DE GRUYTER BE J. Theor. Econ. 2016; 16(1): 303–336

Research Article

Hyeon Park*

Loss Aversion and Consumption Plans with Stochastic Reference Points

DOI 10.1515/bejte-2014-0100
Published online December 5, 2015

Abstract: This paper studies the making of risky choices following loss aversion with endogenous reference expectations under the two schemes of state-independent and state-dependent stochastic reference points. Using a tractable, intertemporal choice model, this paper derives analytic solutions to show that, when loss aversion is high, the reference-dependent decision maker saves a markedly larger amount than is predicted by the standard model. When the loss aversion is low (i.e. the individual is loss-tolerant), the overall result is ambiguous, although the decision maker may deviate into consuming more; if he faces a small level of uncertainty relative to the intensity of his loss aversion, he may even do this by borrowing. Given the same loss aversion level, this study determines that, in the presence of positive state-dependence, the state-independent model generates greater deviation than the state-dependent one. Finally, this paper derives a two-period general equilibrium result with two agents who have different attitudes toward loss.

Keywords: loss aversion, reference-dependent preference, stochastic reference point, precautionary saving
JEL Classification: D81, D91, C60, E21

1 Introduction

Individuals' attitudes toward loss affect their rational choices in the presence of uncertainty. As argued by Köszegi and Rabin (2006), individuals may not evaluate utilities in absolute level of outcomes, but instead in gains or losses of the outcomes relative to their reference expectations. The reference-dependent utility, a concept that was originally introduced by Kahneman and Tversky

***Corresponding author: Hyeon Park,** Department of Economics and Finance, Manhattan College, 4513 Manhattan College Pkwy, Riverdale, NY 10471, USA, E-mail: hyeon.park@manhattan.edu

(1979), has been widely confirmed in lab experiments.[1,2] The key features of their theory are reference dependence and loss aversion.

This paper, following the expectations approach by Köszegi and Rabin (2006, 2007, 2009), studies a stochastic intertemporal choice model for individuals who exhibit a reference-dependent preference and gain-loss utility. Unlike Köszegi and Rabin, however, we here posit that the attitudes of decision makers toward loss are not limited to loss aversion, but extend to the cases in which decision makers are tolerant of loss,[3] thereby illuminating the existence of decision makers with loss tolerance. This is important because a reference-dependent general equilibrium model of consumption and savings may not have an equilibrium interest rate without the existence of these consumers. In this paper, we analyze the behavior of both types of agents under both the state-independent[4] and state-dependent stochastic reference points, the latter of which are considered more empirically plausible in many applications including the field of finance according to Giorgi and Posty (2011). We specifically compare the results of this analysis with the results from a standard model under the condition of uncertainty. The general equilibrium result of the model shows that, at a given uncertainty level, both the borrowings and savings are smaller in the state-dependent model than in the state-independent one for all levels of interest rates. Moreover, the results of this study support the idea that, given a symmetric value of loss aversion/tolerance intensity, the state-dependent model agrees more with an even distribution of borrowers and savers in an economy for a wide range of income uncertainty.

It is well known that the standard lifecycle model alone cannot explain some puzzling features in consumption data, such as lifecycle consumption (which is *hump-shaped*) and the aggregate consumption growth (which is both *excessively smooth* and *excessively sensitive*).[5] Many researchers argue that a behavioral approach into consumption-saving models can solve these puzzles by introducing

---

1 The original work is *prospect theory* by Kahneman and Tversky (1979), who develop the theory further in Tversky and Kahneman (1992). The main features of their theory are (1) reference dependence, (2) gain-loss utility with more weights put on loss (*loss aversion*), and (3) diminishing sensitivity.

2 This theory can help us understand why standard economic theory cannot explain such findings as the failure of the independence axiom in expected utility.

3 We might think of the usual loss aversion as *positive* loss aversion and of loss tolerance as *negative* loss aversion.

4 As will be specified later, the models used by Köszegi and Rabin are based on state-independent reference points, which are said to have a more limited application than state-dependent ones.

5 Specifically, it is excessively smooth relative to current income growth and excessively sensitive regarding lagged income growth. See Ludvigson and Michaelides (2001).

either time-inconsistent tastes or time-inconsistent expectations regarding future income.[6] In fact, the expectations-based reference-dependent preferences are thought to combine both of these elements, one through gain-loss utility (preferences) and the other through the reference points (expectations).[7] It is important, here, to ask what the reference point would be in the stochastic intertemporal choice model of reference expectation. Unlike in a deterministic intertemporal model of reference dependence wherein the reference points are the solutions to the dynamic optimization over time for a forward-looking agent, with uncertainty, it may be necessary to define the reference point *over different states of the world* for a state-contingent decision maker.[8]

This paper examines the making of intertemporal choices following loss aversion with stochastic reference expectations. We specifically want to study the consumption-saving *behavior* of decision makers who evaluate the utility in gains or losses of the outcomes relative to their reference expectations. The model that we analyze here is different from the majority of models of reference-dependent preferences. Unlike Chetty and Szeidl (2010), we develop this model with forward-looking expectations. Unlike Bowman, Minehart, and Rabin (1999), we construct its gain-loss utility in terms of reference-dependent expected utility instead of consumption-based habit formation. Also, avoiding the utility specification of exponential or HARA class that is used by Pagel (2013), we derive closed-form solutions[9] from CRRA.[10] In a method that differs from that used by many researchers, including Köszegi and Rabin (2009), we develop the consumption

6 For example: *hyperbolic discounting* models are based on time-inconsistent tastes, while *short-term planning* models are based on time-inconsistent expectations regarding future income. For a detailed explanation of this, see Park (2015).

7 In standard rational-expectations lifecycle-models, consumption at a point in time is expressed as a fraction of the future income expectation at the time. With rational-expectation-based reference points formed on future income, a dynamic (time-varying) reference-dependent model can display time-inconsistent income expectations, which help induce a hump-shaped consumption profile. Also, the loss aversion parameter in the reference-dependent preferences plays a role similar to the one that the extra time-discounting parameter plays (but in the opposite direction) in hyperbolic discounting models and thus can yield time-inconsistent tastes.

8 Although it is possible to define the reference point over both state and time, the main focus of this paper (i.e. how uncertain outcomes are affected by loss aversion) makes this unnecessary here. Regarding elicitation of risk and time preferences, see Andersen et al. (2008).

9 The closed-form solutions are obtained with a two-period model.

10 Precautionary savings have been well studied in the CRRA class because marginal utility is convex (a positive third derivative). An increase in uncertainty raises the expected value of the marginal utility, which raises consumers' motivation to save. Both CARA and CRRA exhibit this property but in the case of the exponential utility, it is known that uncertainty may lead to negative initial consumption.

and saving model in the world of state-dependency. Moreover, we introduce a novel situation in which decision makers are allowed to have *loss tolerance*.

Our choice of expectation for the stochastic reference point follows from the recent developments in reference-dependent utilities models. One of these developments, studied by Cillo and Delquie (2006) and used by Köszegi and Rabin (2006, 2007, 2009), involves state-independent reference expectations. The other one involves state-dependent reference expectations, analyzed by Sugden (2003) and Giorgi and Posty (2011). The former posits that a decision maker evaluates every possible outcome of a prospect relative to all possible states of the reference point, while the latter assumes that the decision maker evaluates possible outcomes only in relation to the same state. Therefore, the decision maker should experience feelings of loss if the outcome of the prospect in a state falls short of any one of the reference outcomes in the other states in the state-independent world; in the state-dependent world, on the other hand, losses are only experienced when they occur within the same state.

Using these two reference plans, we derive closed-form solutions for the intertemporal choice of consumption and precautionary saving for various decision makers with different attitudes toward losses. A reference-dependent decision maker derives utility from making comparisons to the reference status, which may be a gain or loss to a reference point. A loss is assumed to be more important to a decision maker than a gain of the same size. Also, the intensity of loss aversion generates deviations in behavior among decision makers from that posited by the standard model of consumption-saving under uncertainty. Under the state-independent reference plan, these deviations come from a feeling of loss relative to the expected outcome of other states; under the state-dependent reference plan, they come from a feeling of loss relative to the expected outcome of the same state. However, in both plans, this study finds that decision makers with low loss aversion might want to overconsume, even via borrowing or dissaving. Unlike those who have high loss aversion, these individuals have relatively low loss feeling for a future outcome involving a bad state such as low income. Since the loss is not too painful than the gain is pleasant, their overall utilities would not be lowered by much even when a bad outcome is realized. Because they have little incentive to compensate for their feelings of loss by saving, they might be tempted to overconsume.

In the model we assume that a decision maker's utility has two preference components.[11] One is the usual consumption utility (absolute level), and the

11 This specification is found in the works of Köszegi and Rabin. Classical prospect theory posits only comparison utility (gain-loss). Krähmer and Stone (2013) also propose a model with two components: the comparison and the intrinsic characteristics of utility.

other is the reference-dependent utility (contrast level); these two payoffs occurring over time interact with each other through intertemporal optimization in deterministic models. In the two-period consumption-saving model, however, because it is assumed that uncertainty arises only at the second period, the gain-loss utility relative to the reference state occurs at the second period. Our analysis shows that, when loss aversion is *low* relative to the standard model, it is possible for the decision maker to deviate from the usual precautionary saving behavior and engage in more consumption during the first period. When the loss aversion is *high* relative to the standard, however, the decision maker saves more than the standard model predicts. This results in a lesser amount of consumption in the first period,[12] but with a different magnitude according to which of the two state-related reference schemes is being used. Given the same intensity of loss aversion, this study finds that, when there is positive state-dependence, the state-independent model generates greater deviation than the state-dependent model from the standard behaviors involving both consumption and precautionary saving.[13] Finally, this study derives a two-period general equilibrium result of precautionary saving with two agents who have different attitudes toward loss.

2 Related Literature

By exploring forward-looking reference expectations and introducing a novel idea of loss tolerance, this paper contributes to the literature on consumption-savings model with reference dependence. The loss aversion in individuals' consumption-saving behavior is well studied by Bowman, Minehart, and Rabin (1999) who develop a two-period consumption-savings model with loss aversion and show that when they are under sufficient income uncertainty, consumers may fail to reduce their consumption in response to adverse income shocks. This delayed response to shocks is due to loss aversion and reflection effects related to income uncertainty. Reducing their current consumption will lower consumers' consumption below the reference point, which makes them feel a loss. The reflection effect implies that a risk-loving attitude toward loss makes consumers choose a lottery over a certain prospect. Thus consumers who would rather take a gamble will hesitate before reducing their consumption. However, the reference dependence as formulated in their model is not strictly different from the

12 This implies that, given the same uncertainty parameters, a decision maker with high loss aversion saves more than one with low loss aversion.

13 Therefore, the opposite is true for negative state-dependence.

consumption-based habit, which might also be called reference dependence of consumption,[14] in that the reference point for the gain-loss utility is assumed to depend on *past consumption*.[15]

Chetty and Szeidl (2010) propose a model of forward-looking endogenous reference points, advancing an argument apparently similar to that of Köszegi and Rabin (2009) regarding *recent expectations*. Their model incorporates adjustment costs as the key mechanism for generating reference-dependent preferences whose reference points are consumption commitments. However, unlike the forward-looking mechanism used by Köszegi and Rabin (2009), their model is in fact backward-looking, in that the recent expectations are reflected in recent consumption choices and the current reference point is related to consumption in the recent past.

