### Research Article

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# Welfare-Improving Effect of a Small Number of Followers in a Stackelberg Model

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**Abstract:** We investigate a Stackelberg oligopoly model in which m leaders and N-m followers compete. We find an important welfare effect that relates to antimonopoly policies when we move from the Cournot model (m=N) to the Stackelberg model: Exchanging a small number of Cournot firms for Stackelberg followers always improves welfare under moderate conditions. This contrasts with the welfare effect that can reduce welfare when a small number of Cournot firms are exchanged for Stackelberg leaders. The key result behind this asymmetry is the contrasting limit results in the cases where m converges to N and m converges to N. We also discuss the optimal number of leaders and the integer constraint for the number of firms.

**Keywords:** multiple leaders, Stackelberg, Cournot, limit result, convex cost **JEL classification numbers:** D43, L13, L40

## 1 Introduction

Cournot and Stackelberg models are important models in the literature concerning imperfect competition, and the welfare implications of these models have been discussed intensively in the literature. Many works have considered Stackelberg duopolies in which one firm, called the Stackelberg leader, competes against the other, called the Stackelberg follower. However, duopoly markets are rare, and it is important to generalize the welfare analysis to multi-leader and multi-follower cases.

Daughety (1990) and Ohkawa and Okamura (2003) formulate models in which m Stackelberg leaders and N-m Stackelberg followers compete in a

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homogeneous goods market<sup>1</sup> and in which all firms are identical in all respects except for their roles (each is either a leader or a follower). Using linear demand and linear cost functions, these works identify an inverse U-shaped relationship between m and economic welfare: Welfare is minimized when m = 0 and m = N, where both conditions correspond to the Cournot model. The inverse U-shaped relationship between m and economic welfare indicates that the Stackelberg model always yields larger welfare than the Cournot model does regardless of the number of leaders.

However, more generally for the case when marginal cost is increasing, Levin (1988) constructs an example with a single leader and multiple Stackelberg followers in order to show that the Stackelberg model can yield a smaller welfare than the Cournot model can. Ino and Matsumura (2012) extend this example to the case with multiple leaders and point out that this can happen depending on the number of leaders.<sup>2</sup> Such an inferiority of Stackelberg models can be obtained for the following reason. When the marginal cost is increasing, total production costs are minimized when all firms produce the same output as in the Cournot model. If some firms adopt leadership roles, it yields a difference in the production levels between leaders and followers, reducing production efficiency.<sup>3</sup> On the other hand, the Stackelberg model yields a higher total output and consumer surplus than the Cournot model does under moderate conditions. This is the output-expansion effect that improves welfare. In general, the former welfare-reducing effect can dominate the latter welfare-improving effect.<sup>4</sup> Thus, it is important to consider how these welfare effects of leadership work depending on the number of leaders.

In this study, we find that when the Stackelberg model is compared to the Cournot one, exchanging a small number of Cournot firms for Stackelberg followers always improves welfare under moderate conditions. Interestingly, this result contrasts the one showing that exchanging a small number of Cournot firms for Stackelberg leaders does not always improve welfare. The intuition is that in the case with a single leader, the increase in total production is took on by a single firm whereas in the case with a single follower, the increase in production is spread across N-1 firms. Therefore, market structures

<sup>1</sup> Sherali (1984) provides the same kind of multiple-leader Stackelberg model and a sufficient condition for the existence of an equilibrium, although such a welfare analysis is not this study's interest.

**<sup>2</sup>** Both of these papers use a linear demand and a quadratic cost in some numerical settings.

<sup>3</sup> This effect does not exist when the marginal cost is constant.

<sup>4</sup> A similar trade-off between the two effects has been discussed in many contexts. See, among others, Lahiri and Ono (1988, 1998).

with a few followers are more likely to increase welfare. By investigating limit results in the cases where  $m \to N$  (a small number of followers) and  $m \to 0$ (a small number of leaders), we provide a rigorous and clear foundation for this tendency: with many leaders in the limit, the gains by output expansion always exceed the production-inefficiency losses;<sup>5</sup> but with a few leaders, not always even in the limit.6

These contrasting results neatly indicate that the welfare effects of leadership depend on a prominent market structure, that is, the number of leaders in the market. Thus, a market structure with a small number of followers has quite different implications from that with a small number of leaders. For instance, it might be reasonable for antitrust departments to focus on markets with small numbers of leaders but not to care about markets with small numbers of followers.7 Also from a theoretical viewpoint, it is important to investigate the market structure with a small number of followers. This is because a Stackelberg model with a single follower is the equilibrium outcome in an endogenous timing game played by N firms (see Matsumura 1999).8

Our results also contain implications on the relationship between the Herfindahl-Hirschman Index (HHI) and economic welfare. When all firms are identical except for the timing of production, the Stackelberg model yields a higher HHI than the Cournot model does, indicating a higher degree of concentration of the market. In Daughety's result, because the Stackelberg model always yields a larger welfare than the Cournot model does regardless of the number of leaders, beneficial concentration always occurs. Our results suggest that the robustness of this result (in the sense that it does not depend on the linear demand or the constant marginal cost) depends on the market structure: Beneficial concentration occurs when the number of the followers is small, but it may not occur when the number of leaders is small.<sup>9</sup>

<sup>5</sup> In other words, this main result for a small number of followers indicate that a marginal exchange of the number of Cournot firms for Stackelberg followers improves welfare.

<sup>6</sup> We have to note that the result for a small number of leaders has been essentially found in Levin (1988) and Ino and Matsumura (2012) in numerical examples. However, the present paper considers the limit property to explain the contrast.

**<sup>7</sup>** For the cover stories, see the second paragraph of Section 5.

<sup>8</sup> Pal (1998) also presents a result in which only one firm (the public firm) becomes the follower in a mixed oligopoly. For a recent discussion on the endogeneity of Cournot and Stackelberg competitions in oligopolies, see also Breitmoser (2012).

<sup>9</sup> Also in the context of Stackelberg competition with endogenous entry of followers that has recently been extensively discussed, it is worth revisiting our analysis because the model with a fixed number of followers can be regarded as a short-run case that should be compared with the model with an endogenous number of followers. When the entry of followers is endogenous, Etro (2007, 2008) shows that the existence of a single leader always improves welfare as

The remainder of this paper is organized as follows. In Section 2, we formulate the model. In Section 3, we provide our main results. In Section 4, we further elaborate on the relationship between welfare and the number of leaders considering the optimal number of leaders and the integer constraint of the number of firms by supposing a linear demand function and a quadratic cost function.

## 2 The Model

We formulate an N-firm oligopoly model (N > 1). Firms produce a homogeneous product with an identical cost function,  $C: \mathbb{R}_+ \mapsto \mathbb{R}_+$ . The (inverse) demand function is given by  $P: \mathbb{R}_+ \mapsto \mathbb{R}_+$ . Each firm's payoff is its own profit. Firm i's profit associated with output tuple  $(x_i)_{i=1}^N$  is given by  $P(X)x_i - C(x_i)$ , where  $x_i$  is firm i's output and  $X = \sum_{i=1}^{N} x_i$ . First of all, we assume the relevant production levels to which we should restrict our attention.

