is unaware of road work as in Figure 1. The payoffs remain unchanged from the previous game and we just note that Alice's payoff when Carol texts her is now 3 (and Bob's -2) if she produces and delivers on Tuesday no matter if Bob calls her or not, since she can now invoke the unforeseen contingency. In fact, we can imagine a situation where her payoff is even higher if Bob does not call her especially since she is aware that he is aware of the road work¹.

The game form illustrates the nested nature of higher order view points. In this game the green decision points and actions constitute a subtree hence naturally correspond to a game tree. Furthermore, there is unawareness of a player (Bob is unaware of Carol). The reasoning can be applied similarly as before, Carol will text Alice who will then become aware of the full scope of the game with unawareness and will choose to invoke the contingency and deliver on Tuesday. Bob will not call Alice being unaware of Carol and he would expect Alice to try and ship on Monday.

The extensive form game with unawareness is a collection of standard extensive form games corresponding to how Alice perceives the game at her decision points, how Alice perceives how Bob perceives the game at his decision points in the game she considers, how she perceives the game he perceives that she perceives in her decision point in the game she perceives he is considering, and so on.

¹ This game is a slightly simplified version of the game we defined in Feinberg (2012). In that game Alice could deliver the goods on Monday, Tuesday and Wednesday and we used that setting to allow Alice to get an even higher payoff if she was aware of the road work and Bob decided not to call her. A similar setting can be achieved here by increasing the payoff for Alice and decreasing it for Bob at that particular outcome.

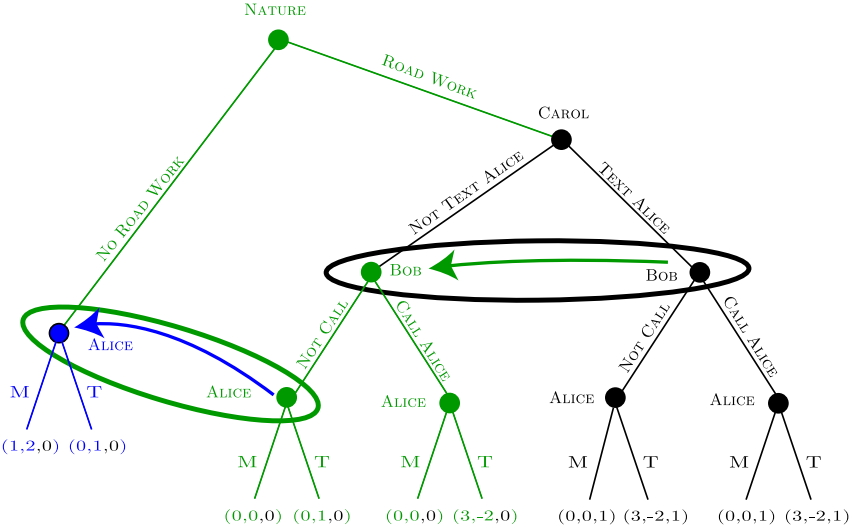


Figure 3: With Carol’s E-mail Decision. This is the game as seen by the modeler as well as by Alice once Carol texts her. The game in green and blue indicates the game as Bob is aware of once he learns about the road work, it is also how Alice views Bob’s view of the game once she knows about the road work. The game in blue is the game that Alice plays if she is unaware of the road work, what Bob thinks she plays if he doesn’t call her and what Alice thinks Bob thinks she thinks the game is if he does not call her.

These nested games will have consistency conditions that assure principles such as Alice being aware of what she thinks Bob is aware of, as well as making sure that the view from each decision point is indeed itself a game with unawareness. As with the extension of normal form games, solution concepts can be extended by applying reasoning principles to the decision points taking into account that these will be applied to the nested extensive form games as they are perceived in those decision points.

1.2 Results and Related Literature

The examples above are meant to illustrate a number of features of our model. First, the definition of games with unawareness for all game forms is based on a collection of standard games, games that describe players’ viewpoints, views of other players views of the game and so on. In the case of extensive form games, viewpoints correspond to decision points, in incomplete information games defined below they will correspond to types. We can also see the nested features of

these games where the actions and players that Alice views Bob is considering become part of the game that Alice is considering. Hence the game form satisfies the property that if Alice thinks Bob is aware of an aspect of the game she is also aware of that aspect. The examples also demonstrate how solutions can come about in these games as they illustrate the best response reasoning as Alice considers the actions of Bob in the game she perceives he is considering.

After defining the extended forms for normal, repeated, incomplete information and extensive form games with unawareness, our structural result, Proposition 1, shows that all the definitions satisfy the property that each player's view of the game, view of others, and so on, itself is a game with unawareness satisfying the exact same conditions as we, the modelers, are considering. In particular, each view of each other player induces a game with unawareness and so on at higher levels of iterative views. Hence we have a common perception that all players are modeling a game with the same framework. Next we extend rationalizability and Nash equilibria to normal form games with unawareness and prove existence in these possible infinite structures using the generalizations of equilibria existence conditions due to Glicksberg (1952) and Fan (1952). We similarly define extended BNE for incomplete information games with unawareness and extend the notion of assessment to extensive form games with unawareness allowing the definition of assessment based equilibria (such as sequential equilibria). We then demonstrate that all extended solution concepts are non-empty (assuming standard conditions for the underlying perceived games) and that the extended solutions coincide with the standard solutions when there is no unawareness. Games with unawareness may have infinite hierarchies of the perceptions of the game yielding an infinite number of different viewpoints within a single game with unawareness. Using a result by Higman (1952) we provide sufficient conditions for a game with unawareness to induce only a finite collection of viewpoints and hence have a finite representation.

One of the main questions that arises with any new game form is whether it can tell us something that the standard formulations cannot. With restricted perceptions it is particularly important to study whether these situations can be modeled using probability zero rather than unawareness. We show that indeed normal form games with unawareness can be represented as games with incomplete information. Moreover, there is a natural mapping (in the sense that the mapping is independent of the exact payoffs in the game with unawareness) to incomplete information games that fully captures all the relevant views in the game. But while the structure can be mapped between game forms the extended NE in games with unawareness does not map to any known solution of Bayesian games. In fact, it seems that generating such a solution in the Bayesian game equivalent would essentially amount to reconstructing the normal form game

with unawareness. We show that the extended NE is a strict refinement of the Bayesian NE of the incomplete information representation, while, interestingly, the latter coincides with extended rationalizability for normal form games with unawareness.

There are two general types of studies that relates to multi-person unawareness in economic theory. The first is a logic and foundational perspective. One of the earliest works providing foundations for unawareness is Fagin and Halpern (1988) who provide a host of logics for unawareness and study their syntax and semantics. They mention the “semantic naturalness” of their modeling as it provides a clear interpretation of the notion of awareness. Halpern (2001) and Halpern and Rego (2008, 2009, 2013) expand the analysis as well as provide more insights on the properties of unawareness, with the latter two papers providing important insights on the foundational question of modeling of awareness of unawareness. Modica and Rustichini (1994, 1999) have developed a decision theoretic treatment of unawareness based on the notion of event-based reasoning (partition models). The work of Heifetz, Meier and Schipper (2006, 2008) took an event-based approach to interactive unawareness. This semantic approach provided a natural framework for interactive unawareness that also allows to bypass the impossibility results of the classical partition model framework. Li (2008a, 2009) provided an alternative product based construction for modeling unawareness and Sadzik (2006) developed a novel framework for the foundations of probability and unawareness. Board and Chung (2007) propose an alternative approach and an axiomatization that also touches on awareness of unawareness and Walker (2014) provides an alternative semantic approach.

The second type of studies considers the modeling of games with unawareness, the extension of existing solution concepts and exploration of new ones as well as applications to particular economic settings. Many of the foundational papers were expanded or include the development of tools for modeling strategic interaction with unawareness. Copic and Galeotti (2005) consider awareness action equilibrium in the context of interactive awareness. Li (2006) considers unawareness in dynamic games. Mengel, Tsakas and Vostroknukov (2009) model and study unawareness in repeated games. Grant and Quiggin (2013) provide an alternative model for extensive form game and Halpern and Rego (2014) extend their framework to extensive form game and define Nash equilibria while Rego and Halpern (2012) define sequential equilibria. Meier and Schipper (2014) provide a model for Bayesian games with unawareness, Heifetz, Meier and Schipper (2013a) apply their foundational approach to strategic introductions and specifically explore characteristics of the no-trade theorem. In Feinberg (2004a) a syntax based approach to interactive unawareness is introduced and applied to extensive form

games. Heifetz, Meier and Schipper (2013b) consider the application to dynamic games and extend extensive form correlated rationalizability to these games. Board and Chung (2009) explore the application of their axiomatic model to interactive unawareness and Sasaki (2014) defines subjective rationality in games where players may misspecify the game. Additional studies both model and apply unawareness in interactive settings to particular classes of economic problems. Chen and Zhao (2013) explore unawareness in principle-agent models and von Thadden and Zhao (2012) study incentive schemes in the presence of unawareness and in von Thadden and Zhao (2013) explore multi dimensional principal agent problems with unawareness, Zhao (2011) studies the framing of contingencies in contracts, Filiz-Ozbay (2012) studies unawareness modeling in contract theory, Grant, Kline and Quiggin (2012) explore different notions of awareness and ambiguity in contracts and Zhao (2008) studies moral hazard with unawareness. Our approach extends the conceptual principle of modeling the subjective reasoning of players from the perspective of every decision instance which we developed in Feinberg (2005b, 2005c). Finally, the work in Feinberg (2005a) contains initial results regarding modeling normal form games with unawareness and is subsumed by the current paper. For more related work on unawareness also consult the bibliography set up by Burkhard Schipper at <https://www.econ.ucdavis.edu/faculty/schipper/unaw.htm>.

The use of standard games as the primitive building block for games with unawareness allows us to keep the definitions relatively short and simple with four conditions applied to each game form and reduces the variation between the game form being used. The nested structure of standard games also provides a relatively clear extension of existing solution concepts to games with unawareness. Using hierarchies of perceptions allows the proof of consistency of the representation in the sense that the modeler, players, and higher order players' view of other players, all have the same form as games with unawareness. The importance of this characteristic is evident if we view our modeling as prescriptive for reasoning agents and, therefore, want its descriptive nature to coincide with how the reasoning agents are assumed to model their strategic environment.

2 Modeling Games with Unawareness

A game with unawareness is defined by describing the set of players, actions and payoffs in the strategic situation, how each player views these, how they view others' views and so on, while allowing these views to be restricted – exclude players or actions from such perceived games. The main principle for defining these games is that every view point is described by a standard game of the corresponding form.

We extend four game forms to games with unawareness: normal, repeated, incomplete information and extensive form. We postulate four conditions that are adapted to each game form:

Condition 1. The decision maker Alice views the decision maker Bob to be relevant for the strategic situation if and only if she views Bob to be a player in the game she is considering.

Condition 2. Every action or player that Alice perceives that Bob is modeling in the game, is also part of Alice's perception of the game – Alice can consider the aspects of the strategic situation that she thinks Bob is considering.

Condition 3. Alice's view of Bob's perception of the game coincides with her view of Bob's view of his own perception of the game.

Condition 4. Even if Alice is unaware of Bob's participation in the game, the outcomes she considers must agree with the outcomes of the game given one of Bob's actions.

We generalize these conditions to high order reasoning in the formal definitions below. The rest of this section is organized as follows. We begin with a definition of normal form games with unawareness which is followed by a discussion of the hierarchy structure of the game form. The following subsections present the definitions of the other game forms concluding with the proposition stating that our model provides a consistent representation of games with unawareness in the sense that every player's view of the strategic situation is itself a well defined game with unawareness, as is a player's view of the view of others, and so on, and that these forms coincide with our description of the game as modelers.

2.1 Normal Form Games with Unawareness

In standard normal form games the modeler describes the set of players, their possible actions and payoffs for action profiles. The modeler's normal form game is our starting point: $G = \left(I, \prod_{i \in I} A_i, \{u_i\}_{i \in I} \right)$ where I is a finite set of players, A_i is the finite set of actions available for each player and the functions u_i associate the utility for action profiles in $\prod_{j \in I} A_j$. These are the set of players and actions that the modeler is considering whether or not the players are aware of each other or of some of the actions. Each player may have a restricted view of the game. Hence for a player $v \in I$ we consider a normal form game $G_v = \left(I_v, \prod_{i \in I_v} (A_i)_v, \{(u_i)_v\}_{i \in I_v} \right)$. Similarly, a player considers how each of the players that appear in her game models the game. In general, a finite sequence of players $v = (i_1, \dots, i_n)$ is associated with a normal form

game $G_v = (I_v, \prod_{i \in I_v} (A_i)_v, \{(u_i)_v\}_{i \in I_v})$ where I_v is the set of players that i_1 finds that i_2 finds that ... that i_n is considering, and similarly for the sets of actions $(A_i)_v$ and payoffs $(u_i)_v$ defined on the set of action profiles $\prod_{i \in I_v} (A_i)_v$. We call v an *iterated view*, or in short a *view*. Throughout, $v = \phi$ corresponds to the modeler's view, i.e., $G_\phi = G$. Note that this is the modeler's view of the relevant players and their actions and payoffs not of the unawareness of players. Correspondingly, we say that G_v is the situation as viewed, or perceived at v , that $(A_j)_v$ is the set of j 's actions as viewed from v and so on. We denote an action profile in G_v by $(a)_v$. The singletons $(v = i)$ corresponding to a player's view are called *viewpoints* and the set of viewpoints (players, in the case of normal form games) is denoted V with a typical element v .

We denote the concatenation of two views $\bar{v} = (i_1, \dots, i_n)$ followed by $\tilde{v} = (j_1, \dots, j_m)$ as $v = \bar{v} \wedge \tilde{v} = (i_1, \dots, i_n, j_1, \dots, j_m)$. The set of all potential views is denoted $\bar{V} = \bigcup_{n=0}^{\infty} (I)^{(n)}$ where $I^{(n)} = \prod_{j=1}^n I$ and $I^{(0)} = \phi$.

Definition 1. A collection $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ where G_v are normal form games and $\mathcal{V} \subset \bar{V}$ is a collection of finite sequences of players is called a normal form game with unawareness and the collection of views $\mathcal{V}$ is called its set of relevant views if the following properties hold:

C1 For every $v \in \mathcal{V}$ we have

$$v \wedge v \in \mathcal{V} \text{ if and only if } v \in I_v \quad (5)$$

The first condition requires that the set of relevant views $\mathcal{V}$ is closed under the set of players considered in the game perceived at a relevant view and that viewpoints of non-players are irrelevant. There would be no impact on our results if players were to consider the views of players not participating in the game, however we find such redundancy unpleasing, as with any scientific modeling. The other direction of this condition is crucial for our setting since if Alice models Bob as one of the players in the game, it is required that Alice should find Bob's view of the game to be relevant.

