DE GRUYTER BE J. Theor. Econ. 2016; 16(2): 515–543

Research Article

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The Rationality of Expectations Formation

DOI 10.1515/bejte-2014-0082
Published online February 11, 2016

Abstract: Rational expectations are not required to follow from beliefs that explain well history, but just to correctly foresee the future. As a consequence, at a rational expectations equilibrium, the agents' expectations may follow from beliefs that explain poorly the observed history, even among those rationalizing their choices. This paper shows, firstly, that if agents hold *rationally formed expectations* instead – in the sense of following from beliefs that explain history better than any other beliefs justifying their choices – then allocations unsupported by rational expectations can be shown to be equilibrium outcomes. By means of this result, it is established, secondly, that adding the common knowledge of the rationality of the *formation* of expectations to that of the rationality of choices and of market clearing, still does not suffice to guarantee rational expectations. Finally, the rationally formed expectations equilibria produced in this paper exhibit a sunspot-like volatility that, interestingly enough, do not rely on an explicit sunspot mechanism.

Keywords: rationality, expectations, overlapping generations
JEL Codes: D84, D5, E3

1 Introduction

Agents make depend their decisions on their expectations about what may have an impact on their consequences, namely other agents' current actions and future plans of action, as well current and past events. Whether these expectations are rational and whether they are *rationally formed* are related but essentially distinct issues. In fact, what is a rational way of forming expectations depends on what these expectations are about. While in strategic situations common knowledge of rationality can guide an agent in forming his expectations

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about other agents' plans,[1] when it comes to expectations about future events a competing source of information to form expectations is empirical evidence from the past. Thus, in non-strategic situations only history carries weight in forming expectations rationally. Bearing this in mind, we would expect rational agents to hold expectations that follow from beliefs that explain best the past they observe. Indeed, that an agent may make a decision based on beliefs distinct from those that explain best the history he observes amounts to assume that he does not infer rationally from evidence, which must be rejected on the grounds of the agent's rationality.

Does not rational expectations address already the issue of the formation of expectations? No, in fact the rational expectations hypothesis is silent about how expectations are formed:[2] in sequential markets models with an objective stochastic process driving the fundamentals of the economy, it just forbids the agents to hold expectations that lead to systematic forecasting mistakes; more generally, in the general equilibrium literature, it requires the agents' choices to be contingent only to the information available, including the information revealed by prices. At any rate, nothing prevents that at a rational expectations equilibrium an agent holds expectations following from beliefs that do not explain best the history he observes, not *even among those that rationalize his choice*.[3] Thus, if empirical evidence is the only source of information on which agents rationally form their expectations in a non-strategic environment, then rational expectations need not be rationally formed.

This paper proposes instead an equilibrium concept that requires each agent's expectations to follow from beliefs that are not worse at explaining the available evidence than any other beliefs rationalizing his choice. Since this is a condition on what is going on inside the agents' minds when making a choice, should it make no difference in allocative terms, it would then be irrelevant. But the fact is that – as it will be shown below – this requirement does matter for the determination of what can be an equilibrium outcome and what cannot.

1 As a matter of fact, common knowledge of rationality is unnecessarily strong to underpin Nash equilibria: individual rationality and mutual knowledge of everybody else's strategy suffices – see Aumann and Brandemburger (1995).

2 Lucas (1978) argues, nonetheless, that rational expectations equilibria are the asymptotic result of Bayesian learning or boundedly rational learning processes. Lucas (1986) quotes Bray (1982, 1983) and Blume and Easley (1982) along this line. Ben Porath and Heifetz (2010) have spelled out the framework needed to state and prove such a claim.

3 More precisely, at a rational expectations equilibrium agents need not hold as subjective probabilities the asymptotic maximum likelihood estimate (consistent with the optimality of their choices) of the probabilities of the transitions they observe in sequential markets models. This, from an admittedly frequentist viewpoint, is an unsatisfactory feature of the rational

Interestingly, with this equilibrium notion, instances of sunspot-like volatility of prices and trades – i.e. unrelated to shocks to the fundamentals – that cannot be rational expectations equilibria happen to be, nonetheless, rationally formed expectations equilibrium outcomes.

The results of this paper contribute also to the literature seeking to provide a common knowledge foundation to competitive equilibria. Specifically, the result above allows to establish that, in sequential market economies, adding to the common knowledge of rationality and of market clearing also the common knowledge of rationality *in the formation of expectations* is still not enough to guarantee rational expectations. That rational expectations equilibria outcomes need not follow from common knowledge of just rationality and market clearing has been established in Ben-Porath and Heifetz (2010) for finite exchange economies with asymmetric information. Morris (1995) had shown that common knowledge of rationality and market clearing imply rational expectations equilibria only if the agents share a common prior on the set of states of the *world* – which includes the whole system of beliefs and higher order beliefs, on top of the state of *nature* – an admittedly too demanding assumption.