**Assumption 1** P(0) > C'(0) and P(X) > 0 if and only if  $X < \overline{X} \in \mathbb{R}_{++}$ .

Since we are interested in the range where the price is positive from the perspective of economics, this assumption tells us that  $\bar{X}$  is the ceiling of the relevant production levels. Thus, we restrict the domain of the production levels to  $[0, \bar{X}]$ hereafter and make the following standard assumptions on this domain. 10

**Assumption 2** *C* is twice differentiable and  $C'(x) \ge 0$  for all  $x \in [0, \bar{X}]$ .

**Assumption 3** *P* is twice differentiable and P'(X) < 0 for all  $X \in [0, \bar{X}]$ .

**Assumption 4** P'(X) - C''(x) < 0 for all  $X, x \in [0, \bar{X}]$ .

**Assumption 5** P''(X)X + P'(X) < 0 for all  $X \in [0, \bar{X}]$ .

compared to the Cournot model. Ino and Matsumura (2012) extend this result to the case with multiple leaders and show that regardless of the number of leaders, the Stackelberg model always yields a larger welfare than the Cournot model. Thus, this long-run result contrasts with the short-run result when the number of leaders is small, whereas both the short-run and longrun results have similar implications when the number of followers is small.

<sup>10</sup> In other words, our analysis considers the left (right) derivative at  $\bar{X}$  (zero) by truncating on the right (left) of  $\bar{X}$  (zero), which is the meaningless part of domain.

Assumptions 2 and 3 assume that the cost function is increasing and the demand curve is downward-sloping in the relevant range. Assumption 4 is satisfied if (but not only if) the cost function is convex  $C'' \ge 0$  (including constant marginal costs) under Assumption 3.11 Assumption 5 is a well-known stability condition that ensures that each firm's marginal revenue is decreasing in the rivals' outputs and that the reaction curve in the Cournot model has a negative slope (i.e., strategic substitute). It is easy to show that under Assumption 3, Assumption 5 holds if and only if P''(X)x + P'(X) < 0 for all  $X \in [0, \bar{X}]$  and for all  $x \in [0, X]$ .

The game runs as follows. In the first stage,  $m \in [0, N]$  firms (Stackelberg leaders) independently choose their outputs. In the second stage,  $n (\equiv N - m)$ firms (Stackelberg followers) independently choose their outputs after observing the leaders' outputs. When m = 0 or m = N, the model corresponds to the Cournot model, in which N firms produce simultaneously and independently.

It is known that under Assumptions 1–5, Cournot models (m = 0 or m = N)have a unique equilibrium where all the firms choose the identical positive output level.<sup>12</sup> Hence, we denote the equilibrium output of each firm in the Cournot model by  $x_C^* > 0$  and the equilibrium social welfare in the Cournot model by

$$W_C^{\star} \equiv \int_0^{X_C^{\star}} P(q) dq - NC(X_C^{\star}), \qquad [1]$$

where  $X_C^* \equiv Nx_C^*$ .

Regarding Stackelberg models, we call the subgame perfect equilibrium symmetric and positive if all leaders choose the same positive output level and all followers choose the same positive output level in any subgame.<sup>13</sup> We restrict our attention to this positive symmetric equilibrium. Then, the outputs of a leader  $x_L \in (0, \bar{X})$  and of a follower  $x_F \in (0, \bar{X})$  in any positive symmetric Stackelberg equilibrium must satisfy the following system of equations of the two variables,  $F(x_L, x_F) \equiv (F_1(x_L, x_F), F_2(x_L, x_F)) = \mathbf{0}$ , which is constituted by the first-order conditions for leaders eq. [2] and for followers eq. [3]:

$$F_1(x_L, x_F) \equiv (1 + nR'(X_L))P'(X)x_L + P(X) - C'(x_L) = 0,$$
 [2]

<sup>11</sup> Thus, we can allow not only increasing but also decreasing marginal costs to some extent. 12 See, among others, Section 4.2 of Vives (1999), which tells us that under the appropriate smooth demand and cost (Assumptions 2 and 3), P'(X) - C''(x) < 0 (Assumption 4) and P''(X)x + P'(X) < 0 (Assumption 5) are sufficient conditions for the existence and the uniqueness of the Cournot equilibrium. Further, P(0) > C'(0) (Assumption 1) tells us that the equilibrium output of each firm must be positive. Since the symmetric setting of ours indicates that this Cournot equilibrium must be symmetric, there is no other equilibrium than the positive symmetric equilibrium that we focus on.

<sup>13</sup> A leader's output can differ from a follower's even in the symmetric equilibrium.

$$F_2(x_L, x_F) \equiv P'(X)x_F + P(X) - C'(x_F) = 0,$$
 [3]

where  $X \equiv mx_L + nx_F$  and  $X_L \equiv mx_L$ .  $R(X_L)$  represents a follower's reaction function, that is, a follower's output in the unique symmetric Nash equilibrium in any subgame subsequent to the leaders' total output  $X_L$ . <sup>14</sup>  $R(X_L)$  is obtained from the follower's first-order condition,

$$P'(X_L + nR(X_L))R(X_L) + P(X_L + nR(X_L)) - C'(R(X_L)) = 0.$$
 [4]

To facilitate the analysis, we presume that  $\det DF(x_L, x_F)|_{m=0} \neq 0$ , <sup>15</sup> where DF represents the Jacobian matrix of F.

In this study, our results are deduced as the properties that any solution of the system of eqs [2]-[3] satisfies and so the Stackelberg equilibria do.16 More precisely, we focus on a two-valued function of m,  $(x_I^*, x_F^*):(0, N)^2 \mapsto (0, \bar{X})^2$ , that satisfies the following properties. 17

**Assumption 6** For all 
$$m \in (0, N)$$
, (i)  $F(x_L^*(m), x_F^*(m)) = \mathbf{0}$  and (ii)  $mx_L^*(m) + (N - m)x_F^*(m) < \bar{X}$ .

Thus, the results induced under Assumption 6 can apply for all the positive symmetric Stackelberg equilibria that have positive price. It must be noted that we neglect the integer problems for the number of firms, 18 and by regarding  $(x_L^{\star}(m), x_F^{\star}(m))$  that satisfies Assumption 6 as the Stackelberg equilibrium patterns with respect to  $m \in (0, N)$ , we induce the essential tendencies of the welfare effects caused by marginal changes in the number of leaders. 19 Note that

<sup>14</sup> Under Assumptions 1–5, we can show that all subgames in the second stage have a unique equilibrium that is symmetric, as does the entire Cournot model (m = 0 or m = N).

<sup>15</sup> Indeed, this is presumed to simplify the statements and explanations when the number of leaders is small  $(m \to 0)$ . Our welfare implications (Proposition 2) are induced without this presumption. Thus, it is harmless to impose such an almost surely satisfied presumption.

<sup>16</sup> Therefore, even when the Stackelberg models have multiple symmetric equilibria, our analysis can be applied if we arbitrarily take an equilibrium pair of a leader's and a follower's outputs that satisfies Assumption 6.

<sup>17</sup> The range of  $(x_L^{\star}, x_F^{\star})$  is  $(0, \bar{X})^2$  because the Stackelberg equilibria are clearly in this range.

<sup>18</sup> The integer problems are also worthy of discussion. When we restrict our attention to the linear demand and the quadratic production cost, even under the integer constraint  $m \in \{0, 1, \dots, N-1, N\}$ , we obtain an asymmetric welfare implication similar to our main result. See Section 4.3.