C2 For every $v \wedge \tilde{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \quad (6)$$

$$\phi \neq I_{v \wedge \tilde{v}} \subset I_v \quad (7)$$

$$\text{for all } i \in I_{v \wedge \tilde{v}} \text{ we have } \phi \neq (A_i)_{v \wedge \tilde{v}} \subset (A_i)_v \quad (8)$$

The first part of this condition states that if a relevant view's perception of another view is relevant, then the first view must itself be relevant, e.g., if it is relevant to consider Alice's view of Bob's view of Carol, then Alice's view of Bob is also

relevant. Together with condition **C1** this implies that the set of relevant views is exactly the set of views inductively constructed from considering the players that are perceived to be participating in the game. In particular, if there exists any relevant view at all we also have, fortunately, that $\phi \in \mathcal{V}$ – the modeler’s view is relevant.

Condition **C2** extends this principle to the set of players and actions: if Alice finds that Bob is considering a player or an action as part of the game, she herself must consider them to be part of the game. In other words, by the mere fact that you find it relevant that others find some aspect of the game to be relevant enough for modeling, you must model that aspect yourself. Much like we, the modelers, do when considering the reasoning of players. Restating this with the notation of awareness the condition states that: what Alice is aware that Bob is aware of, are things that Alice is aware of as well.

C3 If $v \wedge v \wedge \bar{v} \in \mathcal{V}$ we have

$$v \wedge v \wedge v \wedge \bar{v} \in \mathcal{V} \quad (9)$$

$$G_{v \wedge v \wedge \bar{v}} = G_{v \wedge v \wedge v \wedge \bar{v}} \quad (10)$$

The third condition requires that each player, that is relevant along some view, has a correct perception of their own perceived perception: if Alice perceives that Bob has a certain perception of the game, she also perceives that he perceives to have that perception. With awareness: if Alice is aware that Bob is aware of something, she is also aware that he is aware that he is aware of it. It is important to point out that this does *not* imply that Bob is actually aware of it. We note that when considering a relevant view of the form $v \wedge (i)$ we have that $v \wedge (i) \wedge (i)$ is relevant and by **C1** we have $i \in I_{v \wedge (i)}$ hence all players are aware of their own participation in the game.

C4 For every action profile² $(a)_{v \wedge \bar{v}} = \{a_j\}_{j \in I_{v \wedge \bar{v}}}$ there exists a completion to an action profile $(a)_v = \{a_j, a_k\}_{j \in I_{v \wedge \bar{v}}, k \in I_v \setminus I_{v \wedge \bar{v}}}$ such that

$$(u_i)_{v \wedge \bar{v}}((a)_{v \wedge \bar{v}}) = (u_i)_v((a)_v) \quad (11)$$

Since a view may consider only some of the players considered by another relevant view, or by the modeler, the payoffs may not be uniquely determined by defining the restricted set of players and actions. The fourth condition requires that the payoffs in a restricted game coincide with payoffs in the larger game

² We use a set notation for the action profiles since we have varying set of players considered. This assumes the action sets for players are notated in a distinct manner. Alternatively, one can use vectors for action profiles but note that they are different projections based on the players considered.

with more players by fixing some action profile for these players. We note that for different action profiles of the restricted game we can have different completing actions by players that only appear in the less restricted game³. In other words, a restricted view of the game cannot introduce new payoffs. If one wishes, this condition can easily be generalized to assume missing players are playing mixed strategies, or even correlated strategies, without impact on our results.

These properties are used in all game forms discussed below. While the objects in the game may change (with the addition of histories, types, or game trees) we obtain analogous properties describing the structure of games with restricted reasoning translating conditions **C1-C4** to the appropriate setting. Similarly, we consider standard game forms, e.g., finite normal form games, as the building blocks for our games with unawareness. While many of our results extend to a variety of generalizations this simplifies stating our constructions and results in a unified manner across game forms.

2.2 Hierarchies and Types

Before we define the extension to additional game forms, we briefly discuss our choice for describing a game with unawareness as a collection of standard form games – a “hierarchy” of games, rather than a “type space” – a set of type profiles representing each player’s perception of the game and perception of the types of other players. First we note that a representation a la Harsanyi’s type spaces is possible and actually quite readily obtained and is briefly provided below. Moreover, one can also define a universal type space which corresponds to the incomplete information representation we provide in Section 3.2. Hence, the transition from hierarchies to type spaces and back is quite a standard (if tedious) exercise. However, this brings us to a central reason for choosing hierarchies: in the standard incomplete information framework, the set of fundamentals generating the universal type space is fixed. In particular, all players are implicitly assumed to be reasoning with the same set of fundamentals which, in the Harsanyi’s approach, includes the collection of relevant payoff matrices for the fixed and given set of players and their given sets of actions. This makes the universal type space “common knowledge” in the sense that all players

³ This condition is weaker than the condition we proposed in Feinberg (2005a) where we required that in all the games a certain player is missing, the *same* action of this player will be used to determine the payoffs. We also note that conditions **C1-C3** coincide with the weak axioms for unawareness we postulated in Feinberg (2005a).

would construct the same universal type space if they were modeling the game, see Aumann (1999) for a discussion of this property. In contrast, our starting point is that players may reason using different fundamentals – the building blocks of the game. As such, they would construct different universal type spaces – different from each other and different from the one the modeler uses. Moreover, the players would also attribute different universal type spaces to other players. As such, while we can combine all these to an abstract space as sketched below, working with such a space seems much more difficult than the analysis we did in the examples in the introduction via the hierarchy representation. Working with a type space representation that captures reasoning with differing fundamentals requires extra caution since one needs to figure out at each state how each player perceives the type space. While this is possible, it essentially amounts to reconstructing the hierarchies. Hence, we prefer working directly with hierarchies. This is best exemplified when constructing solution concepts. If we want a player type to play a best response to what other players' types are playing we need to consider the types that correspond to this player's type space, which is not the modeler's type space. In addition, this player will need to evaluate the type spaces that each other player is considering, the spaces that each of them associates to other players (more precisely, the spaces that the first player *perceives* the other players think others are considering), and so on – recreating the hierarchies once again. Finally, the hierarchies allow us to obtain an “equal perception” principle. In the sense that the modeler and the players are easily shown to model the situation in the same manner (as well as model others' perceptions and so on). While they may use different fundamentals for reasoning and attribute different fundamentals to others, the rules governing the reasoning, the relationships between these sets of fundamentals and the definition governing the game form all are of the exact same form. Not only are the games the players consider of the same form, the games they perceive other consider are of the same form and all these coincide with the modeler's game form. This is not a minor issue since, at least in economics, one might consider it desirable that a model used in describing or predicting behavior will not be at odds with its own predictions if the modeled players themselves were to use the exact same modeling approach. When one begins with an abstract modeler type space alone, it is not a priori clear whether this property holds.

We describe a type space representation for normal form games with unawareness with two players that are aware of each other. The generalizations are later described. The type space representation is defined as a product $T_1 \times T_2 \times \Lambda$ and mappings τ_1, G_1, τ_2, G_2 such that for $i, j \in \{1, 2\}$ and $j \neq i$ we have: T_i is player i 's type space, Λ is a set of normal form games, $\tau_i: T_i \rightarrow T_j$ and $G_i: T_i \rightarrow \Lambda$. Hence, each $t_i \in T_i$, is mapped to a member $\tau_i(t_i) \in T_j$ and a game $G_i(t_i) \in \Lambda$, these

correspond to how type t_i views j 's view of the game with unawareness and the game as t_i views it. Note that the hierarchy of how i 's type views j 's view of i 's view etc, is captured via iterations of the mappings τ and G .

We now add the conditions that correspond to conditions **C1-C4** for the type space representation. Condition **C1** will be represented by the requirement that the game $G_i(t_i)$ is a two person game played by i and j . Condition **C3** will hold immediately from the type space representation. Condition **C4** holds for this specific case where there is no unawareness of players. For condition **C2** we will require that the game $G_j(\tau_i(t_i))$ is obtained from the game $G_i(t_i)$ by (at most) eliminating pure strategies.

We need to show that every game with unawareness corresponds to a hierarchy of some type space and mappings which satisfy the above conditions and that for every type space with mappings as above we have that every pair of types generates a hierarchy that corresponds to a game with unawareness. The first part will be shown when we discuss the canonical representation of games with unawareness later on. The second part follows from noting that for a given type t_i , we can define the following hierarchy of games $G_i(t_i)$, $G_j(\tau_i(t_i))$, $G_i(\tau_j(\tau_i(t_i)))$, ..., which corresponds to the views i , $i \wedge j$, $i \wedge j \wedge i$, ... respectively. For every pair of types t_1, t_2 we define G_ϕ as the game that contains the union of pure strategies of $G_1(t_1)$ and $G_2(t_2)$ (choosing arbitrary payoffs for payoff of action profiles that are not already present in either G_1 or G_2). We define the games for all views by extending the above inductively so that $G_{v \wedge v \wedge \bar{v}} = G_{v \wedge v \wedge \bar{v}}$ and we have a game with unawareness satisfying **C1-C4**.

This equivalence extends to more than two players without any change in conditions. For the case where players may be unaware of other players we need to modify τ_i so that it does not necessarily map to a type profile of *all* other players, but only the players that the specific type t_i is aware of. Similarly, $G_i(t_i)$ will include exactly the players that $\tau_i(t_i)$ maps to in addition to player i herself. Finally, we require that for a player j in $G_i(t_i)$ the game $G_j(\tau_i(t_i))$ is obtained from $G_i(t_i)$ by at most eliminating actions and players and preserving the payoff condition as in condition **C4**. Thus, the resulting type space and mappings will represent all normal form games with unawareness. Note that while the representation has a fairly simple space structure, the set of fundamentals considered by different types (even of the same player) can be quite different.

For the most part a compact representation of a game with unawareness can be produced. For example, in the first example in the introduction Alice and Bob are both aware that the game is as in (1), but Alice thinks that Bob is unaware of one of her actions and considers the game as in (2). Bob actually realizes that Alice attributes to him this restricted view. The representation of this game with unawareness can be collapsed to three states $\{x, y, z\}$, two states x, y corresponding

to (1) and the state z to (2). Alice is unaware of the actual state x , at state x she considers state y to be the actual state, at state y Bob would consider state z to be the actual state and he would be unaware of state y . Hence, Alice thinks Bob views the game as (2) while Bob realizes this and actually views the game as (1). For the seminal formal treatment of interactive unawareness and in particular the relationship between models (state space formulation) and the syntax of hierarchies see Fagin and Halpern (1988).

2.3 Repeated Games with Unawareness

Repeated games introduce a dynamic environment where the restricted view of the players may change as the games unfolds. Their view may be widened from directly observing behavior they previously did not consider, from reflecting about behavior, or any other means of discovery. A view can also dynamically narrow from forgetfulness, or by deeming some aspect of the game irrelevant at some point.

As with the normal form games, we consider a collection of repeated games corresponding to relevant views – sequence of viewpoints. This time, however, a player's limited view may depend on the period in which the game is played and, most importantly, it may also depend on the history of play that this player observed. Hence, the collection of potential *viewpoints* includes every player at every period conditional on every possible history of actions by a subset of players.

We define repeated games with unawareness as a collection of standard repeated games satisfying the four consistency conditions as above. The first condition accommodates the added dynamic constraint on relevant viewpoints which now refers to specific histories. For example, if Bob cannot reason about a certain action in the first period of the repeated game, he cannot reason in this first period about his, or Alice's, reasoning in the second period conditional on this action being taken. As such, the viewpoint after the action has occurred is not relevant for views that do not consider that action possible. The third condition is modified to allow for imperfect monitoring and forgetfulness. In these cases a player may not be aware of, or remember, the history of play which requires a modification of the self awareness condition. The other two conditions remain intact.

Let $G(T)$ denote the $T \geq 1$ repeated game with a stage game $G = (I, \prod_{i \in I} A_i, \{u_i\}_{i \in I})$ and payoffs determined by the sum or average of the stage game payoffs (similarly we can consider the infinitely repeated δ -discounted games $G(\delta)$). Let $h^t(J) = (a^1, \dots, a^t)$ denote a history of (restricted) action profiles in the first t periods where for every s we have $a^s \in \prod_{i \in J} A_i$. Hence a^s is the action profile for the given subset $J \subset I$. We denote by $A_i(h^t(J))$ the actions taken by i along the

history $h^t(J)$ assuming $i \in J$, i.e. the set $\{a_i^s | s = 1, \dots, t\}$. Let $H = \{h^t(J) | t = 0, 1, \dots, T-1 \text{ and } \phi \neq J \subset I\}$ be the set of all histories up to length $T-1$ with $h^0(J) = \phi$ for all J .

Each player after each history of play of a subset of players constitutes a possible viewpoint. The set of possible viewpoints is defined as the collection $V = \{(h^t(J), i) | h^t \in H, i \in I, \phi \neq J \subset I\}$, i.e., it is each player's view of the game as a function of the history at the time of observation when considering some of the players in the game. As before, a typical viewpoint is denoted by $v \in V$.

The set of all finite sequences of viewpoints denoted $\bar{V} = \bigcup_{n=0}^{\infty} V^{(n)}$ with $V^{(n)} = \prod_{j=1}^n V$ and the convention $V^{(0)} = \phi$. As before a finite sequence of viewpoints $v = (v_1, \dots, v_n)$ is associated with a repeated game $G_v(T) = (I_v, \prod_{i \in I_v} (A_i)_v, \{(u_i)_v\}_{i \in I_v})$ where I_v is the set of players that v_1 finds that v_2 finds that ... that v_n is considering, and similarly for the sets of actions $(A_i)_v$ and payoffs $(u_i)_v$ defined for the stage game with action profiles $\prod_{j \in I_v} (A_j)_v$ and repeated T times.

Definition 2. A collection $\Gamma = \{G_v(T)\}_{v \in \mathcal{V}}$ where $G_v(T)$ are repeated games and $\mathcal{V} \subset \bar{V}$ is a set of relevant views is called a repeated game with unawareness if the following properties hold:

CR1 For every $v \in \mathcal{V}$, $v = (h^t(J), i)$ we have

$$v \wedge v \in \mathcal{V} \text{ if and only if } i \in I_v, J = I_v, A_j(h^t(J)) \subset (A_j)_v \text{ for all } j \in J \quad (12)$$

The viewpoints that are considered relevant from the view v are exactly those that correspond to players and histories taken from the stage game $G_v(T)$.

