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Tenacious Selection of Nash Equilibrium

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Abstract: We propose a complexity measure and an associated refinement based on the observation that best responses with more variations call for more precise anticipation. The variations around strategy profiles are measured by considering the cardinalities of players' pure strategy best responses when others' behavior is perturbed. After showing that the resulting selection method displays desirable properties, it is employed to deliver a refinement: the tenacious selection of Nash equilibrium. We prove that it exists; does not have containment relations with perfection, properness, persistence and other refinements; and possesses some desirable features.

Keywords: anticipation, complexity, refinements of Nash equilibrium, perfect, proper, persistent

JEL Classification: C72

1 Introduction

The concept of Nash equilibrium (henceforth to be abbreviated as NE) is central in the theory of games, and as put by Myerson (1978), "it is one of the most important and elegant ideas in game theory". On the other hand, Nash's pointwise stability may create multiplicity of equilibria some of which do not satisfy local stability and produce outcomes that can be criticized on grounds of not corresponding to intuitive notions about how plausible behavior should look like. In order to alleviate these problems, important refinements of NE have been developed: perfection by Selten (1975), properness by Myerson (1978), and persistence by Kalai and Samet (1984), among others, have been standards in the theory of games.

However, complex equilibrium anticipation may still be needed. The following game with three players has three NE, $s^1 = (I, I, I)$, $s^2 = (II, II, I)$, and $s^3 = (II, II, II)$:

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3							
ı					II		
1 \2	1	II	1 \2	1	11		
I II	(1, 1, 1) (0, 0, 0)	(0,0,0) (0,0,0)	I II	(0,0,0) (0,0,0)	(0,0,0) (1,1,0)		

Only s^1 and s^2 are perfect and proper as s^3 involves a weakly dominated strategy. The behavior in s^2 corresponds to a coordination failure, hence, is undesirable; and "very specific set of trembles is needed to justify" this equilibrium (Kalai and Samet 1984): for II to appear in player 1's perturbed best response, player 1 has to anticipate that the mistake of player 2 about choosing I instead of II has to be strictly less than the mistake of player 3 about choosing II instead of I. This is a clear display of the serious requirements imposed on players' anticipation capacities: even when every player is making mistakes about his own choices, his assessment about the magnitudes of others' mistakes needs to be correct.

On the other hand, s^1 is more desirable on account of involving less complex anticipation: every players' only best response to any one of the others' strategies that are sufficiently close and possibly equal to the one given by s^1 , is as given by s^1 . Hence, when the approximation is sufficiently precise, the local behavior of each player's best response around s^1 does not involve any variations. Then, each player needs only minimal anticipation capacities. The numbers of actions that appear in best responses around s^2 are given by 2 for player 1, 2 for player 2, and 1 for player 3 even with arbitrarily precise approximation; and these numbers are given by 1 for player 1, 1 for player 2, and 2 for player 3 when considering s^3 .

In order to formalize these ideas, the current study proposes the notion of tenacious selection: given any strategy profile and any player, we consider the number of pure strategies that may appear in that player's best response when others may choose a strategy vector that is either arbitrarily close or equal to the one specified. By employing the upper hemi continuity of best responses, we show that this integer attains a limit, a lower bound greater or equal to one, before the approximation terms reach zero. We refer to this as the t-index of the given strategy and player, and the t-index of a strategy profile is a vector of

¹ Considering a perturbation around s^2 with $s_{\varepsilon}^2 = (\varepsilon_1 I + (1 - \varepsilon_1) II, \varepsilon_2 I + (1 - \varepsilon_2) II, (1 - \varepsilon_3) I + \varepsilon_3 II)$ with $\varepsilon_i > 0$ for i = 1, 2, 3, one can observe that s_{ε}^2 is an ε -perfect equilibrium only if $\varepsilon_3 > \varepsilon_2$ and $\varepsilon_3 > \varepsilon_1$.

t-indices where each coordinate is associated with the t-index of the corresponding player of that given strategy profile. Given any set of strategies, one of its elements belongs to its tenacious selection whenever there is no other element of the same set which has a t-index less than or equal to and not equal to that of the strategy under consideration.

The method of tenacious selection is a low–cost notion of complexity aversion. A higher t–index of a given strategy and player implies that player's optimal plan of action displays more variations around that strategy, hence, demands more accurate anticipation of others' behavior. So strategy profiles involving lower t–indices are more appealing on grounds of complexity aversion. The identification of such strategies involves the simple act of counting the relevant actions while more complicated methods are also available. Indeed, the demonstration that this low–cost method displays a solid performance, we think, is noteworthy.