**<sup>19</sup>** Strictly speaking,  $(x_L^*(m), x_F^*(m))$  is not the Stackelberg equilibrium if m is not an integer. However, the marginal effects of the number of firms have been often discussed to determine essential properties of welfare in oligopoly models. See Mankiw and Whinston (1986) and Suzumura and Kiyono (1987), who provide the well-known "excess entry" theorem, which implies that a marginal reduction in the number of firms improves welfare.

because  $(x_L^*(m), x_F^*(m))$  is defined on  $m \in (0, N)$  (excluding m = 0, N) here, there is nothing required a priori concerning the variation of the equilibrium outcomes between the Stackelberg models  $(m \in (0, N))$  and the Cournot model (m = 0, N).

# 3 Limit Properties and Welfare

In this section, we compare the welfare yielded in the Cournot model with that yielded in the Stackelberg model when the numbers of followers (leaders) is small by investigating marginal changes in m from m = N (m = 0).

When m = N or m = 0, eqs [2] and [3], respectively, correspond to the first-order condition of the Cournot model with N firms; that is, each is reduced to a equation of single variable x:

$$P'(Nx)x + P(Nx) - C'(x) = 0.$$
 [5]

The Cournot outcome,  $x = x_C^*$ , is a solution of this equation. Although it is well known that the Cournot equilibrium is unique under our assumptions, we show the uniqueness of the solution of this equation here because it is used for our analysis later.<sup>20</sup>

**Lemma 1** Suppose that Assumptions 1–5 are satisfied. Then, eq. [5] never has multiple solutions in  $[0, \bar{X}/N]$ .

Q.E.D.

Furthermore, although eqs [3] and [2] has no economic meaning when m = N (m = 0), we can solve the system of equations as seen in the following lemma.

**Lemma 2** Suppose that Assumptions 1–6 are satisfied. Then, (i)  $\lim_{m\to N}(x_L^*(m), x_F^*(m))$  exists and  $(x_L, x_F) = \lim_{m\to N}(x_L^*(m), x_F^*(m))$  is a solution of the system of eqs [2]–[3] when m = N; (ii) if  $\det DF(x_L, x_F)|_{m=0} \neq 0$ ,  $\lim_{m\to 0}(x_L^*(m), x_F^*(m))$  exists and  $(x_L, x_F) = \lim_{m\to 0}(x_L^*(m), x_F^*(m))$  is a solution of the system of eqs [2]–[3] when m = 0.

**<sup>20</sup>** It must be noted that Lemma 1 states nothing about existence of the solution and the eq. [5] is applicable only for the positive symmetric equilibrium. Thus, the lemma just indicates that the positive symmetric Cournot equilibrium is unique if it exists. Indeed, it is guaranteed in our setting that the Cournot equilibrium uniquely exists and it is positive and symmetric (see Footnote 12).

Proof See Appendix.

Q.E.D.

Hence, we extend  $(x_L^*, x_F^*) : [0, N]^2 \mapsto [0, \bar{X}]^2$  by defining

$$(x_L(0), x_F(0)) \equiv \lim_{m \to 0} (x_L^*(m), x_F^*(m)),$$

$$(x_L(N), x_F(N)) \equiv \lim_{m \to N} (x_L^{\star}(m), x_F^{\star}(m)).$$

Then, the following lemma ensures that our operation works well.

**Lemma 3** Suppose that Assumptions 1–6 are satisfied. Then, (i)  $x_L^*(m)$  and  $x_F^*(m)$  are first-order differentiable at m = N; (ii) if  $\det DF(x_L, x_F)|_{m=0} \neq 0$ ,  $x_L^*(m)$  and  $x_F^*(m)$  are first-order differentiable at m = 0.

Proof See Appendix.

Q.E.D.

Proposition 1 presents an important limit property that helps us understand the asymmetric welfare implication presented in Proposition 2. More concretely, Proposition 1 states that both the Stackelberg leader's output and the follower's output converge to the Cournot output when m is close to N, whereas the leader's output does not converge to the Cournot output (the follower's output converges to the Cournot output) when m is close to 0.

**Proposition 1** Suppose that Assumptions 1–6 are satisfied. Then,

(i) 
$$x_C^* = x_L^*(N) \equiv \lim_{m \to N} x_L^*(m) = \lim_{m \to N} x_F^*(m) \equiv x_F^*(N)$$
 and

(ii) if 
$$\det DF(x_L, x_F)|_{m=0} \neq 0$$
,  $x_L^*(0) \equiv \lim_{m \to 0} x_L^*(m) > \lim_{m \to 0} x_F^*(m) \equiv x_F^*(0) = x_C^*$ .

**Proof** First, we show that  $x_C^* = x_F^*(0)$ . When m = 0, eq. [3] is reduced to eq. [5] with  $x = x_F$ . Note that  $x_C^* \in (0, \bar{X}/N)$  in the positive Cournot equilibrium. Further, because  $x_F^*(m) > 0$  and  $mx_L^*(m) + (N-m)x_F^*(m) < \bar{X}$  for all  $m \in (0, N)$  by Assumption 6(ii),  $x_F^*(0) = \lim_{m \to 0} x_F^*(m)$  and  $Nx_F^*(0) = \lim_{m \to 0} (mx_L^*(m) + (N-m)x_F^*(m))$  imply that  $x_F^*(0) \in [0, \bar{X}/N]$ . Therefore, Lemma 1 implies that  $x_C^* = x_F^*(0)$  because both  $x_C^*$  and  $x_F^*(0)$  are solutions of eq. [5] by the definition of the Cournot equilibrium and Lemma 2.

We can similarly obtain  $x_C^* = x_L^*(N)$ . When m = N (n = 0), because nR' vanishes, eq. [2] is also reduced to eq. [5] with  $x = x_L$ . Then, the rest of proof is similar to that in the previous paragraph.

Next, we show that  $x_L^*(0) > x_C^*$ . Hereafter, we also use the notation  $X_C^* \equiv Nx_C^*$  to save space. We assume that  $x_L^*(0) \le x_C^*$  and derive a contradiction. Taking m = 0, we

will compare the LHS of eq. [5] (eq. [3] taking m=0) and eq. [2]. We have  $P'(X_C^\star)x_C^\star + P(X_C^\star) - C'(x_C^\star) \le P'(X_C^\star)x_L^\star(0) + P(X_C^\star) - C'(x_L^\star(0))$  because  $x_L^\star(0) \le x_C^\star$  and  $P'(X_C^\star) - C''(x) < 0$  for all  $x \in [x_L^\star(0), x_C^\star]$ . Assumption 4 is used here because  $x_C^\star < X_C^\star < \bar{X}$  in the positive Cournot equilibrium with N > 1. Differentiating eq. [4] yields

$$R'(X_L) = -\frac{P'(X_L + nR) + P''(X_L + nR)R}{n(P'(X_L + nR) + P''(X_L + nR)R) + (P'(X_L + nR) - C''(R))}.$$
 [6]

Evaluating eq. [6] at m = 0 (n = N) yields

$$R'\big|_{m=0} = -\frac{P'(X_C^{\star}) + P''(X_C^{\star}) x_C^{\star}}{N(P'(X_C^{\star}) + P''(X_C^{\star}) x_C^{\star}) + (P'(X_C^{\star}) - C''(x_C^{\star}))}.$$

Hence, by Assumptions 4 and 5, which are applicable here because  $x_C^\star < X_C^\star < \bar{X}_C$ , we have  $NR'\big|_{m=0} \in (-1,0)$ . Thus, we have  $P'(X_C^\star)x_L^\star(0) + P(X_C^\star) - C'(x_L^\star(0)) < (1+NR'\big|_{m=0})P'(X_C^\star)x_L^\star(0) + P(X_C^\star) - C'(x_L^\star(0))$ . Therefore, if  $x=x_C^\star$  satisfies eq. [5], the LHS of eq. [2] must be positive even when the solution at m=0,  $(x_L,x_F)=(x_L^\star(0),x_C^\star)$  where  $x_C^\star=x_F^\star(0)$ , is substituted, which is a contradiction.

