DE GRUYTER BE J. Theor. Econ. 2016; 16(2): 767–795

Notes

Ahmed Doghmi*

On Nash Implementability in Allotment Economies under Domain Restrictions with Indifference

DOI 10.1515/bejte-2015-0091
Published online June 1, 2016

Abstract: In this paper we give a full characterization of Nash implementability of social choice correspondences (SCCs) in allotment economies on preference domains with private values and different types of indifference. We focus on single-peaked/single-plateaued preferences with worst indifferent allocations, single-troughed preferences and single-troughed preferences with best indifferent allocations. We begin by introducing a weak variant of no-veto power, called I^**-weak no-veto power*, which form together with unanimity and a stronger version of Maskin monotonicity a sufficient condition for Nash implementation in general environment. We apply this result to the above preference domains and we prove that any SCC that has full range is Nash implementable if and only if it satisfies Maskin monotonicity. We examine the implementability of some well-known correspondences. We give examples of SCCs that are monotonic in the unrestricted domains and also monotonic in our setup, and we provide exemples of SCCs that are not monotonic in the unrestricted domains, but monotonic and therefore Nash implementable in our context. Finally, we give examples of SCCs that are not monotonic in the restricted domains and also not monotonic in our area, and therefore not Nash implementable.

Keywords: allotment economies, domain restrictions with indifference, nash implementation
JEL Classification: C72, D71

***Corresponding author: Ahmed Doghmi,** University of Rabat, Mohammadia School of Engineering, the QSM Laboratory, Avenue Ibn Sina B.P. 765 Agdal, 10100 Rabat, Morocco; Center for Research in Economics and Management (UMR CNRS 6211), University of Caen, 19 Rue Claude Bloch 14032 Caen, Cedex, France, E-mail: ahmeddoghmi@hotmail.com

1 Introduction

In games and social choice theories, many domain restrictions have been used in order to obtain positive results. In most cases, these results were obtained when special conditions were imposed on preference profiles. For example, limiting the preference of agents to linear orders or allowing some very limited indifference classes simplifies the expression of the conditions considerably, as well as the proofs of results based on restricting the domains of social choice rules. Among these conditions, the well-known notions are *single-peaked preferences* that were introduced by Black (1948) and *single-dipped preferences* that were provided by Inada (1964). In addition to these properties, other important conditions have been imposed on preference profiles and they are useful tools that obtain interesting results.[1]

Nevertheless, many authors have considered that some of the conditions imposed on preferences profiles are very strong. Thus, they have tried to relax these conditions to wider classes. For example, the fact that they have only one maximal element in a single-peaked domain, without admitting the indifference of at least one other alternative, is not always considered natural. Hence, many enlargements to this domain have been explored to allow either many best indifferent elements, which are now known as *single-plateaued domains*, or many worst indifferent elements, which were introduced by Cantala (2004) and they are referred to as *outside options*.[2] We also cite the extension of the single-dipped preferences domain to a wider class of preferences where agents can express their indifference among multiple worst alternatives, termed *single-trouthed preferences*, or multiple best alternatives, which were introduced by Manjunath (2011) and Aragón and Caramuta (2011).

The motivation for this article is to study the implications of these extensions in the implementation literature that has a close connection with social choice theory. Before explaining the problematic and providing our contribution in this direction, we start by giving a brief definition of implementation theory. This theory studies the problem of a society that chooses a social choice correspondence (SCC) to represent the social welfare of individuals or desired outcomes and the planner confronts the difficulty of how to implement this rule. This occurs because the planner does not know the exact preferences of individuals regarding these outcomes. Thus, in order to participate with a lower cost

1 For a more detailed survey, see Gaertner (2001).

2 For further details see, for example, Moulin (1984), Ehlers (2002), Barberà (2007), Berga (1998), Berga (2006), Berga and Moreno (2009), Bossert and Peters (2013), Doghmi and Ziad (2013a, 2013b) and Klaus and Bochet (2013).

(e. g., in the case of the construction of public goods), the agents may falsify their preferences. To ensure that they disclose their true preferences, the planner will organize a non-cooperative game among these individuals. In a given solution context, when the set of equilibrium outcomes of this game coincides with the set of socially desired alternatives, we can say that an SCC is implementable in this given solution context.

To achieve this objective, some conditions should be imposed on the SCCs that the planner hopes to implement. Thus, Maskin (1977, 1999) was the first to show that there is an important connection between the implementability of SCCs and a property now known as *Maskin monotonicity*. He proved that this requirement is a necessary condition for an SCC to be Nash implementable, and it becomes sufficient together with an additional property called *no-veto power*. Since this latter requirement fails to be satisfied for many examples of Maskin monotonic SCCs in economic and political applications like, among others, the core correspondence, the individually rational correspondence in voting problems and assignment of indivisible goods, the stable rule in matching problems, and the no-envy correspondence, the individually rational correspondence from equal division in private good economies, many studies have proposed either alternative conditions or different mechanisms to overcome this deficiency. For instance, Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Danilov (1992) provided full characterizations whereas many other authors proposed either necessary conditions, such as Ziad (1997, 1998), sufficient conditions, such as Yamato (1992) and Doghmi and Ziad (2008a), or different mechanisms like the one recently proposed by Doğan (2015) to implement the no-envy correspondence on general domains of economies.

In relation to the standard domain restrictions with private values of single-peaked and single-dipped preferences, Thomson (1990, 2010), Doghmi and Ziad (2008b), and Doghmi (2013a) tried to apply some of these theoretical results to implement many examples of solutions of the problem of fair division. On single-peaked domain, Thomson (1990, 2010) proved that only the Pareto correspondence can be implemented by Maskin's theorem (1999). To implement the no-envy correspondence, the individually rational correspondence from equal division and their intersection, he applied Yamato's theorem (1992). For the implementability of the intersections of the Pareto correspondence with the no-envy correspondence, and with the individually rational correspondence from equal division, these tools do not work and hence he appealed to Sjöström's algorithm (1991). On single-peaked and single-dipped domains, Doghmi and Ziad (2008b), and Doghmi (2013a) applied the results they developed in Doghmi and Ziad (2008a) and they examined the implementability of all these correspondences in an easy way. They proved that only Maskin monotonicity is a necessary and sufficient condition to implement these unanimous solutions.

In these domains of private good economies when we allow different types of indifference between alternatives, the properties used in these previous findings do not always apply, and they are only a few studies which are interested in this topic. To our knowledge, only the recent works of Doghmi (2013b) and Doghmi and Ziad (2013a, 2013b, 2015) are concerned with this issue. They studied Nash-implementability of SCCs when preferences profile are single-plateaued, and single-dipped with best indifferent allocations. Thus, they extended the results of Doghmi and Ziad (2008b), Thomson (1990, 2010), and Doghmi (2013a) on single-peaked and single-dipped domains in considering preference profiles which admit *multiple best indifferent elements*. However, these studies do not cover preference profiles which contain *multiple worst indifferent elements*.

Our objective in this article is to fill this gap and to contribute to the understanding of the implications of implementability in these domains of economies which admit different types of indifference between alternatives. We examine in particular the following domains.

1. *Single-peakedness with worst indifferent allocations*. This class of preferences extends the standard concept of single-peaked preferences by allowing multiple worst indifferent alternatives. It was introduced by Cantala (2004)[3] and explored recently by Klaus and Bochet (2013) to study the relationship between monotonicity and strategy-proofness.
2. *Single-plateauedness with worst indifferent allocations*. This class of preferences contains two types of indifference between alternatives: multiple best indifferent elements and multiple worst indifferent elements when generalizing the well-known domains of single-peaked and single-plateaued preferences.
3. *Single-troughedness*. This class of preferences is a natural enlargement of the domain of single-dipped preferences, which requires that each agent is indifferent among several worst alternatives. These worst alternatives are called the "trough". More recently, Bossert and Peters (2014) examined the notion of single-troughedness in a choice-theoretic setting.[4] This domain of preferences has also been used by Manjunath (2011) to characterize all rules that satisfy unanimity and strategy-proofness properties.

3 In Cantala (2004), single-peaked preferences with worst indifferent allocations are referred to as single-peaked preferences with outside options.

4 In Bossert and Peters (2014), single-troughed preferences are called single-basined preferences.

4. *Single-troughedness with best indifferent allocations*. This domain is an enlargement of the domain of single-dipped preferences with best indifferent allocations, which was explored recently by Aragón and Caramuta (2011) and Doghmi (2013b), but also an extension of the single-troughed domain examined by Bossert and Peters (2014), and by Manjunath (2011).

In this work we give a full characterization of Nash-implementability of SCCs on these four domains of economies. We introduce a weak variant of no-veto power, namely $I^{\star}$-weak no-veto power, and we prove that this property together with unanimity and a stronger version of Maskin monotonicity, termed I-monotonicity, developed in Doghmi and Ziad (2015), are sufficient for Nash-implementability in general environments. Using this result, which differs from the previous findings, we show that on the above preference domains any SCC that has full range is Nash implementable if and only if it satisfies Maskin monotonicity. To derive this characterization, we first prove that the new property of $I^{\star}$-weak no-veto power is implied by unanimity. In this auxiliary result, the role of our new requirement is central given that the different variants of no-veto power in the previous literature do not work. Second, we demonstrate that Doghmi and Ziad's property of I-monotonicity continue, together with the full range requirement, to involve the property of unanimity in the preference domains with different types of indifference under consideration here. Moreover, I-monotonicity becomes equivalent to Maskin monotonicity. Given this full characterization, we inspect the implementability of various well-known SCCs. Firstly, we show that the monotonic solutions in the unrestricted domains, which violate no-veto power, like the no-envy correspondence, the individually rational correspondence from equal division and their intersection, are Nash implementable in our setup. Hence, we deduce that these solutions that are implementable in the domains of strict preferences persist and keep their implementability when we allow different types of indifference. Secondly, we give examples of correspondences that are not monotonic in the unrestricted domains but monotonic in our context, like the strong Pareto indifferent correspondence and its intersections with the no-envy correspondence, with the individually rational correspondence from equal division, and with the Pareto correspondence and therefore Nash implementable. Finally, we provide examples of correspondences which do not satisfy Maskin monotonicity in general domain, like the Pareto correspondence and its intersections with the no-envy correspondence and with the individually rational correspondence from equal division, and we prove that these correspondences continue to violate this property in our setup, and hence they are not Nash implementable. Thus, we conclude that these correspondences, which are Nash implementable in some domains of strict preferences like in single-peaked domains, lose their implementability when we extend the preference domains to allow for indifferences.

The rest of this paper is organized as follows. In Section 2, we introduce notations and definitions for the general setup. In Section 3, we introduce a weaker version of no-veto power and we provide a sufficient condition for Nash implementation in general environment. In Section 4, we present our main results on preference domains with private values and indifference; we discuss the challenges for these domains compared to the previous findings, we give a full characterization for Nash implementation, and we study the implementability of various well-known SCCs. In Section 5, we provide a conclusion.