CR2 For every $v \wedge \bar{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \quad (13)$$

and

$$\phi \neq I_{v \wedge \bar{v}} \subset I_v \quad (14)$$

as well as

$$\phi \neq (A_i)_{v \wedge \bar{v}} \subset (A_i)_v \quad (15)$$

for every $i \in I_{v \wedge \bar{v}}$.

CR3 If $v \wedge v \wedge \bar{v} \in \mathcal{V}$ where $v = (h^t(J), i)$ then there exists some $\tilde{v} = (\tilde{h}^t(\tilde{J}), \tilde{i})$ such that

$$G_{v \wedge v \wedge \bar{v}}(T) = G_{v \wedge v \wedge \tilde{v} \wedge \bar{v}}(T) = \dots = G_{v \wedge v \wedge \tilde{v} \wedge \dots \wedge \bar{v} \wedge \bar{v}}(T) \quad (16)$$

and $v \wedge v \wedge \tilde{v} \wedge \dots \wedge \bar{v} \wedge \bar{v} \in \mathcal{V}$.

This refines condition **C3** by allowing a viewpoint to consider itself with respect to the histories it can reason about. While a view v may consider $v = (h^t(J), i)$ because the *view* finds the history $h^t(J)$ possible, it will recognize that the decision maker i may not be aware of this history due to imperfect monitoring or forgetfulness. Hence, he might associate himself with other viewpoints such as $\bar{v} = (\bar{h}^t(\bar{J}), i)$.

CR4 For every action profile $(a)_{v \wedge \bar{v}} = \{a_j\}_{j \in I_{v \wedge \bar{v}}}$ there exists a completion to an action profile $(a)_v = \{a_j, a_k\}_{j \in I_{v \wedge \bar{v}}, k \in I_v \setminus I_{v \wedge \bar{v}}}$ such that

$$(u_i)_{v \wedge \bar{v}}((a)_{v \wedge \bar{v}}) = (u_i)_v((a)_v) \quad (17)$$

The dynamic features of repeated games with unawareness prompt us to consider the properties that relate a player's past and present views. As demonstrated in the example of a repeated game with unawareness in the introduction, what players learn about the perceptions of others may be quite intricate. This opens up the possibility of modeling a variety of "learning" dynamics, where learning fits a discovery of facets of the game previously not considered, included higher level of perceptions. We find this to be more close to the casual use of the word, to quote Alvin Roth: "One of the most general things that experiments demonstrate is that subjects adjust their behavior as they gain experience and learn about the game they are playing and the behavior of other subjects." (Roth 1995 p.327).

Memory For $v = (i_1(h_{t_1}), \dots, i_k(h_{t_k}), \dots, i_n(h_{t_n}))$ and $\bar{v} = (i_1(h_{t_1}), \dots, i_k(h_{t_{k+l}}), \dots, i_n(h_{t_n}))$ where $h_{t_{k+l}}$ is a continuation of h_{t_k} and such that $v, \bar{v} \in \bar{V}$ we have that $\alpha_v \subset \alpha_{\bar{v}}$. Memory assumes that awareness is monotonic, in the sense that what a player (i_k) is aware of after a history (h_{t_k}) he will still be aware of after some continuation of the game. This property states that a player remembers not only actions but also what he was aware of others' awareness, and it also assumes that others are aware the player remembers. Our results hold both with and without this assumption.

To get a better feel for the potential complexity of repeated games with unawareness consider the game depicted in Figure 1 repeated twice. In period 1 Alice and Bob are viewing each other's perception at the current period as before: Alice and Bob are aware of the actions $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, and Alice perceives that Bob is unaware of her action a_3 , i.e., she views him as viewing the game as depicted in Figure 2 as in our original example. But how Alice views (at either period) the viewpoint of Bob in period 2 will depend on the realization of play in period 1. If Alice played a_3 in period one, she may safely assume that Bob will remember this action and she will attribute to him the awareness he actually already has. Moreover, if Bob plays b_3 in the first period, Alice might deduce that Bob must have

been aware of a_3 in the first period (and if he remembers, in the second period as well) even if she did not choose a_3 in the first period. Hence, Alice's perception of Bob's awareness may change not only from the revelation of actions that were considered secretive, but also from behavior that may best be explained by a different scope of awareness Alice should attribute to Bob. In particular, Alice here may realize that her perception was limited (she did not consider Bob considering a_3) and revise her perception. In this case, Alice may actually reason at period 1 about how she would reason in period 2 about Bob's reasoning in period 1 if she observes b_3 . The important feature of the game form illustrated here is that the choice of relevant views may depend on how perception changes with observed behavior – a choice we usually associate with a solution manifests here in the game form. For more on this game and how communication impacts strategic interaction with unawareness see Feinberg (2007).

2.4 Incomplete Information Games with Unawareness

Incomplete information games with unawareness allow us to model uncertainty about the awareness of players in addition to uncertainties about the payoffs as well as high order uncertainties about both. They also allows us to model views that do not consider all possible types. As with other game forms we begin with the modeler's view which begins with a standard Bayesian game with a prior $G(B) = (I, \prod_{i \in I} A_i, \Theta_0 \times \prod_{i \in I} \Theta_i, P, \{u_i\}_{i \in I})$ where I is a finite set of players, A_i is player i 's finite actions set, $\Theta = \Theta_0 \times \prod_{i \in I} \Theta_i$ is a set of type profiles where Θ_i is the set of player i 's types and Θ_0 is the set of states of nature, the u_i 's are the players' utilities defined for a realization of Θ and action profiles, and P is a probability distribution over Θ . The distribution P is the probability over the type profiles as seen by the *modeler*. Recall that in a standard incomplete information game each type has a distribution over the other players' type profiles. This will be captured in our setting when we describe the game as viewed by each type which will include a probability distribution corresponding to that type's beliefs.

The set of viewpoints is the set of all possible types $V = \bigcup_{i \in I} \Theta_i$ and a typical viewpoint is denoted by $v = \theta_i \in V$. The set of all finite sequences of viewpoints is $\bar{V} = \bigcup_{n=0}^{\infty} V^{(n)}$. A finite sequence of viewpoints $v = (v_1, \dots, v_n)$ is associated with a game $G_v(B) = (I_v, \prod_{i \in I_v} (A_i)_v, (\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v, P_v, \{(u_i)_v\}_{i \in I_v})$ where I_v is the set of players that v_1 finds that v_2 finds that ... that v_n is considering, and similarly for the sets of actions $(A_i)_v$, states of nature $(\Theta_0)_v$, types $(\Theta_i)_v$, with $\Theta_v = (\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v$, the distribution P_v over the viewed type space $\Theta_v = (\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v$, and payoffs $(u_i)_v$ defined for the viewed states of nature,

types and action profiles $(\Theta_0)_v \times \prod_{i \in I_v} (\Theta_i)_v \times \prod_{j \in I_v} (A_j)_v$. The conditions defining the extension to unawareness are almost identical to those used for the normal form games.

Definition 3. A collection $\Gamma = \{G_v(B)\}_{v \in \mathcal{V}}$ where $G_v(B)$ are games as above and $\mathcal{V} \subset \overline{\mathcal{V}}$ is a set of relevant views is called an incomplete information game with unawareness if the following properties hold:

CI1 For every $v \in \mathcal{V}$, $v = \theta_i \in \Theta_i$ we have

$$v \wedge v \in \mathcal{V} \text{ if and only if } i \in I_v, \theta_i \in (\Theta_i)_v \quad (18)$$

CI2 For every $v \wedge \bar{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \quad (19)$$

and

$$\phi \neq I_{v \wedge \bar{v}} \subset I_v \quad (20)$$

as well as

$$\phi \neq (\Theta_i)_{v \wedge \bar{v}} \subset (\Theta_i)_v \quad (21)$$

$$\phi \neq (\Theta_0)_{v \wedge \bar{v}} \subset (\Theta_0)_v \quad (22)$$

$$\phi \neq (A_i)_{v \wedge \bar{v}} \subset (A_i)_v \quad (23)$$

for every $i \in I_{v \wedge \bar{v}}$.

CI3 If $v \wedge v \wedge \bar{v} \in \mathcal{V}$ we have

$$G_{v \wedge v \wedge \bar{v}}(B) = G_{v \wedge v \wedge \bar{v}}(B) \quad (24)$$

and $v \wedge v \wedge \bar{v} \in \mathcal{V}$.

CI4 For every state (nature and type profile) and action profile pair $(\theta, a)_{v \wedge \bar{v}} \in (\Theta_0)_{v \wedge \bar{v}} \times \prod_{i \in I_{v \wedge \bar{v}}} (\Theta_i)_{v \wedge \bar{v}} \times \prod_{j \in I_{v \wedge \bar{v}}} (A_j)_{v \wedge \bar{v}}$ there exists a completion to a pair $(\theta, a)_v \in (\Theta)_v \times \prod_{j \in I_v} (A_j)_v$ that agrees with the appropriate coordinates of $(\theta, a)_{v \wedge \bar{v}}$ such that

$$(u_i)_{v \wedge \bar{v}}((\theta, a)_{v \wedge \bar{v}}) = (u_i)_v((\theta, a)_v) \quad (25)$$

Note that the notion of a type in an incomplete information game with unawareness differs from a type in a standard incomplete information game. The difference is that the beliefs a type has over the type space (as he perceives it) need not be held in common knowledge. Moreover, types are allowed to perceive different type spaces. For example, a type θ_i may have a belief P_{θ_i} yet type θ_j may conceive type θ_i 's belief to be different, $P_{\theta_j \theta_i} \neq P_{\theta_i}$ and even defined on a different space.

It is natural to ask whether an analog to Harsanyi's consistency condition – the common prior assumption – can be found for these games with unawareness. The obvious prior candidate is the modeler distribution $P = P_\emptyset$, however, since a type θ_i may be unaware of the whole space Θ his distribution P_{θ_i} may be defined on a different set of types $(\Theta)_{(\theta_i)}$, and if θ_i is unaware of some players then $(\Theta)_{(\theta_i)}$ may not be a subset of Θ but rather a subset of a projection. Although this implies that P_{θ_i} cannot be merely a conditional of P , it indicates that a projection of the prior might do. Letting P_{θ_i} be exactly the conditional probability over $(\Theta)_{(\theta_i)}$ of the marginal of P with respect to $\Theta_0 \times \prod_{j \in I_{\theta_i}} \Theta_j$, and extending this definition to iterated relevant views provides a candidate for an extended common prior condition.

Definition 4. *We say that consistency (in the sense of Harsanyi) holds for an incomplete information game with unawareness if for every relevant $v \wedge \tilde{v} \in \mathcal{V}$ we have*

$$P_{v \wedge \tilde{v}} = \underset{(\Theta_0)_v \times \prod_{j \in I_{v \wedge \tilde{v}}} \Theta_j}{\text{Marg}} \left((\Theta_j)_v P_v \middle| (\Theta_0)_{v \wedge \tilde{v}} \times \prod_{j \in I_{v \wedge \tilde{v}}} (\Theta_j)_{v \wedge \tilde{v}} \right) \quad (26)$$

Note that this is the marginal of P_v with respect to $(\Theta_0)_v \times \prod_{j \in I_{v \wedge \tilde{v}}} (\Theta_j)_v$ and then conditional on the types from the perspective $v \wedge \tilde{v}$, i.e., conditional on the potential subsets of types considered $(\Theta_0)_{v \wedge \tilde{v}} \times \prod_{j \in I_{v \wedge \tilde{v}}} (\Theta_j)_{v \wedge \tilde{v}}$. Copic and Galeotti (2005) have independently modeled incomplete information games with unawareness in quite a similar manner. The difference is that they consider players and actions as commonly known and modeled unawareness of types and their beliefs. While our model above is more general, we find it reassuring that the structure of the two definitions is essentially the same.

2.5 Extensive Form Games with Unawareness

The final, and in some sense most general, class of games we consider are extensive form games. This game form captures both uncertainties and dynamics. An extensive form game is composed of a game tree capturing decision points, actions, nature moves, information sets, probabilities for nature moves and payoffs. An extensive form game is denoted $G(D) = ((W), I, A_0 \times \prod_{i \in I} A_i, \{F_i\}_{i \in I}, P, \{u_i\}_{i \in I})$ where $(W, <)$ is a finite tree (infinite games vary in definition and unawareness can be extended accordingly) with a disjoint union of vertices (a partition) $W = V_0 \cup \cup_{i \in I} V_i \cup Z$ where V_0 is the set of natures moves, V_i denotes the set of player i 's decision points and Z is the set of terminal vertices and the order $w' < w$ denotes that w' occurs before w on the tree.

Denote by $Pred(w)$ and $Succ(w)$ the (immediate) predecessor, and respectively successor, of w – formally it is the maximal vertex smaller than w and the minimal larger than w respectively. A pair $e = (w, Succ(w))$ of a vertex and its successor is called an *edge* and the set of edges emanating from w is denoted $E(w)$. We assume that every two vertices in the tree are connected with a finite path – a finite sequence of edges. Hence, every vertex has a (single) predecessor except for the root that has none and the set of terminal vertices $Z \subset W$ is the set of vertices that have no successor. The set of players is I . An edge $e = (w, w')$ belongs to player i , resp. Nature, when $w \in V_i$, resp. $w \in V_0$. The mappings $A_i(w, w')$ are defined for player i 's edges, $w \in V_i$, $w' = Succ(w)$, $A_0(w, w')$ when (w, w') is a nature move, and associate an action with each edge in $E(w)$. The partitions F_i of the sets V_i correspond to the information sets of player i and $f_i(w) \in F_i$ denotes the partition member containing a vertex $w \in V_i$. We require that for every w the function $A_i(w, \cdot)$ of the successors of w is one to one, that for every pair $w \neq w'$ the set of values that $A_i(w, \cdot)$ and $A_i(w', \cdot)$ obtain are disjoint unless $w' \in f_i(w) \in F_i$ in which case they are identical. The mapping P associates a probability distribution over the edges following each of nature's vertices. For every $w \in V_0$ we denote by $P(w)$ the probability distributions over $E(w)$. Finally, $u_i : Z \rightarrow \mathbb{R}$ are the utilities of players defined for terminal vertices.