Tenacious selection of strategies with the best response property is of particular interest, and leads us to the *tenacious selection of Nash equilibrium* (TSNE hereafter): every NE in the TSNE involves less complex anticipation by all the players and this holds strictly for least one of the players when compared with those of the NE that are not in the TSNE.

After proving that tenacious selection of any nonempty set of strategy profiles exists, we analyze the TSNE of finite normal–form games and display that it is an idiosyncratic refinement of NE as it does not have any containment relations with the notions of perfection, properness, persistence, among other refinement concepts.³ In fact, the TSNE equals the set of strict NE whenever there is one.⁴ In such cases, apart from containing neither mixed nor weakly dominated NE while being lower hemi continuous, the TSNE does not display a weaker refinement performance in comparison with perfection and properness and persistence and settledness because it is their subset and this relation may be strict. And our further findings indicate that the TSNE is not logically related to these notions even when attention is restricted to games that have no strict NE and neither redundant nor weakly dominated actions. Moreover, we show that the TSNE does not get affected by the elimination of strictly dominated

² Our notion involves complexity of implementation rather than that of computation, and philosophical aspects of various complexity formulations are not addressed in the current study.

³ These are regular equilibrium (Harsanyi 1973), essential equilibrium (Wu and Jia-He 1962), strongly stable equilibrium (Kojima, Okada, and Shindoh 1985), and settled equilibrium (Myerson and Weibull 2013).

⁴ A NE is strict if and only if deviations strictly hurt the deviators. Clearly, strict NE must be pure.

strategies, hence, it is immune to the criticisms of Myerson known as imperfections of perfection which were directed to perfection (Myerson 1978). However, when there is no strict NE, both a pure NE and a mixed NE may be in the TSNE; it may contain a weakly dominated NE, and is not lower hemi continuous.⁵

The notion that is most closely related with the TSNE is persistent equilibrium (PE, henceforth). When there is no strict NE interesting distinctions between these notions surface. The TSNE involves "local" considerations: whether or not the behavior in a specified NE is plausible is judged only with pure strategies which can appear in players' best responses when fine perturbations are considered. On the other hand, the minimality requirement of the essential Nash retracts in the definition of the PE implies that considerations of whether or not equilibrium behavior is plausible may have to incorporate the whole game, hence, they are rather "global". 6 We take the stand that the actions considered to be relevant in the determination of the plausibility of behavior in an equilibrium should involve only the pure strategies that can appear in players' best responses when fine perturbations are considered. This enables us to present the notion of the TSNE not only as a concept based on complexity aversion, but also as one that has a similar motivation as the PE but with the novel feature of evaluating plausibility of equilibria through local considerations. But while global considerations help persistence to tackle weak domination, the local evaluation measure of the TSNE does not discriminate between weakly dominated NE and mixed NE. As a result, weakly dominated NE may be elements of the TSNE.

It is useful to emphasize that the trembles employed in the current paper are due to players' imprecise anticipation of their opponents' actions. Hence, our approach is immune to the arguments of Kreps (1990) advocating that classical refinements literature is flawed because there is no explanation for the trembles (see also Fudenberg, Kreps, and Levine 1988 and Dekel Fudenberg 1990. Moreover, while considerations with approximate common knowledge (Monderer and Samet 1989) and employing incomplete information settings to formulate higher order beliefs (Kajii and Morris 1997b; Kajii and Morris 1997a) are very interesting, the current study lies within the framework of common knowledge and complete information.

⁵ In order to dismiss weak domination, one may consider the notion of tenacious selection of undominated NE. Indeed, using our techniques, it is easy to show that all our results continue to hold. Another alternative is to consider the tenacious selection of perfect (alternatively, proper) equilibrium.

⁶ For the details and formal presentation please see the discussion following example 2 on page 8.

The next section presents the preliminaries and the method of tenacious selection and Section 3 the important properties of the TSNE. Section 4 concludes.