More specifically, I consider a deterministic overlapping generations exchange economy of agents that hold *rationally formed expectations* in the sense that no other expectations consistent with their choices follow from beliefs implying a higher likelihood for the history they observe. In such a setup I argue, first, that the belief that the observed history is Markovian can never be falsified when the agents' memory (or the history itself) is finite. Then, for Markovian beliefs, I establish that when the agents' memories are finite there exist rationally formed expectations equilibria that no rational expectations equilibrium can match (Proposition 2). Very importantly, note that assuming rationally formed expectations is *not* to assume that agents form their expectations maximizing the likelihood of the observed history, since the agents' expectations must justify their choices, which depend on the expectations themselves. As a matter of fact, rationally formed expectations typically do *not* maximize the *unconstrained* likelihood of observed history because of this latter condition. The actual formation of expectations itself is left un-modeled here, but the rationality condition on their formation introduced *does* restrict the expectations formation process nonetheless. There is nonetheless a formal link between the way agents are deemed to form expectations "rationally" here and the literature on the empirical likelihood (EL) estimator (Owen (1988)) that has

expectations equilibria since the maximum likelihood estimator achieves the Cramér-Rao lower bound for the asymptotic mean squared error among unbiased estimators, while any other estimator converging to a rational-expectations limit does not.

been shown to exhibit better higher order properties than the GMM estimator (Newey and Smith (2004)) embedding estimation methods such as ordinary least squares, maximum likelihood, two-stage least squares and instrumental variables. The empirical likelihood estimator was developed to address the small sample bias of the two-step GMM estimator.

It is worth noticing also that limited memory and communication between agents is essential for the existence of rationally formed expectations equilibria distinct from a rational expectations equilibrium: with infinite memory rationally formed expectations equilibria are allocationally equivalent to rational expectations equilibria (Proposition 1). With limited memory, instead, different agents can hold different expectations at a rationally formed expectations equilibrium. This diversity of beliefs follows from the fact that different generations observe different bits of the same history and therefore form their beliefs using different information – the limited memory itself captures the bounded computing abilities of actual agents. Thus, in spite of the result in Geanakoplos and Polemarchakis (1982) showing that unrestricted communication allows agents with a common prior but different information to agree in finite time on a common posterior, the limited communication implied by the demographic structure of the economy (a sequence of overlapping generations) allows for the agents to disagree.

Specifically, a rationally formed expectations equilibrium will consist of, for each agent in each generation and for each history of prices he may observe, (1) a belief that prices follow a particular stochastic process, and (2) consumption decisions (contingent to future prices for future consumptions), such that, for any history of prices up to any date, (i) the allocation is feasible, (ii) the agents' consumptions maximize their expected utilities given the price process they believe they face, and (iii) *the agents' beliefs about the price process (and hence their expectations) are formed rationally* – i.e. their beliefs are not falsified by history and attach to the latter a likelihood not smaller than any other beliefs justifying their choices.

Note that the equilibrium concept leaves open the question of how the actual history of prices is determined: it just requires that history does not falsify the agents' beliefs. Therefore, no objective process is needed to be assumed for prices and, as a consequence, there is no room for agents to mistake a price process they supposedly face – which would be the ultimate rationality test under rational expectations. In case not specifying such process may be disconcerting, it is important to note that, *in the absence of shocks to the fundamentals*, assuming that some objective sunspot signal drives the prices, amounts to postulate implicitly a particular price formation theory – separate from the equilibrium conditions – that acts as a selecting device within the set of possible price processes. Given the obvious difficulties in justifying the causation from sunspots all the way to prices and, more importantly, given that it is superfluous here, I do away with it. The results in this paper establish thus the existence of equilibria akin to sunspot equilibria but without the need to make an explicit reference to sunspots – i.e. "sunspot equilibria" without sunspots, so to speak. This shows that the introduction of sunspot mechanisms is not essential to account for pure expectations-driven fluctuations.

The remainder of the paper is organized as follows: Section 2 presents the main ideas by means of a leading example conveying the intuition driving the result: it produces constructively rationally formed expectations equilibria exhibiting fluctuations distinct from those of any rational expectations equilibrium. Section 3 generalizes the setup, provides a more general definition of a rationally formed expectations equilibrium for a deterministic overlapping generations economy, and establishes the existence of equilibria of this type exhibiting fluctuations that no rational expectations equilibrium can generate (Proposition 2). Section 4 embeds the notion of rationally formed expectations equilibrium within an epistemic model specifying an interactive system of beliefs and higher order beliefs in order to show that, very much like common knowledge of rationality and market clearing does not necessarily imply rational expectations in finite economies (see Ben-Porath and Heifetz [2010]), common knowledge of rationality, market clearing, *and beliefs formation rationality* needs not lead to rational expectations equilibria either. It moreover shows that the existence argument provided for the setup in Section 3 can be extended to the setup in Section 4 (Proposition 3). Finally, since the constructive argument used in Proposition 2 reveals a high level of degrees of freedom to produce rationally formed expectations equilibria, I establish in Section 5 the important fact that not anything can be a rationally formed expectations equilibrium in that setup. Section 5 also briefly discusses some related literature.