Pagel (2013) adapts the expectation-based, reference-dependent preferences used by Köszegi and Rabin (2009) to develop a lifecycle model based on news utility. Like that of Bowman et al. (1999), Pagel's model demonstrates a delayed response to income shocks. This is because the lifecycle utility at a given point in the lifetime depends on layers of consumption expectations via the forward-looking formation of reference points through expectations about future income. This specification is closely related to the subject of the current paper, in that it posits that expectation comes from a forward-looking mechanism. However, Pagel's model has two potential drawbacks: first, it assumes reference points that are state-independent. Second, its closed form solution is obtained only using unrealistic exponential utility functions.

Siegmann (2002) also uses reference-dependent preferences to construct a two-period consumption-savings model. The utility specification of his reference-dependence model is a one-sided value function (for losses) similar to the one used by Aizenman (1998), who also shows a positive relation between precautionary saving and uncertainty via disappointment aversion. Siegmann emphasizes the role of precautionary savings for a loss-averse agent by

14 This term may refer to the dependence that arises from differences in consumption as is seen in habit formation. More recent developments in reference dependence have been formulated on the basis of utility difference (the reference dependence of utility). These include the works of Koszegi and Rabin, of Pagel (2013), and this paper. Some lifecycle researchers argue that habit formation is not capable of resolving a number of puzzling features of consumption data: the preference itself cannot produce a consumption hump in lifecycle models, although it may solve the problem of excess smoothness (but only under the condition of very high wealth accumulation).

15 In their model, overall utility depends on both reference utility and gain–loss utility. The reference point is a linear combination of the last period's reference point and the last period's consumption.

demonstrating that an increase in the uncertainty of future wealth has a non-decreasing effect on savings regardless of the expected return. His emphasis on the relationship between uncertainty and precautionary saving related to loss aversion is one element that the topic of the current paper has in common, although his approach does not provide a clear decomposition of loss aversion and risk aversion. He focuses specifically on the distribution of the return on savings and finds non-linearity in savings out of wealth.[16] Related with these, the classical literature on precautionary savings is: the importance of precautionary savings relative to the borrowing frictions of the model (Feigenbaum 2009); income uncertainty and precautionary savings (Carroll 1997; Gourinchas and Parker 2002; Feigenbaum 2008).

Two papers make cases for alternative positions regarding the model we are studying of the formulation of reference points. Sugden (2003),[17] who generalizes the subjective expected utility theory into a reference-dependent model, notices that an agent's reference point may depend on the state of the world, but he adopts the status quo practice for the reference points. An agent's reference point can simply be interpreted as the agent's current endowments, letting the reference points be *state-dependent*. This is one way of modeling an agent's reference point so that it is a function of the agent's expectations. Another way to construct a reference point out of expectations may be the *equilibrium* approach suggested by Köszegi and Rabin (2006, 2007). On this view, the reference point is the optimal expectation, the one that maximizes an agent's utility given any expectation generated by a strategy. Furthermore, these authors endogenize the reference point by allowing agents' beliefs to follow rational expectations.[18] In the case of stochastic reference points, they assume that an agent's beliefs fully reflect the true distribution of outcomes[19] and that gain-loss utility comes from comparisons made between the consumption lottery and a reference lottery, which are assumed to be independent. Köszegi and Rabin (2009) extend their model to a dynamic case that they construct by specifying that the agent should meet the rational consistency condition: any expectation

16 Siegmann (2002) finds that the sign of the real return on savings determines the specific saving pattern.

17 Sugden (2003)'s model is called reference-dependent subjective expected utility and is based on regret theory. He proposes a simple form of the theory, according to which subjective expected utility is represented by a tractable functional form. In this way he is able to explain some systematic features of observed behavior that are not explained by the conventional theory.

18 In addition, they use consumption utility as an added component to the total utility, as is also specified in the model we study.

19 See Köszegi and Rabin (2007).

generated by a dynamic strategy will maximize the agent’s utility in each period as long as the continuation strategies are consistent with rationality.

Other readings on the conceptual development about loss aversion include Rabin (2000), who shows that reference dependence may be an important factor in an agent's attitudes toward risk; many people have argued that aversion to small gambles is due in considerable part to loss aversion. Köbberling and Wakker (2005) demonstrate this by decomposing the risk attitude into three components: utility, probability weighting, and loss aversion. Krähmer and Stone (2013) discuss anticipated regret as an explanation of uncertainty aversion. They posit that the reference point depends on the agent's ex post beliefs about what he should have done ex ante (if he had the wisdom of hindsight), and thus in their model the loss aversion arises endogenously. Choosing a compound lottery for an uncertain option reveals information, and this changes the agent's ex post evaluation of the best decision to have made. They are in fact searching for the psychological foundations of uncertainty aversion. The work of Halevy and Felkamp (2005) takes a similar line.

Regarding the applicability of reference-dependence to economic models, Freeman (2013) claims that standard economic data can be consistent with the testable implications of the expectations-based reference-dependence models.[20] Because expectations do not appear in the data, reference points obtained from rational expectations may not properly reveal preferences. Freeman provides axiomatic foundations characterizing the testable implications of those models. A testable application of reference dependence in financial market can be found in Pasquariello (2014), who studies loss aversion and risk seeking in losses in terms of market quality.

3 Reference-Dependent Preferences

Following Köszegi and Rabin (2006), a riskless reference-dependent utility is defined by

$$U(c|r) \equiv u(c) + \mu(c|r) \qquad [1]$$

where $u(c)$ is classical consumption utility which is increasing (concave or quasi-concave) and differentiable. The next term $\mu(c|r)$ is gain-loss utilities additively separable across $k-$dimensions, related to deviation from a reference point r. A gain-loss utility is specified by

20 This opposes the idea that these models lack meaningful implications regarding revealed preferences.

$$\mu_k(c_k|r_k) \equiv \sum_k \mu_k(u_k(c_k) - u_k(r_k)) \qquad [2]$$

where k is the dimension of the state space. For all k, the gain-loss utility $\mu_k(x|r)$ satisfies the following:[21]

(A0) Continuous, differentiable except at $x = 0$: $\mu_k(0) = 0$.
(A1) Strictly increasing.
(A2) If $y > z > 0$, then $\mu_k(y) + \mu_k(-y) < \mu_k(z) + \mu_k(-z)$.
(A3) $\mu_k''(x) < 0$ for $x > 0$ and $\mu_k''(x) > 0$ for $x < 0$.
(A4) $lim_{x \to 0+} \frac{\mu'(-x)}{\mu'(x)} \equiv \lambda_k > 1$.

In a *state-independent* stochastic world, its outcome may be evaluated according to expected utility as suggested by the Von Neumann-Morgenstern preferences. This is one of the points at which the Köszegi and Rabin model is different from *prospect theory* by Kahneman and Tversky (1979). In prospect theory, a subjective weight function is used to evaluate a lottery. Reference-dependent expected utility implies a weighted average of the utilities of each of its possible outcomes relative to each possible realization of a stochastic reference point. Therefore, given a stochastic reference point $r \in Y$ following a distribution G, the utility of a stochastic outcome $c \in X$, which follows F is

$$EU(c|r) = \int\int U(x|y)dG(y)dF(x) \qquad [3]$$

Assume further that there is only one dimensional state space for $U(c|r)$ so that $k = 1$. Then the reference-dependent utility of the decision maker who has a functional form of CRRA for the consumption utility $u(c)$ is

$$U(c|r) = \frac{c^{1-\gamma}}{1-\gamma} + \eta\mu(c|r) \qquad [4]$$

where η is the weight of gain-loss utility relative to consumption utility. It is noticeable that if η is very small, then the gain-loss effect is negligible.[22] The gain-loss utility related to deviation in outcome from a reference point is defined by[23]

21 The A0-A4 properties are from Bowman, Minehart, and Rabin (1999), corresponding to the original idea by Kahneman and Tversky (1979).
22 Many researchers simply set $\eta = 1$.
23 Following Kőszegi and Rabin (2009) we posit a *linear* gain-loss function to get closed form solutions. Other candidates for the gain-loss function may be power functions and exponential functions.

$$\mu(c|r) = \begin{cases} u(c) - u(r) & \text{if } u(c) - u(r) \geq 0 \\ \lambda[u(c) - u(r)] & \text{if } u(c) - u(r) < 0 \end{cases} \qquad [5]$$

in which $\lambda > 1$ is the coefficient of loss aversion. The total utility is specified as follows, assuming the decision maker has a reference point $u(r) \equiv u(c^{\star}) = \frac{c^{\star 1-\gamma}}{1-\gamma}$,

$$U(c|c^{\star}) = \begin{cases} \dfrac{c^{1-\gamma}}{1-\gamma} + \eta\left(\dfrac{c^{1-\gamma}}{1-\gamma} - \dfrac{c^{\star 1-\gamma}}{1-\gamma}\right) & \text{if } \dfrac{c^{1-\gamma}}{1-\gamma} - \dfrac{c^{\star 1-\gamma}}{1-\gamma} \geq 0 \\ \dfrac{c^{1-\gamma}}{1-\gamma} + \eta\lambda\left(\dfrac{c^{1-\gamma}}{1-\gamma} - \dfrac{c^{\star 1-\gamma}}{1-\gamma}\right) & \text{if } \dfrac{c^{1-\gamma}}{1-\gamma} - \dfrac{c^{\star 1-\gamma}}{1-\gamma} < 0 \end{cases} \qquad [6]$$

With intertemporal decision making, it is necessary to modify the gain-loss utility to incorporate the decision maker's *psychological weight* on the gain-loss utilities over time.[24] Also, it is natural to posit that the weighting is likely to decay as its effect fades. To simplify the model for this, we assume that the initial strength of the concern for loss relative to gain is $\omega > 0$ and that the decay follows standard time-discounting.[25]

4 Stochastic Reference Points

In this section we study a parametric model of reference-dependent utility with loss aversion. Specifically, we examine the consumption and precautionary saving plan of a reference-dependent decision maker who lives for two periods but faces uncertainty in the second period. Regarding the reference point, we first consider the case in which a decision maker evaluates every possible outcome of a prospect with all possible outcomes of the reference points, which effectively makes the reference point state-independent. In Section 4.2, we consider the case in which a decision maker evaluates all the possible outcomes of a prospect only in the same state. In this state-dependent situation, the decision maker experiences a loss only if the outcome of a prospect falls short of the reference point in the same state.

24 Köszegi and Rabin (2009) propose, for $\tau = t, \ldots, T$, $u_t = m(c_t) + \sum_{\tau=t}^{T} \varphi_{t,\tau} n(F_{t,\tau}|F_{t-1,\tau})$, in which $F_{t-1,\tau}$ represents fixed beliefs, inherited from last period and $F_{t,\tau}$ represents new beliefs that decision maker forms. $\varphi_{\tau,\tau} \geq \varphi_{\tau-1,\tau} \geq \ldots \geq \varphi_{0,\tau} \geq 0$ are weights on gain-loss utilities.