Lastly, we show that  $x_F^*(N) = x_C^*$ . From Lemma 2 and  $x_C^* = x_L^*(N)$ ,  $(x_L, x_F) = (x_C^*, x_F^*(N))$  must satisfy eq. [3] when m = N:

$$P'(Nx_C^*)x_F^*(N) + P(Nx_C^*) - C'(x_F^*(N)) = 0.$$
 [7]

Then,  $x_F^\star(N)$  satisfies this equation if  $x_F^\star(N) = x_C^\star$  because  $x = x_C^\star$  satisfies eq. [5] (eq. [2] taking m = N). Because  $P'(X_C^\star) - C''(x) < 0$  for  $X_C^\star < \bar{X}$  and all  $x \in [0, \bar{X}]$  by Assumption 4, the LHS of eq. [7] is decreasing in  $x_F^\star(N) \in [0, \bar{X}]$ . Therefore, given  $x_C^\star$ , the  $x_F^\star(N)$  that satisfies eq. [7] is unique. These imply that  $x_F^\star(N) = x_C^\star$ . Q.E.D.

We explain the intuition behind the asymmetry in this proposition; that is, why the leader's output converges to the Cournot output when m approaches N, but does not converge to the Cournot output when m approaches 0. When m = N, no follower exists. Thus, leaders do not make a strategic commitment against the other firms. Exchanging a small number of Cournot firms for Stackelberg followers makes the firms that become leaders take such a strategic action. However, the number of leaders is large whereas the number of followers is small. Thus, the strategic effect is small, and each leader slightly expands its output. This is why  $x_L^*(m)$  approaches  $x_C^*$  when m approaches N. The same principle cannot be applied to the case where m approaches 0. Exchanging a small number of Cournot firms for Stackelberg leaders evokes the leaders'

strategic commitments. Because the number of leaders is small and the number of the firms that become followers is large, the strategic effect affects each leader's output significantly, resulting in a non-negligible difference between  $x_L^{\star}(m)$  and  $x_C^{\star}$ . This is why  $x_L^{\star}(m)$  does not converge to  $x_C^{\star}$  when m approaches 0.

The equilibrium social welfare (the consumer surplus plus the profits of all firms)  $W^*$  is given by

$$W^{\star}(m) \equiv \int_{0}^{X^{\star}(m)} P(q)dq - mC(x_{L}^{\star}(m)) - nC(x_{F}^{\star}(m)),$$
 [8]

where  $X^*(m) \equiv mx_I^*(m) + nx_F^*(m)$ . Furthermore, let the equilibrium profit of a leader be  $\pi_L^*(m) \equiv P(X^*(m))x_L^*(m) - C(x_L^*(m))$  and that of a follower be  $\pi_F^*(m) \equiv P(X^*(m)) \chi_F^*(m) - C(\chi_F^*(m))$ . Despite the asymmetry of the limit quantities that is shown in Proposition 1,  $W^*(m)$  converges to the Cournot equilibrium welfare  $W_C^*$  not only when m approaches N but also when m approaches O. The following lemma clarifies this fact formally.

**Lemma 4** Suppose that Assumptions 1–6 are satisfied. Then,  $\lim_{m\to N} W^*(m) =$  $W^{*}(N) = W_{C}^{*}$  and if  $\det DF(x_{L}, x_{F})|_{m=0} \neq 0$ ,  $\lim_{m\to 0} W^{*}(m) = W^{*}(0) = W_{C}^{*}$ .

**Proof**  $\lim_{m\to 0} W^*(m) = W^*(0) = W_C^*$  stems from the following manipulation:

$$\begin{split} \lim_{m \to 0} W^{\star}(m) &= \int_{0}^{X^{\star}(0)} P(q) dq - 0C(x_{L}^{\star}(0)) - NC(x_{F}^{\star}(0)) \equiv W^{\star}(0) \\ &= \int_{0}^{X^{\star}(0)} P(q) dq - NC(x_{F}^{\star}(0)) \\ &= \int_{0}^{X_{C}^{\star}} P(q) dq - NC(x_{C}^{\star}) \equiv W_{C}^{\star}, \end{split}$$

where the first line uses  $\lim_{m\to 0} x_L^*(m) = x_L^*(0)$  and  $\lim_{m\to 0} x_F^*(m) = x_F^*(0)$  by the definitions and thus  $\lim_{m\to 0} X^*(m) = 0x_L^*(0) + Nx_F^*(0) \equiv X^*(0)$ . Note the calculation in the limit is valid here since  $\int_0^X P(q)dq$  and C(x) are continuous in X and x, respectively. The last line uses  $x_F^*(0) = x_C^*$  shown by Proposition 1(ii).

 $\lim_{m\to N} W^{\star}(m) = W^{\star}(N) = W_{C}^{\star}$  is obtained by a similar manipulation. **Q.E.D.** 

Thanks to this lemma, we can interpret a marginal decrease (increase) in m at m = N (m = 0) as "exchanging a small number of Cournot firms for Stackelberg followers (leaders)."

Our main result for the welfare comparison between the Cournot and the Stackelberg models follows from the difference in each production shown in Proposition 1 and the number of firms shouldering that difference. When a small number of Cournot firms are exchanged for Stackelberg followers  $(m \to N)$ , the negative and positive welfare effects are associated with a *small* decrease in production  $(x_F^*(N) = x_C^*)$  took on by a *small* number of firm  $(N - m \to 0$  followers) and a *small* increase in production  $(x_L^*(N) = x_C^*)$  that nevertheless spreads across *almost all* the firms  $(m \to N)$  leaders), respectively. Whereas, when a small number of Cournot firms are exchanged for Stackelberg leaders  $(m \to 0)$ , the negative and positive welfare effects are associated with a *small* decrease in production  $(x_F^*(0) = x_C^*)$  that spreads across *almost all* the firms  $(N - m \to N)$  followers) and a *non-small* increase in production  $(x_L^*(0) > x_C^*)$  took on only by a *small* number of firm  $(m \to 0)$  leaders), respectively. As the result, we obtain Proposition 2, which indicates that (i) exchanging a small number of Cournot firms for Stackelberg followers always improves welfare<sup>21</sup> while (ii) exchanging a small number of Cournot firms for Stackelberg leaders does not always improve welfare.

**Proposition 2** Suppose that Assumptions 1–6 are satisfied. Then,

- (i) we always have  $\frac{\partial W^*}{\partial m}\Big|_{m=N} < 0$  and
- (ii) we can have both the case where  $\frac{\partial W^*}{\partial m}\Big|_{m=0} < 0$  and the case where  $\frac{\partial W^*}{\partial m}\Big|_{m=0} > 0$ .