2 The Leading Example

2.1 What is a Rationally Formed Expectations Equilibrium?

Consider an overlapping generations economy with a 2-period-lived representative agent born at each period t – with preferences on consumption profiles $\left(c_t^t, c_{t+1}^t\right)$ represented by a utility u and endowments (e_1, e_2) – facing a single intertemporal budget constraint.

In general, at a competitive equilibrium of this economy

(i) every agent maximizes his utility under his budget constraint, and
(ii) individual consumption decisions are compatible.

Such an economy is known to have a continuum of non-stationary equilibria and, under some conditions, a continuum of stationary equilibria as well, in particular of the following type.

Definition: *A k-state (Markovian)* **Stationary Sunspot Equilibrium** *(or k-SSE) is any collection of prices p^i, consumption plans $\left(c_1^i, \left(c_2^j\right)_{j=1}^k\right)$, for all $i = 1, \ldots, k$, and beliefs about a Markovian price process $(\pi^{ij})_{i,j=1}^k$ – according to which an agent facing when young a price p^i expects it to be p^j with probability π^{ij} when old, for $i, j=1, \ldots, k$ – such that*

(i) *the consumption plan $\left(c_1^i, \left(c_2^j\right)_{j=1}^k\right)$ of agents born when p^i is the solution to*

$$\begin{aligned} &\max_{c_1^i, (c_2^j)_j} \sum_{j=1}^k \pi^{ij} u\left(c_1^i, c_2^j\right) \\ &p^i c_1^i + p^j c_2^j = p^i e_1 + p^j e_2, \quad \forall j \end{aligned} \qquad [1]$$

and

(ii) *the allocation of resources is feasible, that is to say,*

$$c_1^i + c_2^i = e_1 + e_2 \qquad [2]$$

for all $i = 1, \ldots, k$.

Conditions for the existence of k-SSE, i.e. of prices p^i, consumptions (c_1^i, c_2^j), and probabilities π^{ij}, for $i, j = 1, \ldots, k$, such that eqs. [1] and [2] hold for all $i = 1, \ldots, k$, are well known. For instance, there exists a continuum of them arbitrarily close to a steady state that is indeterminate in the perfect foresight equilibrium dynamics. More generally, see Azariadis (1981), Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1988, 1989), Guesnerie (1986) for existence characterizations. In such equilibria, the allocation and prices fluctuate randomly according to the Markov chain with probabilities of transition π^{ij}, even if the fundamentals of the economy are constant.

Note however that, as soon as $k \geq 3$, a given choice $(c_1^i, (c_2^j)_{j=1}^k)$ can follow from different beliefs about the probabilities $\pi^{i1}, \ldots, \pi^{ik}$ of transition from a price p^i to any other p^j. Indeed, at any such equilibrium each vector $\left(\pi^{i1}, \ldots, \pi^{ik}\right)$ of probabilities of transition from each state $i = 1, \ldots, k$, must satisfy the two linear equations consisting of (i) being in the unit simplex in $\mathbb{R}^k$ and (ii) satisfying the equation

$$\sum_{j=1}^{k} \pi^{ij} \left[u_1\left(c_1^i, c_2^j\right)\left(c_1^i - e_1\right) + u_2\left(c_1^i, c_2^j\right)\left(c_2^j - e_2\right) \right] = 0 \qquad [3]$$

(where u_i stands for the partial derivative of u with respect to c_i). These conditions characterize necessarily – and sufficiently, for the relative prices implicitly defined by the optimal consumption plan through the budget constraint – the solution to eq. [1] under standard assumptions. As a consequence, they leave k–2 degrees of freedom for each row $\left(\pi^{i1}, \ldots, \pi^{ik}\right)$ of the Markov matrix $\left(\pi^{ij}\right)_{i,j=1}^{k}$ of believed probabilities of transition between prices, as illustrated in Figure 1 below for the case $k = 3$, where $\pi^{i} \equiv \left(\pi^{i1}, \ldots, \pi^{ik}\right)$ and $A^{ij} \equiv u_1(c_1^i, c_2^j)\left(c_1^i - e_1\right) + u_2(c_1^i, c_2^j)(c_2^j - e_2)$.

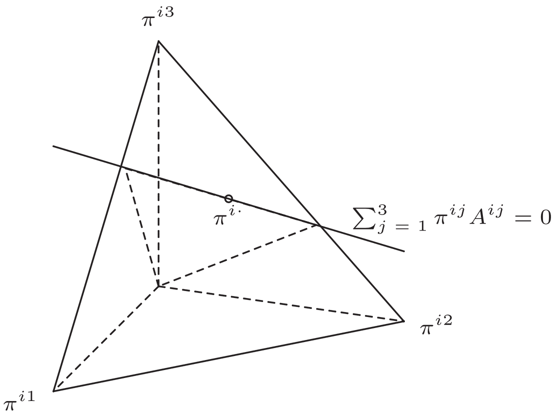

Figure 1: Intersection of the FOC and the simplex, containing the rationalising beliefs.