25 This is because the psychological weight may or may not follow the usual time-discounting denoted by β henceforth. When $\omega = 0$, there is no gain-loss utility and the model returns to the standard one.

In Section 5, we derive a closed form solution in a simplified version of the model for both of the specifications. Based on the solution, in Section 6, we derive a two-period general equilibrium result with two agents who differ from each other in their attitudes toward loss. In partial equilibrium, where the interest rate is fixed, the policy variables such as consumption and bond demand are derived given any parameter values. However, in general equilibrium, the bond demand is a function of the equilibrium interest rate and the net bond supply should be equal to zero in equilibrium: $b_1 + b_2 = 0$ is the market clearing condition for the economy of two agents who have bond holdings, b_1 and b_2, respectively. The candidates for the two different agents of general equilibrium are many.[26] Because our focus in the model is on the strength of concern about loss, the first step may be to build a general equilibrium model with two different types of decision makers with respect to their degrees of loss aversion.

4.1 State-Independent Reference Points

To introduce formal models of stochastic reference points, let us define several notations first. Let $S = \{1, 2, \ldots, s\}$ be the state-space with finitely many elements. Let χ be the collection of feasible prospects $X : S \to \mathbb{R}$ and $\mathbb{P}[E]$ for the probability that an event $E \subset \Omega$ occurs. Following the state-independent expectations formed in Köszegi and Rabin (2006, 2007, 2009), let us define a state-independent expected utility of a prospect $X \in \chi$ given a reference point $Y \in \chi$ as follows:

$$EU = \int_{-\infty}^{\infty} u(x) dF_X(x) \tag{7}$$

$$+ \int_{-\infty}^{\infty} \eta \left(\int_{-\infty}^{x} (u(x) - u(y)) dF_Y(y) + \omega\lambda \int_{x}^{\infty} (u(x) - u(y)) dF_Y(y) \right) dF_X(x)$$

where $F_X(x) = \mathbb{P}[X \le x]$ and $F_Y(y) = \mathbb{P}[Y \le y]$ are cumulative distribution functions of X and Y, respectively. The second term shows both gain utility $u(x) - u(y) > 0$ and loss utility $u(x) - u(y) < 0$. For a discrete case:

$$EU = \sum_{s \in S} p(s) u(X(s)) \tag{8}$$

$$+ \eta \sum_{s \in S} p(s) \left(\sum_{s' \in S}^{X > Y} p(s')[u(X(s)) - u(Y(s'))] + \omega\lambda \sum_{s' \in S}^{X < Y} p(s')[u(X(s)) - u(Y(s'))] \right)$$

26 Agents with i) two different *time preference* (β), ii) two different *risk attitudes* (γ), iii) two different *weights for loss* (λ), iv) two different *wealth or income positions* (y).

This specification of state-independence implies that every outcome of a prospect may act as a reference point. By this definition, a decision maker should have a feeling of loss whenever the outcome of a prospect in *any state* falls short of the one of the reference point in *any other states*. Let us assume that the decision maker (DM hereafter) lives for two periods $t = \{0, 1\}$ and is subject to face uncertainty at the second period. Thus the DM receives income $\tilde{y}_1$ following a known probability distribution of $F(y_1)$ for the second period, but receives certain income y_0 at $t = 0$. Having the consumption utility from different states of the world as his reference points, the DM forms an expectation about the optimal consumption with the probability distribution F. Then his utility is described by the reference-dependent expected utility formalized above. Let the maximization problem for the DM who solves for the optimal consumption and saving for the future be

$$Max\ u(c_0) + \beta E\left[u(\widetilde{c}_1) | u(c_1^r)\right] \qquad [9]$$

subject to

$$c_0 + b_1 = y_0$$

$$c_1 \quad = \tilde{y}_1 + Rb_1$$

where b_1 is bond holdings (saving or borrowing) for the next period. Likewise, c_0 and c_1 represent current consumption and future consumption, respectively. Also, assume that there is no borrowing constraint and the agent can borrow or lend freely at the market gross interest rate R. If there are only two states of the world with respect to second period income $(H$ or $L)$ with probability p and $1 - p$, then the DM expects either high consumption or low consumption with the probability and these two consumption outcomes serve as the reference points: $u(c_{1h})$ and $u(c_{1l})$. The expected (contemporaneous) gain-loss utility relative to these reference points arises in the second period. When the expected consumption utility is high, the DM has a feeling of gain relative to the low possible consumption utility that he might have, but zero feeling of gain relative to the high consumption utility since the expectation is met. Likewise, if the expected consumption utility is low, the DM has a feeling of loss relative to the high possible consumption utility, but zero relative to the low one. Also, the gain-loss utility in the second period comes along with psychological weighting ω, as described in the previous section. Finally, there is no (prospective) gain-loss utility in the first period because there is no uncertainty and the DM learns no new information. Thus, the total utility is given by

$$EU(c|p) = \quad [10]$$

$$\frac{c_0^{1-\gamma}}{1-\gamma} + \beta\left(p\frac{c_{1h}^{1-\gamma}}{1-\gamma} + (1-p)\frac{c_{1l}^{1-\gamma}}{1-\gamma}\right)$$

$$+\beta\eta p\left(p\left[\frac{c_{1h}^{1-\gamma}}{1-\gamma} - \frac{c_{1h}^{1-\gamma}}{1-\gamma}\right] + (1-p)\left[\frac{c_{1h}^{1-\gamma}}{1-\gamma} - \frac{c_{1l}^{1-\gamma}}{1-\gamma}\right]\right)$$

$$+\beta\eta(1-p)\left(\omega\lambda p\left[\frac{c_{1l}^{1-\gamma}}{1-\gamma} - \frac{c_{1h}^{1-\gamma}}{1-\gamma}\right] + (1-p)\left[\frac{c_{1l}^{1-\gamma}}{1-\gamma} - \frac{c_{1l}^{1-\gamma}}{1-\gamma}\right]\right)$$

Notice that only two terms survive from the four expected contemporaneous gain-loss utility: there will be one gain utility and one loss utility at $t=1$, while at $t=0$, there is no gain-loss utility. Solving for these values according to the model returns the consistent consumption and bond holding for the DM in partial equilibrium given an interest rate and an income distribution.

4.2 State-Dependent Reference Points

In this subsection, we examine the case in which the reference points are state-dependent and may be chosen endogenously due to this dependency. Let us assume the same state space as in Section 4.1 and consider the following state-dependent model of reference-dependent preferences (RDP hereafter). For jointly continuous random variables, a risky prospect $X \subseteq \chi$ and a reference point $Y \subseteq \chi$, the state-dependent RDP valuation of X relative to Y is given by

$$EU^{\star} = \int_{-\infty}^{\infty} u(x) f_X(x)dx \quad [11]$$

$$+\eta\int_{-\infty}^{\infty}\int_{-\infty}^{x}(u(x)-u(y))f_{X,Y}(x,y)dydx$$

$$+\eta\omega\lambda\int_{-\infty}^{\infty}\int_{x}^{\infty}(u(x)-u(y))f_{X,Y}(x,y)dydx$$

where $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$ is the joint density from the joint c.d.f $F_{X,Y}(x,y) = P[X \leq x; Y \leq y]$, and $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$ is the marginal density of X from the joint density. In case of discrete random variables, $X(s)$ and $Y(s')$, where they are completely specified by a joint probability mass function $p_{X,Y}(s,s')$, the expected utility is defined as follows:

$$EU^{\star} = \sum_{s \in S} p_X(s) u(X(s)) \\ + \eta \sum_{s \in S} \sum_{s' \in S}^{X > Y} p_{X,Y}(s, s')[u(X(s)) - u(Y(s'))] \\ + \eta \omega \lambda \sum_{s \in S} \sum_{s' \in S}^{X < Y} p_{X,Y}(s, s')[u(X(s)) - u(Y(s'))] \quad [12]$$

where $p_X(s) = \sum_{s' \in S} p_{X,Y}(s, s')$ is the marginal p.m.f. of X from the joint p.m.f. $p_{X,Y}(s, s')$. As seen above, with state-dependent RDP, the expected utility $EU^{\star}$ is obtained by their joint distribution, through which the state-dependency may be implied. Unlike the state-independence model, this specification implies that the decision maker experiences gain-loss utility not for each possible outcome relative to all, but those within the same state.[27] Because when any two variables are independent, the joint distribution is the product of the two individual distributions, it is implied that $EU^{\star} = EU$ if X and Y are independent.

In the example of the two-period consumption-saving model, in which the consumption and reference points are positively related, the total state-dependent RDP utility is given by

$$EU^{\star}(c|p) = \\ \frac{c_0^{1-\gamma}}{1-\gamma} + \beta\left(p\frac{c_{1h}^{1-\gamma}}{1-\gamma} + (1-p)\frac{c_{1l}^{1-\gamma}}{1-\gamma}\right) \\ + \beta\eta\left(p\left[\frac{c_{1h}^{1-\gamma}}{1-\gamma} - \frac{c_{1l}^{1-\gamma}}{1-\gamma}\right] + \omega\lambda(1-p)\left[\frac{c_{1l}^{1-\gamma}}{1-\gamma} - \frac{c_{1h}^{1-\gamma}}{1-\gamma}\right]\right) \quad [13]$$

Comparing with EU, it is clear that the gain utility arises with p instead of $p(1-p)$ and loss utility arises with $(1-p)$ instead of $(1-p)p$. This is because the $EU^{\star}$ is obtained from the joint distribution of consumption states and reference states, and the DM only considers the gain-loss prospect over outcomes in the same state.[28] However, just as in the state-independent case, only two terms survive from the four contemporaneous gain-loss utility components.

27 This implies $EU^{\star} = \sum_{s \in S} p(s)u(X(s)) + \eta \sum_{s \in S}^{X>Y} p(s)[u(X(s)) - u(Y(s))] + \eta\omega\lambda \sum_{s \in S}^{X<Y} p(s)[u(X(s)) - u(Y(s))]$.

28 In a state-dependent world the DM expects to feel a gain or a loss should his consumption be greater or less than the reference point, but only in the same state. For example, if the state of the world is High and Low with equal probability, then the gain utility occurs with $p = 1/2$ when the DM expects high consumption in the Low state, while his feeling of loss occurs when

5 An Analytic Solution

In this section, we analyze a simplified two-period model of stochastic reference points that can be solved analytically. We first present the state-independent RDP model, then a state-dependent one, and then we compare the two. The purpose of this step is to show the role of gain-loss utility in the intertemporal decision-making and disentangle the implication of partial equilibrium from the one of general equilibrium. Consider an economy populated by different types of DMs who receive the same stochastic incomes in their second periods but differ from one another in their attitudes toward loss. We specifically focus on the case in which a DM deviates from the standard model due to loss aversion that is higher $(\omega\lambda > 1)$ or lower $(\omega\lambda < 1)$ than the standard $(\omega\lambda = 1)$ following the reference-dependent expected utility.