2 Notations and Definitions

Let A be a set of alternatives, and let $N = \{1, ..., n\}$ be a set of individuals, with generic element i. Each individual i is characterized by a preference relation R_i defined over A, which is a complete, and transitive relation in some class $\Re_i$ of admissible preference relations. Let $\Re = \Re_1 \times ... \times \Re_n$. An element $R = (R_1, ..., R_n) \in \Re$ is a preference profile. The relation R_i indicates the individual i's preference. For $a, b \in A$, the notation aR_ib means that the individual i weakly prefers a to b. The asymmetrical and symmetrical parts of R_i are denoted respectively by P_i and $\sim_i$.

A social choice correspondence (SCC) F is a multi-valued mapping from $\Re$ into $2^A \backslash \{\emptyset\}$, that associates with every R a nonempty subset of A. For all $R_i \in \Re_i$ and all $a \in A$, the lower contour set for agent i at alternative a is denoted by: $L(a, R_i) = \{b \in A \mid aR_ib\}$. The strict lower contour set and the indifference lower contour set are denoted respectively by $LS(a, R_i) = \{b \in A \mid aP_ib\}$ and $LI(a, R_i) = \{b \in L(a, R_i) \mid a \sim_i b\}$.

A mechanism (or a game form) is given by $\Gamma = (S, g)$ where $S = \Pi_{i \in N} S_i$; S_i denotes the strategy set of the agent i and g is a function from S to A. The elements of S are denoted by $s = (s_1, s_2, ..., s_n) = (s_i, s_{-i})$, where $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$. When $s \in S$ and $b_i \in S_i$, $(b_i, s_{-i}) = (s_1, ..., s_{i-i}, b_i, s_{i+1}, ..., s_n)$ is obtained after replacing s_i by b_i, and $g(S_i, s_{-i})$ is the set of results which agent i can obtain when the other agents choose s_{-i} from $S_{-i} = \Pi_{j \in N, j \neq i} S_j$.

A Nash equilibrium of the game (Γ, R) is a vector of strategies $s \in S$ such that for any i, $g(s)R_ig(b_i, s_{-i})$ for all $b_i \in S_i$, i. e. when the other players choose s_{-i}, the player i cannot deviate from s_i. Given $N(S, g, R)$ the set of Nash equilibria of the game (Γ, R), a mechanism $\Gamma = (S, g)$ implements a SCC F in Nash equilibria if for all $R \in \Re$, $F(R) = g(N(S, g, R))$. We say that a SCC F is implementable in Nash equilibria if there is a mechanism which implements it in these equilibria.

A SCC F satisfies unanimity if for any $a \in A$ and any $R \in \Re$, if for any $i \in N$, $L(a, R_i) = A$, then $a \in F(R)$.

A central theorem of the implementatability of SCCs is provided by Maskin (1977/1999). It is based on the following properties:

Monotonicity: A SCC F satisfies monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $i \in N$, $L(a, R_i) \subseteq L(a, R'_i)$, then $a \in F(R')$.

No-veto power: A SCC F satisfies no-veto power if for each i, each $R \in \Re$, and each $a \in A$, if $L(a, R_j) = A$ for all $j \in N \backslash \{i\}$, then $a \in F(R)$.

Maskin (1977/1999) proved that any Nash implementable correspondence must satisfy Maskin monotonicity and, when the number of agents is at least three, any SCC that satisfies Maskin monotonicity and no-veto power is Nash implementable.

3 A Weak Property of No-veto Power for Sufficiency

Recently, Doghmi and Ziad (2015) provided a full characterization for the implementability of SCCs in private good economies with single-plateaued preferences. They showed that any SCC that has full range is Nash implementable if and only if it satisfies Maskin monotonicity. To derive this result, they provided a new sufficient condition in general environment. Nevertheless, this result of general setup does not apply to domains of economies which admit multiple worst indifferent alternatives. To fill this gap and generalize these results for different types of indifference, we introduce a weak version of no-veto power that covers a large domain of preferences with indifference. To define this property, we start with the following definition.

Definition 1: *(Indifferent options subset)*
For any agent i's preference R_i, any alternative $a \in F(R)$ for some singleton "operator" $\{o\} \in LI(a, R_i)$ with $o \neq a$, the indifferent options subset is the subset $I(a, o, R_i) = \{b \in A \backslash \{a, o\}$ s.t. $a \sim_i b \sim_i o\}$. If $|LI(a, R_i)| \geq 3$, then $I(a, o, R_i) \neq \emptyset$, otherwise $I(a, o, R_i) = \emptyset$.

Now, we introduce the weak property of no-veto power that we call $I^{\star}$*-weak no-veto power*. Before describing this property, we give the following key notation. The arguments of the minimum of a set of alternatives A at an individual i's preference R_i, denoted $Argmin(R_i, A)$, are the set of the bottom-ranked alternatives for an agent i at a preference profile R; i. e., $Argmin(R_i, A) = \{a|$ for all $b \in A : bR_i a\}$.

Definition 2: *($I^{\star}$-weak no-veto power)*
An SCC F satisfies $I^{\star}$-weak no-veto power if for each i, each $R \in \Re$, each $a \in F(R)$, and each $b \in A$, if for each $R' \in \Re$, $b \in LS(a, R_i) \cup I(a, o, R_i) - argmin(R'_i, A) \subseteq L(b, R'_i)$ and $L(b, R'_j) = A$ for all $j \in N \backslash \{i\}$, for some $o \in LI(a, R_i) \backslash \{a\}$, then $b \in F(R')$.

Loosely speaking, the property of $I^{\star}$-weak no-veto power means that if an alternative a is socially chosen in a profile R and, for an agent i, if an alternative b is an element of the union of the subset of the alternatives which are ranked strictly below a and the subset of the *indifferent options*, in excluding all alternatives of i that are bottom-ranked in a new profile R', improves its ranking in R'_i and becomes top-ranked for all $j \neq i$ in R'_j, then it must be socially chosen in R'. To illustrate this property, we give the following example.

Example 1: Let $A = \{a, b, c, d\}$, $N = \{1, 2, 3\}$ and $\Re = \{R, R'\}$ such that:

***R*:**	R_1	R_2	R_3
	a	c	b
	b,c,d,e	a	a,d
		b,d	c
		e	e

***R'*:**	R'_1	R'_2	R'_3
	a	a,c	b,c
	d	b,d	a,d
	e	e	e
	b		
	c		

Let $F(R) = \{a, b\}$ and $F(R') = \{a\}$. In this example, the $I^{\star}$-weak no-veto power condition is satisfied. In profile R' we have $L(c, R'_{i=2,3}) = A$, and in profile R, we have two optimal alternatives: $\{a, b\} \subseteq F(R)$. For $b \in F(R)$, we have for player 1 $I(b, o = d, R_1) = \{c, e\}$, $I(b, o = c, R_1) = \{d, e\}$ or $I(b, o = e, R_1) = \{c, d\}$. For $I(b, o = d, R_1) = \{c, e\}$, we have $c \notin LS(b, R_1) \cup I(b, o = d, R_1) - argmin(R'_1, A) = \{e\} \nsubseteq L(c, R'_1) = \{c\}$. For $I(b, o = c, R_1) = \{d, e\}$, we have $c \notin LS(b, R_1) \cup I(b, o = c, R_1) - argmin(R'_1, A) = \{d, e\} \nsubseteq L(c, R'_1) = \{c\}$. For $I(b, o = e, R_1) = \{c, d\}$, we have $c \notin LS(b, R_1) \cup I(b, o = e, R_1) - argmin(R'_1, A) = \{d\} \nsubseteq L(c, R'_1) = \{c\}$. Now, for $a \in F(R)$, we have $I(a, o, R_1) = \{\emptyset\}$ and $c \notin LS(a, R_1) \cup I(a, o, R_1) - argmin(R'_1, A) = \{b, d, e\} \nsubseteq L(c, R'_1) = \{c\}$; thus $c \notin F(R')$.

$I^{\star}$-weak no-veto power lies between the standard properties of no-veto power and unanimity, it is implied by the former, but it has no logical relationship with the latter, as we show in the following observation.

Observation 1: *i) The I^*-weak no-veto power property is implied by no-veto power, but the converse does not always hold. ii) There is no logical relationship between I^*-weak no-veto power and unanimity.*

Proof. For the part (*i*), it is trivial that no-veto power implies I^*-weak no-veto power. For the converse, it follows from Example 1 that I^*-weak no-veto power is satisfied, but no-veto power is not.

Concerning the part (*ii*), to prove that I^*-weak no- veto power does not imply the unanimity condition, it suffices to have an alternative b ranked at the top and strictly preferred to an optimal outcome $a \in F(R)$, in this case I^*-weak no-veto power is satisfied, but unanimity is not. For the converse, we just replace in Example 1 the preference order for agent 1 at profile R'; we make $aP_i'dP_i'c\sim_i' eP_b'$ instead of $aP_i'dP_i'eP_i'bP_c'$. In this case, it is easy to see that the unanimity property is satisfied, but I^*-weak no-veto power is not. We have $b \in F(R)$, $I(b, o = d, R_1) = \{c, e\}$, $c \in LS(b, R_1) \cup I(b, o = d, R_1) - Argmin(R_1', A) = \{c, e\} \subseteq L(c, R_1') = \{b, c, e\}$ and $L(c, R_{i=2,3}') = A$, but $c \notin F(R')$. Q.E.D

The second property, called I-monotonicity, that we consider for sufficiency has been introduced recently by Doghmi and Ziad (2015). An SCC F satisfies I-monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $i \in N$, $LS(a, R_i) \cup I(a, o, R_i) \cup \{a\} \subseteq L(a, R_i')$ for some $o \in LI(a, R_i) \backslash \{a\}$, then $a \in F(R')$. This property can be equivalently reformulated as follows.

Definition 3: *(I-monotonicity)*
An SCC F satisfies I-monotonicity if for all $R, R' \in \Re$, for any $a \in F(R)$, if for any $i \in N$, $[LS(a, R_i) \cup I(a, o, R_i) - argmin(R_i', A)] \cup \{a\} \subseteq L(a, R_i')$ for some $o \in LI(a, R_i) \backslash \{a\}$, then $a \in F(R')$.

I-monotonicity says that if an alternative a is selected for some profile of preferences R, and if for all agents, the alternatives produced by the union of the subset of those that are ranked strictly below a and the subset of *indifferent options*, in excluding all bottom-ranked alternatives in a new profile R' (including a) remain ranked below a (in large sense) in R', then the alternative a must be chosen for the new profile R'.

The next result provides a sufficient condition for an SCC to be Nash implementable. We shall use this result, which differs from the previous findings in the literature, as a principle tool to prove the main result in the next section.

Theorem 1: *Let $n \geq 3$. Any I-monotonic and unanimous SCC satisfying I^*-weak no-veto power can be implemented in Nash equilibria.*

Proof. See appendix.