2 Definitions and Auxiliary Results

Let $\Gamma = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite normal-form game where A_i is a finite nonempty set of actions (alternatively, pure strategies) of player $i \in N$ and $u_i : \times_{i \in N} A_i \to \mathbb{R}$ is agent i's von Neumann Morgenstern utility function. We keep the standard convention that $A = \times_{i \in N} A_i$ and $A_{-i} = \times_{j \neq i} A_j$. A mixed strategy of player i is represented by $s_i \in \Delta(A_i) \equiv S_i$ where $\Delta(A_i)$ denotes the set of all probability distributions on A_i and $s_i(a_i) \in [0,1]$ denotes the probability that s_i assigns to $a_i \in A_i$ with the restriction that $\sum_{a_i \in A_i} s_i(a_i) = 1$. S_i denotes the interior of S_i and its members are referred to as totally mixed strategies. A strategy profile is denoted by $s \in \times_{i \in N} S_i \equiv S$. We let $S_{-i} \equiv \times_{j \neq i} S_j$, $S_i \equiv \times_{i \in N} S_i$, and $S_{-i} \equiv \times_{j \neq i} S_j$. We say that a game has no redundant actions whenever for all $i \in N$ we have $(u_i(a_i, a_{-i}))_{a_{-i} \in A_{-i}} \neq (u_i(a_i', a_{-i}))_{a_{-i} \in A_{-i}}$ for any $a_i, a_i' \in A_i$ with $a_i \neq a_i'$. Let $\mathfrak G$ be the set of finite normal-form games, and $\mathfrak G_{\mathfrak R} \subset \mathfrak G$ be those without redundant actions.

The *best response* of player i to s_{-i} is defined by $\mathcal{BR}_i(s_{-i}) \equiv \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \text{ for all } s_i' \in S_i\}.$ s^* is a NE if for every $i \in N$, $s_i^* \in \mathcal{BR}_i(s_{-i}^*)$. The set of NE of Γ is denoted by $\mathcal{N}(\Gamma) \subset S$. A NE, s^* , is *strict* whenever $u_i(s_i^*, s_{-i}^*) > u_i(s_i', s_{-i}^*)$ for all $i \in N$ and for all $s_i' \in S_i \setminus \{s_i^*\}$. $\mathcal{N}_s(\Gamma) \subset A$ denotes the set of strict NE of Γ .

Let F be a correspondence mapping X into Y where X and Y are both finite dimensional Euclidean spaces. We say that F is *lower hemi continuous* if for all $x \in X$ and all sequences $(x_n)_{n \in \mathbb{N}}$ in X converging to x and for every $y \in F(x)$ there exist a sequence $(y_n)_{n \in \mathbb{N}}$ in Y with $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ and $y_n \to y$. Insisting on the additional requirement that Y is compact and F is a nonempty and compact valued correspondence, we say that F is *upper hemi continuous* if for all $x \in X$ and all sequences $(x_n)_{n \in \mathbb{N}}$ in X with $x_n \to x$ and every sequence $(y_n)_{n \in \mathbb{N}}$ in Y with $y_n \to y$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ implies $y \in F(x)$.

⁷ A related solution concept, proposed by Harsanyi (1973), is $quasi-strict\ equilibrium$: A NE s^* is quasi-strict if for all $i\in N$ and for all $a_i,b_i\in A_i$ with $s_i^*(a_i)>0$ and $s^*(b_i)=0$, $u_i(a_i,s_{-i}^*)>u_i(b_i,s_{-i}^*)$. That is, all pure strategy best responses are required to be chosen with strictly positive probabilities.

An action $a_i \in A_i$ is *strictly dominated* for player i, if there exists $a_i' \in A_i \setminus \{a_i\}$ with $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$ for all $a_{-i} \in A_{-i}$. The game obtained from Γ by the elimination of strictly dominated strategies is referred to as the strict dominance *truncation* of Γ and is denoted by $\mathcal{D}(\Gamma)$. We say that s in Γ and \tilde{s} in $\mathcal{D}(\Gamma)$ are equivalent under strict domination, and denote it by $s = \tilde{s}$, whenever for all $i \in N$ it must be that $s_i(a_i) = \tilde{s}_i(a_i)$ for any $a_i \in A_i$ that is not strictly dominated. Moreover, for a given $K \subset S$ in Γ and $\tilde{K} \subset \tilde{S}$ in $\mathcal{D}(\Gamma)$, we say that $K \stackrel{\mathcal{D}}{=} \tilde{K}$ whenever for every $s \in K$ there exists $\tilde{s} \in \tilde{K}$ with $s \stackrel{\mathcal{D}}{=} \tilde{s}$ and for every $\tilde{s}' \in \tilde{K}$ there exists $s' \in K$ with $s' \stackrel{\mathcal{D}}{=} \tilde{s}'$. Clearly, $\mathcal{N}(\Gamma) \stackrel{\mathcal{D}}{=} \mathcal{N}(\mathcal{D}(\Gamma))$.

An action $a_i \in A_i$ is *weakly dominated* for player i if there exists $a_i' \in A_i$ with $u_i(a_i, a_{-i}) \le u_i(a'_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ and this inequality holds strictly for some $a_{-i} \in A_{-i}$. A strategy profile $s \in S$ is undominated if $s_i(a_i) = 0$ for any $a_i \in A_i$ that is weakly dominated.