First, we construct a simple model in which the DM faces uncertainty of his income in the second period. Assume that $y_0 = 1$ and $y_{1h} = 1 + \varepsilon$ and $y_{1l} = 1 - \varepsilon$ with probability $p = 1/2$. For simplicity, we assume $\gamma = 1$ in CRRA utility function and the gross interest rate is fixed at $1/\beta$ in partial equilibrium. Also, the DM has consumption utility for both periods: the consumption utility in the first period is deterministic, but in the second period it is an expected value. The DM has expected gain-loss utility in the second period and as described above, the reference point comes from the outcome of the other states of the world in the state-independent RDP. The situation for the DM who has gain-loss utility due to uncertain outcome is summarized by

$$Max\ \ln c_0 + \beta\left(\frac{1}{2}\ln c_{1h} + \frac{1}{2}\ln c_{1l}\right) \tag{14}$$

$$+\beta\eta\left\{\frac{1}{2}\frac{1}{2}(\ln c_{1h} - \ln c_{1l}) + \frac{1}{2}\frac{1}{2}\omega\lambda(\ln c_{1l} - \ln c_{1h})\right\}$$

subject to

$$c_0 + b_1 = y_0$$

$$c_1 \qquad = \tilde{y}_1 + Rb_1$$

he expects low consumption in the High state, which is $1 - p = 1/2$. This is different from the case of state-independence, in which the DM feels a loss even if the outcome of the prospect in a given state falls below the outcome of the reference point in other states: $p(1-p) = 1/4$ or $(1-p)p = 1/4$.

The second term is expected consumption utility and the third and fourth terms are expected contemporaneous gain and loss utilities for the second period. The optimality condition leads to

$$c_0^{-1} = R\beta\left\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1} + \eta(\omega\lambda - 1)\frac{1}{2}\frac{1}{2}\left(c_{1l}^{-1} - c_{1h}^{-1}\right)\right\}. \qquad [15]$$

Excepting the third term, this condition coincides with the standard Euler equation. The first two terms of *RHS* represent expected *consumption marginal utility*. For those who have high loss aversion ($\omega\lambda > 1$), the positive *gain-loss marginal utility* (the third term) causes imbalance in the *total marginal utility* of the equation. This leads to consumption adjustment. To equate the total marginal utilities over the two periods, the DM has to reduce his consumption in the first period.[29] In fact, the last term reflects the DM's intention to save more due to loss aversion. The DM may have a sense of gain when a high outcome is realized, but a feeling of loss when a low outcome is realized. Because the loss is more painful than the gain is pleasant, his overall utility is negative when the bad outcome occurs. Thus, he saves more to compensate for this feeling of loss in a bad situation. This explains how the precautionary saving is obtained in the reference-dependence model. However, we can also see that when the loss aversion is low enough to bring $\omega\lambda < 1$, the opposite could happen, –that is, a lower level of loss aversion can induce the DM to increase his consumption in the first period. As we see from the analytic solution in the next two subsections, the DM does deviate from the standard prediction and consume more (thus save less) if the uncertainty level is modest. Before solving this further, let us turn to the case in which the DM follows the standard model under uncertainty. In the next subsection, we look for the closed-form solution in the standard model.

5.1 Precautionary Saving in A Standard Model

Following Deaton (1991), let us define *cash on hand* (x) as the sum of current income and financial income: $x_t = y_t + Rb_t$. For the standard model, the two-period maximization problem above should be modified to

$$Max\ \ln c_0 + \beta\left(\frac{1}{2}\ln c_{1h} + \frac{1}{2}\ln c_{1l}\right) \qquad [16]$$

assuming the same budget constraint. To compare this with the above reference-dependence model, we posit the same assumption of $R = 1/\beta = 1$. Likewise, the

29 This is because the marginal utility is decreasing.

second period income is given by $y_{1h} = 1 + \varepsilon$ and $y_{1l} = 1 - \varepsilon$. The optimal consumption path that the DM follows in the standard world implies $c_0^{-1} = R\beta\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1}\}$. Because $b_1 = y_0 - c_0 = 1 - c_0$, the optimality condition leads to[30]

$$\frac{2}{c_0} = \frac{1}{2+\varepsilon-c_0} + \frac{1}{2-\varepsilon-c_0}. \qquad [17]$$

Then the solution to the maximization problem in the standard model is

$$c_0 = \frac{3}{2} - \frac{1}{2}\sqrt{1+2\varepsilon^2} < 1 \quad \text{if } \varepsilon > 0, \qquad [18]$$

which is decreasing as the uncertainty parameter ε increases. The bond demand is

$$b_1 = 1 - c_0 = -\frac{1}{2} + \frac{1}{2}\sqrt{1+2\varepsilon^2} > 0 \quad \text{if } \varepsilon > 0, \qquad [19]$$

which is increasing as the uncertainty parameter ε increases. How about the expected consumption in the second period, i.e. c_1? Since $c_1 = y_1 + Rb_1$,

$$c_1 = \frac{1}{2} \pm \varepsilon + \frac{1}{2}\sqrt{1+2\varepsilon^2}. \qquad [20]$$

Thus, it is clear that the last term in the above equation has a greater effect on the expected consumption under the condition of uncertainty ($\varepsilon > 0$) than under certainty ($\varepsilon = 0$). When there is uncertainty, the DM consumes less at the first period and save more for the consumption in the second period. This property of consumption under uncertainty is well known, and the saving is called *precautionary saving*. The precautionary saving may be defined better with cash on hand, which is $x = y_0 + Rb_0$. Then the consumption and bond demand over x (cash on hand) are

$$c_0 = \frac{3(x+1)/2}{2} - \frac{1}{2}\sqrt{(x^2+2x+1)/4+2\varepsilon^2} \qquad [21]$$

$$b_1 = \frac{x-3}{4} + \frac{1}{2}\sqrt{(x^2+2x+1)/4+2\varepsilon^2} \qquad [22]$$

$$c_1 = \frac{1+x}{4} \pm \varepsilon + \frac{1}{2}\sqrt{(x^2+2x+1)/4+2\varepsilon^2}. \qquad [23]$$

One may notice that both the direction and the magnitude of consumption and saving remain the same in terms of ε, while an increase in cash on hand yields a mixed result. But, in case of saving, it unambiguously increases in both ε and x. Also, it is apparent that

30 $c_{1h} = y_{1h} + Rb_1 = 2 + \varepsilon - c_0$ and $c_{1l} = y_{1l} + Rb_1 = 2 - \varepsilon - c_0$.

$$c_0(x;\varepsilon>0)<c_0(x;\varepsilon=0)=\frac{1+x}{2} \quad [24]$$

$$b_1(x;\varepsilon>0)>b_1(x;\varepsilon=0)=\frac{x-1}{2} \quad [25]$$

$$c_1(x;\varepsilon>0)>c_1(x;\varepsilon=0)=\frac{1+x}{2}. \quad [26]$$

Equation [25] implies that given a level of cash on hand, the bond demand is always larger with uncertainty than without it. Using this fact, the precautionary saving is defined by $P(x;\varepsilon)=b_1(x;\varepsilon)-b_1(x;\varepsilon=0)$. Thus,

$$P(x;\varepsilon)=\frac{-(x+1)}{4}+\frac{1}{2}\sqrt{(x^2+2x+1)/4+2\varepsilon^2} \quad [27]$$

The precautionary saving increases as income uncertainty increases. However the effect of cash on hand on the precautionary saving is mixed: from eq. [27], when x increases, it increases the second term but decreases the first term. For an overall result we need to examine its derivative *w.r.t.* x: we find $\frac{\partial P(x;\varepsilon)}{\partial x}=-\frac{1}{4}+\frac{x+1}{8\sqrt{(x^2+2x+1)/4+2\varepsilon^2}}<0$ whenever $\varepsilon>0$ because $x+1<\sqrt{(x+1)^2+8\varepsilon^2}$. Figure 1 shows the precautionary saving $P(x;\varepsilon)$ (for $x>0$) as a function of ε for various x. The figure describes that given ε, the precautionary saving motive will be lowered if the consumer has a high level of cash on hand. However, given a level of cash on hand, higher income uncertainty increases the precautionary

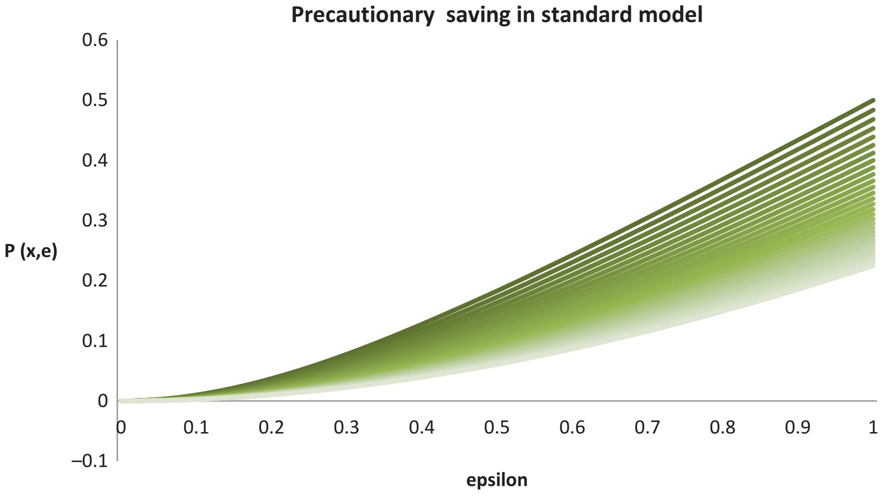

Figure 1: Precautionary Saving in standard model $(\omega\lambda=1)$: Darker (higher from below) lines represent the savings when the cash on hand (x) is lower.

saving motive. In the following subsection, we go back to the RDP-maximization problem and solve for these values in the RDP model.

5.2 RDP Precautionary Saving

In the reference-dependence model, the consumption and saving profiles are affected by the gain-loss parameters. The loss aversion arises here with respect to bad outcomes due to uncertainty. Thus, as explained by eq. [15], high loss aversion ($\omega\lambda > 1$) implies that people save more than in the standard uncertainty model ($\omega\lambda = 1$). Likewise, low loss aversion ($\omega\lambda < 1$) implies that people save less than in the standard model. The closed form state-independent RDP consumption and saving functions are

$$c_0^{RDP} = \frac{3(1+x)}{4} + \frac{\varepsilon\eta(\omega\lambda - 1)}{2} - \frac{1}{2}K(x, \varepsilon, \eta\lambda\omega) \qquad [28]$$

$$b_1^{RDP} = \frac{x-3}{4} - \frac{\varepsilon\eta(\omega\lambda - 1)}{2} + \frac{1}{2}K(x, \varepsilon, \eta\lambda\omega) \qquad [29]$$

$$c_1^{RDP} = \frac{1+x}{4} \mp \frac{\varepsilon\eta(\omega\lambda - 1)}{2} + \frac{1}{2}K(x, \varepsilon, \eta\lambda\omega) \qquad [30]$$

$$K(x, \varepsilon, \eta\omega\lambda) = \sqrt{\{3(1+x)/2 + \varepsilon\eta(\omega\lambda - 1)\}^2 - 2[1 + 2x + x^2 - \varepsilon^2]} \qquad [31]$$

The following (Figures 2, 3, 4, 5) show c_0 and b_1 as functions of x or ε for different values of ε or x. Each figure demonstrates either the consumption or

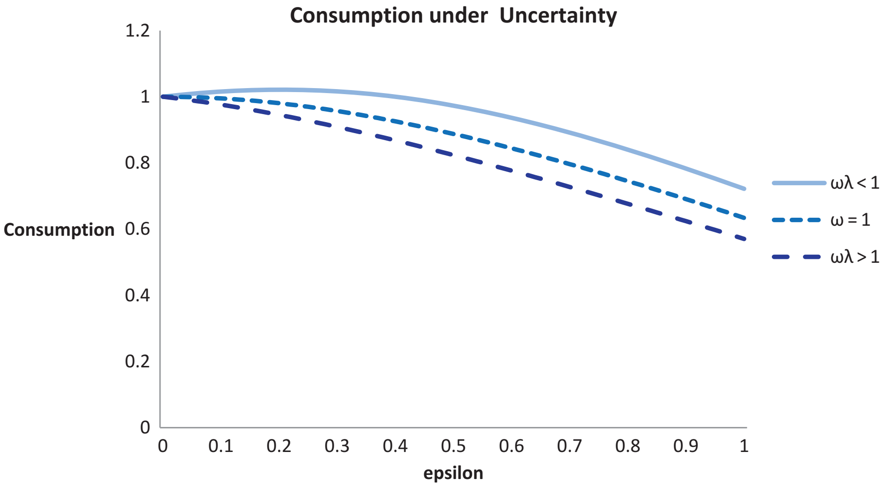

Figure 2: RDP plots of c_0 when $x = 1$. The loss aversion parameters are $\omega\lambda = 0.8$, $\omega\lambda = 1$, and $\omega\lambda = 1.2$.