Thus eqs [2] and [3] may still hold true – i.e. (i) everyone behaves rationally *given his beliefs* and (ii) markets clear – even if different agents within and across generations hold different beliefs about the probabilities of transition π^{ij}. In other words, for a given $(c_1^i, c_2^j)_{i,j=1}^{k}$ satisfying eqs [2] and [3] for some Markov matrix $\left(\pi^{ij}\right)_{i,j=1}^{k}$, there exist continuum of Markov matrices solution to eq. [3] for all $i = 1, \ldots, k$, so that across and within generations the agents may hold different beliefs about the probabilities of transition between prices, and they would still be optimizing while choosing a feasible allocation.

Of course, this possibility is excluded if the agents are supposed to hold rational expectations, since in that case all the agents must share the same "true" π^{ij}'s. Note however that, in the definition of a k-SSE above, no mention has been made yet of a "true" objective process from which this "true" π^{ij}'s would stem, but rather of the agents' *expectations* about future prices instead, according to their *beliefs*. That is because, in a k-SSE, the probability π^{ij} with which the agent *expects* the transition from a price p^i to a price p^j to happen is implicitly assumed to be the actual probability with which such transition does happen as a result of the agents' choices based on a never falsified belief in a perfect correlation between some sunspot signal and prices, so that beliefs are self-fulfilling. Note that this amounts to assuming implicitly a price formation mechanism separate from the equilibrium notion that acts as a selection device, very much like an ad hoc choice of a particular equilibrium out of a multiplicity of them in, for instance, an Edgeworth box.

More specifically, the rational expectations hypothesis imposes the additional conditions that

(i) *all* agents' expectations follow from a common belief in the same price process, and
(ii) this process is the objective process driving prices.

Note that, in terms of the equilibrium eqs [2] and [3], condition (ii) above has no bite in the sunspot case, since it can just be dropped without any consequence for the set of solutions to the equations. It is precisely in this sense that the assumption of an objective process driving prices is an implicit, ad hoc selection device in the absence of shocks to the fundamentals. But condition (i) without (ii) becomes arbitrary, and raises difficult questions regarding the spontaneous coordination of every agent within and across the infinity of generations on a particular belief, i.e. on a particular Markov matrix $\left(\pi^{ij}\right)_{i,j=1}^{k}$. Accordingly, both conditions (i) and (ii) can arguably be dropped and the rationale of the rational expectations hypothesis be questioned in the absence of shocks to the fundamentals.

Thus, since plenty of beliefs are compatible with the agents' behavior at a k-SSE, there is room for alternative consistency conditions on the agents' expectations at equilibrium, distinct from the rational expectations hypothesis. The most natural one is to require that the agents' expectations are rationally inferred from the information available at the time they make their decisions. Indeed, to allow for an agent's decision to follow from expectations derived from beliefs that do not make the likelihood of the history of prices he observes as big as possible – among all the expectations that would have led to the same decision – amounts to assume that the agent formed his expectations using inefficiently the available information.

In Figure 2 below, for the case $k = 3$, the agent's rationally formed expectations about these probabilities of transition from a price p^i (if he believes the

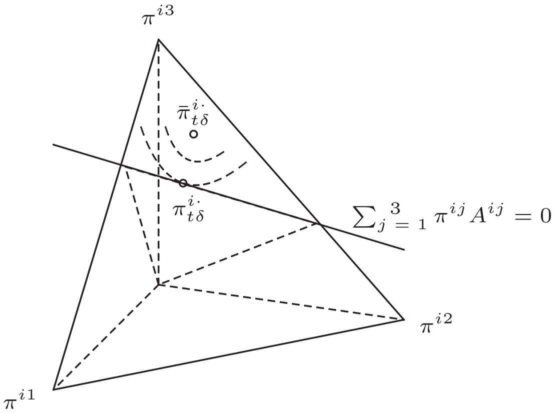

Figure 2: Empirical distribution of transitions $\bar{\pi}^{i\cdot}_{t\delta}$ and highest likelihood distribution of transitions $\pi^{i\cdot}_{t\delta}$ rationalising the choice.

prices follow a Markov process) would be the point $\pi^{i\cdot}_{t\delta}$ (where t stands for the date up to which the generation t can observe a history of prices given by a sequence denoted by δ)[4] attaining the highest likelihood level curve on the unit simplex *among those consistent with the first-order condition* satisfied by the agent's decision – represented by the plane intersecting the unit simplex in Figure 2. Note that the empirical frequencies of transitions starting from p^i (the number of observed transitions from price p^i to each price p^j over the number of times p^i has realized, depicted as $\bar{\pi}^{i\cdot}_{t\delta}$ in Figure 2) are the beliefs that best explain the observed history if no consistency with the agent's choice was required, but such expectations will typically not be consistent with the agent's behavior.