The set of viewpoints is the players' set of decision points $V = \bigcup_{i \in I} V_i$ and a typical viewpoint is denoted by v . The set of all finite sequences of viewpoints is $\overline{V} = \bigcup_{n=0}^{\infty} V^{(n)}$. A finite sequence of viewpoints $v = (v_1, \dots, v_n)$ is associated with an extensive form game $G_v(D) = ((W_v, <), I_v, (A_0)_v \times \prod_{i \in I} (A_i)_v, \{(F_i)_v\}_{i \in I_v}, P_v, \{(u_i)_v\}_{i \in I_v})$ where $W_v = (V_0)_v \cup \bigcup_{i \in I_v} (V_i)_v \cup Z_v$ and similarly for all other ingredients of the game in accordance with the description above. We use the same ordering $<$ since W_v will be subsets of W .

Definition 5. A collection $\Gamma = \{G_v(D)\}_{v \in \mathcal{V}}$ where $G_v(D)$ are games as above and $\mathcal{V} \subset \overline{V}$ is a set of relevant views is called an Extensive Form Game with unawareness if the following properties hold:

CD1 For every $v \in \mathcal{V}$, $v \in V_i$ we have

$$v \wedge v \in \mathcal{V} \text{ if and only if } i \in I_v, v \in (V_i)_v \quad (27)$$

CD2 For every $v \wedge \tilde{v} \in \mathcal{V}$ we have

$$v \in \mathcal{V} \quad (28)$$

$$\phi \neq W_{v \wedge \tilde{v}} \subset W_v \quad (29)$$

$$\phi \neq I_{v \wedge \bar{v}} \subset I_v \quad (30)$$

as well as for all $i \in I_{v \wedge \bar{v}}$, $w \in (V_i)_{v \wedge \bar{v}}$

$$(V_i)_{v \wedge \bar{v}} = (V_i)_v \cap (W_{v \wedge \bar{v}} \setminus Z_{v \wedge \bar{v}}) \quad (31)$$

$$(V_0)_{v \wedge \bar{v}} = (V_0)_v \cap (W_{v \wedge \bar{v}} \setminus Z_{v \wedge \bar{v}}) \quad (32)$$

$$(F_i)_{v \wedge \bar{v}} = \{f \cap (W_{v \wedge \bar{v}} \setminus Z_{v \wedge \bar{v}}) \mid f \in (F_i)_v\} \quad (33)$$

and

$$(A_i)_{v \wedge \bar{v}}(w, w') = (A_i)_v(w, w'') \quad (34)$$

for the unique successor w'' of w in W_v such that $w'' < w'$, where w' is the successor of w in $W_{v \wedge \bar{v}}$.

CD3 If $v \wedge v \wedge \bar{v} \in \mathcal{V}$ with $v \in V_i$ then we have $f_i(v) \cap (V_i)_{v \wedge v} \neq \phi$ and for every $\bar{v} \in f_i(v) \cap (V_i)_{v \wedge v}$ we have

$$G_{v \wedge v \wedge \bar{v}}(T) = G_{v \wedge v \wedge \bar{v} \wedge \bar{v}}(T) = \dots = G_{v \wedge v \wedge \bar{v} \dots \bar{v} \wedge \bar{v}}(T) \quad (35)$$

and $v \wedge v \wedge \bar{v} \dots \bar{v} \wedge \bar{v} \in \mathcal{V}$.

The third condition states that a viewpoint must consider its information set to be relevant. Moreover, the viewpoints it considers in its information set are assumed to find themselves relevant and model the game in the same manner. Otherwise, a player at an information set would be able to distinguish decision points based on differing views at the decision points.

CD4 Let $v \wedge \bar{v} \in \mathcal{V}$. For every terminal vertex $w \in Z_{v \wedge \bar{v}}$ there exists a vertex $w' \in Z_v$ such that $w < w'$ and

$$(u_i)_{v \wedge \bar{v}}(w) = (u_i)_v(w') \quad (36)$$

Note that throughout our constructions and conditions the ordering $<$ for a subset of the histories is inherited from the ordering in the original standard game. As with incomplete information games we did not constrain the subjective probabilities that a viewpoint, or a view, associates with nature moves. If a restricted view of the game omits some nature moves one may still impose an analog for common priors:

Definition 6. We say that Harsanyi consistency holds for a extensive form game with unawareness representing an incomplete information game with unawareness, if for every relevant $v \in \mathcal{V}$ we have at every $w \in (V_0)_v$ that

$$P_v(w) = P(w)|E_v(w). \quad (37)$$

We note that the definition of condition **CD3** is stronger than the repeated games version **CR3** as it requires that a viewpoint not only see itself as a viewpoint in the same information set and agree with it, but also that it will agree with all its viewpoints in the restricted information set. This definition follows the interpretation that every player at an information set perceives it as representing indistinguishable information *and* indistinguishable awareness. In other words, in an information set of a game with unawareness the relevant player will perceive all their decision points to have the same perception of the game.

Our main structural result states that games with unawareness are consistent in the sense that from every view of the game is seen as a game with unawareness satisfying the exact same conditions as the modeler's game.

Consider a game with unawareness $\Gamma = \{G_v(\cdot)\}_{v \in \mathcal{V}}$ where $G_v(\cdot)$ has one of the four forms: normal, repeated, incomplete information or dynamic. For every relevant $v \in \mathcal{V}$ we define the relevant views as seen from v as: $\mathcal{V}^v = \{\bar{v} \in \bar{\mathcal{V}} \mid v \wedge \bar{v} \in \mathcal{V}\}$. For each relevant view $\bar{v} \in \mathcal{V}^v$ we define the game $G_{\bar{v}}^v(\cdot) = G_{v \wedge \bar{v}}(\cdot)$ and the game with unawareness as seen from v is defined as $\Gamma^v = \{G_{\bar{v}}^v(\cdot)\}_{\bar{v} \in \mathcal{V}^v}$.

Proposition 1. *For every game with unawareness Γ with a relevant view $v \in \mathcal{V}$ the game Γ^v is a game with unawareness with relevant views $\mathcal{V}^v$.*

The proof of this as well as all other propositions in this paper appears in the Appendix.

We note that this game form allows for imperfect recall, i.e., players may not only discover aspects of the game they were not aware of, they can also forget past actions. If one would rather maintain perfect recall two extra conditions are required. First the actual game with all the players and actions must be a game of perfect recall. Furthermore, at every decision point a player must consider a game that includes the game they considered in the past, i.e., no player may have their view of the game contract over time. Imposing the latter condition on higher order views (everyone views everyone's view ... to include all past views) will allow the conditioning on perfect recall at any reasoning level. Note that this condition still allows a player to revise their view of what other players view of the game is. For example, Alice can observe an action by Bob that may lead her to think that Bob is less aware than she previously thought. This does not imply that she believes Bob forgot, it is a revision of Alice's view of Bob's view at a *given* decision point for Bob. Once she revises this view, she may well assign a restricted view for Bob in *future* decision points. Still, she will assume that Bob has perfect recall as she revises all of his views to be more restrictive.

3 Solutions for Games with Unawareness

In defining solution concepts for games with unawareness we follow the same principle used in constructing the games: Each relevant view considers equilibrium behavior in a manner consistent with the modeler's definition of equilibrium behavior. There are some degrees of freedom when taking this approach. Obviously, one needs to select the solution concept for standard games whose behavior is being generalized, moreover, there may be some flexibility in the extension of behavior to games with unawareness. The first solutions we analyze are NE and Rationalizability (R) for normal form games with unawareness. The definition and analysis of some solutions for the other game forms follow.

3.1 Rationalizability and Nash Equilibrium in Normal Form Games with Unawareness

In order to define the solutions for games with unawareness we need to associate behavior with each possible view.

Definition 7. Let $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ be a normal form game with unawareness. An extended strategy profile ESP in this game is a collection of strategy profiles $\{(\sigma_v)_v\}_{v \in \mathcal{V}}$ where $(\sigma_v)_v$ is a strategy profile in the game G_v such that for every $v \wedge v \wedge \bar{v} \in \mathcal{V}$ we have

$$(\sigma_v)_v = (\sigma_v)_{v \wedge \bar{v}} \quad (38)$$

in the sense that the same pure strategies are assigned the same probabilities in the two games G_v and $G_{v \wedge \bar{v}}$, as well as

$$(\sigma)_{v \wedge v \wedge \bar{v}} = (\sigma)_{v \wedge v \wedge \bar{v}} \quad (39)$$

The first condition requires that the strategy that the view v associates with player v in the game G_v is the same strategy that the view v finds the player playing in the game as he is seen to see it, i.e., in the game $G_{v \wedge \bar{v}}$. We abuse the notation in the equalities in the definition since $(\sigma_v)_v$ and $(\sigma_v)_{v \wedge \bar{v}}$ may reside in different pure strategy product spaces, however the whole point is that one is a product of subsets of the other where the support of the strategies must lie and the probabilities coincide. Consider, for example our repeated game from the introduction, Alice perceives that Bob is not aware of non-cooperation hence she will not consider he would play it as she considers his strategy in the game with non-cooperation actions that she is aware of. The interpretation is that whenever a strategy is assigned to a player from a given perspective while taking into account the player's

perception, that player is indeed assumed to be playing the strategy from that perspective. The second condition follows the same logic behind condition **C3** in stating that the behavior associated by a player to the games he views is identical to the behavior he reasons about himself associating to games, and that this principle holds from every view. In other words, Bob's strategy associated with the game that Alice perceives that Bob perceives is the strategy that Bob is assumed to play in the game as Alice perceives it. In addition Bob's view of his own view of the strategy coincides with his view of the strategy and this is commonly understood at every view.

It is worthwhile noting that the definition of an extended strategy restricts the *behavior* of players to actions they are aware of. In the definition of a game with unawareness we allowed the possibility that a player may have an action he is unaware of, i.e., a view v may perceive a player v as having an available action a , i.e., $a \in (A_v)_v$, while at the same time v may perceive that v is unaware of a , i.e., $a \notin (A_v)_{v \wedge v}$. In this case the right-hand side of (38) is defined on a strictly smaller set of actions and we assume that the right-hand side support is in that set, hence the ESP is defined such that $(\sigma_v)_v$ assigns 0 probability to the pure strategy a when the player is unaware of it.

We begin by defining rationalizability in games with unawareness. As expected, rationalizability corresponds to playing a best response in the perceived game to perceived strategies that are themselves best responses in how it is perceived the corresponding players view the game, and so on. This extends rationalizability from normal form games to normal form games with unawareness.

Definition 8. An ESP $\{(\sigma_v)\}_{v \in \mathcal{V}}$ in a game with unawareness is called *extended rationalizable* if for every $v \wedge v \in \mathcal{V}$ we have that $(\sigma_v)_v$ is a best response to $(\sigma_{-v})_{v \wedge v}$ in the game $G_{v \wedge v}$.

The principle governing the extension of NE to games with unawareness follows the epistemic foundation of the solution concept. A NE requires rationalizability – players play a best response to conjectures, and some form of truth – knowledge of conjectures, or agreement on strategies. These correspond in our setting to strategies that are best responses at every view, and to strategies that coincide when the views of the game coincide, respectively. The first property corresponds to rationality in the sense of playing a best response to conjectures. The second property requires that the conjectures, or best responses, are the same when reasoning about the same game. In other words, when players have the same perceptions about the game (with unawareness) they share the same conjectures on behavior – agreement on strategies.

For a game with unawareness two views $v, \bar{v}$ share the same perception of the game if they agree on how all other views consider the game, i.e., they consider the same game *with unawareness* $\Gamma^v = \Gamma^{\bar{v}}$.

Definition 9. An ESP $\{(\sigma)_v\}_{v \in \mathcal{V}}$ in a game with unawareness is called an extended Nash equilibrium (ENE) if it is rationalizable and for all $v \wedge, \bar{v} \in \mathcal{V}$ such that $\Gamma^v = \Gamma^{\bar{v}}$ we have that $(\sigma)_v = (\sigma)_{\bar{v}}$.

Note that an ENE assigns the same behavior in games corresponding to the concatenation of views once the perceptions of the game coincide, i.e., $\Gamma^v = \Gamma^{\bar{v}}$ implies that for all $\bar{v}$ such that $v \wedge \bar{v} \in \mathcal{V}$ we have $(\sigma)_{v \wedge \bar{v}} = (\sigma)_{\bar{v} \wedge \bar{v}}$. This follows from noting that $\Gamma^v = \Gamma^{\bar{v}}$ implies that $(\Gamma^v)^{\bar{v}} = (\Gamma^{\bar{v}})^{\bar{v}}$.

We justify the use of the term “extended” with the following result. This result states that when all views see the game in the exact same manner – there is no unawareness – then the extended solutions coincide with their standard counterparts for the normal form game at hand. More generally, at every view such that the game is seen to have no unawareness the extended solution coincides with the standard one.

Proposition 2. Let G be a normal form game and Γ a normal form game with unawareness such that for some $v \in \mathcal{V}$ we have $G_{v \wedge \bar{v}} = G$ for every $\bar{v}$ such that $v \wedge \bar{v} \in \mathcal{V}$. Let σ be a strategy profile in the normal form game G then

1. σ is rationalizable for G if and only if $(\sigma)_v = \sigma$ is part of an extended rationalizable profile in Γ .
2. σ is a NE for G if and only if $(\sigma)_v = \sigma$ is part of an ENE for Γ and this ENE also satisfies $(\sigma)_v = (\sigma)_{v \wedge \bar{v}}$.

While a game with unawareness may correspond to an infinite collection of games, the structure does support the existence of an equilibrium:

Proposition 3. Every normal form game with unawareness where the modelers view has a finite number of players that have a finite set of pure actions has an ENE. Hence, the weaker extended rationalizability solution is also non-empty.

Condition **C2** guarantees that every view of how a viewpoint perceives the game is a restriction of the original view of the game. However, a game with unawareness Γ may still incorporate an infinite number of differing views of the game with unawareness, i.e., the set of games with unawareness $\{\Gamma^v\}_{v \in \mathcal{V}}$ could have an infinite number of distinct members:

Example 1. Consider three players denoted 1, 2, 3 and let G be a normal form game where player 1 has three actions and players 2 and 3 have a single action each. Let F be a normal form game obtained from G by removing one of player 1's

actions, and let E be a normal form game obtained from F by removing one of the two remaining actions of player 1. We define the game with unawareness Γ as follows.

All views are relevant, i.e., $\mathcal{V} = \overline{\mathcal{V}}$. Consider a view $v = v_1, v_2, \dots, v_n$ such that $v_i \neq v_{i+1}$. It suffices to define G_v for such views since $G_{v'}$ for a view v' with consecutive repetitions of v_i is uniquely derived according to **C3**.