4 Main Result on Preference Domains with Private Values and Indifference

In this section, we present a model of allotment economies in preference domains with private values and different types of indifference, which generalize single-peaked and single-dipped preference domains. We show that $I^{\star}$-weak no-veto power is implied by unanimity and I-monotonicity with full range property imply the two properties of $I^{\star}$-weak no-veto power and unanimity. Thus, we conclude that I-monotonicity with the full range property form a sufficient condition for Nash implementation, and moreover I-monotonicity becomes equivalent to Maskin monotonicity, which imply that an SCC with full range is Nash implementable if and only if it satisfies Maskin monotonicity. We use this result to study the implementability of some well-known correspondences, such as the Pareto correspondence, the no-envy correspondence, the individually rational correspondence from equal division, the strong Pareto indifferent correspondence and some of their intersections.

4.1 Model and Preference Domains

An amount $\Omega \in \mathbb{R}_{++}$ of certain infinitely divisible goods is to be allocated among a set $N = \{1, \ldots, n\}$ of n agents. The preference of each agent $i \in N$ is represented by a continuous[5] relation R_i over $[0, \Omega]$ for the following types of preference.[6]

Single-peaked preferences with worst indifferent allocations: A preference relation R_i is *single-peaked* if there is a number $p(R_i) \in [0, \Omega]$ such that for all $x_i, y_i \in [0, \Omega]$ if $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, then $x_i P_i y_i$. We call $p(R_i)$ the *peak* of R_i.

A preference relation R_i is *single-peaked with worst indifferent allocations* if there exists an interval $[a, b] \subseteq [0, \Omega]$ and a peak $p(Ri) \in]a, b[$ such that (i) R_i is single-peaked on $[a, b]$; (ii) for all $x_i \in]a, b[$ and $y_i \in [0, \Omega] \backslash [a, b]$, $x_i P_i y_i$; and (iii) for all $x_i, y_i \in [0, \Omega] \backslash]a, b[$, $x_i \sim_i y_i$. This definition provides an ordinal representation of Cantala's (2004) class of preferences.

5 In this case, continuous means that if $[a, b[\cup]b, c] \subseteq L(x_i, R_i')$ for some a, b, c, x and R', then $[a, c] \subseteq L(x_i, R_i')$.

6 For all $x_i, y_i \in [0, \Omega]$, $x_i R_i y_i$ means that it is as good for agent i to consume a share x_i as it is to consume the quantity y_i. The asymmetrical part is written as P_i and the symmetrical part as $\sim_i$.

The class of all single-peaked preference relations with worst indifferent allocations is represented by $\Re^{wia}_{sp_i} \subseteq \Re_i$. Let $\Re^{wia}_{sp} = \Re^{wia}_{sp_1} \times \ldots \times \Re^{wia}_{sp_n}$ be the domain of single-peaked preferences with worst indifferent allocations. For a single-peaked preference relation with worst indifferent allocations $R_i \in \Re^{wia}_{sp_i}$, the function r_i is defined as follows: if $x_i \in [0, a]$, then, $r_i(x_i = [b, \Omega]$ if such a interval (or a number) exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \in [b, \Omega]$, then, $r_i(x_i) = [0, a]$ if such a interval (or a number) exists or $r_i(x_i) = 0$ otherwise; if $x_i \in]a, b[$, then we have the same definition to that of single-peaked preference relations.

Single-plateaued preferences with worst indifferent allocations: A preference relation R_i is *single-plateaued* if there are two numbers $\underline{x}_i, \overline{x}_i \in [0, \Omega]$ such that $\underline{x}_i \le \overline{x}_i$ and for all $x_i, y_i \in [0, \Omega]$: (*i*) if $y_i < x_i \le \underline{x}_i$ or $\overline{x}_i \le x_i < y_i$, then $x_i P_i y_i$; (*ii*) if $x_i, y_i \in [\underline{x}_i, \overline{x}_i]$, then $x_i \sim_i y_i$. We call $pl(R_i) \equiv [\underline{x}_i, \overline{x}_i]$ the *plateau* of R_i, $\underline{x}_i$ is the left end-point of the plateau of R_i, and $\overline{x}_i$ is the right end-point.

A preference relation R_i is *single-plateaued with worst indifferent allocations* if there exists an interval $[a, b] \subseteq [0, \Omega]$ and a plateau $[\underline{x}_i, \overline{x}_i] \subseteq]a, b[$ such that (i) R_i is single-plateaued on $[a, b]$; (ii) for all $x_i \in]a, b[$ and $y_i \in [0, \Omega] \backslash [a, b]$, $x_i P_i y_i$; and (iii) for all $x_i, y_i \in [0, \Omega] \backslash]a, b[$, $x_i \sim_i y_i$.

The class of all single-plateaued preference relations with worst indifferent allocations is represented by $\Re^{wia}_{spl_i} \subseteq \Re_i$. Let $\Re^{wia}_{spl} = \Re^{wia}_{spl_1} \times \ldots \times \Re^{wia}_{spl_n}$ be the domain of single-plateaued preferences. A single-plateaued preference relation with worst indifferent allocations $R_i \in \Re^{wia}_{spl_i}$ is described by the function $r_i : [0, \Omega] \to [0, \Omega]$ which is defined as follows: if $x_i \in [0, a]$, then, $r_i(x_i = [b, \Omega]$ if such a interval (or a number) exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \in [b, \Omega]$, then, $r_i(x_i) = [0, a]$ if such a interval (or a number) exists or $r_i(x_i) = 0$ otherwise; if $x_i \in]a, b[$, then we have the same definition to that of single-plateaued preference relations.

This super-domain is an enlargement of single-peaked, single-plateaued, and Cantala's preferences (2004), i. e., single-peaked preferences with worst indifferent allocations. It lies between these preferences domains and that of Dasgupta and *al.* (1979), where there is a top plateau of indifferent best elements with additional plateaus.[7]

Now, we present the inverse configuration of the above preference domains.

7 A preference relation R_i is *top-single-plateaued with additional plateaus* if there are two numbers $\underline{x}_i, \overline{x}_i \in [0, \Omega]$ such that $\underline{x}_i \le \overline{x}_i$ and for all $x_i, y_i \in [0, \Omega]$: (*i*) if $y_i < x_i \le \underline{x}_i$ or $\overline{x}_i \le x_i < y_i$, then $x_i R_i y_i$; (*ii*) We call $[\underline{x}_i, \overline{x}_i]$ the *top-plateau* of R_i, $\underline{x}_i$ is the left end-point of the plateau of R_i, and $\overline{x}_i$ is the right end-point.

Single-troughed preferences: A preference relation R_i is *single-troughed* if there are two numbers $\underline{x}_i, \overline{x}_i \in [0, \Omega]$ such that $\underline{x}_i \leq \overline{x}_i$ and for all $x_i, y_i \in [0, \Omega]$: (*i*) if $< x_i < y_i \leq \underline{x}_i$ or $\overline{x}_i \leq y_i < x_i$, then $x_i P_i y_i$; (*ii*) if $x_i, y_i \in [\underline{x}_i, \overline{x}_i]$, then $x_i \sim_i y_i$. We call $t(R_i) \equiv [\underline{x}_i, \overline{x}_i]$ the *trough* of R_i, $\underline{x}_i$ is the left end-point of the trough of R_i, and $\overline{x}_i$ is the right end-point. A preference relation R_i is *single-dipped* if $\underline{x}_i = \overline{x}_i$.

The class of all single-troughed preference relations is represented by $\Re_{st_i} \subseteq \Re_i$. Let $\Re_{st} = \Re_{st_1} \times \ldots \times \Re_{st_n}$ be the domain of single-troughed preferences. For $\Re_{st}$, let $t(R) = (t(R_1), \ldots, t(R_n))$ be the profile of troughs. A single-troughed preference relation $R_i \in \Re_{st_i}$ is described by the function $r_i : [0, \Omega] \to [0, \Omega]$ which is defined as follows: $r_i(x_i)$ is the consumption of the agent i on the other side of the trough which is indifferent to x_i (if it exists), else, it is 0 or Ω. Formally, if $x_i \leq t(R_i)$, then, $r_i(x_i) \geq t(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \geq t(R_i)$, then, $r_i(x_i) \leq t(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = 0$ otherwise.

Single-troughed preferences with best indifferent allocations: A preference relation R_i is *single-troughed with best indifferent allocations* if there exists an interval $[a, b] \subseteq [0, \Omega]$ and a trough $t(R_i) = [\underline{x}_i, \overline{x}_i] \subseteq]a, b[$ such that (i) R_i is single-troughed on $[a, b]$; (ii) for all $y_i \in]a, b[$ and $x_i \in [0, \Omega] \backslash [a, b]$, $x_i P_i y_i$; and (iii) for all $x_i, y_i \in [0, \Omega] \backslash]a, b[$, $x_i \sim_i y_i$.

The class of all single-troughed preference relations with best indifferent allocations is represented by $\Re^{bia}_{st_i} \subseteq \Re_i$. Let $\Re^{bia}_{st} = \Re^{bia}_{st_1} \times \ldots \times \Re^{bia}_{st_n}$ be the domain of single-troughed preferences with best indifferent allocations. For $R_i \in \Re^{bia}_{st_i}$, the function r_i is defined as follows: if $x_i \in [0, a]$, then, $r_i(x_i = [b, \Omega]$ if such a interval (or a number) exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \in [b, \Omega]$, then, $r_i(x_i) = [0, a]$ if such a interval (or a number) exists or $r_i(x_i) = 0$ otherwise; if $x_i \in]a, b[$, then we have the same definition in standard single-troughed preference relations. This domain is a super-domain of the single-dipped preferences domain, the single-troughed preferences domain and of the single-dipped preferences domain with best indifferent allocations.

We denote these four domain restrictions with indifferent allocations by $\Re^{IA}_{DR} = \{\Re^{wia}_{sp}, \Re^{wia}_{spl}, \Re_{st}, \Re^{bia}_{st}\}$. Hence, without loss of generality, when we say $R \in \Re^{IA}_{DR}$, it implies that $R \in \Re^{wia}_{sp}$, or $R \in \Re^{wia}_{spl}$ or $R \in \Re_{st}$ or $R \in \Re^{bia}_{st}$.