For any given $\varepsilon > 0$, a totally mixed strategy $s \in \overset{\circ}{S}$ is an ε -perfect equilibrium if for all $i \in N$ and $a_i \in A_i$, $a_i \notin \mathcal{BR}_i(s_{-i})$ implies $s_i(a_i) \le \varepsilon$. On the other hand, a totally mixed strategy $s \in S$ is an ε -proper equilibrium if for all $i \in N$ and $a_i, a_i' \in A_i, u_i(a_i, s_{-i}) < u_i(a_i', s_{-i})$ implies $s_i(a_i) \le \varepsilon s_i(a_i')$. s^* is perfect (proper) if there exists $(\varepsilon^k)_{k\in\mathbb{N}}$ and $(s^k)_{k\in\mathbb{N}}$ with the property that $\lim_k \varepsilon^k = 0$ and s^k an ε^k -perfect (ε^k -proper, respectively) equilibrium for each k and $\lim_k s^k = s^*$. It is well-known that every proper equilibrium is automatically perfect.⁸

R is a *retract* of *S* if $R = \times_{i \in N} R_i$ where for any $i \in N$, R_i is a nonempty convex and closed subset of S_i . For any given $K \subset S$, it is said that R absorbs K if for every $s \in K$ and for any $i \in N$ it must be that $\mathcal{BR}_i(s_{-i}) \cap R_i \neq \emptyset$. Any retract absorbing itself is a Nash retract, and a retract is an essential Nash retract if it absorbs a neighborhood of itself. It is said to be a persistent retract if it is an essential Nash retract and is minimal with respect to this property, $s \in S$ is a PE if it is a NE contained in a persistent retract.

The linearity of the expected utility functions and the upper hemi continuity of the best responses imply: Player i's pure strategy best responses to some $s'_{-i} \in S_{-i}$ where $|s_{-i} - s'_{-i}| < \varepsilon$ with $\varepsilon > 0$, equals her pure strategy best responses to s_{-i} , $\mathcal{PB}_i(s_{-i}) \equiv \{a_i \in A_i : a_i \in \mathcal{BR}_i(s'_{-i})\}$, whenever her anticipation of the opponents' behavior becomes sufficiently precise (i. e. ε is sufficiently small).

⁸ The notion of regularity implies strong stability and the latter essentiality which in turn implies strict perfection, hence, perfection (Kojima, Okada, and Shindoh 1985). And properness is implied by strong stability (van Damme 1991, Section 2.4) and by settledness (Myerson and Weibull 2013). Moreover, due to Jansen (1981, Theorem 7.4) and van Damme (1991, Theorem 3.4.5) a NE of a finite two-player game is regular if and only if it is essential and all players utilize each of their pure strategy best responses.

Thus, defining $\mathbb{T}_i(s) \equiv |PB_i(s_{-i})|$ delivers a measure of complexity aversion discussed in the introduction and enables us to present the following:

Definition 1: For any given strategy profile $s \in S$, we define the t-index of s by $\mathbb{T}(s) \equiv (\mathbb{T}_i(s))_{i \in \mathbb{N}}$. Moreover, for any nonempty $K \subseteq S$, s is said to be in the tenacious selection of K, denoted by $\mathcal{T}(K)$, if there is no $s' \in K$ with $\mathbb{T}(s) \geq \mathbb{T}(s')$ and $\mathbb{T}(s) \neq \mathbb{T}(s')$. Finally, the TSNE of a finite normal-form game Γ equals $\mathcal{T}(\mathcal{N}(\Gamma))$.

The following existence result is immediate.

Theorem 1: $\mathcal{T}(K)$ is nonempty for any given nonempty $K \subset S$.

Proof: For any given $K \subset S$, let $s \in K$ and notice that $\mathbb{T}_i(s) \leq |A_i|$, and hence, $\cup_{s \in K} \mathbb{T}(s)$ is a finite set in \mathbb{N}^N , so $\mathcal{T}(K)$ is nonempty.

An observation that may be helpful when employing the method of tenacious selection as a bounded rationality measure involves the requirements on the knowledge of rationality among players: all that is needed is that every player knows that he himself is rational.

3 Tenacious Selection of Nash Equilibrium

This section presents our findings about the TSNE which exists due to Theorem 1.

3.1 Idiosyncrasy

In what follows, examples 1–3 establish that even when attention is restricted to games without redundant and weakly dominated actions, the TSNE does not have any containment relations with perfection and properness and persistence whenever there is no strict NE.⁹

Example 1: In the following game player 1 chooses rows, 2 columns, and 3 matrices:

⁹ With redundant and/or weakly dominated actions the identification of the desired examples becomes easier.