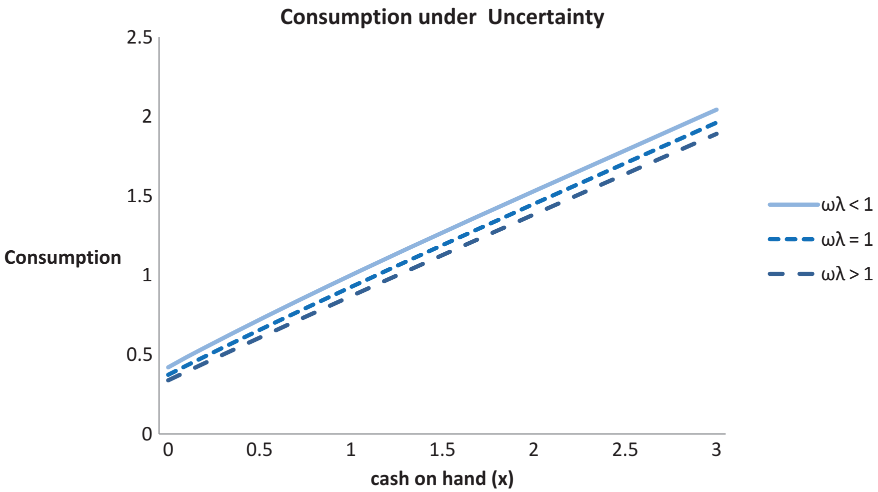

Figure 3: RDP plots of c_0 when $\varepsilon = 0.4$. The loss aversion parameters are $\omega\lambda = 0.8$, $\omega\lambda = 1$, and $\omega\lambda = 1.2$.

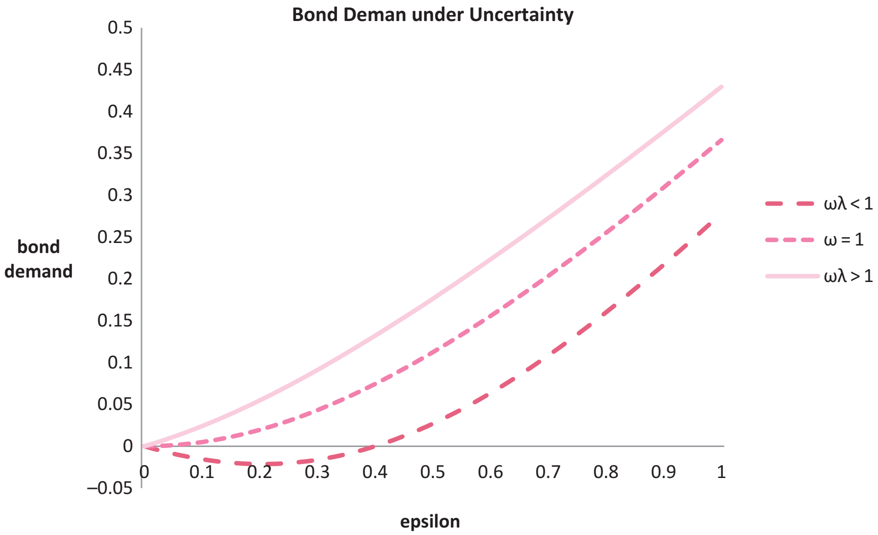

Figure 4: RDP plots of b_1 when $x = 1$. The loss aversion parameters are $\omega\lambda = 0.8$, $\omega\lambda = 1$, and $\omega\lambda = 1.2$. When $\omega\lambda < 1$, the bond demand can be negative.

the saving behavior of the reference-dependent consumers with three different specifications of the loss aversion parameter $(\omega\lambda \lesseqqgtr 1)$. When $\omega\lambda = 1$, this represents the standard model. Figure 3 displays the first-period consumption for the three different models. In the figure, all the models predict higher consumption

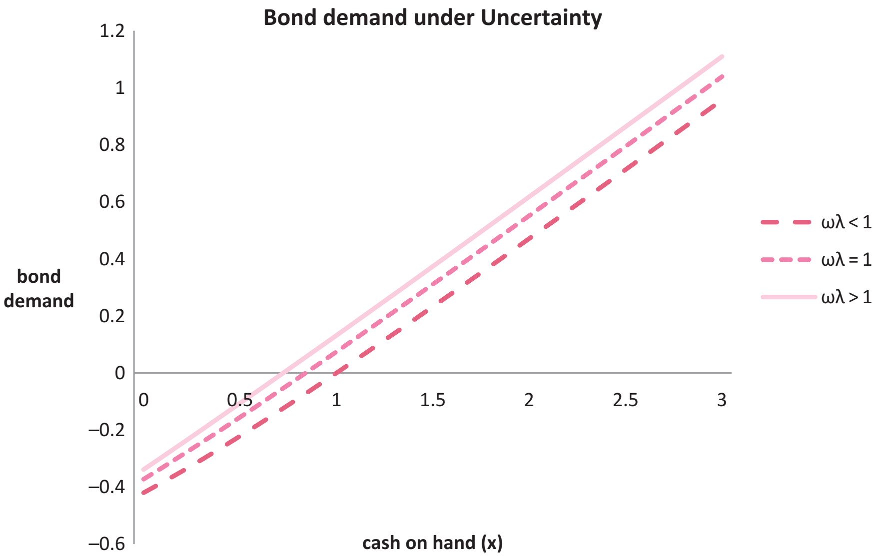

Figure 5: RDP plots of b_1 when $\varepsilon = 0.4$. The loss aversion parameters are $\omega\lambda = 0.8$, $\omega\lambda = 1$, and $\omega\lambda = 1.2$. Each bond demand cuts $b_1 = 0$ at $x \le 1$.

as the DMs have more cash on hand. But given the cash on hand, the DM with the least loss aversion would consume more in the first period by saving less for the future. Figure 5 suggests the same reasoning with respect to saving because saving is defined by $b_1 = y_0 - c_0$, and current consumption and saving move in opposite directions. Figures 2 and 4 jointly show that if the uncertainty level is not severe, the DM with low loss aversion $(\omega\lambda < 1)$ may want to overconsume $(c_0 > 1)$ even by borrowing or dissaving $(b_1 < 0)$. Unlike those who have high loss aversion, these consumers have a relatively low feeling of loss when a low outcome is realized. Because the loss is not more painful than the gain is pleasant, their overall utilities are not lowered much even when the bad outcome happens. Thus, they have little incentive to save to compensate for their feeling of loss in a bad situation and are sometimes tempted to consume more by borrowing. The precautionary saving is

$$P^{RDP}(x;\varepsilon) = -\frac{(1+x)}{4} - \frac{\varepsilon\eta(\omega\lambda-1)}{2} + \frac{1}{2}K(x,\varepsilon,\eta,\omega\lambda) \qquad [32]$$

The next two figures (Figures 6 and 7) show $P^{RDP}(x;\varepsilon)$ and P^{RDP}/x (for $x > 0$) as functions of x or ε. Figure 6 describes that given ε, the precautionary saving motive will be lowered if the consumer has a high level of cash on hand. However, given a level of cash on hand, higher income uncertainty increases

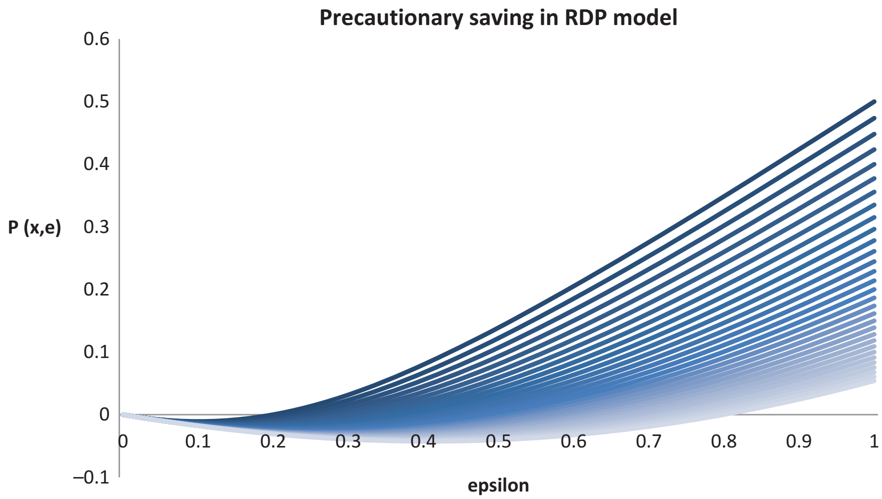

Figure 6: RDP($\omega\lambda = 0.8$) Precautionary Saving: the darker (higher from below) lines are from lower cash on hand (x).

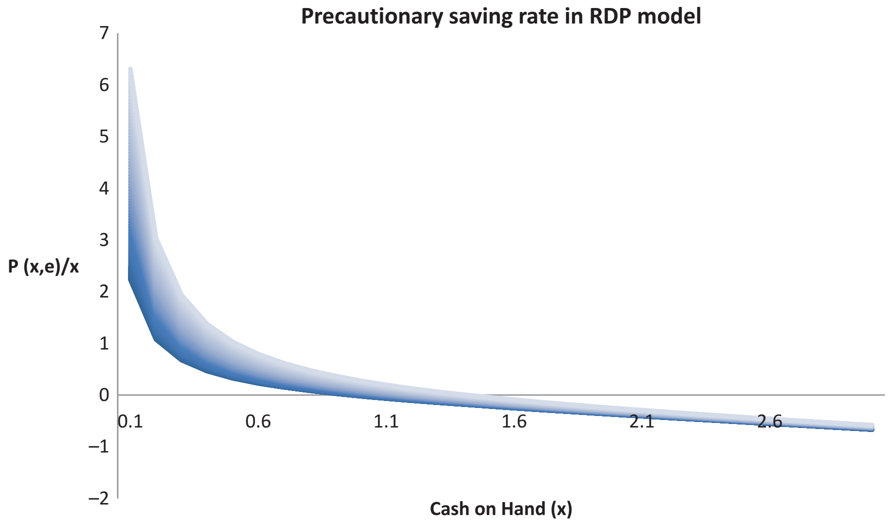

Figure 7: RDP($\omega\lambda = 0.8$) Precautionary saving rate: the darker (closer to the origin) lines are from lower epsilon (ε).

the precautionary saving motive. But with DM who has a low loss aversion parameter, the precautionary saving can be negative (borrowing) under minor income shocks. When either $\omega\lambda = 1$ or $\eta = 0$, the RDP precautionary saving function collapses to the standard saving function under uncertainty above.