A feasible allocation for the economy $(R, \Omega) \in (\Re^{IA}_{DR}, \mathbb{R}_{++})$ is a vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = \Omega$ and X is the set of the feasible allocations. We note that the feasible allocations set is $X \subseteq [0, \Omega] \times \ldots \times [0, \Omega]$. Thus, $L(x, R_i) = X$ is equivalent to $L(x_i, R_i) = [0, \Omega]$. For the set $L(x, R_i) = X$, xR_iy for all $y \in X$ implies that $x_iR_iy_i$. Thus, the agents preferences are defined over individual consumption spaces, not over allocation space. Then the properties of implementation theory, presented in general setup in Section 2, become as follows. An SCC F is a multi-valued mapping from $\Re^{IA}_{DR}$ into X. A SCC F satisfies monotonicity if for all $R, R' \in \Re_{DR}^{IA}$, for any $x \in F(R)$, if for any $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R_i')$, then $x \in F(R')$. A SCC F satisfies I-monotonicity if for all $R, R' \in \Re_{DR}^{IA}$, and for any $x \in F(R)$, if for any $i \in N$, $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} \subseteq L(x_i, R_i')$ for some $z \in LI(x, R_i) \setminus \{x\}$, then $x \in F(R')$. A SCC F satisfies no-veto power if for each i, each $R \in \Re_{DR}^{IA}$, and each $x \in X$, if $L(x_j, R_j) = [0, \Omega]$ for all $j \in N \backslash \{i\}$, then $x \in F(R)$. A SCC F satisfies $I^{\star}$-weak no-veto power if for each i, each $R \in \Re_{DR}^{IA}$, each $x \in F(R)$, and each $y \in X$, if for $R' \in \Re_{DR}^{IA}$, $y_i \in LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega]) \subseteq L(y_i, R_i')$ and $L(y_j, R'_j) = [0, \Omega]$ for all $j \in N \backslash \{i\}$, for some $z \in LI(x, R_i) \setminus \{x\}$, then $y \in F(R')$. A SCC F satisfies unanimity if for any $x \in X$ and any $R \in \Re_{DR}^{IA}$, if for any $i \in N$, $L(x_i, R_i) = [0, \Omega]$, then $x \in F(R)$. We note that the free disposability of the good is not assumed.

4.2 The Challenges for the $\Re_{DR}^{IA}$ Domains Compared to the Previous Findings

Before providing our main result in the next subsection, we discuss here the challenges of the implementability of SCCs on the domain restrictions with indifferent allocations $\Re_{DR}^{IA}$ compared to the previous findings of Thomson (1990, 2010), Doghmi and Ziad (2008b, 2013a, 2013b, 2015), and Doghmi (2013a, 2013b) for the restricted domains of single-peaked, single-dipped and single-plateaued preferences, and single-dipped preferences with best indifferent allocations. On single-peaked domain, Thomson (1990, 2010) applied the different techniques developed in Maskin (1999) and Yamato (1992) to examine the implementability of a family of solutions of the problem of fair division. He concluded that these tools do not always work. Hence, he appealed to Sjöström's algorithm (1991), which is not simple. In an easy way, Doghmi and Ziad (2008b) and Doghmi (2013a) applied the results of Doghmi and Ziad (2008a) and proved that Maskin monotonicity is a necessary and sufficient condition to implement these solutions on single-peaked and single-dipped domains without appealing to the different techniques used by Thomson (1990, 2010). Although this latter result is encouraging, a difficulty occurs when we allow indifference between alternatives. To overcome this difficulty, Doghmi and Ziad (2015) provided the new properties of I-monotonicity and I-weak no-veto power[8] to implement the

8 A SCC F satisfies I-weak no-veto power if for i, $R \in \Re$, and $a \in F(R)$, if for $R' \in \Re$, $b \in LS(a, R_i) \cup I(a, o, R_i) \subseteq L(b, R_i')$ and $L(b, R'_j) = A$ for all $j \in N \backslash \{i\}$, for some $o \in LI(a, R_i) \setminus \{a\}$, then $b \in F(R')$. This variant implies $I^{\star}$-weak no-veto power, but the converse does not always hold.

unanimous SCCs in general environment. Then, they showed that in single-plateaued domains (and in single-dipped domain with best indifferent allocations by Doghmi (2013b)), (*i*) full range property and I-monotonicity implies the two properties of unanimity and I-weak no-veto power, and (*ii*) unanimity implies I-weak no-veto power, and hence I-monotonicity with the full range property form together a sufficient condition for implementation. Moreover, I-monotonicity becomes equivalent to Maskin monotonicity, which imply that an SCC with full range is Nash implementable if and only if it satisfies Maskin monotonicity. Nevertheless, this result does not manage the issue of the $\Re_{DR}^{IA}$ domains, which include *multiple worst indifferent allocations*. The reason is that in this area the variant of weak no-veto power provided by Doghmi and Ziad (2015) is not implied by the property of unanimity when preference profile admit indifference between the worst allocations, and hence the analysis proposed by Doghmi and Ziad (2015) to achieve a full characterization does not work. Here, our new property of $I^{\star}$-weak no-veto power has a key role to play in filling this gap. We prove in the next subsection that $I^{\star}$-weak no-veto power is implied by unanimity, and I-monotonicity together with the full range property imply unanimity, and hence they also imply $I^{\star}$-weak no-veto power. Therefore, we conclude that I-monotonicity maintains its robustness and remains a sufficient condition in this context of preference profiles with multiple worst indifferent elements. Moreover, it also remains equivalent to Maskin monotonicity, and hence it constitutes with the full range property a necessary and sufficient condition for an SCC to be Nash implementable. We shall detail and explain this result in a formal way in the next subsection.

4.3 Full Characterization for Nash Implementation on $\Re_{DR}^{IA}$

To provide a full characterization for an SCC to be Nash implementable on the domain restrictions with indifferent allocations ($\Re_{DR}^{IA}$), we first give some useful tools. We begin by introducing the following Lemma.

Lemma 1: *Let $R, R' \in \Re_{DR}^{IA}$, $x, y, z \in X$, and $i \in N$. If $y_i \in LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega]) \subseteq L(y_i, R_i')$, then $L(y_i, R_i') = [0, \Omega]$.*

The proof of Lemma 1 is omitted because it follows the same reasoning as Lemma 1 in Doghmi and Ziad (2008b).

The following proposition shows that our new property of $I^{\star}$-weak-no-veto power is implied by unanimity. This result can not be achieved by earlier versions of no-veto power, and here where it appears the usefulness of this new requirement.

Proposition 1: *In the private goods economies under domain restrictions with indifferent allocations $\Re_{DR}^{IA}$, any unanimous SCC satisfies $I^{\star}$-weak no-veto power.*

The proof of this proposition is omitted, it is immediate from Lemma 1. Next, we provide the definition of the full range property.

Definition 4: *(Full range)*
An SCC F satisfies the full range property (or citizen sovereignty) if for each $x \in X$, there is a profile $R \in \Re_{DR}^{IA}$ such that $x \in F(R)$.

The full range requirement is satisfied vacuously for all SCCs considered in this study on $\Re_{DR}^{IA}$.

The following result extends Lemma 2 of Doghmi and Ziad (2015) from single-plateaued preferences domain to a large domain of preferences with indifference.

Proposition 2: *In the private goods economies under domain restrictions with indifferent allocations $\Re_{DR}^{IA}$, any I-monotonic SCC that has full range satisfies unanimity.*

Proof. We consider the domain restriction $\Re_{sp}^{wia}$. Assume that I-monotonic SCC does not satisfy unanimity. Let $x \in X$ and any $\widetilde{R} \in \Re_{sp}^{wia}$, for any $i \in N$, $[0, \Omega] = L(x_i, \widetilde{R}_i)$, and $x \notin F(\widetilde{R})$. By the full range requirement, for all $x \in X$, there is a profile $R \in \Re_{sp}^{wia}$ such that $x \in F(R)$ and thus for all $i \in N$, $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} \subseteq [0, \Omega] = L(x_i, \widetilde{R}_i)$. By I-monotonicity, $x \in F(\widetilde{R})$, a contradiction. We follow the same reasoning for the domain restrictions $\Re_{spl}^{wia}$, $\Re_{st}$, and $\Re_{st}^{bia}$. Q.E.D.

By Propositions 1 and 2, we obtain the following corollary.

Corollary 1: *In the private goods economies under domain restrictions with indifferent allocations $\Re_{DR}^{IA}$, any I-monotonic SCC that has full range satisfies $I^{\star}$-weak no-veto power.*

From Theorem 1, Propositions 1–2, and Corollary 1 we give the following result.

Corollary 2: *Let $n \geq 3$. In the private goods economies under domain restrictions with indifferent allocations $\Re_{DR}^{IA}$, any SCC that has full range satisfying I-monotonicity can be implemented in Nash equilibria.*

In the following proposition, we show that I-monotonicity is not only sufficient but is also necessary provided that it becomes equivalent to Maskin monotonicity.

Proposition 3: *In the private goods economies under domain restrictions with indifferent allocations* $\Re^{IA}_{DR}$, *the I-monotonicity condition becomes equivalent to Maskin monotonicity.*

Proof. Let $R, R' \in \Re^{IA}_{DR}$, $x, y, z \in X$, and $x \in F(R)$. *i*) I-monotonicity implies Maskin monotonicity; this first implication is immediate from Observation 1 of Doghmi and Ziad (2015). *ii*) Maskin monotonicity implies I-monotonicity; in this case, assume that $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} \subseteq L(x_i, R_i')$ for some $z \in LI(x, R_i) \backslash \{x\}$ (1). We have four situations to study.

Situation 1: We consider the domain restriction $\Re^{wia}_{sp}$. We suppose that $x_i \le p(R_i)$ (we follow the same reasoning for $x_i > p(R_i)$). We have three cases to consider.

i) If $x_i \in]a, p(R_i)]$ then $I(x_i, z_i, R_i) = \emptyset$ and so we follow the same statement of Lemma 1 in Doghmi and Ziad (2008b).

ii) If $x_i \in [0, a]$, and $\exists\{z_i\} \in [0, a]$ s.t. $0 < z_i < a$ then we have $I(x_i, z_i, R_i) \neq \emptyset$ and hence we have the following possibilities:

ii.a – If $x_i < z_i$ with $x_i \neq 0$, then $\emptyset \neq I(x_i, z_i, R_i) = [0, x_i[\cup]x_i, z_i[\cup]z_i, a] \cup [b, \Omega]$ and $LS(x_i, R_i) = \emptyset$. For a profile R', we have $argmin(R_i', [0, \Omega]) = [0, a'] \cup [b', \Omega]$, in this case we have either $x_i \in [0, \Omega] \backslash]a', b'[$ or $x_i \in]a', b'[$. If $x_i \in [0, \Omega] \backslash]a', b'[$, then by (1) $[a, b] \supseteq [a', b']$ and thus $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} = \{x_i\} \subseteq L(x_i, R_i') = [0, a'] \cup [b', \Omega]$. It is clear that $L(x_i, R_i) = [LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} \cup [0, x_i[\cup]x_i, a] \cup [b, \Omega] \subseteq L(x_i, R_i') = [0, a'] \cup [b', \Omega]$. By the continuity of preferences, $L(x_i, R_i) = [0, a] \cup [b, \Omega] \subseteq L(x_i, R_i') = [0, a'] \cup [b', \Omega]$. By Maskin monotonicity, $x \in F(R')$. If $x_i \in]a', b'[$, then from (1) we have $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} =]a', z_i[\cup]z_i, a] \cup [b, b'[\subseteq L(x_i, R_i') = [0, \Omega]$ if $b < b'$ (or $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} =]a', z_i[\cup]z_i, a] \subseteq L(x_i, R_i') = [0, \Omega]$ if $b \ge b'$, in this case we follow the same reasoning). By the continuity of preferences, $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} =]a', a] \cup [b, b'[= L(x_i, R_i) - argmin(R_i', [0, \Omega]) \subseteq L(x_i, R_i') = [0, \Omega]$ (2). From (1) and (2) we have $L(x_i, R_i) - argmin(R_i', [0, \Omega]) \subseteq L(x_i, R_i')$. Therefore, $L(x_i, R_i) \subseteq L(x_i, R_i') \cup argmin(R_i', [0, \Omega])$. Since we have $L(x_i, R_i') \cup argmin(R_i', [0, \Omega]) = L(x_i, R_i')$, then $L(x_i, R_i) \subseteq L(x_i, R_i')$. By Maskin monotonicity, $x \in F(R')$.