**Proof** (i) We show that for any situation satisfying Assumptions 1–6,  $\partial W^*(N)/\partial m < 0$ . By Lemma 3(i), we can differentiate eq. [8] at m = N. Then, we obtain

$$\left. \frac{\partial \boldsymbol{W}^{\star}}{\partial \boldsymbol{m}} \right|_{\boldsymbol{m}=\boldsymbol{N}} = \left[ \pi_{L}^{\star}(\boldsymbol{N}) - \pi_{F}^{\star}(\boldsymbol{N}) \right] + N[P(\boldsymbol{X}^{\star}(\boldsymbol{N})) - C'(\boldsymbol{x}_{L}^{\star}(\boldsymbol{N}))] \frac{\partial \boldsymbol{x}_{L}^{\star}}{\partial \boldsymbol{m}} \right|_{\boldsymbol{m}=\boldsymbol{N}}.$$
 [9]

Because  $x_L^{\star}(N) = x_F^{\star}(N) = x_C^{\star}$  from Proposition 1(i), the first brackets on the RHS is canceled out. Therefore, if the last term in the RHS is negative,  $\partial W^{\star}(N)/\partial m < 0$ . Letting  $X_C^{\star} \equiv Nx_C^{\star}$ , the comparative statics using eqs [2] and [3] on the basis of Lemma 3(i) yield

$$\frac{\partial x_L^*}{\partial m}\bigg|_{m=N} = \frac{P'(X_C^*)x_C^*R'\big|_{m=N}}{N(P'(X_C^*) + P''(X_C^*)x_C^*) + (P'(X_C^*) - C''(x_C^*))} < 0,$$
 [10]

where  $R'|_{m=N} = -(P'(X_C^\star) + P''(X_C^\star)x_C^\star)/(P'(X_C^\star) - C''(x_C^\star)) < 0$ , which is induced from eq. [6]. We use  $x_L^\star(N) = x_F^\star(N) = x_C^\star$  (Proposition 1(i)) to induce eq. [10]. Note

**<sup>21</sup>** Note that we use "exchanging a small number of firms for followers" to imply that m decreases marginally from N.  $\partial W^*/\partial m|_{m=N}$  represents how  $W^*$  moves when m increases marginally from N. Thus, the exchange of a small number of firms for followers enhances welfare if the sign of  $\partial W^*/\partial m|_{m=N}$  is negative.

that Assumptions 3–5 are applicable here because we have  $x_C^* < X_C^* < \bar{X}$  in the positive Cournot equilibrium with N>1. Equation [10] implies that the last term in the RHS is negative since  $P(X_C^*) - C'(x_C^*) > 0$  by the first-order condition [5].

(ii) The example wherein P(X) = 1 - X,  $C(x) = x^2$  yields<sup>22</sup>

$$\frac{\partial W^*}{\partial m}\bigg|_{m=0} = \frac{N(27+3N-N^2)}{(3+N)^3(9+2N)^2},$$

which is positive when  $N \le 6$  and negative when  $N \ge 7$ .

Q.E.D.

We explain the intuition behind the asymmetry between (i) and (ii) in Proposition 2.<sup>23</sup> The following decomposition helps us understand Proposition 2(ii):<sup>24</sup>

$$\frac{\partial W^*}{\partial m}\Big|_{m=0} = \left[\pi_L^*(0) - \pi_F^*(0)\right] + N[P(X^*(0)) - C'(x_F^*(0))] \frac{\partial x_F^*}{\partial m}\Big|_{m=0}.$$
 [11]

The change in welfare caused by the marginal exchange for leaders is decomposed into two parts. The first term is the direct effect of increasing the profit of a firm that changes its role (i.e., from a follower to a leader). Proposition 1(ii) implies that this effect is positive (the first-mover advantage remains significant even when  $m \to 0$ ). Concurrently, the new leader steals the other firms' business, resulting in the reduction of the other firms' (followers') outputs. Because the price is higher than the marginal cost, the welfare change induced by this indirect effect (second term) is negative. The change in welfare can be either negative or positive depending on the relative size of these two effects.

Proposition 2(i) is elucidated by the decomposition eq. [9] in the proof. Similarly, the change in welfare by the marginal exchange for followers is decomposed into two parts. In contrast to the previous case, Proposition 1(i) implies that the first term is zero; that is, the first direct effect (the reduction in the profit of the firm that becomes a follower) is insignificant when  $m \to N$ . Therefore, the indirect effect on the other firms (the second term) alone determines the change in welfare. In this case, this effect is business-augmenting: Exchanging a small number of firms for followers (a marginal reduction of m) increases the rivals' (leaders') output, which implies that the second term is negative. Thus, eq. [9] is unambiguously negative (a marginal reduction of mincreases welfare).

<sup>22</sup> The proof of Proposition 3 provides a more detailed calculation to induce the following equation, because this example is a special case of Proposition 3.

<sup>23</sup> Mankiw and Whinston (1986) explain the "excess entry" theorem in a similar fashion. The terms "business stealing" and "business augmenting" are adopted from their paper.

<sup>24</sup> The relevant differentiability is shown by Lemma 3(ii).

**Remark 1** Proposition 2(i) implies that

$$0, N \notin \underset{m \in [0, N]}{\operatorname{argmax}} W^*(m),$$

that is, Cournot is never best for welfare. In other words, we obtain a corollary that a market with both leaders and followers is most desirable.<sup>25</sup> The HHI is minimized in the Cournot case because there is no asymmetry in this case. Thus, this corollary implies that beneficial concentration can generally occur. We further discuss the properties of the optimal number of leaders in Section 4.2 by specifying the demand and cost functions.

**Remark 2** By differentiating the total profit of the firms,  $\Pi^*(m) = m\pi_L^*(m) + (N-m)\pi_E^*(m)$ , we obtain<sup>26</sup>

$$\left. \frac{\partial \Pi^{\star}(m)}{\partial m} \right|_{m=N} = N(N-1)P'(X^{\star}(N))x_{L}^{\star}(N) \left. \frac{\partial x_{L}^{\star}}{\partial m} \right|_{m=N} > 0.$$

Thus, exchanging a small number of Cournot firms for Stackelberg followers (a marginal reduction of m) always decrease the total profit. This implies that the welfare improvement stated in proposition 2(i) is completely caused by an expansion of the consumer surplus. Thus, the market structure with a small number of followers is acceptable even for policymakers who are concerned only with consumer welfare.

## 4 Linear Demand and Quadratic Cost

In this section, we specify the demand and cost functions and further elaborate on the relationship between welfare and the number of leaders. All the proofs for this section are relegated to the Appendix. Suppose that the inverse demand

<sup>25</sup> This result contrasts with the result for free entry markets. When the number of followers is endogenously determined and the leader can commit to their production before the followers enter, the optimal number of leaders is such that no follower is allowed to enter. See Ino and Matsumura (2012).

**<sup>26</sup>** To induce this, we use Proposition 1(i) and the first-order condition of the Cournot equilibrium [5] for the first equality, and eq. [10] for the last inequality.

<sup>27</sup> This result also contrasts with the result for free entry markets, in which the welfare improvement by the leaders is caused by expanding the firms' profits. See Ino and Matsumura (2012).

**<sup>28</sup>** In numerous countries, a number of law researchers believe that antitrust departments should and do care about consumer welfare only.