Therefore, a rationally formed expectations equilibrium in which prices are believed to follow a Markov chain can be defined as follows.

Definition: *A (k-state Markovian)* **Rationally Formed Expectations Equilibrium** *is any collection of prices p^i, consumption plans $(c_1^i, (c_2^j)_{j=1}^k)$, for all $i = 1, \ldots, k$, and beliefs about a Markovian price process $(\pi^{ij}_{t\delta})_{i,j=1}^k$, for every history δ of prices and up to every date t, such that*

(i) *the consumption plan $c_1^i, (c_2^j)_{j=1}^k$ of agents born when p^i, after observing a history* δ *up to t, is the solution to*

$$\max_{c_1^i, (c_2^j)_j} \sum_{j=1}^{k} \pi^{ij}_{t\delta} u\left(c_1^i, c_2^j\right) \qquad [4]$$
$$p^i c_1^i + p^j c_2^j = p^i e_1 + p^j e_2, \quad \forall j$$

4 Each term δ_t has coordinates $\delta_t^i = 1$ if the price at t is p^2, and 0 otherwise.

(ii)

$$c_1^i + c_2^i = e_1 + e_2 \tag{5}$$

for all $i = 1, \ldots, k$, *and*

(iii) *for any other probabilities in*eq. [4] *for which* $c_1^i, (c_2^j)_{j=1}^k$, *is the solution to* eq. [4], *the likelihood of the transitions starting from* p^i *according to history* δ *up to* t *is not higher than the likelihood implied by* $(\pi_{t\delta}^{ij})_{j=1}^k$, *for every history* δ *up to every date* t *and any* $i = 1, \ldots, k$.

Intuitively, as this example illustrates, at a rationally formed expectations equilibrium the expected probabilities $\pi_{t\delta}^{i\cdot}$ will typically be different for different generations, since they will have access to histories of different length or span, and hence the observed empirical frequencies of transition $\bar{\pi}_{t\delta}^{i\cdot}$ will be different for different t's even for a given history δ. Similarly, in the case in which generations are heterogeneous, the need for the agents' expected probabilities to be consistent with their respective choices leaves room for the agents' beliefs to differ among them as well, as soon as their choices differ. The important question now is whether this room for different expectations and beliefs allows for new equilibria that are not rational expectations equilibria.

2.2 Rationally Formed Expectations Equilibria Distinct From Rational Expectations Equilibria

From eq. [5] and the FOCs characterizing necessarily and sufficiently, under standard assumptions, the solution to eq. [4], i.e.

$$\sum_{j=1}^{k} \pi_{t\delta}^{ij} \left[u_1\left(c_1^i, c_2^j\right)\left(c_1^i - e_1\right) + u_2\left(c_1^i, c_2^j\right)\left(c_2^j - e_2\right)\right] = 0 \tag{6}$$

for all $i = 1, \ldots, k$, – that differ from those of a sunspot equilibrium [2] and [3] only in that they make expectations history dependent – one could be tempted to suspect that any rationally formed expectations equilibrium should converge to a sunspot equilibrium, given that in the case in which an objective sunspot process is supposed to drive prices the empirical frequencies of transition between prices would eventually converge to the actual probabilities of transition. As a consequence, there would not be any allocational difference in the long run between sunspot equilibria and rationally formed expectations equilibria, if that was the case. Nevertheless, this is not so: there do exist rationally formed expectations equilibria whose allocations are not sunspot equilibria, nor rational expectations equilibrium allocations of any other kind.

In order to show that rationally formed expectations equilibria do not replicate rational expectations equilibria (in particular from the allocations viewpoint), I will illustrate in this framework the existence of rationally formed expectations equilibria in which prices and consumptions fluctuate between a finite number of values, as in a k-SSE, even if there is no k-SSE with those prices and consumptions.

The argument is constructive, starting from a given k-SSE – whose existence is well understood (see, for instance, Chappori and Guesnerie 1989) – of an overlapping generations economy with a representative agent with utility function u and endowments $e=(e_1, e_2)$. Specifically, consider, for all $i=1, \ldots, k$, a price p^i, first and second period consumptions c_1^i and c_2^i and a Markov matrix of probabilities of transition $\left(\pi^{ij}\right)_{i,j=1}^k$ such that the conditions [2] and [3] for a k-SSE above are satisfied. Hence, the probabilities of transition π^{ij} satisfy, for all $i=1, \ldots, k$, the equation

$$\sum_{j=1}^{k} \pi^{ij} A^{ij} = 0 \tag{7}$$

where the $A^{ij} \equiv u_1(c_1^i, c_2^j)\left(c_1^i - e_1\right) + u_2(c_1^i, c_2^j)(c_2^j - e_2)$ are determined by the consumption plans $(c_1^i, (c_2^j)_{j=1}^k)$ of the given k-SSE. Figure 1 above shows for $k=3$ the linear constraint on the simplex that the equilibrium equations impose on the probabilities of transition from any given price p^i.