Now let us set $x = 1$ to get the consumption and saving profile of the DM with RDP at partial equilibrium:

$$c_0^{RDP} = \frac{3}{2} + \frac{\varepsilon\eta(\omega\lambda - 1)}{2} - \frac{1}{2}\sqrt{1 + 2\varepsilon^2 + 6\varepsilon\eta(\omega\lambda - 1) + \varepsilon^2\eta^2(\omega\lambda - 1)^2} \quad [33]$$

$$b_1^{RDP} = -\frac{1}{2} - \frac{\varepsilon\eta(\omega\lambda - 1)}{2} + \frac{1}{2}\sqrt{1 + 2\varepsilon^2 + 6\varepsilon\eta(\omega\lambda - 1) + \varepsilon^2\eta^2(\omega\lambda - 1)^2} \quad [34]$$

$$c_1^{RDP} = \frac{1}{2} \mp \frac{\varepsilon\eta(\omega\lambda - 1)}{2} + \frac{1}{2}\sqrt{1 + 2\varepsilon^2 + 6\varepsilon\eta(\omega\lambda - 1) + \varepsilon^2\eta^2(\omega\lambda - 1)^2} \quad [35]$$

First, let us examine the bond demand. If $\omega\lambda = 1$, i.e. the standard model, the bond demand is $b_1 = \frac{1}{2}[\sqrt{1 + 2\varepsilon^2} - 1] > 0$ and the value of this function increases as ε increases. How about the case when $\omega\lambda > 1$? When $\omega\lambda > 1$, it is true that a high ε decreases the second term, but increases the third term. Because $6\varepsilon\eta(\omega\lambda - 1) > 0$, the bond demand increases unambiguously more in the reference-dependence model than in the standard model. This implies that given a value of uncertainty parameter ε, the DM, with loss aversion to bad situation, saves more than the DM without this feeling. If, however, $\omega\lambda < 1$, the overall effect is ambiguous depending on the magnitude of ε and $\omega\lambda$. Second, let us look at the first period consumption c_0^{RDP} with respect to ε. If $\omega\lambda = 1$, then a high ε decreases current consumption. If $\omega\lambda > 1$, then a high ε increases the second term but decreases the third term. By the same argument as in the bond demand case, current consumption decreases unambiguously more in the reference-dependent utility model than in the standard model. If $\omega\lambda < 1$, then for a given value of η, a high ε induces lower current consumption c_0 when $\varepsilon > 6/(1 - \omega\lambda)$ but higher current consumption c_0 when $\varepsilon < 6/(1 - \omega\lambda)$.[31] This suggests that for the case of typical income uncertainty without huge fluctuation, the effect is positive when $\omega\lambda < 1$. Under modest income uncertainty, the RDP effect on consumption is positive for this case. That is, those who care less about the pain from future loss may deviate from typical saving for more present consumption.

31 With the assumption of $y = 1$, one may infer that this condition is satisfied more often than the one with opposite inequality. For example, if $\lambda\omega = 0.4$, then $\varepsilon > 6/(1 - \lambda\omega)$ requires $\varepsilon > 10$, which is not very realistic, while $\varepsilon < 6/(1 - \lambda\omega)$ requires only $\varepsilon < 10$.

5.3 RDP-Dep Precautionary Saving

In the example of the two-period consumption-saving model, in which the consumption and reference points are positively related, the state-dependent RDP maximization problem is given by

$$Max \ln c_0 + \beta\left(\frac{1}{2}\ln c_{1h} + \frac{1}{2}\ln c_{1l}\right) \qquad [36]$$

$$+\beta\eta\left\{\frac{1}{2}(\ln c_{1h} - \ln c_{1l}) + \omega\lambda\frac{1}{2}(\ln c_{1l} - \ln c_{1h})\right\}$$

subject to

$$\begin{aligned} c_0 + b_1 &= y_0 \\ c_1 &= \tilde{y}_1 + Rb_1 \end{aligned}$$

Unlike the state-independent case, the gain or loss utility arises with $p = 1/2$ instead of $p = 1/4$. The optimality condition leads to

$$c_0^{-1} = R\beta\left\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1} + \eta(\omega\lambda - 1)\frac{1}{2}\left(c_{1l}^{-1} - c_{1h}^{-1}\right)\right\}. \qquad [37]$$

Excepting the third term, this optimality condition is equal to that of the state-independent RDP in previous subsection. Compared to state-independent RDP, for those who have a high loss aversion ($\omega\lambda > 1$), the DM's intention to save more due to loss aversion is slightly weaker with state-dependent RDP. The state-dependent consumption-saving profile is

$$c_0^{RDP(dep)} = \frac{3x}{2} + \frac{\varepsilon\eta(\omega\lambda - 1)}{4} - \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda) \qquad [38]$$

$$b_1^{RDP(dep)} = -\frac{x}{2} - \frac{\varepsilon\eta(\omega\lambda - 1)}{4} + \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda) \qquad [39]$$

$$c_1^{RDP(dep)} = \frac{2 - x}{2} \mp \frac{\varepsilon\eta(\omega\lambda - 1)}{4} + \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda) \qquad [40]$$

$$K(x, \varepsilon, \eta\omega\lambda) = \sqrt{x^2 + 3x\varepsilon\eta(\omega\lambda - 1) + \{\varepsilon\eta(\omega\lambda - 1)/2\}^2 + 2\varepsilon^2}. \qquad [41]$$

Likewise, the precautionary saving is obtained from

$$P^{RDP(dep)}(x; \varepsilon) = -\frac{x}{2} - \frac{\varepsilon\eta(\omega\lambda - 1)}{2} + \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda). \qquad [42]$$

However, it is still true that if $\omega\lambda = 1$ or $\eta = 0$, this indicates the case of the standard model. The consumption and saving at $x = 1$ are

$$c_0^{RDP} = \frac{3}{2} + \frac{\varepsilon\eta(\omega\lambda - 1)}{4} - \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda) \tag{43}$$

$$b_1^{RDP} = -\frac{1}{2} - \frac{\varepsilon\eta(\omega\lambda - 1)}{4} + \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda) \tag{44}$$

$$c_1^{RDP} = \frac{1}{2} \mp \frac{\varepsilon\eta(\omega\lambda - 1)}{4} + \frac{1}{2}K(x, \varepsilon, \eta\omega\lambda) \tag{45}$$

$$K(x, \varepsilon, \eta\omega\lambda) = \sqrt{1 + 3\varepsilon\eta(\omega\lambda - 1) + \{\varepsilon\eta(\omega\lambda - 1)/2\}^2 + 2\varepsilon^2}. \tag{46}$$

Figure 8 displays RDP consumption when $x = 1$ for both state-independent and state-dependent reference specifications. As seen in the figure, a state-independent model implies lower consumption in the first period due to stronger precautionary saving motivation.[32] Figure 9 displays RDP bond demands when $x = 1$ for both state-independent and state-dependent models. From the figure, it is clear that state-dependent RDP model generates less savings or smaller over-consumption intensity than in the state-independent RDP world.

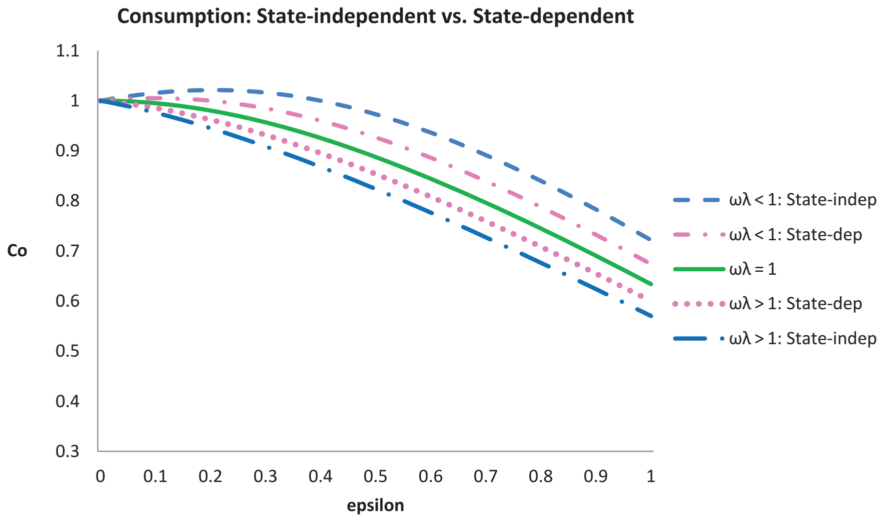

Figure 8: RDP plots of c_0 when $x = 1$ for both state-independent and state-dependent specifications. The loss aversion parameters are $\omega\lambda = 0.8$, $\omega\lambda = 1$, and $\omega\lambda = 1.2$.

32 This is the case for $\omega\lambda > 1$. In the case of $\omega\lambda < 1$, the opposite is true.

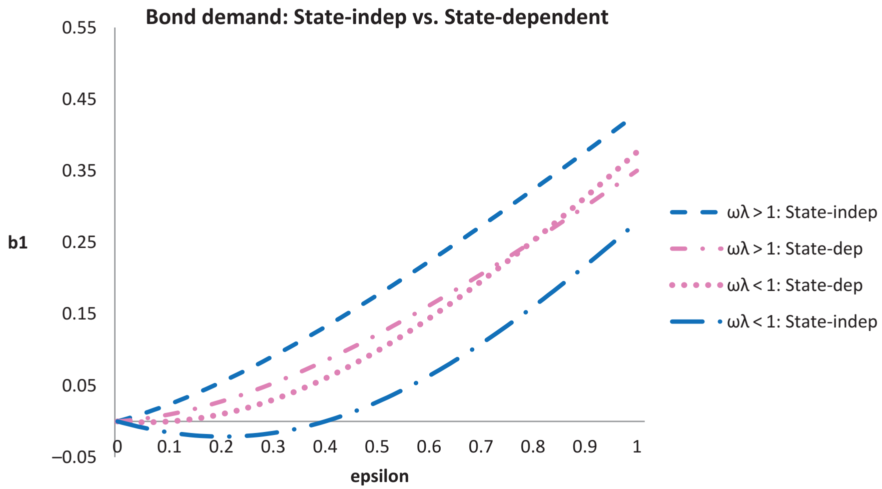

Figure 9: RDP plots of b_1 when $x = 1$ for both state-independent and state-dependent specifications. The loss aversion parameters are $\omega\lambda = 0.8$ and $\omega\lambda = 1.2$.