3						
1					11	
1 \2	1	11	1 \2	1	II	
I	(1, 1, 0)	(1, 0, 1)	I	(1, 0, 1)	(0,1,0)	
II	(1, 1, 1)	(0,0,1)	II	(0, 1, 0)	(1, 0, 0)	

The NE are $s^p = ((1-p)I + pII, I, I)$ where $p \ge 1/2$ and $s^2 = (I, 1/2I + 1/2II, 1/2I + 1/2II)$ (while only s^2 is perfect and proper). Both s^1 and s^2 are in the TSNE because $\mathbb{T}(s^p)$ equals (2,1,1) if p > 1/2 and (2,1,2) when p = 1/2, and $\mathbb{T}(s^2) = (1,2,2)$.

The game of example 1, possessing neither any redundant actions nor weakly dominated actions nor a strict NE, also displays that the TSNE is not an impassable barrier to mixed strategies: both a pure NE and a mixed NE are in the TSNE. ¹¹

Example 2: The following is a coordination game where one of the pure actions in which players are not coordinated is replaced by a matching pennies:

1\2	I	II	III
I	(1, 1)	(2, -2)	(-2,2)
II	(1, 1)	(-2, 2)	(2, -2)
III	(0,0)	(1, 1)	(1, 1)

Here, the NE, perfect equilibria, proper equilibria, and the PE coincide: $s^1 = (1/4I + 1/4II + 1/2III, 1/2I + 1/4II + 1/4III)$, $s^2 = (III, 1/4II + 3/4III)$, $s^3 = (III, 3/4II + 1/4III)$, $s^4 = (3/4I + 1/4II, I)$, $s^5 = (1/4I + 3/4II, I)$. Because that the t-index of s^1 is given by (3,3) and the others' by (2,2), s^1 is not in the TSNE.

¹⁰ Perfection follows because (1) regardless of the magnitudes of player 2 and 3's strictly positive mistakes around mixing I and II with equal probabilities action I for player 1 is the only best response; and (2) every finite normal form game has to possess a perfect equilibrium which has to be Nash.

¹¹ Example 1 implies further conclusions of idiosyncrasy when comparing the TSNE with the notions of quasi-strict equilibrium (see footnote 7) and the concepts of regularity, essentiality, and strong stability (see footnote 8): $s^{0.50}$ is a quasi-strict equilibrium that is not in the TSNE while s^1 is in the TSNE but is neither quasi-strict nor perfect; the totally mixed strategy NE of the battle of the sexes (Kalai and Samet 1984, p.131) is regular but not in the TSNE.

This game has no redundant and weakly dominated actions and no strict NE. Additionally, it displays an important distinction between persistence and our concept: the former, as opposed to the latter, entails that whether or not behavior in a specified equilibrium is plausible may depend on the presence or absence of pure strategies that do not appear in players' best responses when fine perturbations are considered. In other words, while the TSNE employs "local" performance measures when evaluating the performances of NE, the method of evaluation of persistence is rather "global". To see this, it suffices to restrict attention to s^1 and s^2 . First, observe that s^1 is a PE and the persistent retract it is contained in is *S*. Moreover, $\mathcal{PB}_i(s_{-i}^1) = \{I, II, III\}, i = 1, 2$. Second, with persistence (unlike the TSNE) s^1 is not eliminated by s^2 because of the following: $R = \times_{i=1,2} R_i$ and $R_i = \{s_i^2\}$ for i=1,2, is a Nash retract but not essential because it cannot absorb a neighborhood of itself which is due to both players being indifferent between II and III in s^2 . Indeed, $\mathcal{PB}_i(s^2) = \{II, III\}, i = 1, 2$. Yet, the Nash retract defined by $R' = x_{i=1,2}R'_i$ with $R'_i = \{(0, x_i, 1-x_i) : x_i \in [0,1]\}$ is not essential (due to the inherent matching pennies feature) because for $\varepsilon > 0$ sufficiently small $(\varepsilon, 1-2\varepsilon, \varepsilon)$ is a point in the neighborhood of R_2' to which player 1's corresponding best response calls for (1,0,0) and $(1,0,0) \cap R'_1 = \emptyset$. Hence, R' is not a persistent retract due to the pure strategy I even though $I \notin \mathcal{PB}_1(s_{-1}^2)$. So with persistence s^1 is not eliminated by s^2 due to I, an action which does not appear in player 1's best responses when the other is choosing a strategy either close or equal to s_2^2 . Similarly, s^k , k = 3, 4, 5, do not eliminate s^1 with persistence.