ii.b – If $x_i < z_i$ with $x_i = 0$, then $\emptyset \neq I(x_i, z_i, R_i) =]0, z_i[\cup]z_i, a] \cup [b, \Omega]$, thus we follow the same reasoning in (*ii.a*).

ii.c – If $z_i < x_i$ with $x_i \neq a$ or $z_i < x_i$ with $x_i = a$, then we follow the same reasoning in (*ii.a*).

iii) If $x_i \in [0, a]$, and $\exists\{z_i\} \in [b, \Omega]$ s.t. $b < z_i < \Omega$, $\emptyset \neq I(x_i, z_i, R_i) = [0, x_i[\cup]x_i, a] \cup [b, z_i[\cup]z_i, \Omega]$ and $LS(x_i, R_i) = \emptyset$. For a profile R', we have $argmin(R_i', [0, \Omega]) = [0, a'] \cup [b', \Omega]$, hence we have either $x_i \in [0, \Omega] \backslash]a', b'[$ or $x_i \in]a', b'[$. For the two cases, we follow the same statement in (*ii.a*).

Situation 2: We consider the domain restriction $\Re_{st}$. We have three cases to consider.

i) If $x_i \notin [\underline{x}_i, \overline{x}_i]$, then $I(x_i, z_i, R_i) = \emptyset$ and $LS(x_i, R_i) =]x_i, r_i(x_i)[$. In this case, we have either $x_i \in t(R') = [\underline{x}'_i, \overline{x}'_i]$ or $x_i \notin [\underline{x}'_i, \overline{x}'_i]$. If $x_i \in t(R') = [\underline{x}'_i, \overline{x}'_i]$ with $t(R') \supset t(R)$ (we follow the same reasoning to prove all other configurations), then by (1), $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} = \{x_i\} \subseteq L(x_i, R'_i) = [\underline{x}'_i, \overline{x}'_i] = [\underline{x}'_i, r_i(x_i)_i]$ if $r_i(x_i)$ exists (if the numbers $r_i(x_i)$ and $r'_i(x_i)$ do not exist, we follow the same reasoning and we obtain the same result). We have $L(x_i, R_i) = [LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} \cup]x_i, r_i(x_i)]$. By the continuity of preferences, $L(x_i, R_i) = [LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} \cup] x_i, r_i(x_i)] = [x_i, r_i(x_i)] \subseteq L(x_i, R'_i) = [\underline{x}'_i, r_i(x_i)]$. By Maskin monotonicity, $x \in F(R')$. If $x_i \notin [\underline{x}'_i, \overline{x}'_i]$. By (1) we have $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} = [x_i, \underline{x}'_i[\cup]\overline{x}'_i, r_i(x_i) \; [\subseteq L(x_i, R'_i) = [x_i, r'(x_i)]$ if $t(R') \subset [x_i, ri(x_i)]$ (we follow the same reasoning to prove all other configurations). It is clear that $L(x_i, R_i) = [LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} \cup]\underline{x}'_i, \overline{x}'_i[\cup]r_i(x_i), r_i(x_i)] = [x_i, \underline{x}'_i [\cup]\underline{x}'_i, \overline{x}'_i[\cup]\overline{x}'_i, r_i(x_i)[\cup]r_i(x_i), r_i(x_i)] \subseteq L(x_i, R'_i) = [x_i, r'(x_i)]$. By the continuity of preferences, $L(x_i, R_i) = [x_i, r_(x_i)] \subseteq L(x_i, R'_i) = [x_i, r'(x_i)]$. By Maskin monotonicity, $x \in F(R')$.

ii) If $x_i \in [\underline{x}_i, \overline{x}_i]$, then $I(x_i, z_i, R_i) \neq \emptyset$ and hence we have the following possibilities:

ii.a – If $x_i < z_i$ with $x_i \neq \underline{x}_i$, then $\emptyset \neq I(x_i, z_i, R_i) = [\underline{x}_i, x_i[\cup]x_i, z_i[\cup]z_i, \overline{x}_i]$ and $LS(x_i, R_i) = \emptyset$. In this case, we have by (1) $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} = \{x_i\} \subseteq L(x_i, R'_i) = [\underline{x}'_i, \overline{x}'_i]$. It is clear that $L(x_i, R_i) = [LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R'_i, [0, \Omega])] \cup \{x_i\} \cup [\underline{x}_i, x_i[\cup]x_i, \overline{x}_i] \subseteq L(x_i, R'_i)$, i. e., $L(x_i, R_i) = \{x_i\} \cup [\underline{x}_i, x_i[\cup]x_i, \overline{x}_i] \subseteq L(x_i, R'_i)$. By the continuity of preferences, $L(x_i, R_i) = [\underline{x}_i, \overline{x}_i] \subseteq L(x_i, R'_i)$. By Maskin monotonicity, $x \in F(R')$.

ii.b – If $x_i < z_i$ with $x_i = \underline{x}_i$, then $\emptyset \neq I(x_i, z_i, R_i) =]x_i, z_i[\cup]z_i, \overline{x}_i]$ and $LS(x_i, R_i) = \emptyset$ and so we follow the same reasoning in (*ii.b*).

ii.c If $z_i < x_i$ with $x_i \neq \overline{x}_i$ or $z_i < x_i$ with $x_i = \overline{x}_i$ then we follow the same reasoning in (*ii.a*).

Situation 3: We consider the domain restriction $\Re^{wia}_{spl}$. We suppose that $x_i \leq \overline{x}_i$ (we follow the same reasoning for $x_i > \underline{x}_i$). We have four cases to consider.

i) If $x_i \in]a, \underline{x}_i[$ then $I(x_i, z_i, R_i) = \emptyset$ and so we follow the same statement of Proposition 4 in Doghmi and Ziad (2008b).

ii) If $x_i \in [0, a]$, and $\exists \{z_i\} \in [0, a]$ s.t. $0 < z_i < a$ then we have $I(x_i, z_i, R_i) \neq \emptyset$ and so we follow the same reasoning in (*ii*) of Situation 1.

iii) If $x_i \in [0, a]$, and $\exists \{z_i\} \in [b, \Omega]$ s.t. $b < z_i < \Omega$, then we have $I(x_i, z_i, R_i) \neq \emptyset$ and so we follow the same reasoning in (*iii*) of Situation 1.

iv) If $x_i \in [\underline{x}_i, \overline{x}_i]$ then we follow the same statement of Proposition 2 in Doghmi and Ziad (2013b).

Situation 4: We consider the domain restriction $\Re_{st}^{bia}$. Suppose that $x_i \leq \overline{x}_i$ (similar reasoning can be followed for $x_i > \overline{x}_i$. We have three cases to consider.

i) If $x_i \in [0, a]$ and $\exists\{z_i\} \in [0, a]$ s.t. $0 < z_i < a$, then we have the following possibilities to study.

i.a – If $x_i < z_i$ with $x_i \neq 0$, then $\emptyset \neq I(x_i, z_i, R_i) = [0, x_i[\cup]x_i, z_i[\cup]z_i, a] \cup [b, \Omega]$ and $LS(x_i, R_i) =]a, b[$. For a profile R', assume that $t(R_i') \supseteq t(R)$ with $a' < a < \underline{x}'_i < \underline{x}_i$ and $\overline{x}_i < b < \overline{x}'_i < b'$. By (1), $[LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin\ (R_i', [0, \Omega])] \cup \{x_i\} = [0, z_i[\cup]z_i, \underline{x}'_i[\cup]\overline{x}'_i, \Omega] \subseteq L(x_i, R_i') = [0, \Omega]$. It is clear that $L(x_i, R_i) = [LS(x_i, R_i) \cup I(x_i, z_i, R_i) - argmin(R_i', [0, \Omega])] \cup \{x_i\} \cup \{z_i\} \cup [\underline{x}'_i, \overline{x}'_i]$, i. e., $L(x_i, R_i) = [0, z_i[\cup \{z_i\} \cup]z_i, \underline{x}'_i[\cup [\underline{x}'_i, \overline{x}'_i] \cup]\overline{x}'_i, \Omega]$. By the continuity of preferences, $L(x_i, R_i) = [0, \Omega] \subseteq L(x, R_i') = [0, \Omega]$. By Maskin monotonicity, $x \in F(R')$. We follow the same statement for all other configurations.

i.b – If $x_i < z_i$ with $x_i = 0$, then $\emptyset \neq I(x_i, z_i, R_i) =]x_i, z_i[\cup]z_i, a] \cup [b, \Omega]$ and so we follow the same reasoning in (*i.a*).

i.c – If $z_i < x_i$ with $x_i \neq a$ or $z_i < x_i$ with $x_i = a$, then we follow the same reasoning in (*i.a*).

ii) If $x_i \in [0, a]$ and $\exists\{z_i\} \in [b, \Omega]$, then we follow the same statement in in case (*i*).

iii) If $x_i \in]a, \overline{x}_i]$, then we follow the same statement of the proof of Situation 2. Q.E.D.

By Corollary 2 and Proposition 3 we complete the proof of the following main result in this work.

Theorem 2: *Let $n \geq 3$. In the private goods economies under domain restrictions with indifferent allocations $\Re_{DR}^{IA}$, any SCC that has full range is Nash implementable if and only if it satisfies Maskin monotonicity.*

This result also applies to the domains restrictions of single-peaked preferences ($\Re_{sp}$), single-dipped preferences ($\Re_{sd}$), single-plateaued preferences ($\Re_{spl}$), and single-dipped preferences with best indifferent allocations ($\Re_{sd}^{bia}$). Thus, it follows from Theorem 2, Theorem 1 of Doghmi (2013b), and Theorem 2 of Doghmi and Ziad (2015), the next corollary.

Corollary 3: *Let $n \geq 3$. In the private goods economies under the domain restrictions of $\Re_{sp}$, $\Re_{sd}$, $\Re_{spl}$, $\Re_{sd}^{bia}$, and $\Re_{DR}^{IA}$, any SCC that has full range is Nash implementable if and only if it satisfies Maskin monotonicity.*

In the next subsection, we provide some applications of Theorem 2 by examining the implementability of some well-known SCCs.