Define for every prime $p > 2$

$$v'_p = \underbrace{1, 2, 1, 2, \dots, 1, 2, 1, 3}_p \quad (40)$$

$$v''_p = v'_p \wedge v'_p = \underbrace{1, 2, 1, 2, \dots, 1, 2, 1, 3}_p, \underbrace{1, 2, 1, 2, \dots, 1, 2, 1, 3}_p \quad (41)$$

For v with no consecutive identical viewpoints we define:

$$G_v = \begin{cases} E & v = v''_p \bar{v} \text{ for some prime } p > 2 \text{ and some view } \bar{v} \\ F & \text{if } G_v \text{ is not defined above and } v = v'_p \bar{v} \text{ for some prime } p > 2 \text{ and view } \bar{v} \\ G & \text{otherwise.} \end{cases} \quad (42)$$

For v with consecutive identical viewpoints we define G_v to be the same as the view obtained by replacing each string of consecutive identical viewpoints with a single representative.

Claim 4. Γ is a normal form game with unawareness.

To prove this claim we need to show that $\Gamma = \{G_v\}_{v \in \overline{\mathcal{V}}}$ satisfies the conditions of a game with unawareness. Since for v with some consecutive identical viewpoints we define G_v according to **C3** this condition holds by definition. Since all players participate in all games we have that all views are relevant and condition **C1** holds as well. Similarly, since all players participate in each of the viewed games, the payoffs are well defined and condition **C4** holds as well. Similarly, parts (6) and (7) in condition **C2** hold. It remains to show that (8) holds, or in other words, whenever $G_{v \wedge \bar{v}} = F$ then $G_v \neq E$ and that whenever $G_{v \wedge \bar{v}} = G$ then $G_v = G$. As before, we can assume that $v, v \wedge \bar{v}$ have no consecutive identical viewpoints. In the first case, if by way of contradiction $G_v = E$ then there is a prime $p > 2$ and some $\bar{v}$ such that $v = v''_p \wedge \bar{v}$. In particular, $v \wedge \bar{v} = v''_p (\bar{v} \wedge \bar{v})$ which implies that $G_{v \wedge \bar{v}} = E$ – a contradiction. In the second case, we have that if $G_v \neq G$ then $v = v'_p \wedge \bar{v}$ for some prime $p > 2$ and view $\bar{v}$ (note that this holds whether G_v equals F or E). Hence $v \wedge \bar{v} = v'_p \wedge \bar{v} \wedge \bar{v}$ and we must have $G_{v'_p \bar{v}} \neq G$ which completes the proof that Γ is a well defined normal form game with unawareness.

Claim 5. The set of normal form games with unawareness $\{\Gamma^{v'_p}\}_{p > 2 \text{ prime}}$ are all different.

The proof of this claim follows from observing that for every prime $p > 2$ we have that $G_v^{v'_p} = F$ for $v = \emptyset, v = 1, v = 12, \dots, v = \underbrace{121\dots 21}_p$ and $G_{v'_p}^{v'_p} = G$. Hence for all prime q such that $q > p$ we have that $G_{v'_p}^{v'_p} = F \neq G_{v'_p}^{v'_p}$ which implies that every member of the set differs from all the following members assuring that no two members coincide.

Claim 5 illustrates that the existence of an ENE follows from the fact that at each view each viewpoint is considered as playing a best response to strategies in some finite game, so although we have an infinite number of games (one for each view) the fixed point conditions are satisfied since payoffs for each view are determined in a finite game. However, we would like to study conditions under which the normal form game with unawareness is finite, in the sense that there is only a finite set of games with unawareness associated with the views in the game. This is particularly important if one wishes to represent the game in a state space approach with a finite set of states. The following results demonstrate such a condition.

Consider a view $v = (v_1, \dots, v_n) \in \overline{V}$, each view $v' = (v_{k_1}, \dots, v_{k_m})$ with $1 \leq k_1 < \dots < k_m \leq n$ is called a *sub-word* of v and we denote the order induced by sub-words as $v' \leq v$. The stronger version of condition **C2** is stated as follows:

S-C2 For every $v \in \mathcal{V}$ we have that for every $v' \leq v$

$$v' \in \mathcal{V} \quad (43)$$

and

$$\phi \neq I_v \subset I_{v'} \quad (44)$$

as well as

$$\phi \neq (A_i)_v \subset (A_i)_{v'} \quad (45)$$

for all $i \in I_v$.

We have

Proposition 6. *If a normal form game with unawareness Γ satisfies condition **S-C2** then the set of games with unawareness $\{\Gamma^v\}_{v \in \mathcal{V}}$ is finite.*

Condition **S-C2** is quite strong as it requires that if Alice models Bob's perception of Carol reasoning about some action then Bob's model of the game *must* indeed include Carol's reasoning about that action. Furthermore, Alice model of Carol's perceptions of the game must assume that Carol is reasoning about this action. The interpretation of this condition is that the players do not "get it wrong", in the sense that when they reason about someone else's perception they cannot attribute to

that person an ability that he does not actually posses. While we do not advocate setting condition **S-C2** on the same level as the more intuitive assumption **C2**, we point out that our introductory examples do satisfy this condition. We note that condition **S-C2** also implies that if two views of the game see it as a standard game with no unawareness, then it must be the *same* standard game:

Proposition 7. *For a normal form game with unawareness Γ which satisfies condition **S-C2** and such that Γ^v and $\Gamma^{v'}$ are both standard games with $v \wedge v', v' \wedge v \in \mathcal{V}$, we have that $\Gamma^v = \Gamma^{v'}$, and in particular we have the same ENE behavior in both.*

We point out that even with condition **S-C2** the length of views at which there is a change of the perception of the game need not be bounded.

Example 2. Consider three players denoted 1, 2, 3 and let G be a normal form game where player 1 has two actions and players 2 and 3 have a single action each. Let F be a normal form game obtained from G by removing one of player 1's actions. We define the game with unawareness Γ as follows.

All views are relevant, i.e., $\mathcal{V} = \overline{\mathcal{V}}$. For a view $v = v_1, v_2, \dots, v_n$ such that for all i we have $v_i \neq 3$ we define $G_v = G$ and otherwise we set $G_v = F$.

We leave it to the reader to check that the example satisfies all required conditions and note that for any view v that does not contain player 3 the consideration of that player changes the game, i.e., $G_v \neq G_{v \wedge 3}$.

We conclude this section with the observation that any NE of a standard game G such that all its actions are held in common awareness, corresponds to an ENE of the normal form game with unawareness. The definition of an ENE readily implies:

Claim 8. *Let Γ be a normal form game with unawareness with $G_\emptyset = G$. Assume that σ is a NE of G and that every action in the support of σ is held in common awareness, i.e., every view finds all the players in G and the actions in the support of σ to be part of the game, then $(\sigma)_v = \sigma$ for all v is an ENE.*

3.2 Representing Games with Unawareness as Games with Incomplete Information

The definition of a game with unawareness maps the strategic situation to a collection of standard games – associating one standard game with each possible relevant view of the situation. On the other hand the notion of unawareness or, “leaving some aspect out of the modeled game”, begs the comparison to the notion of assigning zero probability to an event. After all, one plausible interpretation of zero probability events (in a finite setting) can be “things that the player is sure will

not occur” and hence the player could safely operationally ignore. So it is natural to ask why a novel structure is required, why not represent situations with unawareness with incomplete information games with zero probabilities replacing unawareness. Indeed, such a construction is feasible, moreover there is a canonic mapping of normal form games with unawareness to standard games with incomplete information – canonic in the sense that varying the payoffs in the game, renaming actions, or players, does not alter the state space under this mapping. We provide this mapping below. Furthermore, the extended rationalizable solution exactly coincides with BNE under this mapping. However, what this mapping does not preserve is the extended NE solution. Hence, while normal form games with unawareness can be mapped to corresponding games with incomplete information, Nash equilibrium reasoning in games with unawareness requires some additional structure without which it differs from the solutions of incomplete information games. In other words, the extended NE solution in games with unawareness does not correspond to an existing solution in games with incomplete information.

With respect to the other forms of games with unawareness the imbedding becomes more complicated, but not impossibly so. For example, incomplete information games with unawareness can be mapped to standard incomplete information games, yet the problem of mapping the solutions of the unawareness form becomes more severe. The reason is that behavior in incomplete information games with unawareness distinguishes between probability zero and unawareness, a distinction that disappears in the representation with games with incomplete information. Since our solution dictates behavior with characteristics unique to unawareness the elimination of this distinction in the standard incomplete information form hinders the reproduction of the solution to games with unawareness. Similar difficulties arise when trying to imbed dynamic and repeated games with unawareness which require the combination of extensive and incomplete information games.

Consider a normal form game with unawareness $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ with a set of relevant views $\mathcal{V}$ with $G_\phi = G = (I, \prod_{i \in I} A_i, \{u_i\}_{i \in I})$. We define the following I -player incomplete information game $G_{\text{Bayesian game}} = (I, \prod_{i \in I} A_i, \Theta_0 \times \prod_{i \in I} \Theta_i, \{P_i\}_{i \in I}, \{u_i\}_{i \in I})$ where A_i agree with the game G , $\Theta_0 = \overline{\mathcal{V}} = \bigcup_{n=0}^{\infty} I^{(n)}$, for every $i \in I$ the players types are defined by $\Theta_i = \{v \wedge i \mid v \in \overline{\mathcal{V}}\}$, the players beliefs are $P_i(v \wedge i, (v \wedge i \wedge j)_{j \in I} \mid v_i) = 1$ hence all other $(\bar{v}, (\bar{v}_j)_{j \in I})$ are assigned zero probability by the type $v \wedge i$. Finally, the payoffs u_i are defined as

$$u_i((v, (\tilde{v}_j \wedge j))_{j \in I})((a_j)_{j \in I}) = \begin{cases} (u_i)_v & \text{when } (a_j)_{j \in I} \text{ are an extension where } (u_i)_v \text{ is defined} \\ -\infty & \text{otherwise.} \end{cases} \quad (46)$$

where $(u_i)_v$ are the payoffs to i as defined in G_v . Note that the payoffs are determined by the state of nature in Θ_0 and that, following Harsanyi, we set the payoffs for actions that are not modeled in G_v at $-\infty$. We also emphasize that this mapping is not one to one since a game with unawareness where a player has a single action according to some view v is mapped to the same game as a game with unawareness for which at the same view v the player with the single action is not part of the description of the game and the payoffs are determined according to the single action, yet this minute difference has no impact on behavior. We also note that the completion of the action set follows the spirit of Harsanyi's games of incomplete information. In our case it plays a structural role in the definition of the mapping and when we consider whether solutions are preserved it retains the same set of best responses (for any order of reasoning) by allocating $-\infty$ to the augmented actions. As with any mapping one wishes to minimize any increase in the requirements on players reasoning as seen in Hu and Stuart (2001) who provided the foundation for a mapping for the Harsanyi construction.

An equivalent representation of a normal form game with unawareness can be provided via a formulation where the set of states is V and each player possesses an information partition of the state space. For every player i and every $v \in V$ the player's partition member includes two states $v, v \wedge i$ and player i assigns the probabilities 0 and 1 to the two states respectively. This corresponds to the above game in that it adds a partition, but preserves all probabilities. This mapping also provides a graphic representation of games with unawareness which is depicted in Figure 4. The tree representation indicates the relationship dictated by the second condition (for each game form) where the arrow from a view v to a view $v \wedge v$ indicates how the viewpoint v perceives the game according to v as a more restricted version of the game $G_v(\cdot)$.

A modification of the tree in Figure 4 can also capture unawareness of players – the relevant views – by trimming the tree at views that are not relevant. Such elimination of views still leaves us with a tree (and well defined partitions) since the second condition implies that if a view is irrelevant so are all the views following it on the tree. We note that all these alternative representations will preserve extended rationalizability as Bayesian Nash, but will map ENE to a new solution which refines Bayesian Nash.

We note that the representation does not depend on the payoffs of the games, it also generates the same state space and beliefs if the names of the actions, the extent of unawareness of actions and the names of the players are changed. In fact, there will only be a change if the unawareness of players changes and this is only in

profiles of Γ coincide with the BNE profiles of $G(b)$ at the state of the world corresponding to $v = \phi$.

The claim follows by observing that at every state the relevant player is playing a best response to the state he assigns probability 1 to, where other players play according to how he views their perception, in turn this applies for iteration of views and as can be seen in Figure 4 the conditions for extended rationalizability coincide with the Bayesian NE conditions. Claim 9 then implies that ENE is mapped to a strict subset of the BNE. In particular, this is a subset where at various states players must play the same strategy if the *games* at the states they assign probability 1 to, they assign probability 1 to others assigning probability 1, and so on, have exactly the same trees of payoffs corresponding to these iterated beliefs. We note that trying to gage these higher order beliefs about payoffs, amounts to the construction of the game with unawareness to begin with as a collection of normal form games corresponding to a set of views. Hence, games with unawareness and the ENE provide a novel solution to settings with restricted perceptions.

3.3 Equilibria of Incomplete Information Games with Unawareness

The definition of extended BNE is constructed exactly in the same manner as the normal form case. Every view considers a strategy in the game with incomplete information such that every type plays a best response to the strategies in the game which the type perceives. In turn, these strategies correspond to how the type perceives the other players' types play in the game as the first type perceives the other types are considering, and so on. As with ENE we also require that if two views perceive the same incomplete information game with unawareness – have the same perception of the incomplete information game, the same perception of how the types in the game view the incomplete information game, and so on – then they prescribe the same behavior.

Definition 10. Let $\Gamma = \{G_v(B)\}_{v \in \mathcal{V}}$ be an incomplete information game with unawareness. An extended strategy profile ESP in this game is a collection of strategy profiles $\{(\sigma_v)_v\}_{v \in \mathcal{V}}$ where $(\sigma_v)_v$ is a strategy profile in the game $G_v(B)$ such that for every $v \wedge v \wedge \bar{v} \in \mathcal{V}$ we have

$$(\sigma_v)_v = (\sigma_v)_{v \wedge \bar{v}} \quad (47)$$

and

$$(\sigma)_{v \wedge v \wedge \bar{v}} = (\sigma)_{v \wedge v \wedge v \wedge \bar{v}} \quad (48)$$

Recall that in this case each viewpoint v corresponds to one type of a player in an incomplete information setting. As with normal form games we extend the solutions:

Definition 11. An ESP $\{(\sigma)_v\}_{v \in \mathcal{V}}$ in an incomplete information game with unawareness is said to be extended rationalizable if for every $v \wedge v \in \mathcal{V}$ we have that $(\sigma_v)_v$ is a best response to $(\sigma_{-v})_{v \vee}$ in the game $G_{v \vee}$.

and

Definition 12. An ESP $\{(\sigma)_v\}_{v \in \mathcal{V}}$ in an incomplete information game with unawareness is called an extended Bayesian Nash equilibrium EBNE if it is rationalizable and for all $v, \bar{v} \in \mathcal{V}$ such that $\Gamma^v = \Gamma^{\bar{v}}$ we have that $(\sigma)_v = (\sigma)_{\bar{v}}$.