Example 3: Consider the following four player game: Players 1 and 2 play the game on the left in the following table independent of the choices of players 3 and 4; players 3 and 4 play the game in the middle when players 1 and 2 choose (I, I) or (II, II) and the game on the right when 1 and 2 choose (I, II) or (II, II).

1 \2	1	II	3\4	III	IV	3∖4	III	IV
I	(1, 1)	(0,0)	III	(1, 0)	(0,1)	III	(2, 2)	(2,0)
II	(0,0)	(1, 1)	IV	(0, 1)	(1, 0)	IV	(0, 2)	(0, 0)

Here, s = (1/2I + 1/2II, 1/2I + 1/2II, III, III) is in the TSNE, but is not persistent.¹²

¹² In order to observe that s is in the TSNE, note that there is no pure strategy NE in this game (hence, the set of strict NE is empty). Note further that there is no NE where only one player mixes among his strategies. Therefore, in all NE at least two players randomize. Hence, this game involves t-indices with at least two numbers strictly exceeding 1. Now, considering s we have that $\mathbb{T}_i(s) = 2$, i = 1, 2, and $\mathbb{T}_j(s) = 1$, j = 3, 4: players i = 1, 2 are randomizing in NE, and j = 3, 4

Here, not every element in the TSNE is a PE even when there are neither redundant nor weakly dominated actions and no strict NE. In fact, the essence of the distinction between persistence and the TSNE in the context of this game is the very same as that in the context of the game of example 2. However, this time global considerations of persistence help to eliminate s which is not eliminated by employing local concerns of the TSNE.

3.2 Domination and Strict NE

The following presents properties of the TSNE in relation with domination and strict NE:

Theorem 2: *The following hold:*

- The TSNE does not change with strict dominance truncations. 1.
- The TSNE equals the set of strict NE whenever the game possesses a strict 2. NE.

We wish to discuss direct implications of and issues about this theorem before its proof.

First, it should be emphasized that when evaluating the performance of our notion against strict domination, Theorem 2-1 ensures that we do not encounter the type of problems often cited in the discussion of "imperfections of perfection" (see myerson (1978)). To see this, consider the following:

Example 4: First, consider the strict dominance truncation of the following game:

are choosing III and if player 1 and player 2 make small mistakes player 3 and 4's best responses will still be III (due to the strict dominance in the game on the right). Hence, $\mathbb{T}(s) = (2, 2, 1, 1)$ which is the best *t*-index that can be achieved in this game. But s is not a PE. For any $\varepsilon > 0$, the best response of player 1 against the following perturbation is II: $(((1/2-\varepsilon)I+(1/2+\varepsilon)II),((1/2-\varepsilon)I+(1/2+\varepsilon)II),(1-\varepsilon)III+\varepsilon IV,(1-\varepsilon)III+\varepsilon IV)$. Similarly, I is player 1's only best response when this perturbation is reversed. Moreover, for the retract defined by $\Delta(\{I,II\})$ for players 1 and 2 and III for the others is not persistent. Because when players 1 and 2 choose (I, I) (or (II, II)), the persistent retract in the middle game (of the above table) is $(\Delta(\{III,IV\}))^2$. Therefore, there is no persistent retract which includes this NE other than the whole game. Also, note that there is a persistent retract: neighborhoods around I for player 1 and 2, and $\Delta(\{III,IV\})$ for players 3 and 4. For any strategy in this retract, the best response of first and second players are still I. Third and fourth players best responses to this tremble will be trivially be in this retract as well. So there is a persistent retract other than the whole game which contains the NE given by (I, I, 1/2I + 1/2II, 1/2I + 1/2II), but not s.

13 Kohlberg's Example is also based on a similar observation (Kohlberg and Mertens 1986).

1\2	1	II	III
I	(1, 1)	(0,0)	(1, -1)
II	(0,0)	(0,0)	(2, -1)
III	(-1,1)	(-1, 2)	(-1, -1)

The NE are s = (I, I) and s' = (II, II), but only s is perfect. But considering strictly dominated strategies as well results in II not being weakly dominated. NE and perfect equilibria coincide and are equal to s = (I, I) and s' = (II, II). But in both cases $\mathbb{T}(s) = (1, 1)$ and $\mathbb{T}(s') = (2, 2)$, so the TSNE equals $\{s\}$.

The second point concerns our finding that the TSNE exhibits stronger refinement powers than the other refinements of NE whenever the game at hand possesses a strict NE. Indeed, Theorem 2–2 immediately implies (1) the TSNE contains neither weakly dominated NE nor mixed NE and is a strict subset of the lower hemi continuous selection of NE;¹⁴ and (2) the TSNE is a strict subset of the set of perfect equilibrium, proper equilibrium, and PE. While the battle of the sexes shows that the containment relation of the TSNE with perfection and properness is strict, the following example performs the same task with persistence:

Example 5: This game is one that has a strict NE but no redundant and weakly dominated actions, and the TSNE is a strict subset of the set of PE.