Symbol	Value
$\overline{x_{l}^{\star}(m)}$	$a(2k+1)/\beta$
$x_F^*(m)$	$a(4k^2+2k(2+n)+1)/\alpha\beta$
X*(m)	$a(4k^2N + 2k(2+n)N + (mn+N))/\alpha\beta$
$\pi_L^{\star}(m)$	$a^{2}(2k+1)^{2}(2k^{2}+k(3+n)+1)/\alpha\beta^{2}$
$\pi_F^*(m)$	$a^{2}(k+1)(4k^{2}+2k(2+n)+1)^{2}/\alpha^{2}\beta^{2}$

**Table 1:** Results under the linear demand and quadratic cost functions:  $\alpha = (2k + n + 1)$ ,  $\beta = (4k^2 + 2k(2 + N) + (1 + m))$  and N = m + n.

function is linear (i.e., P(X) = a - X) and the cost function is quadratic (i.e.,  $C(x) = kx^2$ , where  $k \ge 0$ ). Note that if k = 0 (k > 0), we have a constant (increasing) marginal cost. The results obtained by solving the model under these specifications are summarized in Table 1, where n = N - m. We can compute the equilibrium social welfare in this specific case by substituting the results of Table 1 into

$$W^{*}(m) = \frac{1}{2} (X^{*}(m))^{2} + m\pi_{L}^{*}(m) + n\pi_{F}^{*}(m).$$
 [12]

The asymmetry of the limit quantities stated in Proposition 1 is surely observed in this example. The Cournot output is easily calculated as  $x_C^* = a/(2k+N+1)$ . From Table 1, we can check that  $x_L^*(N)$ ,  $x_F^*(N)$ , and  $x_L^*(0)$  are equal to  $x_C^*$ . However,

$$\chi_L^{\star}(0) = \frac{a(2k+1)}{4k^2 + 2k(2+N) + 1} = \frac{a(2k+1)}{(2k+1)(2k+N+1) - N} > \chi_C^{\star}.$$

## 4.1 The Total Number of Leaders and Followers

We demonstrate that leadership reduces welfare under increasing marginal costs if the total number of firms is sufficiently large.

**Proposition 3** Assume linear demand and quadratic cost. If k>0, there exists N'>1 such that for all N,  $\frac{\partial W^*}{\partial m}\big|_{m=0} \leq 0$  if and only if  $N \geq N'$ .

This proposition implies that when the number of leaders is small and the total number of firms is large (i.e., when the proportion of leaders in the industry is small), leadership is more likely to be harmful.

## 4.2 The Optimal Number of Leaders

We discuss the properties of the optimal number of leaders solving the following problem:

$$\max_{m\in[0,N]}W^*(m).$$
 [13]

**Proposition 4** Assume linear demand and quadratic cost. Then, there uniquely exists a solution  $m^*$  of prob. [13], and  $m^* = N/2$  if k = 0 and  $m^* \in (N/2, N)$  if k > 0.

We conclude from the asymmetric result in Proposition 2 that welfare can be decreasing in m when m is small; thus welfare must increase in m for some m, and finally decrease again in m when m is close to N (typically, a rotated S-shaped relationship, as shown in Figure 1). It is then plausible that welfare tends to be maximized when m is closer to N than to 0. Proposition 4 states that this is true under linear demand and quadratic cost: The optimal number of leaders is larger than N/2 (biased towards N).

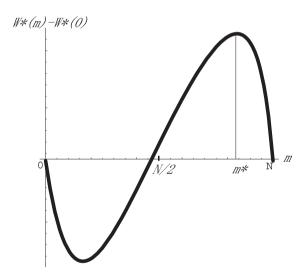


Figure 1: Welfare comparison when the number of leaders changes for fixed N = m + n. When  $m = m^*$ , welfare is maximized.

## 4.3 Integer Constraint

In this subsection, we consider the integer constraint for the number of firms. We show that if the model has linear demand and quadratic cost, we obtain results similar to those of Propositions 2 and 3 even under this integer constraint. Proposition 5(i) suggests that exchanging *one* Cournot firm for a

Stackelberg follower always improves welfare as in Proposition 2(i). Proposition 5(ii) indicates that exchanging one Cournot firm for a Stackelberg leader may reduce welfare, as in Proposition 2(ii), specifically when the total number of firms in the industry is large as in Proposition 3.

**Proposition 5** Assume linear demand and quadratic cost. Suppose that  $N \in \{2, 3, \dots\}$ . (i)  $W^*(N-1) - W^*(N) > 0$  for all N. (ii) If k > 0, there exists N' > 2such that for all N,  $W^*(1) - W^*(0) \leq 0$  if and only if  $N \geq N'$ .

# 5 Concluding Remarks

Comparing the multi-leader and multi-follower Stackelberg model with the Cournot model, we showed that exchanging a small number of Cournot firms for Stackelberg leaders may reduce welfare but that exchanging a small number of Cournot firms for Stackelberg followers always improves welfare. This result is obtained because while a leader's output never converges to the Cournot output when the number of leaders becomes smaller, it converges to the Cournot output when the number of followers becomes smaller.

While the market structures that have a single dominant firm (i.e., a Stackelberg leader) among other firms (i.e., multiple Stackelberg followers) often faces antitrust issues, it is also important to consider the circumstances concerning anti-monopoly policies on the contrasting market structures that have a single Stackelberg follower and multiple Stackelberg leaders. In reality, since it often takes so substantial time to enter a market that the entries occurs one by one, a market structure where a single new entrant competes with some incumbents is often the case, at least in the short-run, after the entry. Consider the circumstance where all the incumbents can take leadership roles and behave aggressively toward the new entrant in order to deter this newcomer from expanding its market share (Stackelberg-fashion competition with a small number of followers). In such a market, we can observe cases where new entrants appeal to the antitrust department to stop, or the antitrust department itself is concerned about stopping, incumbents' aggressive behavior (Cournot-fashion competition).<sup>29</sup> According to our result, it is better to let the incumbents be

<sup>29</sup> For instance, in the market for broadband Internet access service, incumbents (e.g., AT&T, Time Warner, and Comcast in the U.S. market) have already commit to their supply capacities, because wiring of homes is time-consuming and expensive. Thus, a new provider (e.g., Verizon in the last decade) must be a follower for a while. Shinkai (2000) also provides an interesting example of DRAM (system memory) as a market where a Stackelberg-fashion competition occurs by nature for similar reasons concerning inflexible capacity investment. Whether the

leaders. Thus, it is difficult to rationalize the government's intervention as long as there is no special issue other than the market structure.

In this study, we assumed that all firms have an identical cost function and that only the difference in their roles yields the asymmetry among firms. If we introduce cost differences among firms, the analysis becomes considerably more complicated. Leadership by a firm with a lower (higher) marginal cost is more likely to improve (deteriorate) welfare. However, it is not easy to find a more meaningful condition for welfare improvement. Extending our analysis in this direction is left for future research.<sup>30</sup>

Furthermore, this study focuses on symmetric equilibria in Stackelberg models. We believe that this restriction is meaningful because we consider the slight change from the Cournot equilibrium that must be symmetric. However, it is known that asymmetric equilibria sometimes arise in the Stackelberg model even if all firms are identical.<sup>31</sup> If the equilibrium pattern chosen by the firms suddenly switches from being symmetric type to being asymmetric type when m changes from the Cournot model, our approach is not applicable. To solve this problem, more sophisticated analysis is required. This issue also remains for future research.