4.4 Examples of Some Monotonic and Nonmonotonic Well-known SCCs under Domain Restrictions with Indifferent Allocations

To apply the result of Theorem 2 to the problem of the fair allocation of an infinite divisible commodity among a group of agents, we introduce a variety of well-known solutions to this problem and we inspect whether these correspondences are implementable on the $\Re_{DR}^{IA}$ domains.

4.4.1 Some Well-known Correspondences

We begin by defining the no-envy solution that was introduced by Foley (1967).

No-envy correspondence, *NE*: This correspondence selects the feasible allocations for which each agent prefers his own share than the shares of the other agents. It is defined as follows: Let $R \in \Re_{DR}^{IA}$, $NE(R) = \{x \in X$ if $x_i R_i x_j$ for all $i, j \in N\}$.

Individually rational correspondence from equal division, I_{ed}: This correspondence selects the feasible allocations for which each agent prefers his own share to the average one. It is defined as follows: Let $R \in \Re_{DR}^{IA}$, $I_{ed}(R) = \{x \in X : x_i R_i (\Omega/n)$ for all $i \in N\}$.

Pareto correspondence, *P*: This solution selects the feasible allocations which are not weakly dominated by an other allocation for all agents and not strictly dominated for at least one agent. It is defined as follows: Let $R \in \Re_{DR}^{IA}$, $P(R) = \{x \in X : = \nexists x' \in X$ such that for all $i \in N$, $x'_i R_i x_i$, and for some $i \in N$, $x'_i P_i x_i\}$.

Pareto indifferent, *PI*: This solution selects the feasible allocations which are indifferent for all agents. It is defined as follows: Two allocations $x, y \in X$ are *Pareto indifferent* under R if for all $i \in N$, $x_i \sim_i y_i$.

In the next definition, we assume that for all agent the share of an allocation x is not only indifferent to one share but to a set of shares.

Strong Pareto indifferent, *SPI*: An allocation x is strongly Pareto indifferent if there exists a subset $Y \subset X$ with $Y \neq \emptyset$ and $Y \backslash \{x\} \neq \{y\}$ such that for all $i \in N$, $x_i \sim_i y_i$ for all $y \in Y$.[9]

9 The notation $Y \backslash \{x\} \neq \{y\}$ means that if $x \in Y$ then $Y \backslash \{x\}$ is different to a singleton. If $Y \backslash \{x\}$ is a singleton, the solution of SPI reduces to the PI correspondence, which violates Maskin monotonicity on the $\Re_{DR}^{IA}$ domains.

In addition to these correspondences, we also introduce some solutions produced by intersections. We consider the intersection of the no-envy correspondence with the individually rational correspondence from equal division ($NE \cap I_{ed}$), and the intersections of the no-envy correspondence and the individually rational correspondence from equal division with the Pareto correspondence ($NE \cap P, I_{ed} \cap P$). We also consider the intersections of strong Pareto indifferent with the no-envy correspondence ($SPI \cap NE$), with the individually rational correspondence from equal division ($SPI \cap I_{ed}$), and with the Pareto correspondence ($SPI \cap P$).

4.4.2 Examples of Monotonic SCCs under Domain Restrictions with Indifferent Allocations $\Re_{DR}^{IA}$

In this subsection, we first give examples of SCCs that are monotonic in the unrestricted domains and also monotonic in the current setup, and we second provide exemples of SCCs that are not monotonic in the unrestricted domains, but monotonic and therefore Nash implementable in our context.

4.4.2.1 Examples of SCCs that are monotonic in unrestricted domains

The monotonic SCCs that we examine their implementability are the no-envy correspondence, the individually rational correspondence from equal division and their intersection. We give the following observation.

Observation 2: *In the private goods economies under domain restrictions with indifferent allocations $\Re_{DR}^{IA}$, the no-envy correspondence, the individually rational correspondence from equal division and their intersection all satisfy Maskin monotonicity.*

Since the no-envy correspondence, the individually rational correspondence and their intersection are monotonic in unrestricted domains, it is trivial that these correspondences are monotonic in our area. Thus, the proof of Observation 2 is omitted.

It follows from Thomson's results (1990, 2010) and Proposition 8 of Doghmi (2013a) that the no-envy solution, the individually rational correspondence from equal division and ($NE \cap I_{ed}$) correspondence all fail to satisfy no-veto power on domain restrictions with indifferent allocations. Thus, Maskin's theorem is silent about their implementability in this environment. Therefore, from Observation 2 we appeal to Theorem 2 and we give the following corollary.

Corollary 4: *In the private goods economies under domain restrictions with indifferent allocations* $\Re_{DR}^{IA}$, *the no-envy correspondence, the individually rational correspondence from equal division and their intersection are all Nash implementable by Theorem 2.*

4.4.2.2 Examples of SCCs that are nonmonotonic in unrestricted domains

In Example 2 we show that the *SPI* correspondence violates Maskin monotonicity in unrestricted domains. To illustrate this, we begin by adapting its notation to the general context provided in Section 2. An alternative a is strongly Pareto indifferent if there exists a subset $B \subset A$ with $B \neq \emptyset$ and $B \backslash \{a\} \neq \{b\}$ such that for all $i \in N$, $a \sim_i b$ for all $b \in B$.

Example 2: Let $A = \{a, b, c, d, e, f\}$, $B = \{b, e, f\} \subset A$, $N = \{1, 2, 3\}$ and $\Re = \{R, R'\}$ such that:

R:	R_1	R_2	R_3
	a,b,e,f	c	c
	d	a,b,e,f	a,b,e,f
	c	d	d

R′:	R'_1	R'_2	R'_3
	a	a	a
	b,c,e,f	b,c,e,f	b,c,e,f
	d	d	d

From the definition of *SPI* we have $SPI(R) = \{a, b, e, f\}$ and $SPI(R') = \{b, c, e, f\}$. It is clear that the *SPI* correspondence violates Maskin monotonicity, and hence it is not Nash implementable in unrestricted domains.

As mentioned in Subsection 4.4.1, we construct new solutions produced from the intersections of the SPI correspondence with the individually rational correspondence from equal division ($SPI \cap I_{ed}$), with the no-envy correspondence ($SPI \cap NE$), and with the Pareto correspondence ($SPI \cap P$). We show that these solutions that violate Maskin monotonicity in general domains become monotonic in our context.

To prove that the ($SPI \cap I_{ed}$) correspondence is nonmonotonic in unrestricted domains, we fix the definition of I_{ed} in this context. Let a_0 and b be elements of the set of alternatives A, and let $R \in \Re$. We denote by I the notion of I_{ed} in general setup, $I(R) = \{a \in A : aR_i a_0 \text{ and } a \sim_i b \text{ for all } i \in N\}$. In Example 2, let $a_0 = d$, therefore $(SPI \cap I)(R) = \{a, b, e, f\}$ and $(SPI \cap I)(R') = \{b, c, e, f\}$. We have $L(a, R_i) \subseteq L(a, R'_i)$ for $i = \{1, 2, 2\}$, but $a \notin (SPI \cap I)(R')$, hence the ($SPI \cap I$) correspondence violates Maskin monotonicity.

Now, we consider the same general of economies used in Doğan (2015) and we show in Example 2 that the ($SPI \cap NE$) correspondence does not satisfy Maskin monotonicity.

Example 3: Let $N=\{1,2,3\}$ and $X=\{x,y,z,w\}$ with $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$, $z=(z_1,z_2,z_3)$ and $w=(w_1,w_2,w_3)$. Let $Y=\{y,z,w\}\subset X$ and $\Re=\{R,R'\}$ such that:

R:	R_1	R_2	R_3
	x_1,y_1,z_1,w_1	x_2,y_2,z_2,w_2	x_3,y_3,z_3,w_3
	x_2,y_3,z_1	x_1,y_3,z_1	x_1,y_2,z_2
	x_3,y_2,z_3	x_3,y_1,z_3	x_2,y_1,z_1
	w_2,w_3	w_1,w_3	w_1,w_2

R′:	R'_1	R'_2	R'_3
	x_1	x_2,y_2,z_2,w_2	x_3,y_3,z_3,w_3
	y_1,z_1,w_1	x_3,y_3,z_1	x_2,y_1,z_1
	x_2,y_3,z_2,w_2	x_1,y_1,z_3	x_1,y_2,z_2
	x_3,y_2,z_3,w_3	w_1,w_3	w_1,w_2

From the definition of $(SPI\cap NE)$, we have $(SPI\cap NE)(R)=\{x,y,z,w\}$ and $(SPI\cap NE)(R')=\{y,z,w\}$. In this example, we have $x\in(SPI\cap NE)(R)$, and for $i=\{1,2,3\}$, $L(x_i,R_i)\subseteq L(x_i,R'_i)$, but $x\notin(SPI\cap NE)(R')$.

Finally, the $(P\cap SPI)$ correspondence that we consider does not satisfy Maskin monotonicity in unrestricted domains as long as the Pareto correspondence violates this property in these general domains.

In Proposition 4 that follows, we show that the correspondences of SPI, $SPI\cap I_{ed}$, $SPI\cap NE$, and $SPI\cap P$ satisfy the property of Maskin Monotonicity on the $\Re^{IA}_{DR}$ domains, and hence they are Nash implementable by Theorem 2.

Proposition 4: *In the private goods economies under domain restrictions with indifferent allocations $\Re^{IA}_{DR}$, the correspondence of strong Pareto indifferent and its intersections with the individually rational correspondence from equal division, with the no-envy correspondence, and with the Pareto correspondence do satisfy Maskin monotonicity, and thus they are Nash implementable by Theorem 2.*

Proof: Let $R\in\Re_{st}$, $x\in X$, and $x\in SPI(R)$. Assume that for a preference profile $R'\in\Re_{st}$ we have $L(x_i,R_i)\subseteq L(x_i,R'_i)$ for all $i\in N$ (1). The assumption $x\in SPI(R)$ means that for $R\in\Re_{st}$, there is $Y\subset X$ with $Y\neq\emptyset$ and $Y\backslash\{x\}\neq\{y\}$ such that for all $i\in N$, $x_i{\sim}y_i$ for all $y\in Y$ (2). From the definition of the $\Re_{st}$ preference profile class and (2), we must have $x_i\in[\underline{x}_i,\overline{x}_i]\subseteq[0,\Omega]$ such that $x_i{\sim}y_i$ for all $y_i\in[\underline{x}_i,\overline{x}_i]\}\subseteq Y$. By (1) we obtain $x_i\in[\underline{x}_i,\overline{x}_i]\subseteq[\underline{x'}_i,\overline{x'}_i]$, and hence $x\in SPI(R')$. We follow the same statement for the intersections of SPI with I_{ed}, NE, and P. We proceed in the same way for the preference domains of $\Re^{wia}_{sp}$, $\Re^{wia}_{spl}$, and $\Re^{bia}_{st}$. By Theorem 2 we complete the proof as required.[10] Q.E.D

10 For the case of the $\Re^{wia}_{spl}$ domain, we require that the plateau $[\underline{x}_i,\overline{x}_i]\neq\{x_i\}$, i. e., we can not combine the two domains of $\Re^{wia}_{sp}$ and $\Re^{wia}_{spl}$ to prove the case of the latter. We assume the same requirement for the intervals of best indifferent allocations of $\Re^{bia}_{st}$ domain.