6 General Equilibrium

In this section, we analyze equilibrium savings related to interest rates in a general equilibrium model, in which the economy is populated by two types of agents who are different in their degrees of loss aversion, but identical otherwise. Though it is possible to construct a general equilibrium model between loss-averse agents and standard agents $\{\omega_A\lambda > 1$ and $\omega_B\lambda = 1\}$, it is more useful to assume that $\{\omega_A\lambda > 1$ and $\omega_B\lambda < 1\}$ because in an economy with two types of identical agents, one type of agents must save (lend) while the other must borrow to clear the market. The first specification may not generate this structure. Unlike in the partial equilibrium case, the bond price or the interest rate is not fixed in general equilibrium, and it is determined in the market by the term that the net supply of bonds equals zero: bond prices clear the market at equilibrium.

6.1 State-Independent RDP

From eq. [15], we know that each type of DM meets one of the following optimality condition:

$$c_0^{-1} = R\beta\left\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1} + \eta(\omega_A\lambda - 1)\frac{1}{2}\frac{1}{2}\left(c_{1l}^{-1} - c_{1h}^{-1}\right)\right\} \qquad [47]$$

$$c_0^{-1} = R\beta\left\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1} - \eta(1-\omega_B\lambda)\frac{1}{2}\frac{1}{2}(c_{1l}^{-1} - c_{1h}^{-1})\right\} \qquad [48]$$

How can we compare the two? Because $\omega_A\lambda > 1$ and $\omega_B\lambda < 1$ it is expected that

$$-\eta(1-\omega_B\lambda)\frac{1}{2}\frac{1}{2}(c_{1l}^{-1} - c_{1h}^{-1}) < \eta(\omega_A\lambda - 1)\frac{1}{2}\frac{1}{2}(c_{1l}^{-1} - c_{1h}^{-1}) \qquad [49]$$

This implies that the three gain-loss marginal utilities satisfy: $MU_{G-L}(c_1^{\omega\lambda<1}) < MU_{G-L}(c_1^{\omega\lambda=1}) < MU_{G-L}(c_1^{\omega\lambda>1})$. Thus, the DM with $\omega_B\lambda < 1$ would not save more than the DM with $\omega_A\lambda > 1$ would save. To derive a general equilibrium result for the different types of agents, we need a price to clear the saving/borrowing market. Therefore, it is necessary to determine the bond demand as a function of interest rates. For simplicity we keep assuming $\beta = 1/R$. Under the assumption of $y_0 = 1$, together with $y_{1h} = 1+\varepsilon$ and $y_{1l} = 1-\varepsilon$, the first order condition implies

$$\frac{4[1+R+\varepsilon-Rc_0)][1+R-\varepsilon-Rc_0]}{c_0} \qquad [50]$$

$$= [1+\varepsilon+R-Rc_0][2-4(1-\omega\lambda)] + [1-\varepsilon+R-Rc_0][2+4(1-\omega\lambda)]$$

Solving the equation brings consumption c_0 as a function of the interest rate and thus bond demand is obtained by $b_1(R) = y_0 - c_0(R)$. The bond demand of the DM(A) who is loss averse ($\omega_A\lambda > 1$) is

$$b_1^A = \frac{-(1+R) - 2\varepsilon\eta(\omega_A\lambda - 1)}{2R(R+1)} \qquad (51)$$

$$+\frac{\sqrt{[(2R+1)(1+R) + 2\varepsilon\eta(\omega_A\lambda-1)]^2 - 4(R+1)R[(1+R)^2 - \varepsilon^2]}}{2R(R+1)}$$

while the bond demand of the DM(B) who is tolerant to loss ($\omega_B\lambda < 1$) is

$$b_1^B = \frac{-(1+R) + 2\varepsilon\eta(1-\omega_B\lambda)}{2R(R+1)} \qquad [52]$$

$$+\frac{\sqrt{[(2R+1)(1+R) - 2\varepsilon\eta(1-\omega_B\lambda)]^2 - 4(R+1)R[(1+R)^2 - \varepsilon^2]}}{2R(R+1)}$$

Now let us introduce a market clearing condition for the economy. Assume that there are only two types of agents in the economy and that their intensity of loss aversion/tolerance is symmetric, i.e. $\omega_A\lambda - 1 = 1 - \omega_B\lambda$. Assume further that the

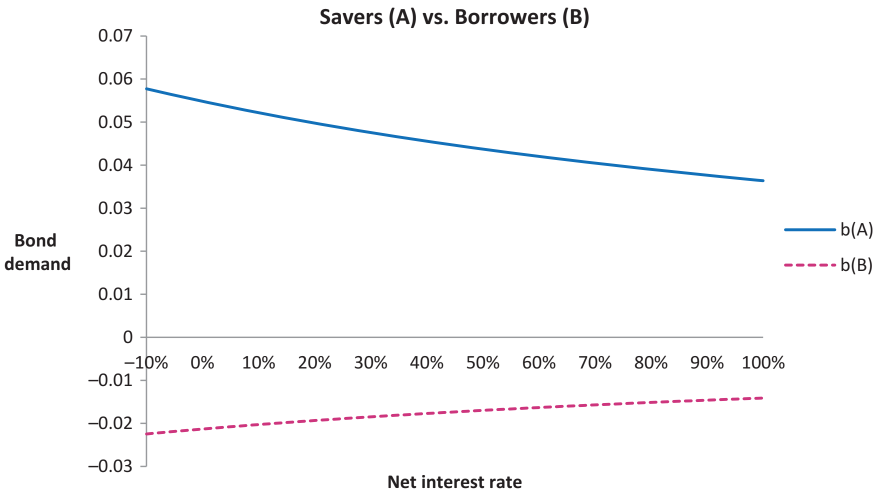

Figure 10: The state-independent bond demand with $\theta = 1/2$: the two agents are represented by $\lambda\omega_A = 1.2$ and $\lambda\omega_B = 0.8$. The uncertainty is given by $\varepsilon = 0.2$.

portion of those who are loss averse (type A) is θ. Initially let us set $\theta = 1/2$. Figure 10 shows the bond demand by the two different DMs. The uncertainty specification is $\varepsilon = 0.2$. With an equal distribution of the two types of agents, the bond market yields positive net savings as seen in the figure and therefore the market is not in equilibrium. Under the assumption of $\theta = 1/2$, together with the symmetric intensity level of loss aversion in the simple model, it is found that positive net savings are persistent[33] even with a very low uncertainty level.[34] Figure 11 summarizes the result when the economy is equally populated by both types of agents.

This implies that the economy should have more of the savers to lower the interest rate to meet the borrowers. In fact, a higher θ about $2/3$, namely, $\theta = 0.659646$ induces zero net savings for the uncertainty level: to clear the bond market, it should be satisfied that

$$\theta b_1^A + (1 - \theta) b_1^B = 0 \qquad [53]$$

33 This is the result for the case in which $\omega_A\lambda - 1 = 1 - \omega_B\lambda = 0.2$. If, however, $\omega_A\lambda - 1 < 1 - \omega_B\lambda$, then negative net savings are obtained when ε is high.

34 If the uncertainty level is high enough so that $\varepsilon > 0.4$, then even the type B agents would save as shown in Figure 9.

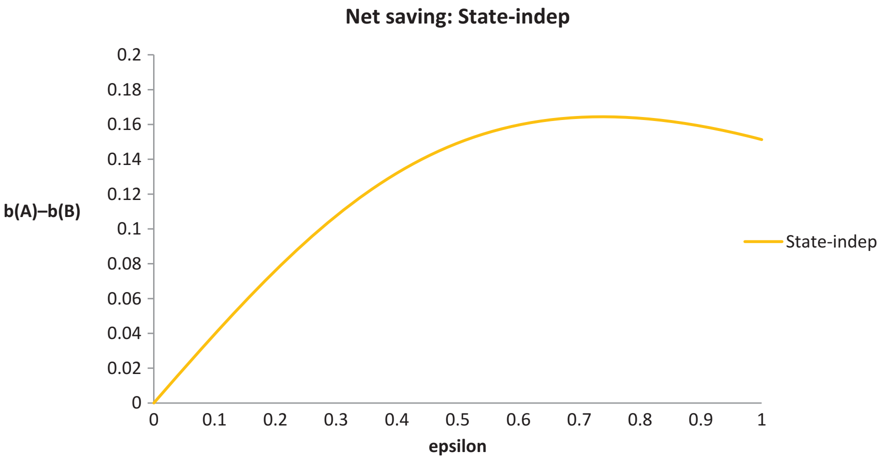

Figure 11: The state-independent net saving of the economy with $\theta = 1/2$: the two agents are represented by $\lambda\omega_A = 1.2$ and $\lambda\omega_B = 0.8$.

where $0 < \theta < 1$. Figure 12 shows the general equilibrium result for two different DMs with asymmetric intensity parameters, $\omega_A\lambda = 2$ and $\omega_B\lambda = 0.2$. The uncertainty specification is $\varepsilon = 0.2$, and θ is adjusted accordingly to set an equilibrium for each level of the gross interest rates.

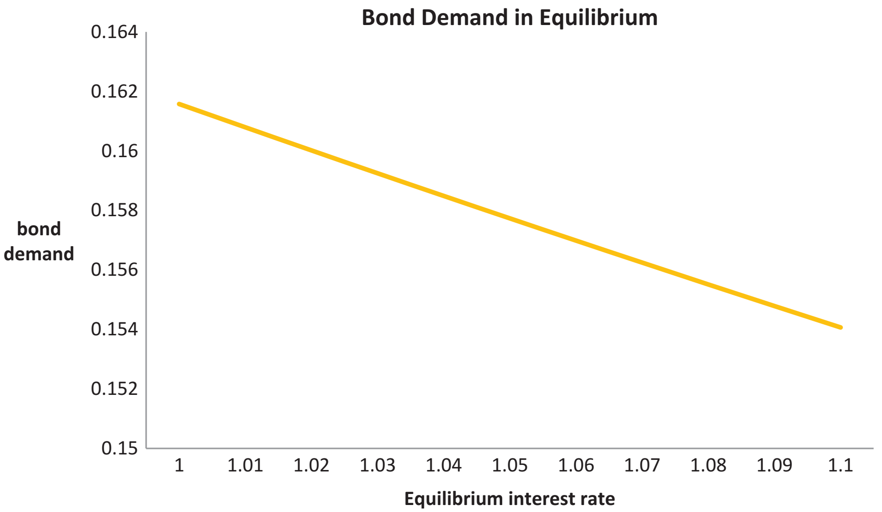

Figure 12: The state-independent equilibrium bond demand: the two agents are represented by $\lambda\omega_A = 2$ and $\lambda\omega_B = 0.2$. The uncertainty is given by $\varepsilon = 0.2$.