Now imagine this was in fact an economy of two *identical* agents a and b per generation, so that u and e are the utility and preferences u^h and e^h of both agents $h=a, b$; and, for all $i, j=1, \ldots, k$, c_1^i and c_2^j are the equilibrium contingent consumptions c_1^{hi} and c_2^{hj} of both $h=a$, b as well. Consider then a nearby economy in which agent b has a utility function u^b that is slightly different from u, while u^a continues to be u. Since u^b is now different from, but close enough to u (in values and, at least, first partial derivatives), then the linear constraints on each row of the Markov matrix generated by the first order conditions of agent b still intersect the simplex but differ from those of agent a. Actually, for some robust perturbations the new linear constraints on the probabilities of transition have no intersection with the old ones *on the unit simplex*, as illustrated in Figure 3 in the case $k=3$.

This implies that for the economies resulting from such perturbations there is no Markov matrix that makes *both* agents a and b choose the *same* contingent consumptions c_1^i and c_2^j whenever facing the prices p^i and p^j, for all $i, j=1, \ldots, k$. Indeed,

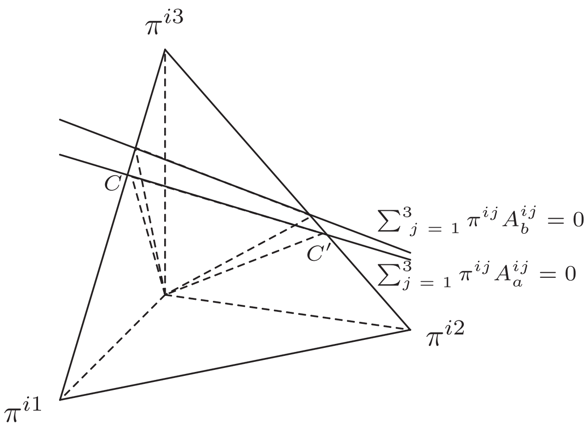

Figure 3: Perturbation in FOC subspace after perturbing the economy.

as long as the perturbation makes the normal vector $A_b^{i\cdot}$ to the second linear subspace in eq. [8] below – following from agent b's first-order conditions – to be

(a) distinct from the vector $A_a^{i\cdot}$ normal to a's linear subspace in eq. [8], and
(b) in the span of $A_a^{i\cdot}$ and the normal vector to the unit simplex $(1, \ldots, 1)$

then the system in $\pi^{i1}, \ldots, \pi^{ik}$

$$\begin{aligned} \sum_{j=1}^{k} \pi^{ij} A_a^{ij} = 0 \\ \sum_{j=1}^{k} \pi^{ij} A_b^{ij} = 0 \end{aligned} \qquad [8]$$

has no solution within the unit simplex – indeed, if $A_b^{i\cdot} = \alpha A_a^{i\cdot} + \beta \mathbf{1}$ with $\beta \neq 0$, then a solution to eq. [8] above would imply $\sum_j \pi^{ij} = 0$. Note that there is a $(k-2)$-dimensional manifold (after normalization) of possible vectors $A_b^{i\cdot}$ satisfying the conditions (a) and (b) above. Of course, any other small enough perturbation of any vector on this manifold would still be such that no k-SSE exists with the given $(c_1^i, c_2^j)_{i,j=1}^k$ as supporting allocation for the corresponding 2-agent overlapping generations economy, so that the property is robust.

Notwithstanding, there do exist rationally formed expectations equilibria over the given support $(c_1^i, c_2^j)_{i,j=1}^k$ for any of the 2-agent overlapping generations economies resulting from such robust perturbations. Indeed, for small enough perturbations the unit simplex still has a nonempty intersection with the linear subspaces following from the agents' first-order conditions and hence, for all h,

δ, and t, there exist probabilities $\left(\pi_{t\delta}^{hij}\right)_{i,j=1}^{k}$ that maximize the likelihood of observing the history δ up to period t – in particular the likelihood $\prod_{j=1}^{k}\left(\pi^{ij}\right)^{\sum_{\tau=1}^{t}\delta_{\tau-1}^{i}\delta_{\tau}^{j}}$ of observed transitions starting from any given p^i, for all $i = 1, \ldots, k$ – among the probabilities of transition in the unit simplex that are consistent with the agents' first-order conditions (the existence, illustrated in Figure 4 below for $k = 3$, is guaranteed by the continuity of the likelihood function and the compactness of the constrained domain).