4.4.3 Examples of Nonmonotonic SCCs under Domain Restrictions when the Preferences are Single-peaked/single-plateaued with Worst Indifferent Allocations ($\Re^{wia}_{sp}$, $\Re^{wia}_{spl}$)

The next proposition shows that the Pareto correspondence and its intersections with the no-envy correspondence, and with the individually rational correspondence from equal division do not satisfy Maskin monotonicity. Therefore, they are not Nash implementable in private goods economies when the preferences are single-peaked/single-plateaued with worst indifferent allocations.[11]

Proposition 5: *In private goods economies, when preferences are single-peaked/single-plateaued with worst indifferent allocations, the Pareto correspondence and its intersections with the no-envy correspondence, and with the individually rational correspondence from equal division do not satisfy Maskin monotonicity, and thus they are not Nash implementable.*

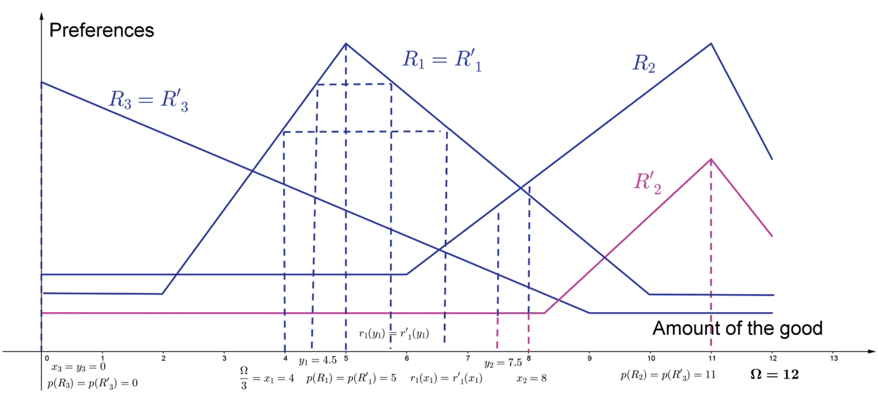

Figure 1: The P, $(P \cap NE)$, and $(P \cap I_{ed})$ correspondences do not satisfy Maskin monotonicity.

Proof: Let $R, R' \in \Re^{wia}_{sp}$, $x = (4, 8, 0), y = (4.5, 7.5, 0) \in X$ and $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = \Omega = 12$. Let $R_1 = R'_1$, $R_3 = R'_3$, and $p(R) = p(R') = (5, 11, 0)$. Figure 1 illustrates such representations.

11 We note that the Pareto correspondence satisfies Maskin monotonicity and no-veto power in *single-peaked* domain without worst indifferent allocations. Hence, it is Nash implementable in this domain by Maskin's original result. For further details, see Thomson (1990, 2010) and Doghmi and Ziad (2008b).

Note that $x \in P(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, for profile R', we have $y_{i=2,3} \sim'_{i=2,3} x_{i=2,3}$ and $y_1 P'_1 x_1$. Therefore, $x \notin P(R')$. We have $x \in (P \cap NE)(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, for profile R', we have $x \notin (P \cap NE)(R')$. We have $x \in (P \cap I_{ed})(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, in profile R', we have $y_{i=2,3} \sim'_{i=2,3} x_{i=2,3}$ and $y_1 P'_1 x_1$. Therefore, $x \notin P(R')$ and so $x \notin (P \cap I_{ed})(R')$. Therefore, P, $P \cap NE$, and $P \cap I_{ed}$ violate Maskin monotonicity. From Theorem 2 of Maskin (1999), they are not Nash implementable in $\Re^{wia}_{sp}$, and in $\Re^{wia}_{spl}$ given that the $\Re^{wia}_{sp}$ domain is a sub-domain of $\Re^{wia}_{spl}$. Q.E.D.

4.4.4 Examples of nonmonotonic SCCs under domain restrictions when the preferences are single-troughed/single-troughed with best indifferent allocations ($\Re_{st}$, $\Re^{bia}_{st}$)

It follows from Proposition 5 of Doghmi and Ziad (2013a) that the Pareto correspondence does not satisfy Maskin monotonicity when we allow multiple worst indifferent allocations. In the following, we show that in contrast to Doghmi's result (2013a), the intersections of the Pareto correspondence with the no-envy correspondence fail to satisfy Maskin monotonicity when we allow multiple best indifferent allocations, as proved in the following proposition.

Proposition 6: *In private goods economies, when preferences are single-troughed/single-troughed with best indifferent allocations, the Pareto correspondence and its intersections with the no-envy correspondence, and with the individually rational correspondence from equal division do not satisfy Maskin monotonicity, and thus they are not Nash implementable.*

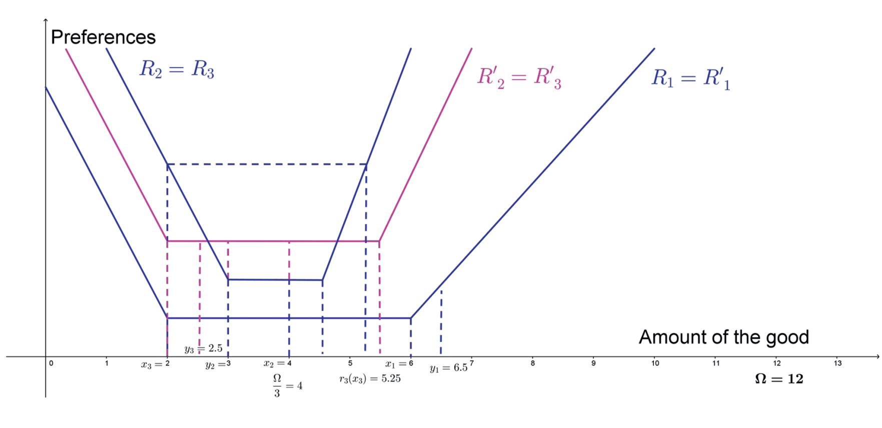

Figure 2: The P, $(P \cap NE)$, and $(P \cap I_{ed})$ correspondences do not satisfy Maskin monotonicity.

Proof: Let $R, R' \in \Re_{st}$, $x = (6, 4, 2), y = (6.5, 3, 2.5) \in X$ and $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = \Omega = 12$. Let $R_1 = R'_1$, $R_2 = R_3, R'_2 = R'_3$, $t(R) = ([2, 6], [3, 4.5], [3, 4.5])$, and $t(R') = ([2, 6], [2, 5.5], [2, 5.5])$. Figure 2 illustrates such representations.

Note that $x \in (P \cap NE)(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, for profile R', we have $y_{i=2,3} \sim'_{i=2,3} x_{i=2,3}$ and $y_1 P'_1 x_1$. Therefore, $x \notin P(R')$ and so $x \notin (P \cap NE)(R')$. Also, we have $x \in P(R)$ and $x_i R_i \frac{\Omega}{3}$ for $i = 1, 2, 3$ and so $x \in (P \cap I_{ed})(R)$. We have for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, for profile R', we have $y_{i=2,3} \sim'_{i=2,3} x_{i=2,3}$ and $y_1 P'_1 x_1$. Therefore, $x \notin P(R')$ and so $x \notin (P \cap I_{ed})(R')$. Therefore, P, $P \cap NE$, and $P \cap I_{ed}$ violate Maskin monotonicity in $\Re_{st}$ domain. From Theorem 2 of Maskin (1999), these solutions are not Nash implementable. Since $\Re_{st}$ is a sub-domain of $\Re_{st}^{bia}$, we complete the proof as required. Q.E.D.

5 Conclusion

In this study, we provided a full characterization of Nash implementation in many restrictive domains by allowing different types of indifference between allocations. First, we extended our characterization from a single-peaked domain to Cantala's domain (2004) by allowing multiple worst indifferent allocations, where we also implemented some well-known correspondences. Second, in an inverse configuration, we enlarged Doghmi's characterization (2013a) from a single-dipped domain to a single-troughed domain by providing some examples that support this result. Finally, we generalized our characterization by allowing multiple best and worst indifferent allocations at the same time. These results can be exploited to examine the connection between strategy-proofness and the implementation literature as in Diss et *al.*(Diss, Doghmi, and Tlidi 2015).

Appendix

Proof. Let $\Gamma = (S, g)$ be a mechanism which is defined as follows: For each $i \in N$, let $S_i = \Re \times A \times \mathbb{N}$, where $\mathbb{N}$ consists of the nonnegative integers. The generic element of strategic space S_i is denoted by: $s_i = (R^i, a^i, m^i)$. Each agent announces a preference profile, an optimal alternative for this profile and nonnegative integer. The function g is defined as follows:

Rule 1: If for each $i \in N$, $s_i = (R, a, m)$ and $a \in F(R)$, then $g(s) = a$.

Rule 2: If for some i, $s_j = (R, a, m)$ for all $j \neq i$, $a \in F(R)$ and $s_i = (R^i, a^i, m^i) \neq (R, a, m)$, then:

$$g(s) = \begin{cases} a^i & \text{if for all } R' \in \Re, a^i \in LS(a, R_i) \cup I(a, o, R_i) - argmin(R_i', A) \neq \emptyset \\ & \quad \text{for some o} \in LI(a, R_i) \setminus \{a\}, \\ a & \text{otherwise.} \end{cases}$$

Rule 3: In any other situation, $g(s) = a^{i^*}$, where i^* is the index of the player of which the number m^{i^*} is largest. If there are several individuals who check this condition, the smallest index i will be chosen.

Let us show that $F(R) = g(N(S, g, R))$. The proof contains two steps:

Step 1. For all $R \in \Re$, $F(R) \subseteq g(N(S, g, R))$.

Let $R \in \Re$ and $a \in F(R)$. For each $i \in N$, let $s_i = (R, a, m)$. Then, by rule 1, $g(s) = a$. We want to show that $s \in N(S, g, R))$. Let us choose any individual i and any strategy $\widetilde{s}_i \in S_i$ such that $\widetilde{s}_i = (\widetilde{R}, a^i, \widetilde{m})$. For all $R' \in \Re$, if $a^i \in LS(a, R_i) \cup I(a, o, R_i) - argmin(R_i', A)$ for some $o \in LI(a, R_i) \setminus \{a\}$, then by rule 2, $g(\widetilde{s}_i, s_{-i}) = a^i$. But, since $LS(a, R_i) \cup I(a, o, R_i) - argmin(R_i', A) \subseteq L(a, R_i)$, then $g(s) R_i g(\widetilde{s}_i, s_{-i})$, thus $s \in N(S, g, R)$. Otherwise, $g(s) = g(\widetilde{s}_i, s_{-i})$, thus $s \in N(S, g, R)$.