We note that, as with ENE for normal form games with unawareness, the EBNE is a new solution concept. In order to capture this solution with standard incomplete information games one would require a fine tailoring of the game that distinguishes the beliefs that stem from uncertainty from those that are generated by unawareness. One would need to choose a particular state space and define the types' beliefs on a case by case basis to capture the EBNE behavior in standard incomplete information games, while in the process essentially mimicking the construction of incomplete information games with unawareness.

3.4 Equilibria of Dynamic Games with Unawareness

Dynamic games provide a host of solution concepts. Many of these consider alternative principles for belief revision. In particular, beliefs and behavior after a deviation from the equilibrium path occurs. The extension of these solutions is no different than the extensions to normal form, or incomplete information, games with unawareness. The extensions require an epistemic foundation for the solution determining the nature of reasoning about beliefs and rationality in a dynamic setting, and will depend on the choice of epistemic characterization. Such characterizations need not be unique as can be seen in the epistemic characterization of sequential equilibria for dynamic games with unawareness in Feinberg (2004).

Definition 13. Let $\Gamma = \{G_v(D)\}_{v \in \mathcal{V}}$ be a dynamic game with unawareness. An extended strategy profile ESP in this game is a collection of strategy profiles $\{(\sigma)_v\}_{v \in \mathcal{V}}$ where $(\sigma)_v$ is a behavior strategy profile in the game $G_v(D)$ such that for every $v \wedge v \wedge \bar{v} \in \mathcal{V}$ we have

$$(\sigma_v)_v = (\sigma_v)_{v \wedge v} \quad (49)$$

and for every $\tilde{v} \in f_i(v) \cap (V_i)_{v \wedge v}$ we have

$$(\sigma)_{v \wedge v \wedge \tilde{v}} = (\sigma)_{v \wedge v \wedge \tilde{v} \wedge \tilde{v}} \quad (50)$$

Recall that a viewpoint v corresponds to a decision point in the game tree. In particular, this could be one of a number of points in an information set. Furthermore, at that decision point the player may perceive the situation as corresponding to one of the other decision points at the same information set due to beliefs about prior strategic behavior. We need to verify that using condition (50) agrees with the definition of a dynamic game and with the interpretation of an information set. Indeed, inductively applying (50) we have that

$$(\sigma)_{v \wedge v \wedge \tilde{v}} = (\sigma)_{v \wedge v \wedge \tilde{v} \wedge \tilde{v}} = (\sigma)_{v \wedge v \wedge \tilde{v} \wedge \dots \wedge \tilde{v} \wedge \tilde{v}} \quad (51)$$

which is a well defined condition according to condition **CD3**.

As with standard dynamic games, rationality in a dynamic setting depends on whether the payoff is calculated ex-ante, or conditional on reaching a decision point. With the latter requiring postulating conditions for belief revision as well as forward, backward and hypothetical rationality analysis. While one can analyze these alternatives as in Feinberg (2004) here we simply extend NE directly without discussion of the epistemic conditions. We do the same for refinements such as sequential equilibria which consider assessments – strategies plus beliefs at information sets.

Definition 14. An ESP $\{(\sigma)_v\}_{v \in \mathcal{V}}$ in a dynamic game with unawareness is called an extended Nash equilibrium ENE if for every $v \wedge v \in \mathcal{V}$ with $v \in (V_i)_v$ we have that the behavior strategy $\{(\sigma_v)_v\}$ such that $v \wedge v \wedge \sim v \in \mathcal{V}$ for player i is a best response to $(\sigma_{-i})_{v \wedge v}$ in the game $G_{v \wedge v}(D)$. In addition, for all $v, \bar{v} \in \mathcal{V}$ such that $\Gamma^v = \Gamma^{\bar{v}}$ we have that $(\sigma)_v = (\sigma)_{\bar{v}}$.

We note that the definition requires that a player's strategy is considered from how a view perceives the perception of a decision point v . Even if the strategy implies that this decision point is not reached and even if the player corresponding to v has a different perception of the game at other decision points. This is not a choice made in defining the solution, but rather a constraint of the definition of games with unawareness since a behavior strategy as defined for player i in the game perceived from v may be beyond the scope of strategies in the game as perceived by $v \wedge v$.

The definition of solutions based on assessments is more straightforward as they already require reasoning at information sets. However, since these solutions involve beliefs at information sets we must define extended assessments as well.

Definition 15. Let $\Gamma = \{G_v(D)\}_{v \in \mathcal{V}}$ be a dynamic game with unawareness. An extended assessment in this game is a an ESP $\{(\sigma_v)\}_{v \in \mathcal{V}}$ and a collection of belief function $\{(\mu_v)_v\}_{v \wedge v \in \mathcal{V}}$ such that $(\mu_v)_v$ is a probability distribution over the information set $f_i(v) \cap (V_i)_{v \wedge v}$ such that for every $v \wedge v \wedge \bar{v} \in \mathcal{V}$ we have

$$(\mu_v)_v = (\mu_v)_{v \wedge v} \quad (52)$$

and for every $\bar{v} \in f_i(v) \cap (V_i)_{v \wedge v}$ we have

$$(\mu)_{v \wedge \bar{v} \wedge \bar{v}} = (\mu)_{v \wedge v \wedge \bar{v} \wedge \bar{v}} \quad (53)$$

We are now set to define the extension of solutions based on assessments and will do so in a general manner.

Definition 16. An extended assessment $\{(\sigma_v), (\mu_v)\}_{v \in \mathcal{V}}$ in a dynamic game with unawareness is called an extended $\star$ -equilibrium if for every $v \wedge v \in \mathcal{V}$ with $v \in (V_i)_v$ we have for any $\sim v \in f_i(v) \cap (V_i)_{v \wedge v}$ the mixed strategy $(\sigma_{\sim v})_{v \wedge v}$ maximizes the expected payoff to player i at $\sim v$ in the game $G_{v \wedge v}(D)$, i.e., the expected payoff to i in $G_{v \wedge v}(D)$ conditional on the information set $f_i(v) \cap (V_i)_{v \wedge v}$ being reached with the probability distribution $(\mu_v)_v$, when other players play according to $(\sigma_{-f_i(v) \cap (V_i)_{v \wedge v}})_{v \wedge v}$. Where $\star$ stands for sequential or variants thereof, and we require that $(\sigma_v)_v$ and $(\mu_v)_v$ satisfy the consistency, or other conditions imposed by refinement $\star$ in the game $G_{v \wedge v}(D)$ with respect to the strategies as perceived from the view $v \wedge v$.

To see how the extended refinement can be implemented to various solution concepts, consider sequential equilibria for example. In this case we can require that $(\mu_v)_v$ agree with the probability distribution generated by $(\sigma_{-f_i(v) \cap (V_i)_{v \wedge v}})_{v \wedge v}$ in $G_{v \wedge v}(D)$ if the information set is reached, and if not, it is the limit of probabilities assigned by best responses ϵ -completely mixed strategies in the game $G_{v \wedge v}(D)$.

We note that any of the $\star$ conditions are applied within a standard game, hence they are not modified, rather they are applied to all the games constituting the game with unawareness using a collection of assessments, one for each perception of the game. Definition 15 simply assures that the views of these assessments assign the same belief within a perceived information set. The added complexity is verifying the properties such as probability assessments are preserved. This is obtained by simply conditioning on the relevant information sets, i.e., the subsets generated by restricted views.

We can now state the generalizations of Propositions 3 and 2 which concludes this paper:

Proposition 10. Assume the normal form, incomplete information and dynamic games serving as building blocks for the games with unawareness we consider all have

*a finite set of players and correspondingly satisfy standard conditions for the existence of equilibria. Then the set of extended solutions for normal form, incomplete information and dynamic games with unawareness is non-empty. Furthermore, each extended solution coincides with the standard solution when there is no unawareness.*⁴

4 Conclusion

This work provides a uniform framework to model and analyze games with unawareness. The framework follows an identical treatment of various game forms and allows for unawareness of participating players and their actions. The basic approach asks that at every decision point (and given any information the player has) the player has a view of what the game is, a view of how each player in that game views the game and so on. Hence, all these views relate to standard games and the collection of standard games constitutes the game with unawareness. Our four conditions guarantee reasoning consistency in the sense that if Alice reasons about Bob reasoning about a player or an action then she can directly reason about them as well, that Alice's view about Bob's view about his own view coincides with how Alice views Bob and that when there is unawareness of a player the outcomes are consistent with one of the player's action.

We emphasize that these games can usually be described much like the examples in the introduction by indicating the relevant view of the game. When higher order views are relevant (as in the games depicted in (1) and (2)) a further state space formulation might be required (see the discussion in Section 2.2). Games with unawareness can be represented as incomplete information games. Indeed, it has been difficult to envision a situation that cannot be presented as simply a game where players assign zero probabilities instead of being unaware. However, this relationship between unawareness and zero probability is still quite open to interpretation (see Li (2008b) and Schipper 2014 for additional discussions and insights). Using the hierarchies approach the mapping to an incomplete information setting allows an explicit formulation of this relationship. Moreover, it also demonstrates why a separate formulation for unawareness may be useful. In particular, it shows that the natural extension of Nash equilibria to games with unawareness does not map to any known solution concept for incomplete information games.

⁴ For solutions requiring additional structures such as assessments, when there is no unawareness we also assume that these assessments coincide for every high order (identical) perception of the game.

Appendix

Proof of Proposition 1

Proposition 1 states that from every relevant view, the game with unawareness defined by considering higher order views, i.e., the game as seen from that view, is itself a game with unawareness. Hence, we need to show that for each relevant view v , the game defined by $\Gamma^v = \{G_{\tilde{v}}^v(\cdot)\}_{\tilde{v} \in \mathcal{V}^v}$ with relevant views $\mathcal{V}^v = \{\tilde{v} \in \overline{V} \mid v \wedge \tilde{v} \in \mathcal{V}\}$ satisfies all the conditions for a game with unawareness.

Recall the notation $G_{\tilde{v}}^v$ denotes the game $G_{v \wedge \tilde{v}}$. Similarly, we denote all the components of the games $G_{\tilde{v}}^v(\cdot)$ accordingly, e.g., the set of players is denoted $I_{\tilde{v}}^v$, player v 's actions by $(A_v)_{\tilde{v}}^v$ and so on.

Consider a normal form game with unawareness $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ and fix a relevant view $v \in \mathcal{V}$. Recall that $\mathcal{V}^v = \{\tilde{v} \in \overline{V} \mid v \wedge \tilde{v} \in \mathcal{V}\}$ and $\Gamma^v = \{G_{\tilde{v}}^v\}_{\tilde{v} \in \mathcal{V}^v}$. Since $\tilde{v} \in \mathcal{V}^v$ is equivalent to $v \wedge \tilde{v} \in \mathcal{V}$ we have from applying **C1** to Γ that

$$v \wedge \tilde{v} \wedge v \wedge v \in \mathcal{V} \text{ if and only if } v \in I_{v \wedge \tilde{v}}. \quad (54)$$

Since the left hand side is identical with $\tilde{v} \wedge v \in \mathcal{V}^v$ and the right hand side coincides with $v \in I_{\tilde{v}}^v$ we have

$$\tilde{v} \wedge v \in \mathcal{V}^v \text{ if and only if } v \in I_{\tilde{v}}^v \quad (55)$$

for all $\tilde{v} \in \mathcal{V}^v$ and the first consistency condition holds for Γ^v .

If $\tilde{v} \wedge \tilde{v} \in \mathcal{V}^v$ we have $v \wedge \tilde{v} \wedge \tilde{v} \in \mathcal{V}$ which implies $v \wedge \tilde{v} \in \mathcal{V}$ by (6) and is equivalent to $\tilde{v} \in \mathcal{V}^v$ proving the first part of condition **C2**. Similarly, we have from (7) and (8) that

$$\phi \neq I_{v \wedge \tilde{v} \wedge \tilde{v}} = I_{\tilde{v} \wedge \tilde{v}}^v \subset I_{v \wedge \tilde{v}} = I_{\tilde{v}}^v \quad (56)$$

and

$$\phi \neq (A_i)_{v \wedge \tilde{v} \wedge \tilde{v}} = (A_i)_{\tilde{v} \wedge \tilde{v}}^v \subset (A_i)_{v \wedge \tilde{v}} = (A_i)_{\tilde{v}}^v \quad (57)$$

for all $i \in I_{v \wedge \tilde{v} \wedge \tilde{v}} = I_{\tilde{v} \wedge \tilde{v}}^v$ which proves the second consistency condition holds for Γ^v .

If $\tilde{v} \wedge v \wedge \tilde{v} \in \mathcal{V}^v$ we have

$$G_{v \wedge \tilde{v} \wedge v \wedge \tilde{v}} = G_{v \wedge \tilde{v} \wedge v \wedge \tilde{v}} \quad (58)$$

and $v \wedge \tilde{v} \wedge v \wedge v \wedge \tilde{v} \in \mathcal{V}$ which is equivalent to

$$G_{\tilde{v} \wedge v \wedge \tilde{v}}^v = G_{\tilde{v} \wedge v \wedge \tilde{v}}^v \quad (59)$$

and $\tilde{v} \wedge v \wedge v \wedge \tilde{v} \in \mathcal{V}^v$ and the third condition holds.