6.2 State-dependent RDP and Comparison

Following the similar steps as those in the state-independent RDP, the optimality condition for the two types of agents in the state-dependent model is given by:

$$c_0^{-1} = R\beta\left\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1} + \eta(\omega_A\lambda - 1)\frac{1}{2}(c_{1l}^{-1} - c_{1h}^{-1})\right\} \tag{54}$$

$$c_0^{-1} = R\beta\left\{\frac{1}{2}c_{1h}^{-1} + \frac{1}{2}c_{1l}^{-1} - \eta(1 - \omega_B\lambda)\frac{1}{2}(c_{1l}^{-1} - c_{1h}^{-1})\right\} \tag{55}$$

The bond demand of the DM(A) who is loss averse ($\omega_A\lambda > 1$) is

$$b_1^A(Dep) = \frac{-(1+R) - \varepsilon\eta(\omega_A\lambda - 1)}{2R(R+1)} \tag{56}$$

$$+ \frac{\sqrt{[(2R+1)(1+R) + \varepsilon\eta(\omega_A\lambda - 1)]^2 - 4(R+1)R[(1+R)^2 - \varepsilon^2]}}{2R(R+1)}$$

while the bond demand of the DM(B) who is tolerant of loss ($\omega_B\lambda < 1$) is

$$b_1^B(Dep) = \frac{-(1+R) + \varepsilon\eta(1 - \omega_B\lambda)}{2R(R+1)} \tag{57}$$

$$+ \frac{\sqrt{[(2R+1)(1+R) - \varepsilon\eta(1 - \omega_B\lambda)]^2 - 4(R+1)R[(1+R)^2 - \varepsilon^2]}}{2R(R+1)}$$

Figure 13 explains the saving and borrowing behaviors when the economy has an equal population of the two types of agents with both state-independent and state-dependent reference dependences. From the figure, it is clear that both borrowing and saving amounts are smaller for all levels of interest rates with the state-dependent model. Figure 14 demonstrates the variation in net savings corresponding to the uncertainty level. From this we may infer that for all levels of income uncertainty, $0 < \varepsilon < 1$, we may achieve equilibrium easily in a state-dependent world. Moreover, in a state-dependent world, a minor adjustment of θ from $1/2$ would cause an equilibrium point to be achieved, which in fact is not expected in the state-independent world.

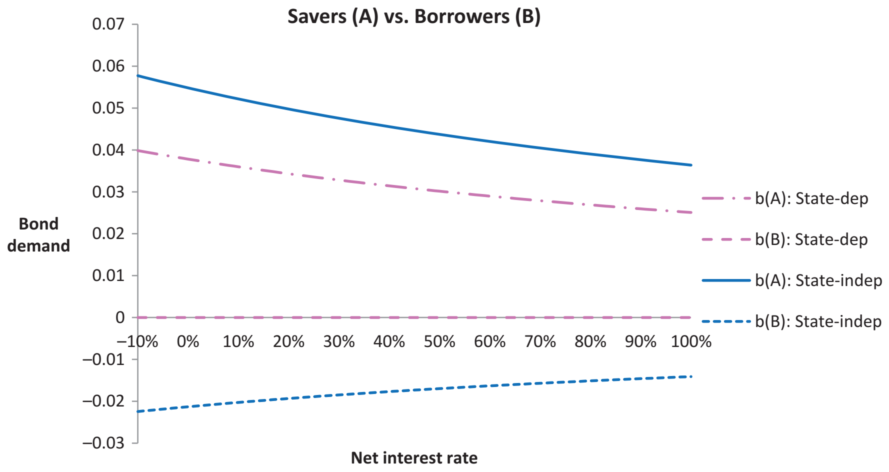

Figure 13: The state-independent and state-dependent bond demands with $\theta = 1/2$: the two agents are represented by $\lambda\omega_A = 1.2$ and $\lambda\omega_B = 0.8$. The uncertainty is given by $\varepsilon = 0.2$.

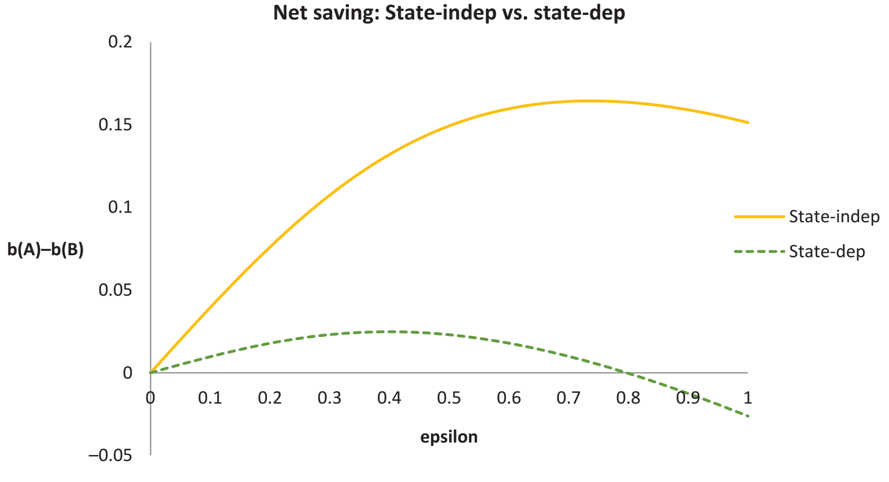

Figure 14: The state-independent and state-dependent net savings of the economy with $\theta = 1/2$: the two agents are represented by $\lambda\omega_A = 1.2$ and $\lambda\omega_B = 0.8$.

7 Conclusion

In this paper, we study a model of reference-dependent preferences with endogenous reference expectations under the two schemes of state-independent and state-dependent stochastic references points. A key issue in reference-dependent utility models is the reference point, because as Pesendorfer (2006) claims, a reference point can be anything and may be selected arbitrarily by researchers. Often the reference point is assumed to be a current status, such as current consumption, position, or endowment. In this paper, we build a tractable, intertemporal choice model following two recent developments regarding expectation-based reference points: state-independent reference expectations, which originated in disappointment theory, and state-dependent reference expectations, which are based on regret theory.

We build the model according to the novel notion that some individuals might be loss-tolerant. Because of this assumption, we can provide a richer analysis that helps us better understand the consumption and precautionary saving behaviors of different types of consumers. Moreover, the model we develop has many merits in terms of economic modeling, in that it departs from typical aspects of reference-dependence, such as habit formation; adapts forward looking expectations; and uses realistic preference specifications.

In the two-period consumption-saving model, we derive analytic solutions to show that, when loss aversion is high, the decision maker saves more than the standard model predicts and thus consumes less in the current period. However, when the loss aversion is low, the overall result is ambiguous, although the decision maker may even want to borrow to consume more for the current period if he faces mild uncertainty relative to the intensity of his loss aversion. Given the same intensity of loss aversion, we find that the state-dependent model generates a smaller deviation than the standard one for both consumption and precautionary saving. Consequently, it is implied that for positively related consumption and its reference points, the state-independent model tends to exaggerate the true outcome of gain-loss feeling and vice versa. In both of the models, the meaning of precautionary saving can be defined in a different way from the standard model under uncertainty, although the size of this effect depends on whether it is state-dependent or not. Throughout this paper, we show that precautionary saving behavior is closely related to the degree of loss aversion, by which the decision makers may deviate for more or less consumption (and thus less or more saving) than what the standard uncertainty model predicts. Finally, from the general equilibrium exercise, we find that at a given uncertainty level, both borrowings and savings are smaller for all levels of interest rates with the state-dependent model. Also for all levels of income uncertainty, the state-dependent model agrees more with the even distribution of borrowers and savers.

References

Aizenman, J. 1998. "Buffer Stocks and Precautionary Savings with Loss Aversion." *Journal of International Money and Finance* 17:931–47.

Andersen, S., G. Harrison, M. Lau, and E. Rutström. 2008. "Eliciting Risk and Time Preferences." *Econometrica* 76:583–618.

Bowman, D., D. Minehart, and M. Rabin. 1999. "Loss Aversion in a Consumption-Savings Model." *Journal of Economic Behavior and Organization* 38:155–78.

Carroll, C. D. 1997. "Buffer-Stock Saving and the Life-Cycle/Permanent Income Hypothesis." *Quarterly Journal of Economics* 112:1–55.

Chetty, R., and A. Szeidl. 2010. "Consumption Commitments: A Foundation for Reference-Dependent Preferences and Habit Formation." Working Paper.

Cillo, A., and P. Delquie. 2006. "Disappointment Without Prior Expectations: A Unifying Perspective on Decision Under Risk." *Journal of Risk and Uncertainty* 33:197–215.

Deaton, A. 1991. "Saving and Liquidity Constraints." *Econometrica* 59:1221–48.

Feigenbaum, J. 2008. "Information Shocks and Precautionary Saving." *Journal of Economic Dynamics and Control* 32:3917–38.

Feigenbaum, J. 2009. "Precautionary Saving Unfettered." Working Paper, University of Pittsburgh.

Freeman, D. 2013. "Revealed Preference Foundations of Expectations-Based Reference-Dependence." Working paper.

Giorgi, E., and T. Posty. 2011. "Loss Aversion with a State-Dependent Reference Point." *Management Science* 57:1094–110.

Gourinchas, P., and J. A. Parker. 2002. "Consumption Over the Life Cycle." *Econometrica* 70:47–89.

Gul, F. 1991. "A Theory of Disappointment Aversion." *Econometrica* 59:667–86.

Halevy, Y., and V. Felkamp. 2005. "A Baysian Approach to Uncertainty Aversion." *Review of Economic Studies* 72:449–66.

Kahneman, D., and A. Tversky. 1979. "Prospect Theory: An Analysis of Decision Under Risk." *Econometrica* 47:263–91.

Kobberling, V., and P. Wakker. 2005. "An Index of Loss Aversion." *Journal of Economic Theory* 122:119–31.

Krähmer, D., and R. Stone. 2013. "Anticipated Regret as an Explanation of Uncertainty Aversion." *Economic Theory* 52:709–28.

Köszegi, B., and M. Rabin. 2006. "A Model of Reference-Dependent Preferences." *Quarterly Journal of Economics* 121:1133–165.

Köszegi, B., and M. Rabin. 2007. "Reference-Dependent Risk Attitudes." *American Economic Review* 97:1047–73.

Köszegi, B., and M. Rabin. 2009. "Reference-Dependent Consumption Plans." *American Economic Review* 99:909–36.

Ludvigson, S. C., and A. Michaelides. 2001. "Does Buffer-Stock Saving Explain the Smoothness and Excess Sensitivity of Consumption?" *American Economic Review* 91:631–47.

Pagel, M. 2013. "Expectations-Based Reference-Dependent Life-Cycle Consumption." Working Paper.

Park, H. 2015. "Bounded Rationality and Lifecycle Consumption." Working paper.

Pasquariello, P. 2014. "Prospect Theory and Market Quality." *Journal of Economic Theory* 149:276–310.

Pesendorfer, W. 2006. "Behavioral Economics Comes of Age: A Review Essay on 'Advances in Behavioral Economics'". *Journal of Economic Literature* 44:712–21.

Rabin, M. 2000. "Risk Aversion and Expected-Utility Theory: A Calibration Theorem." *Econometrica* 68:1281–92.

Siegmann, A. 2002. "Optimal Saving Rules for Loss-Averse Agents Under Uncertainty." *Economics Letters* 77:27–34.

Sugden, R. 2003. "Reference-Dependent Subjective Expected Utility." *Journal of Economic Theory* 111:172–91.

Tversky, A., and D. Kahneman. 1992. "Advances in prospect theory: Cumulative representation of uncertainty." *Journal of Risk and Uncertainty* 5:297–323.