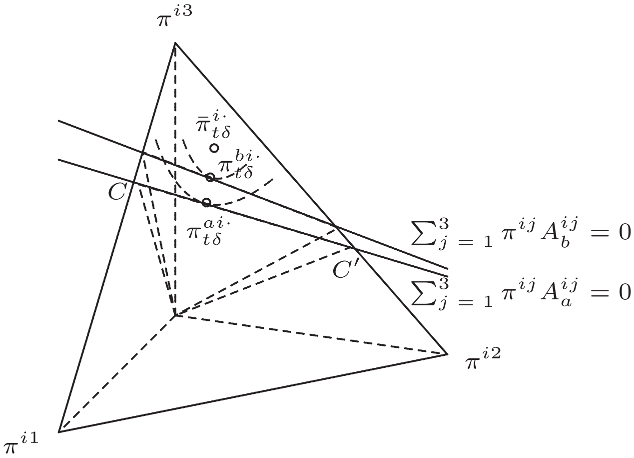

Figure 4: Transition beliefs, consistent with choices, that explain best history.

Thus the allocation, prices, and agent-specific, history-contingent beliefs determined by the perturbed conditions constitute a (k-state Markovian) rationally formed expectations equilibrium whose allocation cannot be that of a rational expectations equilibrium – since, given that the fundamentals are deterministic, if it was a REE allocation it had to be a sunspot equilibrium one, in particular a k-SSE, which cannot be. The next section shows this leading example to be general.

3 Rationally Formed Expectations Equilibria of Overlapping Generations Economies

Consider a deterministic stationary overlapping generations 1-good exchange economy with a representative generation consisting of a number H of

2-period-lived agents with utility function u^h and endowments $e^h = \left(e_1^h, e_2^h\right)$, for all $h = 1, \ldots, H$.

Agents have access to historical records or a memory of length m (maybe infinity), so that they know the price of the good in the last m periods. I will assume more-over, without loss of generality, that the agents *believe* that prices follow a k-state Markov chain over k prices.[5]

In what follows, histories of prices $\{p_t\}_{t \in T}$ (with T either $\mathbb{N}$ or $\mathbb{Z}$) taking at any period any of a finite number k of possible values $p^1, \ldots, p^k$, are denoted by means of a function δ_t^i indicating whether the price p^i has been realized at period t or not. Thus $\delta_t^i = 1$ whenever $p_t = p^i$, and equals 0 otherwise. Since only one price can prevail at any period t, it must hold that $\sum_{i=1}^k \delta_t^i = 1$ for all $t \in T$. Therefore, a history of realizations is a sequence $\delta = \{\delta_t\}_{t \in T}$ of k-tuples of $k-1$ zeros and one 1 at the position of the price realized at that period, that is to say, for all $t \in T$, $\delta_t \in \{0, 1\}^k$ and $\sum_{i=1}^k \delta_t^i = 1$. Let Δ denote the set of such sequences.

A specific instance of a *rationally formed expectations equilibrium* is defined next.

Definition: A (k-state Markovian) **Rationally Formed Expectations Equilibrium** of the deterministic stationary overlapping generations exchange economy with representative generation $\left(u^h, e^h\right)_{h=1}^H$ with memory m consists of

(i) a finite number of positive prices for consumption $p^i > 0$, $i = 1, \ldots, k$[6]

(ii) nonnegative first-period consumptions and contingent plans of second-period consumptions $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ for each agent $h = 1, \ldots, H$ at each possible price when young, i.e. for all $i = 1, \ldots, k$

(iii) beliefs about the probabilities of transition between prices, i.e. a Markov matrix $(\pi_{t\delta}^{hij})_{i,j=1}^k$, for each agent $h = 1, \ldots, H$ and any history of prices $\delta \in \Delta$ up to his date of birth $t \in T$,[7]

5 I will argue below that this assumption is not restrictive, except for the counterfactual case in which memory m is infinite (and there is no first period).

6 This prices are generically distinct for the equilibria shown to exist in Proposition 2 below. Indeed, although a k-SSE can be seen as a k'-SSE with $k' < k$ if some prices are equal, generically in the space of economies, a k-SSE fluctuates between k distinct prices (see Chiappori and Dávila [1996]). Since the existence argument in Proposition 2 for rationally formed expectations equilibria is constructive starting from a k-SSE, this property is inherited by the rationally formed expectations equilibria produced.

7 Note that although with this notation every agent is supposed to hold beliefs about the probabilities of transition after *every* history (i.e. even those beyond his lifespan), only the histories up to the date of his decision are relevant. If memory is finite, the number of histories relevant for the agent's decision is finite, so that he is required to hold only finitely many beliefs. In the infinite memory case this is still the case if there is a first period, but not if there is not one: in that case the number of beliefs would be countable.

such that

(c.1) the allocation is feasible, i.e. for all $i = 1, \ldots, k$

$$\sum_{h=1}^{H} \left(c_1^{hi} + c_2^{hi}\right) = \sum_{h=1}^{H} \left(e_1^h + e_2^h\right) \qquad [9]$$