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government should leave such a market structure without intervening or regulate the dominant providers to set equal-footing competition is under debate. See, for example, "Should the U.S. Regulate Broadband Internet Access as a Utility?" Wall Street Journal, JOURNAL REPORTS: LEADERSHIP published on May 11, 2014. (Available at online: wsj.com/articles/should-the-u-sregulate-broadband-internet-access-as-a-utility-1399645300.)

<sup>30</sup> For the discussion of the welfare effect under cost heterogeneity between a single leader and multiple followers, see Levin (1988). For the limit result when an infinite number of followers enter the market where a single leader has a cost advantage, see Ino and Kawamori (2009). Pal and Sarkar (2001), Mukherjee and Zhao (2009), and Ishida, Matsumura, and Matsushima (2011) also demonstrate that cost heterogeneity plays an important role in strategic commitment games such as Stackelberg oligopolies. The heterogeneity of payoff functions is also extensively discussed in the literature. See Pal (1998), Ino and Matsumura (2010), and Wang and Mukherjee (2012).

<sup>31</sup> For example, Ehrenmann (2004) shows that the multiple-leader Stackelberg model could possess manifold asymmetric equilibria even in the case where the symmetric equilibrium is unique.

# **Appendix**

#### **Proof of Lemma 1:**

Consider two distinct values  $x', x'' \in [0, \bar{x}]$  such that x' > x'', where  $\bar{x} \equiv \bar{X}/N$ . Suppose that both x = x' and x = x'' are solutions of eq. [5]. The LHS of eq. [5] is decreasing in x for all  $x \in (0, \bar{x})$  because differentiating it yields N(P''(Nx)x +P'(Nx) + (P'(Nx) - C''(x)) < 0 by Assumptions 4 and 5. Assumptions 4 and 5 are applicable here because  $x < Nx < \bar{X}$  by  $0 < x < \bar{x}$  and N > 1. Thus,  $P'(Nx')x' + \bar{X}$ P(Nx') - C'(x') < P'(Nx'')x''' + P(Nx'') - C'(x''). This contradicts the fact that both x = x' and x = x'' satisfy eq. [5]. Therefore, eq. [5] never has multiple solutions in  $[0, \bar{x}]$ . Q.E.D.

#### Proofs of Lemmas 2 and 3:

In this proof, after substituting  $(x_L, x_F) = (x_L^*(m), x_F^*(m))$ , we regard the LHS of eq. [2] as the function  $F_1(m; \chi_I^*(m), \chi_F^*(m))$  and the LHS of eq. [3] as the function  $F_2(m; x_I^*(m), x_F^*(m))$  by explicitly including m as an augment.

Because  $(x_L^{\star}(m), x_F^{\star}(m)) \in (0, \bar{X})^2$  for all  $m \in (0, N)$ , there exists a sequence  $\{m^t\}_{t=1}^{\infty}$  with  $m^t > 0$  such that  $\lim_{t \to \infty} m^t = 0$  and  $\lim_{t \to \infty} (x_L^{\star}(m^t), x_F^{\star}(m^t)) \in [0, \bar{X}]^2$ . The continuity of  $F_1(\cdot)$  yields

$$\lim_{t\to\infty} F_1(m^t;x_L^{\star}(m^t),x_F^{\star}(m^t)) = F_1\bigg(0; \ \lim_{t\to\infty} (x_L^{\star}(m^t),x_F^{\star}(m^t))\bigg).$$

On the other hand, because  $F_1(m^t; x_L^*(m^t), x_F^*(m^t)) = 0$  for all  $m^t \in (0, N)$ ,

$$\lim_{t\to 0} F_1(m^t; x_L^*(m^t), x_F^*(m^t)) = 0.$$

Hence,  $F_1(0; \lim_{t\to\infty}(x_L^*(m^t), x_F^*(m^t))) = 0$ . Similarly,  $F_2(0; \lim_{t\to\infty}(x_L^*(m^t), x_F^*(m^t))) = 0$ . Therefore,  $\lim_{t\to\infty}(x_L^{\star}(m^t),x_F^{\star}(m^t))$  is one of the solutions of the system of eqs [2] and [3] when m = 0. Similarly, by taking a sequence  $\{m^s\}_{s=1}^{\infty}$  with  $m^s < N$  such that  $\lim_{s\to\infty} m^s = N$  and  $\lim_{s\to\infty} (x_L^{\star}(m^s), x_F^{\star}(m^s)) \in [0, \bar{X}]^2$ , we can show that  $\lim_{s\to\infty}(x_I^*(m^s),x_F^*(m^s))$  is one of the solutions of the system of eqs [2] and [3] when m = N.

(i) When m = N (n = 0), denoting  $(c_1, c_2) \equiv \lim_{s \to \infty} (x_L^*(m^s), x_F^*(m^s))$ , we have  $F(c_1, c_2)|_{m=N} = \mathbf{0}$  from the previous paragraph and

$$\det DF(c_1, c_2)|_{m=N} = N(P''(Nc_1)c_1 + P'(Nc_1))(P'(Nc_1) - C''(c_1)) + (P'(Nc_1) - C''(c_1))^2 \neq 0.$$

Note that because  $Nc_1 = \lim_{s\to\infty} (m^s x_1^*(m^s) + (N-m^s) x_E^*(m^s)) \le \bar{X}$  by Assumption 6 (ii), we can use Assumptions 4 and 5 to show the last inequality. Thus, by the implicit function theorem, there exist  $\epsilon > 0$  and, uniquely, a two-valued function of m,  $(c_1^\star,c_2^\star):(N-\epsilon,N]^2\mapsto\mathbb{R}^2$  that satisfies  $F(c_1^\star(m),c_2^\star(m))=\mathbf{0}$  for all  $m\in(N-\epsilon,N]$ . Then, Assumption 6(i) and the uniqueness of  $(c_1^\star,c_2^\star)$  imply that  $(x_L^\star(m),x_F^\star(m))=(c_1^\star(m),c_2^\star(m))$  for all  $m\in(N-\epsilon,N)$ . Because  $(c_1^\star,c_2^\star)$  is continuous on  $(N-\epsilon,N]^2$  by the implicit function theorem,  $\lim_{m\to N}(c_1^\star(m),c_2^\star(m))=(c_1^\star(N),c_2^\star(N))=(c_1,c_2)$ . By  $(x_L^\star(m),x_F^\star(m))=(c_1^\star(m),c_2^\star(m))$  on  $m\in(N-\epsilon,N)$ , for any sequence  $\{m^k\}_{k=1}^\infty$  with  $m^k< N$  such that  $\lim_{s\to\infty}m^k=N$ ,  $(x_L^\star(m^k),x_F^\star(m^k))=(c_1^\star(m^k),c_2^\star(m^k))$  for all k. Thus,  $\lim_{m\to N}(x_L^\star(m),x_F^\star(m))=\lim_{m\to N}(c_1^\star(m),c_2^\star(m))=(c_1,c_2)$ . This implies Lemma 2 (i). Because  $(c_1^\star,c_2^\star)$  is first-order differentiable on  $(N-\epsilon,N]^2$  by the implicit function theorem,  $(x_L^\star(m),x_F^\star(m))=(c_1^\star(m),c_2^\star(m))$  on  $m\in(N-\epsilon,N)$  and the definition  $(x_L^\star(N),x_F^\star(N))=(c_1^\star(N),c_2^\star(N))$  implies Lemma 3 (i).