1 \2	I	II	III
I	(10, 10)	(0,0)	(0,0)
II	(0,0)	(3, 1)	(1, 3)
III	(0,0)	(1, 3)	(3, 1)

Here, $s^1 = (I,I)$ is a strict NE which, therefore, is not empty. Thus, the TSNE equals the set of strict NE, hence, does not contain the mixed strategy NE, $s^2 = (1/2II + 1/2III, 1/2II + 1/2III)$. But s^2 is a PE: $R = \times_{i=1,2}R_i$ defined by $R_i = \{(0,x,1-x) : x \in [0,1]\}$, i=1,2, is an essential Nash retract because (1) for $\varepsilon > 0$ sufficiently small the best-responses of agents against $(\varepsilon,x,1-x-\varepsilon)$ do not contain I; (2) it is minimal.

¹⁴ Consider the battle of the sexes and notice that the totally mixed strategy NE is in the lower hemi continuous selection of NE but not in the TSNE.

On the other hand, when there is no strict NE, example 1, a game with no redundant and weakly dominated actions, shows both a pure strategy and a mixed strategy NE may be in the TSNE and this notion may not be lower hemi continuous.15 Moreover, the TSNE may contain weakly dominated NE when the game has no strict NE. This is due to the following:

Example 6: This game has no strict NE and no redundant actions, but two NE: s = (II, II, II) and s' = (1/2I + 1/2II, 1/2I + 1/2II, I).

3							
ı					II		
1 \2	1	II	1 \2	1	II		
I II	(1, 0, 1) (0, 1, 0)	(0,1,1) (1,0,0)	I II	(0,0,0) (0,0,0)	(0,0,0) (1,0,0)		

s, involving a weakly dominated action by player 3, is in the TSNE: $\mathbb{T}(s) = (1,1,2)$ and T(s') = (2, 2, 1).

The TSNE may not eliminate weak domination because when there is no strict NE it may not be able discriminate between weak domination and randomization: in example 6, $\mathbb{T}_3(s) = 2$ because $s_3 = II$ is a weakly dominated action for player 3; $\mathbb{T}_i(s') = 2$ because $s'_i = 1/2I + 1/2II$ for i = 1, 2 is totally mixed. But weak domination is not permitted with persistence (Kalai and Samet 1984, Theorem 4, p.139) due to the minimality requirement of essential Nash retracts. Therefore, the global evaluation measure embedded in persistence results in the elimination of weak domination, while the local means of evaluation with the TSNE does not suffice towards this regard.¹⁶

¹⁵ The lack of lower hemi continuity follows from the example 1 which has two NE $s^1 = (II, I, I)$ and $s^2 = (I, 1/2I + 1/2II, 1/2I + 1/2II)$, and both s^1 and s^2 are in the TSNE. I is player 1's only best response whenever one considers a strategy profile arbitrarily close to s^1 with the requirement that players 2 and 3 are choosing action II with strictly positive probabilities. Hence, we can come up with a sequence of games (each of which does not have any redundant and weakly dominated actions) converging to the one given in example 1 for which the unique TSNE would be only around s^2 .

¹⁶ Therefore, it maybe worthwhile to consider the tenacious selection of undominated NE, the TSUNE. In fact, in all the games handled previously, with the exception of the last one, the TSNE coincides with the TSUNE; and all positive results concerning the TSNE hold with the TSUNE as well.

Proof of Theorem 2: The first item of the above theorem stated formally is: For any $\Gamma \in \mathfrak{G}$, $\mathcal{T}(\mathcal{N}(\Gamma)) \stackrel{\mathcal{D}}{=} \mathcal{T}(\mathcal{N}(\mathcal{D}(\Gamma)))$. Because that for any Γ we have that $\mathcal{N}(\Gamma) \stackrel{\mathcal{D}}{=} \mathcal{N}(\mathcal{D}(\Gamma))$, we prove that for any $s \in \mathcal{N}(\mathcal{D}(\Gamma))$ and $s' \in \mathcal{N}(\Gamma)$ with $s' \stackrel{\mathcal{D}}{=} s$ it must be that $\mathbb{T}_i^{\Gamma}(s') = \mathbb{T}_i^{\mathcal{D}(\Gamma)}(s)$ for all $i \in \mathbb{N}$. Then, the definition of the TSNE implies that $s \in \mathcal{T}(\mathcal{N}(\mathcal{D}(\Gamma)))$ if and only if $s' \in \mathcal{T}(\mathcal{N}(\Gamma))$ where $s' \stackrel{\mathcal{D}}{=} s$. The desired conclusion follows from $\mathcal{PB}_i^{\Gamma}(s'_{-i}) \stackrel{\mathcal{D}}{=} \mathcal{PB}_i^{\mathcal{D}(\Gamma)}(s_{-i})$ because $\mathcal{D}(\Gamma)$ is a strict dominance truncation of Γ , and on account of being a NE s_{-i} and s'_{-i} do not assign strictly positive probabilities to strictly dominated strategies, and player i cannot assign strictly positive probabilities to strictly dominated actions in his best response.