For every action profile $(a)_{\bar{v} \wedge \bar{v}}^v = \{a_j\}_{j \in I_{\bar{v} \wedge \bar{v}}}^v$ we have from (11) a completion to an action profile $(a)_{\bar{v}}^v \wedge \bar{v} = \{a_j, a_k\}_{j \in I_{\bar{v}} \wedge \bar{v} \wedge \bar{v}}, k \in I_{\bar{v} \wedge \bar{v}} \setminus I_{\bar{v} \wedge \bar{v} \wedge \bar{v}}$ hence the existence of an action profile $(a)_{\bar{v}}^v = \{a_j, a_k\}_{j \in I_{\bar{v} \wedge \bar{v}}}^v, k \in I_{\bar{v}}^v \setminus I_{\bar{v} \wedge \bar{v}}^v$ such that

$$(u_i)_{\bar{v} \wedge \bar{v} \wedge \bar{v}}((a)_{\bar{v} \wedge \bar{v} \wedge \bar{v}}) = (u_i)_{\bar{v} \wedge \bar{v}}((a)_{\bar{v} \wedge \bar{v}}) \quad (60)$$

which is equivalent to

$$(u_i)_{\bar{v} \wedge \bar{v}}^v((a)_{\bar{v} \wedge \bar{v}}^v) = (u_i)_{\bar{v}}^v((a)_{\bar{v}}^v) \quad (61)$$

and the fourth condition holds for Γ^v as required completing the consistency proof for normal form games with unawareness.

For a repeated game with unawareness $\Gamma = \{G_v(T)\}_{v \in \mathcal{V}}$ fix a relevant view $v \in \mathcal{V}$. For every $\bar{v} \in \mathcal{V}^v, v = (h^t(J), i) \in V^v$ where the history $h^t(J)$ contains only actions from $(A)^v$ and $i \in I^v, J \subset I^v$ we have from applying (12) to $G_{v \wedge \bar{v}} = G_{\bar{v}}^v$ that

$$\bar{v} \wedge v \in \mathcal{V}^v \text{ if and only if } i \in I_{\bar{v}}^v, J = I_{\bar{v}}^v, A_j(h^t(J)) \subset (A_j)_{\bar{v}}^v \text{ for all } j \in J \quad (62)$$

which is the first consistency condition. The second condition is identical to the normal form games case.

For the third condition let $\bar{v} \wedge v \wedge \bar{v} \in \mathcal{V}^v$ where $v = (h^t(J), i)$ and the history and players are taken from G^v . According to condition CR3 for Γ there exists some $\tilde{v} = (\tilde{h}^t(\tilde{J}), i)$ such that

$$G_{v \wedge \bar{v} \wedge v \wedge \bar{v}}(T) = G_{v \wedge \bar{v} \wedge v \wedge \bar{v} \wedge \bar{v}}(T) = \dots = G_{v \wedge \bar{v} \wedge v \wedge \bar{v} \wedge \dots \wedge \bar{v}}(T) \quad (63)$$

and $v \wedge \bar{v} \wedge v \wedge \bar{v} \wedge \dots \wedge \bar{v} \in \mathcal{V}$. Which is equivalent to

$$G_{v \wedge v \wedge \bar{v}}^v(T) = G_{\bar{v} \wedge v \wedge \bar{v}}^v(T) = \dots = G_{\bar{v} \wedge v \wedge \bar{v} \wedge \dots \wedge \bar{v}}^v(T) \quad (64)$$

and $\bar{v} \wedge v \wedge \bar{v} \wedge \dots \wedge \bar{v} \in \mathcal{V}^v$ and the third condition holds. The fourth condition holds exactly as in the normal form case completing the proof of consistency for repeated games with unawareness.

For an incomplete information game with unawareness $\Gamma = \{G_v(B)\}_{v \in \mathcal{V}}$ and a relevant view v we have $G_{\bar{v}}^v(B) = \left(I_{\bar{v}}^v, \prod_{i \in I_{\bar{v}}^v} (A_i)_{\bar{v}}^v, (\Theta_0)_{\bar{v}}^v \times \prod_{i \in I_{\bar{v}}^v} (\Theta_i)_{\bar{v}}^v, P_{\bar{v}}^v, \{(u_i)_{\bar{v}}^v\}_{i \in I_{\bar{v}}^v} \right)$. The

first three consistency conditions hold in the exact same manner as for normal form games. Note that conditions (21) and (22) are shown to hold in an identical manner as condition (23) which is identical to the normal form case. As for the fourth condition, we note that it too follows a similar proof to the normal form case by replacing partial actions profiles and their completion by partial pairs of state and actions and their completion. More formally, for every state and action profile pair $(\theta, a)_{\bar{v} \wedge \bar{v}}^v \in (\Theta_0)_{\bar{v} \wedge \bar{v}}^v \times \prod_{i \in I_{\bar{v} \wedge \bar{v}}}^v (\Theta_i)_{\bar{v} \wedge \bar{v}}^v \times \prod_{j \in I_{\bar{v} \wedge \bar{v}}}^v (A_j)_{\bar{v} \wedge \bar{v}}^v$ there exists a completion to a pair

$(\theta, a)_{\bar{v}}^v \in (\Theta)_{\bar{v}}^v \times \prod_{j \in I_{\bar{v}}} (A_j)_{\bar{v}}^v$ that agrees with the appropriate coordinates of $(\theta, a)_{\bar{v} \wedge \bar{v}}$ such that

$$(u_i)_{\bar{v} \wedge \bar{v}}^v((\theta, a)_{\bar{v} \wedge \bar{v}}^v) = (u_i)_{\bar{v}}^v((\theta, a)_{\bar{v}}^v) \quad (65)$$

and the proof for incomplete information games with unawareness is complete.

Finally, let $\Gamma = \{G_v(D)\}_{v \in \mathcal{V}}$ be a dynamic game with unawareness and fix a relevant view v and consider the collection $\Gamma^v = \{G_{\bar{v}}^v(D)\}_{\bar{v} \in \mathcal{V}^v}$. As before, the components of $G_{\bar{v}}^v(D)$ are denoted with the superscript v and in particular the game tree is denoted $W_{\bar{v}}^v$ as are its components (nature moves, players' decision points and terminal nodes) and the collection of information sets are denoted $(F_i)_{\bar{v}}^v$.

The first condition holds by noting that $v \in (V_i)_{\bar{v}}^v$ if and only if $v \in (V_i)_{v\bar{v}}$. Similarly, the equivalents of (28), (29) and (30) hold as in the normal form case and (31), (32), (33) and (66) follow from noting that the sets of vertices and ordering coincides between $G_{v\bar{v}}(D)$ and $G_{\bar{v} \wedge \bar{v}}^v(D)$ as do intersections and complements of the sets of vertices and since the order induced from $\leq$ is preserved, i.e.,

$$(A_i)_{\bar{v} \wedge \bar{v}}^v(w, w') = (A_i)_{\bar{v}}^v(w, w'') \quad (66)$$

for the unique successor w'' of w in $W_{\bar{v} \wedge \bar{v}}^v$ such that $w'' \leq w'$, where w' is the successor of w in $W_{\bar{v}}^v$.

If $\bar{v} \wedge v \wedge \bar{v} \in \mathcal{V}^v$ with $v \in V_i^v$ then we have $f_i^v(v) \cap (V_i)_{\bar{v} \wedge v}^v \neq \emptyset$ from condition CD3 applied to Γ as well as for every $v' \in f_i^v(v) \cap (V_i)_{\bar{v} \wedge v}^v$ we have

$$G_{\bar{v} \wedge v \wedge \bar{v}}^v(T) = G_{\bar{v} \wedge v \wedge v' \wedge \bar{v}}^v(T) = \dots = G_{\bar{v} \wedge v \wedge v' \wedge \dots \wedge v' \wedge \bar{v}}^v(T) \quad (67)$$

and $\bar{v} \wedge v \wedge v' \wedge \dots \wedge v' \wedge \bar{v} \in \mathcal{V}^v$ simply by applying (35) for Γ with $v \wedge \bar{v}$ for v and v' for $\bar{v}$.

Similarly, let $\bar{v} \wedge \bar{v} \in \mathcal{V}^v$. For every terminal vertex $w \in Z_{\bar{v} \wedge \bar{v}}^v = Z_{v \wedge \bar{v} \wedge \bar{v}}$ according to CD4 for Γ there exists a vertex $w' \in Z_{\bar{v}}^v = Z_{v \wedge \bar{v}}$ such that $w < w'$ and

$$(u_i)_{\bar{v} \wedge \bar{v}}^v(w) = (u_i)_{v \wedge \bar{v} \wedge \bar{v}}^v(w) = (u_i)_{v \wedge \bar{v}}^v(w') = (u_i)_{\bar{v}}^v(w') \quad (68)$$

and the fourth condition holds as well. We have shown that dynamic games with unawareness are also consistent in the sense that how every view is modeled to perceive the game, is itself a game with unawareness and the proof is complete.

Proof of Proposition 2

Let $G = (I, \prod_{i \in I} A_i, \{u_i\}_{i \in I})$ be a normal form game and Γ a normal form game with unawareness such that for some $v \in \mathcal{V}$ we have $G_{v \wedge \bar{v}} = G$ for every $\bar{v}$ such

that $v \wedge \bar{v} \in \mathcal{V}$. We will assume $v = \phi$ the general case follows from considering Γ^v instead.

Assume a strategy profile σ is rationalizable for the normal form game G . Then for every player $i \in I$ there is a strategy profile for all other players $\bar{\sigma}_{-i}$ such that σ_i is a best response to $\bar{\sigma}_{-i}$ and such that for every $j \in I \setminus \{i\}$ there is a strategy $\bar{\sigma}_j$ such that $\bar{\sigma}_j$ is a best response to it, and so on. For every view v we define a strategy profile inductively as follows: Let $(\sigma)_\phi = \sigma$, for $v = i \in I$ let $(\sigma)_v = (\bar{\sigma}_{-i}, (\sigma_i)_\phi)$ i.e., combining player i 's strategy σ_i with $\bar{\sigma}_{-i}$ to which it is a best response. Given $(\sigma)_v$ we define $(\sigma)_{v \wedge v}$ as the strategy to which $(\sigma_v)_v$ is a best response (augmented by $(\sigma_v)_v$) if the last viewpoint in v does not equal v , and we let $(\sigma)_v = (\sigma)_{v \wedge v}$ if the last viewpoint in v coincides with v .

Since the game associated with all views is G and all views are relevant, the above collection of strategies is a well defined ESP. By definition of rationalizability, every view finds the player is playing a best response to the strategies they are considering according to that view which is the definition of extended rationalizability.

Consider now the same game and let $\{(\sigma)_v\}_{v \in \mathcal{V}}$ be an ESP satisfying extended rationalizability. Since at every view the game corresponds to G we have that all views are relevant. In particular, for every sequence of players $i_1, i_2, \dots, i_n$ we have a sequence of strategies corresponding to $n+1$ views: $\sigma^0 = (\sigma)_\phi$, $\sigma^1 = (\sigma)_{i_1}$, $\sigma^2 = (\sigma)_{(i_1, i_2)}$, $\dots$, $\sigma^n = (\sigma)_{(i_1, i_2, \dots, i_n)}$. These satisfy by extended rationalizability that $\sigma_{i_{k+1}}^k$ is a best response to $\sigma_{-i_{k+1}}^{k+1}$. Hence, we found that for every player $i \in I$ the strategy $(\sigma_i)_\phi$ is a best response to $(\sigma_{-i})_i$ with $(\sigma_j)_i$ each being best responses to $(\sigma_{-j})_{(ij)}$ and so on making $(\sigma)_\phi$ a profile of rationalizable strategies in the normal form game G .

The second part of the proposition follows similarly: Let G be as above and without loss of generality assume the game with unawareness associates G with all views, i.e., we begin with the view ϕ . Let σ be a NE. Then setting $(\sigma)_v = \sigma$ for all views is a well defined ESP. Hence the condition for identical strategies assigned to identical views of the game with unawareness is satisfied. Since σ_i is a best response to σ_{-i} for all $i \in I$ by NE, we have that $(\sigma_i)_v$ is a best response to $(\sigma_{-i})_{vi}$ for all v and every $i \in I$ and the ESP satisfies extended rationalizability and is therefore an ENE.

In the other direction, let σ be such that $(\sigma)_v = \sigma$ for all v is an ENE. In particular σ is a strategy profile in G such that $\sigma_i = (\sigma_i)_v$ is a best response to $\sigma_{-i} = (\sigma_{-i})_{vi}$ making it a NE.

Proof of Proposition 3

The existence of ENE requires the examination of the possibly infinite collection of views of the game assuring that a fixed point exists. Consider a normal form game with unawareness $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ with views $\mathcal{V}$. We define an auxiliary standard normal form game $\mathcal{G}$ as follows:

Let i denote a player in G_ϕ . The set of players in this game is given by

$$\mathcal{N} = \{v \in \mathcal{V} \setminus \{\phi\} \mid v = (i_1, \dots, i_n) \text{ with } i_k \neq i_{k+1} \text{ for all } k\}. \quad (69)$$

The action set for each player $v = (i_1, \dots, i_n) \in \mathcal{N}$, is given by

$$\mathcal{A}_v = (A_{i_n})_v. \quad (70)$$

We define the payoff function for each player $v = (i_1, \dots, i_n) \in \mathcal{N}$ in this game by

$$\mathcal{U}_v(\{(a_j)_{\tilde{v}}\}_{\tilde{v} \in \mathcal{N}}) = (u_{i_n})_v(\{(a_j)_{\tilde{v}}\}_{j \in I_v}) \quad (71)$$

where $(u_{i_n})_v(\{(a_j)_{\tilde{v}}\}_{j \in I_v})$ is the payoff to i_n in the game G_v when the actions played are $\{(a_j)_{\tilde{v}}\}_{j \in I_v}$.

The game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{U})$ constitutes a normal form game with a countable number of players. However, the payoff function for each player depends on the actions of only a finite number of other players. Consider the product space of mixed strategies

$$\Sigma = \prod_{v \in \mathcal{N}} \Delta(\mathcal{A}_v). \quad (72)$$

Since the game at each view has a finite action set we have that $\mathcal{A}$ is compact in the product topology. Considering the best response mapping

$$B(\sigma) = \prod_{v \in \mathcal{N}} B_v(\sigma) \quad (73)$$

where

$$B_v(\sigma) = \left\{ \bar{\sigma}_v \in \Delta(\mathcal{A}_v) \mid E(\mathcal{U}_v(\sigma | \bar{\sigma}_v)) \geq E(\mathcal{U}_v(\sigma | \tilde{\sigma}_v)) \text{ for all } \tilde{\sigma}_v \in \Delta(\mathcal{A}_v) \right\} \quad (74)$$

and $E(\mathcal{U}_v(\sigma | \tilde{\sigma}_v))$ is the expected payoff when considering the strategy σ modified by having player v play $\tilde{\sigma}_v$. The expected payoff is well defined since for every $v \in \mathcal{N}$ there is only a finite number of players influencing the payoffs.