(ii) When m=0, if we presume  $\det DF(x_L,x_F)|_{m=N}\neq 0$ , the implicit function theorem can apply to  $F(\lim_{t\to\infty}(x_L^{\star}(m^t),x_F^{\star}(m^t)))|_{m=0}=\mathbf{0}$ , and the rest of the proofs of Case (ii) in Lemmas 2 and 3 are similar to those in Case (i). **Q.E.D.** 

### **Proof of Proposition 3:**

Substituting the results of Table 1 into eq. [12] and differentiating it yield

$$\frac{\partial W^{*}(m)}{\partial m} = \frac{\alpha^{2}}{\alpha^{3}\beta^{3}}f(m) = \frac{\alpha^{2}}{\alpha^{3}\beta^{3}}\left[k(pm^{4} + qm^{3} + rm^{2} - sm + t) - 2m + N\right],$$
 [14]

where  $\alpha > 0$  and  $\beta > 0$  are defined as in Table 1 and

$$p = -(4k+1) < 0,$$

$$q = 12k^2 + 14(N+1)k + 3N + 4,$$

$$r = 3(8k^3 - 4(2N-3)k^2 - 2(3N^2 + 5N - 3)k - N^2 - 3N + 1),$$

$$s = 64k^4 + 16(3N+10)k^3 - 4(3N^2 - 18N - 40)k^2 - 2(5N^3 + 9N^2 - 18N - 40)k$$

$$-(N^3 + 6N^2 - 6N - 20),$$

$$t = N(32k^4 + 8(3N+10)k^3 + 4(9N+20)k^2 - 2(N^3 + N^2 - 9N - 20)k$$

$$+ (-N^2 + 3N + 10)).$$

Substituting m = 0 into eq. [14], we obtain

$$\frac{\partial W^*}{\partial m}\Big|_{m=0} = \frac{Na^2A}{(2k+N+1)^3(4k^2+2k(2+N)+1)^2},$$
 [15]

where A is given by

$$A = (8k^3 + 12k^2 + 6k + 1) + (2k^2 + k)N - kN^2$$
.

Because the denominator is positive, the RHS of eq. [15] is strictly negative (positive) if and only if A < 0 (A > 0). Because k > 0, A is quadratic function with respect to N and the coefficient of  $N^2$  is negative. Note that when N=1, A>0. Hence, we have A < 0 (A > 0) when N is greater (less) than some number that is greater than 1. Q.E.D.

#### **Proof of Proposition 4:**

There exists a solution of eq. [13] because  $W^*(m)$  is continuous and [0, N] is compact. Furthermore, any solution is in (0,N) because  $\partial W^*(N)/\partial m < 0$  by Proposition 2(ii) and  $W^*(0) = W^*(N)$ .

Because any solution is in (0, N), the first-order condition for eq. [13] is  $\partial W^*(m)/\partial m = 0$ . By using the function f(m) defined in eq. [14], this condition is equivalently expressed as f(m) = 0. Because  $\partial^2 W^*(m)/\partial m^2 = a^2 f'(m)/\alpha^3 \beta^3$  when f(m) = 0, both the first-order and the second-order conditions are met if and only if f(m) = 0 and  $f'(m) \le 0$ .

When k = 0, f(m) is reduced to -2m + N, which is zero only if m = N/2. Because any solution of eq. [13] is in (0, N), it satisfies the first-order condition, that is, it is equal to N/2. Thus, the existence of a solution implies that there uniquely exists a solution  $m^* = N/2$ .

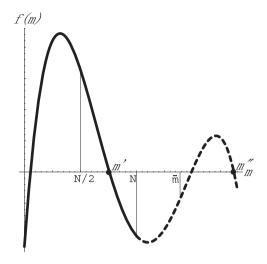
Suppose that k > 0. Then, f(N) < 0 by Proposition 2(ii) and

$$f\left(\frac{N}{2}\right) = \frac{kN^2}{16} \left(96k^3 + 24k^2(6+N) + 4k(18+5N) + N^2 + 4N + 12\right) > 0.$$

Therefore, we have at least one value of m in (N/2, N) that satisfies both the first- and the second-order conditions. From the shape of f(m), as shown in the following paragraph, the values of m that satisfy both the first- and the secondorder conditions are never multiple in (0, N). Thus, there uniquely exists  $m' \in (0, N)$  that satisfies both the first- and the second-order conditions, and  $m' \in (N/2, N)$ . Because any solution of eq. [13] is in (0, N), it satisfies the firstand the second-order conditions, that is, it is equal to m'. Thus, the existence of a solution implies that there uniquely exists a solution  $m^* = m' > N/2$ .

Because f(m) is quadruplicate with respect to m (typically an inverse Wshaped graph), the number of values of m satisfying both the first- and the second-order conditions is at most two (see Figure 2). We have already shown that one solution lies on (N/2, N). We then show that the other lies on  $(N, \infty)$  if it exists. We solve the equation f''(m) = 0 and obtain

$$m = \frac{12k^2 + 14k(N+1) + 3N + 4 \pm (2k+1)\sqrt{D+N^2}}{16k+4},$$



**Figure 2:** Graph of f(m). When m = m'and m = m'', the first-order condition f(m) = 0 and the second-order condition  $f'(m) \le 0$  are simultaneously satisfied.

where  $D = 100k^2 + 4k(5N + 24) + 24 > 0$ . We denote the larger solution by  $\bar{m}$ . Then, we obtain  $\bar{m} > N$  by

$$\bar{m} - N = \frac{12k^2 + 14k + 4 + (2k+1)\left(\sqrt{D+N^2} - \sqrt{N^2}\right)}{16k+4} > 0.$$

Because the other value of m satisfying both the first- and the second-order conditions is greater than or equal to  $\bar{m}$ , it lies on  $(N, \infty)$ . Q.E.D.

#### **Proof of Proposition 5:**

(i) The results in Table 1 and eq. [12] yield the following equation:

$$W^{\star}(N-1) - W^{\star}(N) = \frac{a^2(N-1)B}{8(k+1)^2(2k+N+1)^2(4k^2+2k(2+N)+N)^2},$$

where *B* is given by  $B = (32k^4 + 72k^3 + 52k^2 + 14k + 1) + (16k^3 + 28k^2 + 14k + 3)N > 0$ . (ii) The results in Table 1 and eq. [12] yield the following equation:

$$W^{*}(1) - W^{*}(0) = \frac{a^{2}(N-1)B}{8(2k+N)^{2}(2k+N+1)^{2}(2k^{2}+k(2+N)+1)^{2}},$$
 [16]

where *B* is given by

$$B = (2k+1)^3(4k+1) + (4k(6k^2+9k+4)+3)N + 4kN^2 - 2kN^3.$$

Because the denominator is positive and because  $N \ge 2$ , the RHS of eq. [16] is strictly negative (positive) if and only if B < 0 (B > 0). Because k > 0, B is a cubic function with respect to N, and the coefficient of  $N^3$  is negative. Here, B > 0 when N = 2 and  $\partial B/\partial N > 0$  when N = 0. Hence, we have B < 0 (B > 0) when N is greater (less) than some number that is greater than 2. Q.E.D.

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