In order to prove item 2 we show: Let $\Gamma \in \mathfrak{G}$ be such that $\mathcal{N}_s(\Gamma) \neq \emptyset$; then, $\mathcal{T}(\mathcal{N}(\Gamma)) = \mathcal{N}_s(\Gamma)$. This follows from (1) the observation that for any strict NE, s^* , it must be that $\mathbb{T}_i(s^*) = 1$ for all $i \in N$; and (2) for any NE that is not strict, s', there exists $j \in N$ such that $\mathbb{T}_i(s') > 1$.

This finishes the proof of Theorem 2. ■

4 Concluding Remarks

Our first remark concerns the evaluation of the performances of the TSNE and persistence when attention is restricted to the *unanimity games*. Let the set of actions of every player be given by a finite set C, i. e. $A_i = C$ for all $i \in N$. An action profile $\bar{c} \in C^N$ is called *diagonal* if it is of the form (c, c, \ldots, c) for some $c \in C$. It is assumed that $u_i(a) = 0$ for every $i \in N$ and for all $a \in C^N$ that is not diagonal. And $a' \in C^N$ is *positive* if $u_i(a') > 0$ for every $i \in N$. Naturally, if an action profile is positive, then it is diagonal. Kalai and Samet (1984, Theorem 6) establishes that an action vector is persistent if and only if it is positive provided that the unanimity game at hand has a positive action profile.

Item 2 of Theorem 2 delivers additional insight with the help of Theorem 6 of Kalai and Samet (1984): if the unanimity game has a positive action vector, then the TSNE and the PE and positive action profiles coincide: If $a' \in C^N$ is positive, then it is a strict NE because for every $i \in N$ it must be that $u_i(a') > 0 = u_i(a_i, a'_{-i})$ for every $a_i \in A_i \setminus \{a'_i\}$. So the TSNE equals the set of strict NE, and it is not difficult to see that the set of strict NE equals the set of positive action vectors.

The second remark involves the relation of the TSNE with a recent and elegant refinement, the notion of settled equilibrium due to Myerson-Weibull (2013) (MW hereafter). It is aimed to exclude uncoordinated NE "for more games than persistence, while maintaining general existence of a refined equilibrium that is also proper." Due to space considerations, the definition of this

equilibrium notion is omitted and we refer the reader to MW. Even though our desiderata is similar with MW's, below we display that these refinement concepts are idiosyncratic.

When the game under analysis has a strict NE, it is not surprising to observe that the TSNE is a subset of the set of fully settled equilibrium. Moreover, example 5 shows that this relation may be strict: Both s^1 and s^2 are fully settled while the TSNE equals $\{s^1\}$. Meanwhile, the next example establishes that when the given game does not have a strict NE, then the TSNE and the settled equilibrium are not logically related.

Example 7: This game has no strict NE and neither redundant nor weakly dominated actions, and is obtained by combining two "blocks" consisting of rescaled versions of example 4 of MW and a rock-scissor-paper.

1 \2	Α	В	С	D	1	II	111
Α	(0,5)	(1, 4)	(0,3)	(1, 0)	(-1, -1)	(-1, -1)	(-1, -1)
В	(1, 0)			(0,5)	(-1, -1)	(-1, -1)	(-1, -1)
1	(-1, -1)	(-1, -1)	(-1, -1)	(-1, -1)	(1, 1)	(2,0)	(0, 2)
II	(-1, -1)	(-1, -1)	(-1, -1)	(-1, -1)	(0, 2)	(1, 1)	(2, 0)
III	(-1 , -1)	(-1 , -1)	(-1, -1)	(-1 , -1)	(2,0)	(0, 2)	(1, 1)

It can be verified that here $s^1 = (1/2A + 1/2B, 1/2B + 1/2C)$ is not fully settled while it is in the TSNE; and $s^2 = (1/3I + 1/3II + 1/3III, 1/3I + 1/3III)$ is fully settled but not in the TSNE.

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