Step 2. For all $R \in \Re$, $g(N(S, g, R)) \subseteq F(R)$.

Let $s \in N(g, R, S)$. Let us show that $g(s) \in F(R)$. We study the various possibilities of writing the profile of strategies $s = (s_1, s_2, ..., s_n)$.

Case a: $s = (s_1, s_2, ..., s_n)$. Suppose there exists $(R', a, m) \in \Re \times A \times N$, with $a \in F(R')$, such that s is defined by $s_i = (R', a, m)$ for any $i \in N$. Then, by rule 1, $g(s) = a$.

Let $i \in N$, for all $R \in \Re$, we choose any $b \in [LS(a, R_i') \cup I(a, o, R_i') - argmin(R_i, A)] \cup \{a\}$ for some $o \in LI(a, R_i') \setminus \{a\}$. Let $\widetilde{s}_i = (R', b, m')$. Then, by the rule 2, $g(\widetilde{s}_i, s_{-i}) = b$. Since $s \in N(g, R, S)$, $a = g(s) R_i g(\widetilde{s}_i, s_{-i}) = b$. Therefore, $[LS(a, R_i') \cup I(a, o, R_i') - argmin(R_i, A)] \cup \{a\} \subseteq L(a, R_i)$ for some $o \in LI(a, R_i') \setminus \{a\}$. By I^*-monotonicity, $a \in F(R)$.

Case b: $s = (s_1, s_2, ..., s_n)$. Assume there is $i \in N$, $R' \in \Re$ and $a \in A$ such that $a \in F(R')$. For all $j \neq i$, $s_j = (R', a, m)$ and $s_i = (R'^i, a^i, m^i) \neq s_j$, in this case,

$$g(s) = \begin{cases} a^i & \text{if for all } R \in \Re, a^i \in LS(a, R_i') \cup I(a, o, R_i') - argmin(R_i, A) \neq \emptyset \\ & \quad \text{for some o} \in LI(a, R_i') \setminus \{a\}, \\ a & \text{otherwise.} \end{cases}$$

There are two subcases:

***Subcase* $\mathbf{b_1}$**: If $g(s) = a^i$,

By definition, for all $R \in \Re$, $a^i \in LS(a, R_i') \cup I(a, o, R_i') - argmin(R_i, A) \neq \emptyset$ for some $o \in LI(a, R_i') \setminus \{a\}$. For all $R \in \Re$, take any $b \in LS(a, R_i') \cup I(a, o, R_i') - argmin(R_i, A) \neq \emptyset$ and let $\widetilde{s}_i$ be a deviation by agent i such that $\widetilde{s}_i = (\widetilde{R}, b, \widetilde{m})$. Then, by rule 2, $g(\widetilde{s}_i, s_{-i}) = b$. But, since $s \in N(g, R, S)$, $b \in L(a^i, R_i)$. Hence, we have for all $R \in \Re$, $a^i \in LS(a, R_i') \cup I(a, o, R_i') - argmin(R_i, A) \subseteq L(a^i, R_i)$ for some $o \in LI(a, R_i') \backslash \{a\}$. (1)

Next, for any other deviation $j \neq i$ and any $b \in A$, let $\widetilde{s}_j = (\widetilde{R}, b, \widetilde{m})$ a deviation, $\widetilde{m}$ is the unique greatest integer in the profile $(\widetilde{s}_j, s_{-j})$. By rule 3, $g(\widetilde{s}_j, s_{-j}) = b$. Since $s \in N(g, R, S)$, we have $a^i = g(s) R_i g(\widetilde{s}_j, s_{-j}) = b$. Therefore, for all $j \neq i$, $A \subseteq L(a^i, R_j)$ (2). From (1), (2) and by I^*-weak no-veto power, we have $a^i \in F(R)$.

***Subcase* $\mathbf{b_2}$**: If $g(s) = a$,

By the same reasoning used in case a, we obtain by I^*-monotonicity that $a \in F(R)$.

Case c: $s = (s_1, s_2, ..., s_n)$: $\forall i \in N$, $s_i = (R', a, m)$ with $a \notin F(R')$, $g(s) = a$.

Let $b \in A$, $\widetilde{s}_i = (R', b, m')$, where $m' > m$, then, $g(\widetilde{s}_i, s_{-i}) = b$. As $s \in N(S, g, R)$, then, $a = g(s) R_i g(\widetilde{s}_i, s_{-i}) = b$. Therefore, $A \subseteq L(a, R_i)$ for all $i \in N$. By unanimity, $a \in F(R)$.

Case d: $s = (s_1, s_2, ..., s_n)$: $\exists k_1, k_2, k_3$ where $s_{k_1} \neq s_{k_2}$, $s_{k_1} \neq s_{k_3}$, $s_{k_2} \neq s_{k_3}$, $g(s) = a^l$: m^l is the maximum of the integers m. Let $b \in A$, and $\widetilde{s}_i = (R', b, m^l + 1)$ a deviation. Therefore, $g(\widetilde{s}_i, s_{-i}) = b$. As $s \in N(S, g, R)$, then, $g(s) R_i g(\widetilde{s}_i, s_{-i}) = b$. Thus, $A \subseteq L(g(s), R_i)$ for all $i \in N$. By unanimity, $g(s) \in F(R)$. Q.E.D.

Acknowledgement: I would like to thank Abderrahmane Ziad for providing many and very useful comments. I am very thankful as well to the participants of GATE Lyon Saint-Etienne Seminar for their valuable remarks. I would also thank two anonymous referees for their remarks and suggestions that have improved the quality of the paper considerably. All remaining errors are mine.

References

Aragón, G. -A., and D. -M. Caramuta. 2011. Single-Dipped Preferences with Indifferences: Strong Group Strategy-Proof and Unanimous Social Choice Functions. *Working Paper, Universitat Autnoma De Barcelona*.

Barberà, S. 2007. "Indifferences and Domains Restrictions." *Analysis and Kritik* 29:146–62.

Berga, D. 1998. "Strategy-Proofness and Single-Plateaued Preferences." *Mathematical Social Sciences* 35:105–20.

Berga, D., and B. Moreno. 2009. "Strategic Requirements with Indifference: Single-Peaked Versus Single-Plateaued Preferences." *Social Choice and Welfare* 32:275–98.

Berga, D. 2006. "A Note on Strategy-Proofness and Single-Plateaued Preferences." *Working paper.*

Bossert, W., and H. Peters. 2013. "Single-Plateaued Choice." *Mathematical Social Sciences* 66:134–9.

Bossert, W., and H. Peters. 2014. "Single-Basined Choice." *Journal of Mathematical Economics* 52:162–168.

Cantala, D. 2004. “Choosing the Level of a Public Good When Agents Have an Outside Option.” *Social Choice and Welfare* 22:491–514.

Dasgupta, P., P. Hammond, and E. Maskin. 1979. “The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility.” *Review of Economic Studies* 46:181–216.

Danilov, V. 1992. “Implementation via Nash Equilibrium.” *Econometrica* 60:43–56.

Diss, M., A. Doghmi, and A. Tlidi. 2015. “Strategy Proofness and Unanimity in Private Good Economies with Single-Peaked Preferences.” *Working paper N° 1528, GATE Lyon Saint-Etienne, available at SSRN 2688465*.

Doghmi, A. 2013a. “Nash Implementation in an Allocation Problem with Single-Dipped Preferences.” *Games* 4:38–49.

Doghmi, A. 2013b. “Nash Implementation in Private Good Economies When Preferences Are Single-Dipped with Best Indifferent Allocations.” *Mathematical Economics Letters* 1:35–42.

Doghmi, A., and A. Ziad. 2008a. “Reexamination of Maskin’s Theorem on Nash Implementability.” *Economics Letters* 100:150–2.

Doghmi, A., and A. Ziad. 2008b. “Nash Implementation in Exchange Economies with Singlepeaked Preferences.” *Economics Letters* 100:157–60.

Doghmi, A., and A. Ziad. 2013a. “On Partially Honest Nash Implementation in Private Good Economies with Restricted Domains: A Sufficient Condition.” *The B.E. Journal of Theoretical Economics* 13:1–14.

Doghmi, A., and A. Ziad. 2013b. “Nash Implementation in Private Good Economies with Single-Plateaued Preferences.” *Working paper*.

Doghmi, A., and A. Ziad. 2015. “Nash Implementation in Private Good Economies with Single-Plateaued Preferences and in Matching Problems.” *Mathematical Social Sciences* 73:32–9.

Doğan, B. 2015. “Nash-Implementation of the No-Envy Solution on General Domains of Economies.” *Working paper, available at SSRN 2637965*.

Dutta, B., and A. Sen. 1991. “A Necessary and Sufficient Condition for Two-Person Nash Implementation.” *Review of Economic Studies* 58:121–8.

Ehlers, L. 2002. “Probabilistic Allocation Rules and Single-Dipped Preferences.” *Social Choice and Welfare* 19:325–48.

Foley, D. 1967. “Resource Allocation and the Public Sector.” *Yale Economic Essays* 7:45–98.

Gaertner, W. 2001. *Domain Conditions in Social Choice Theory*. Cambridge, United Kingdom: Cambridge, University Press.

Inada, K. I. 1964. “A Note on the Simple Majority Decision Rule.” *Econometrica* 32:525–31.

Klaus, B., and O. Bochet. 2013. “The Relation Between Monotonicity and Strategy-Proofness.” *Social Choice and Welfare* 40:41–63.

Manjunath, V. 2011. “Strategic and Normative Analysis of Ressource Allocation Problems.” *Phd. thesis, University of Rechester*, New York.

Maskin, E. 1999. “Nash Equilibrium and Welfare Optimality. M.I.T. *Mimeo*, 1977.” *Published 1999 in the Review of Economic Studies* 66:23–38.

Moore, J., and R. Repullo. 1990. “Nash Implementation: A Full Characterization.” *Econometrica* 58:1083–100.

Moulin, H., 1984. “Generelized Condorcet-Winners for Single-Peaked and Single-Plateau Preferences.” *Social Choice and Welfare* 1:127–147.

Sjöström, T. 1991. “On the Necessary and Sufficient Conditions for Nash Implementation.” *Social Choice and Welfare* 8:333–40.

Thomson, W. 1990. "Manipulation and Implementation to Solutions to the Problem of Fair Allocations when Preferences are Single-Peaked." *mimeo*, University of Rochester.
Thomson, W. 2010. "Implementation to Solutions to the Problem of Fair Allocations When Preferences Are Single-Peaked." *Review of Economic Design* 14:1–15.
Yamato, T. 1992. "On Nash Implementation of Social Choice Correspondences." *Games and Economic Behavior* 4:484–92.
Ziad, A. 1997. "On the Necessary and Sufficient Conditions for Nash Implementation." *Economics Letters* 56:209–13.
Ziad, A. 1998. "A New Necessary Condition for Nash Implementation." *Journal of Mathematical Economics* 29:381–7.