The set valued functions B_v are non-empty, compact and convex valued since the payoff functions $\mathcal{U}_v$ are linear continuous functions of $\mathcal{A}_v$. Hence, the product map B is a non-empty, compact and convex valued set function on the compact convex set Σ . B is also upper-hemi continuous since for every sequence $\sigma^n \rightarrow \sigma$ with $\sigma, \sigma^n \in \Sigma$ for $n = 1, 2, \dots$, we have that if $\bar{\sigma}^n \in B(\sigma^n)$ for all n and $\bar{\sigma}^n \rightarrow \bar{\sigma}$ then we must also have $\bar{\sigma} \in B(\sigma)$. This follows from noting that for every $v \in \mathcal{N}$ we have

$$E(\mathcal{U}_v(\sigma^n | \bar{\sigma}_v^n)) \geq E(\mathcal{U}_v(\sigma^n | \tilde{\sigma}_v)) \text{ for all } \tilde{\sigma}_v \in \Delta(\mathcal{A}_v) \quad (75)$$

but since $\mathcal{U}_v$ depends only on the actions of $v = (i_1, \dots, i_l)$ and $v \wedge j$ for $j \in I_v, j \neq i_l$, the convergence of σ^n to σ implies convergence in all coordinates and hence the continuity of the expectation of $\mathcal{U}$ yields

$$E(\mathcal{U}_v(\sigma|\bar{\sigma}_v)) \geq E(\mathcal{U}_v(\sigma|\bar{\sigma}_v)) \text{ for all } \bar{\sigma}_v \in \Delta(\mathcal{A}_v) \quad (76)$$

which holds for all $v \in \mathcal{N}$ and implies $\bar{\sigma} \in B(\sigma)$ as claimed.

The set valued map B satisfies the conditions for the generalized Kakutani fixed point theorem and as shown by Fan (1952) and Glicksberg (1952) there exists a $\sigma \in \Sigma$ such that $\sigma \in B(\sigma)$ hence there exists an equilibrium σ in the auxiliary game $\mathcal{G}$. We now define an ESP for Γ by setting $(\sigma)_v = \sigma_v$ for $v \in \mathcal{N}$ and inductively for $v = (i_1, \dots, i_n) \in \mathcal{V}$ for which $(\sigma)_v$ is defined, we set for $\bar{v} = (i_1, \dots, i_k, i_k, i_{k+1}, \dots, i_n)$ the strategy $(\sigma)_{\bar{v}} = (\sigma)_v$ hence we inductively get an ESP. By definition of a NE for the auxiliary game this ESP is an ENE of Γ as required.

We note that the NE of the auxiliary normal form game $\mathcal{G}$ defined in the proof of Proposition 3 completely characterizes the set of ENE for the game Γ .

Proof of Proposition 6

Assume by way of contradiction that there is an infinite countable sequence of relevant views $\{v_n\}_{n=1}^{\infty}$ that offer distinct views of the game with unawareness. Formally, for every pair $n \neq m$ there is a $\bar{v}$ such that

$$(A)_{v_n \wedge \bar{v}} \neq (A)_{v_m \wedge \bar{v}}. \quad (77)$$

We note that if the games differ in the viewed set of players they will also differ in the set of actions as above.

From Lemma 11 whose proof follows, there exists an infinite countable subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that $v_{n_k} \leq v_{n_{k+1}}$ (the order indicates a substring as in the Lemma). In particular, for all $\bar{v}$ we have $v_{n_k} \wedge \bar{v} \leq v_{n_{k+1}} \wedge \bar{v}$. From condition **S-C2** we have for every k and $\bar{v}$ that

$$(A)_{v_{n_k} \wedge \bar{v}} \supset (A)_{v_{n_{k+1}} \wedge \bar{v}}. \quad (78)$$

From (77) we have for every $k > 1$ there exists some $\bar{v}_k$ such that

$$(A)_{v_{n_{k-1}} \wedge \bar{v}_k} \neq (A)_{v_{n_k} \wedge \bar{v}_k} \quad (79)$$

and for this particular $\bar{v}_k$ we have that

$$(A)_{v_{n_1} \wedge \bar{v}_k} \supset \dots \supset (A)_{v_{n_{k-1}} \wedge \bar{v}_k} \supsetneq (A)_{v_{n_k} \wedge \bar{v}_k}. \quad (80)$$

Since (80) holds for every $k > 1$ we have a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and a sequence $\{\bar{v}_k\}_{k=2}^{\infty}$ such that for all $k = 2, 3, \dots$ we have

$$(A)_{v_{n_k} \wedge \bar{v}_k} \neq (A)_{v_{n_j} \wedge \bar{v}_k} \quad \forall j < k. \quad (81)$$

Using Lemma 11 once more, we can find an infinite subsequence $\{\tilde{v}_{k_l}\}_{l=1}^{\infty}$ such that $\tilde{v}_{k_l} \leq \tilde{v}_{k_{l+1}}$ and considering the same subset of indices for v_{n_k} we have for all l

$$(A)_{v_{n_{k_l}} \tilde{v}_{k_l}} \neq (A)_{v_{n_{k_j}} \tilde{v}_{k_l}} \quad \forall j < l, \quad (82)$$

$$v_{n_{k_l}} \leq v_{n_{k_{l+1}}}, \quad (83)$$

$$\tilde{v}_{k_l} \leq \tilde{v}_{k_{l+1}}. \quad (84)$$

From (83) and (84) and since concatenation with the same word (recall that we termed views as “words” comprised of viewpoints termed as “letters”) preserves the order $\leq$ we have that for all l

$$v_{n_{k_l}} \wedge \tilde{v}_{k_{l+1}} \leq v_{n_{k_{l+1}}} \tilde{v}_{k_{l+1}} \leq v_{n_{k_{l+1}}} \wedge \tilde{v}_{k_{l+2}}. \quad (85)$$

From condition S-C2 and from (85) we have for every l

$$(A)_{v_{n_{k_l}} \wedge \tilde{v}_{k_{l+1}}} \supset (A)_{v_{n_{k_{l+1}}} \wedge \tilde{v}_{k_{l+1}}} \supset (A)_{v_{n_{k_{l+1}}} \wedge \tilde{v}_{k_{l+2}}}. \quad (86)$$

Since every $(A)_v$ is a subset of the finite set A , we conclude that there exists a t such that

$$(A)_{v_{n_{k_t}} \wedge \tilde{v}_{k_{t+1}}} = (A)_{v_{n_{k_{t+1}}} \wedge \tilde{v}_{k_{t+1}}}. \quad (87)$$

Since (87) contradicts (82) we have reached the desired contradiction and the proof is complete.

The proof of the proposition relied on the following lemma due to Higman (1952).

Lemma 11. *For every sequence of views $\{v_k\}_{k=1}^{\infty}$ we can find an infinite countable subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that $v_{n_k} \leq v_{n_{k+1}}$, i.e., each word v_{n_k} can be obtained by deleting some letters of the word $v_{n_{k+1}}$.*

Proof. This Lemma follows immediately from Theorem 4.4 in Higman (1952). Higman shows (as a special case of his finite basis property theorems) that given a finite alphabet I , every set of words X from this alphabet has a finite subset X_0 such that for every word $w \in X$ one can find a word $w_0 \in X_0$ such that the letters of w_0 occur in w in their right order, though not necessarily consecutively. In particular, let $X = \{v_k\}_{k=1}^{\infty}$, from Higman’s theorem there exists a finite subset $X_0 \subset X$ such that from each word in $v \in X$ one can obtain at least one of the words in X_0 by eliminating some members in v . Since X_0 is finite and X is countable there exists a word in X_0 denoted v_{n_1} that can be imbedded in an infinite subsequence of words from $X \setminus X_0$. Hence from every countable sequence of words we can find a subsequence such that the first word in the subsequence can be obtained from *every* word that follows by eliminating some letters. We can now consider the subsequence from

the second word onwards and find a subsequence such that the second word can be imbedded in all the words that follow. Maintaining the same first element v_{n_1} we now have that the first two words can be imbedded in every word that follows. By induction the required subsequence is derived.

The following direct proof of Higman's theorem is due to Michael Ostrovsky:

By induction on k – the number of letters in the alphabet (main induction). For $k = 1$, the claim is obvious. Suppose it is true for k up to n . Let us show that it is also true for $k = n + 1$.

Claim 12. *Any infinite sequence w_i of words (made up of $k = n + 1$ different letters) contains two words, w_{i_1} and w_{i_2} , such that $i_1 < i_2$ and $w_{i_1} \leq w_{i_2}$.*

Proof of claim

By induction on l – the length of the shortest word in the sequence. For $l = 1$ take the one-letter word. Without loss of generality, the letter is A . Eliminate all the words that go before that word from the sequence; we now have $w_1 = A$. If any other word in the remaining sequence contains the letter A , we are done. If not, then the sequence $(w_2, w_3, w_4, \dots)$ is made up of only $n = k - 1$ different letters, and by the assumption of the main induction, this sequence contains an increasing subsequence $(w_{j_1}, w_{j_2}, \dots)$ with any two words, e.g., w_{j_1} and w_{j_2} , satisfying the requirement.

Suppose the claim holds for all l up to m . Let us show that it is also true for $l = m + 1$. Take the shortest word in the sequence. Without the loss of generality, it is the first word in the sequence, and the first letter in this word is A . If there is only a finite number of other words that contain the letter A , then the remaining infinite subsequence is made up of only n different letters and we are done. Otherwise, drop all the words that do not contain the letter A from the sequence. For each remaining word w_i , let L_i be the part of the word that precedes the first occurrence of A in the word, and R_i be the part that follows the first occurrence of A (e.g., if $w_i = BCADCAB$, then $L_i = BC$ and $R_i = DCAB$; if $w_i = ABC$, then L_i is the empty word, and $R_i = BC$). Note that all words in the sequence $(L_1, L_2, \dots, L_i, \dots)$ are made up of only n different letters, and so there exists an increasing subsequence $(L_{i_1}, L_{i_2}, \dots)$ such that for any t , $i_t < i_{t+1}$ and $L_{i_t} \leq L_{i_{t+1}}$. Note also that since we assumed that w_1 is the shortest word and starts with an A , we can let $i_1 = 1$ – the empty word is smaller than any other word.

Now, consider the corresponding sequence $(R_{i_1}, R_{i_2}, \dots)$. The shortest word in this sequence has length of at most m (because R_{i_1} , by construction, has length m), and therefore, by the minor induction assumption there exist u and v such that $u < v$ and $R_{i_u} < R_{i_v}$. But we also know that, by construction, $L_{i_u} \leq L_{i_v}$, and so $w_{i_u} \leq w_{i_v}$, and the claim follows.

We can now complete the proof of the step of the induction of Higman's theorem. Take any sequence of words made up of $k = n + 1$ different letters. Consider all words w_i in this sequence such that there does not exist $j > i$ such that $w_i \leq w_j$. There can be at most a finite number of such words, since otherwise the subsequence formed from these words would be a counter-example to the claim. Let w_h be the last one of these words in the sequence, so that for any $j > h$ there exists $k > j$ such that $w_j \leq w_k$. It is now possible to construct an infinite increasing subsequence, e.g., take the subsequence (w_{i_t}) such that $i_1 = h + 1$ and for all $t > 1$, $i_t = \min_{i > i_{t-1}} \{i | w_{i_{t-1}} \leq w_i\}$ and the proof of Higman's theorem is complete.

Proof of Proposition 7

Let v and v' be such that Γ^v and $\Gamma^{v'}$ are standard games. Hence, for every $\tilde{v}$ such that $v \wedge \tilde{v} \in \mathcal{V}$ we have

$$A_{v \wedge \tilde{v}} = A_v \text{ and } I_{v \wedge \tilde{v}} = I_v \quad (88)$$

as well as for every $\tilde{v}$ such that $v' \wedge \tilde{v} \in \mathcal{V}$ we have

$$A_{v' \wedge \tilde{v}} = A_{v'} \text{ and } I_{v' \wedge \tilde{v}} = I_{v'}. \quad (89)$$

Since $v' < v \wedge v' \in \mathcal{V}$ the condition S-C2 and (88) imply that $A_v = A_{v \wedge v'} \subset A_{v'}$ and using (89) we similarly get $A_{v'} \subset A_v$. Using the same argument we conclude that $I_v = I_{v'}$. Since there is common awareness from these points onward we have that $G_v = G_{v'}$ as well as $G_v = G_{v \wedge \tilde{v}} = G_{v' \wedge \tilde{v}} = G_{v'}$ for all $\tilde{v}$ such that $v \wedge \tilde{v}$ or equivalently $v' \wedge \tilde{v}$ is relevant. Since ENE for games with unawareness where all views share the same standard game coincides with NE of the standard game, the same behavior is dictated by ENE in both Γ^v and $\Gamma^{v'}$.

Proof of Proposition 10

The proof of existence of the extended solution concepts for repeated, incomplete and dynamic games with unawareness follows quite closely the proof of Proposition 3. Similarly, the proof that the solutions coincide with the standard solution once the views agree follows closely the proof of Proposition 2. We detail the required modifications for the application of these proofs.

The proof for incomplete information games with unawareness is identical to the normal form games proof with the exception that type spaces may be infinite. Hence, there may be an uncountable set of views. Still, the critical condition (76) holds as long as the utility of a type is continuous in other players types' strategies, but this amounts to continuity in a standard incomplete information game since all we need is for the strategies to be measurable in the type space. Repeated games and dynamic games with unawareness require a modification of the final part of

the proof of Proposition 3 where the equilibrium of the auxiliary game is extended to the game with unawareness. The difference here is that a viewpoint for a player may consider itself as a different view point, for example, a viewpoint corresponding to one member of an information set may view herself as corresponding to another member of the (same) information set. This is illustrated in condition **CD3** in (35) (and condition **CR3** for repeated games). Constructing a strategy for the game with unawareness from a strategy for the auxiliary game that we construct in the proof of Proposition 3 requires that we extend the strategies defined for a subset of views to all relevant views. The subset of views is determined as iterating views that can be associated with different strategies in the games with unawareness. For dynamic games the subset of views that the auxiliary game considers will not have two views from the same information set. When mapping these strategies to the game with unawareness we complete the set of strategies, for all relevant views, by identifying views that must be associated with the same strategy. In particular, once the strategy for a view $v \wedge v \wedge \bar{v}$ is defined, we assign the same strategy for the views $v \wedge v \wedge \bar{v} \wedge \bar{v}, \dots, v \wedge v \wedge \bar{v} \wedge \dots \bar{v} \wedge \bar{v}$ where $\bar{v}$ is as determined by condition **CD3**. The rest of the proof coincides with the proof for the normal form case.

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