(c.2) for any history $\delta \in \Delta$ and every agent $h = 1, \ldots, H$ born at any date $t \in T$, his first-period consumption and contingent plan of second-period consumptions $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ are optimal, given his beliefs, whenever at t the price is p^i, for any $i = 1, \ldots, k$, so that it solves

$$\begin{aligned} &\max \quad \sum_{j=1}^{k} \pi_{t\delta}^{hij} u^h \left(c_1^i, c_2^j\right) \\ &\text{s.t.} \quad p^i\left(c_1^i - e_1^h\right) + p^j\left(c_2^j - e_2^h\right) = 0, \quad \forall j \end{aligned} \qquad [10]$$

(c.3) for any history $\delta \in \Delta$ and every agent $h = 1, \ldots, H$ born at any date $t \in T$, no other beliefs for which $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ is optimal when at t the price is p^i, for any $i = 1, \ldots, k$, provide a higher likelihood to the history of prices he remembers, i.e. if $\pi^{i\cdot} \in S^{k-1}$ (the $k-1$-dimensional unit simplex) is such that $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ solves

$$\begin{aligned} &\max \quad \sum_{j=1}^{k} \pi^{ij} u^h \left(c_1^i, c_2^j\right) \\ &\text{s.t.} \quad p^i\left(c_1^i - e_1^h\right) + p^j\left(c_2^j - e_2^h\right) = 0, \quad \forall j \end{aligned} \qquad [11]$$

then

$$\prod_{j=1}^{k} \left(\pi^{ij}\right)^{\sum_{\tau=1}^{m} \delta_{t-\tau}^i \delta_{t-\tau+1}^j} \le \prod_{j=1}^{k} \left(\pi_{t\delta}^{hij}\right)^{\sum_{\tau=1}^{t} \delta_{t-\tau}^i \delta_{t-\tau+1}^j} \qquad [12]$$

where $\delta_{t-\tau}^i = 0$ for $\tau \ge t$ if $T = \mathbb{N}$ – i.e. the likelihood of the observed transitions from p^i in history δ up to period t if prices follow the Markov chain $\left(\pi_{t\delta}^{hij}\right)_{i,j=1}^{k}$ (the RHS in eq. [12]), is at least as high as for any other Markov chain $\left(\pi^{ij}\right)_{i,j=1}^{k}$ (the LHS in eq. [12]) – and

(c.4) when $T = \mathbb{Z}$ and $m = \infty$, for any history $\delta \in \Delta$ and every agent $h = 1, \ldots, H$ born at any $t \in T$, his beliefs are not falsified by the information available then, i.e. for all $i, j = 1, \ldots, k$,

$$\pi_{\delta t}^{hij} = \lim_{t' \to -\infty} \frac{\sum_{\tau=t'}^{t-1} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=t'}^{t-1} \delta_\tau^i} \qquad [13]$$

whenever the limit exists.

Some remarks on the definition above are in order. Note first that if the beliefs are constrained to be history and agent independent (so that $\pi_{\delta t}^{hij}$ becomes π^{ij}) and the last conditions (c.3) and (c.4) are dropped, then the definition above becomes that of a stationary rational expectations (sunspot) equilibrium following a k-state Markov chain, or k-SSE.[8] Note that condition (c.4) is trivially satisfied by such a k-SSE but, crucially, (c.3) is not. As a consequence, in a rational expectations equilibrium there exist typically, for every agent, beliefs about the probabilities of transition that are consistent with his consumption choice but that make the history he observes likelier than the equilibrium beliefs do. Of course the discrepancy between the agents' beliefs and those maximizing the likelihood of history while rationalizing the choices vanishes in the limit if, as in the sunspot equilibrium interpretation, the prices are supposed to actually follow a given Markov chain. But the determination of prices by a specific stochastic process is difficult to justify in the absence of shocks to the fundamentals.

Secondly, condition (c.3) is not superfluous. If instead of condition (c.3) only the existence of subjective beliefs rationalizing the agents' choices was required (regardless of history), that would imply a set of equilibrium allocations and prices that is a strict superset of the set of rationally-formed expectations equilibria. In effect, while any rationally-formed expectations equilibrium clearly satisfies the existence of subjective beliefs rationalizing the agents' choices, there are consumption plans, prices and arbitrary, history-independent subjective beliefs rationalizing the agents' choices that are not rationally-formed expectations equilibria, since history-independent beliefs cannot solve the problem [23] in the proof of Proposition 2 below – equivalent to condition (c.3) – *for all realizations of history*.

Finally, note also that, as previously claimed, the restriction to beliefs in Markovian prices is not constraining for finite memory or $T = \mathbb{N}$. In effect, such an assumption cannot be refuted by the agents unless the data available to them is able to falsify it, but for that to be the case it must at least allow to establish that the empirical frequencies are not Cauchy (if the sequence of empirical frequencies of transitions from any p^i to p^j was Cauchy, then completeness

8 See, for instance, Azariadis (1981), Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1988, 1989), Guesnerie (1986). On the notion of sunspot equilibrium see Cass and Shell (1983).

would imply its convergence, which would support the Markovian assumption). That is to say, it must allow to conclude that the distance between any two empirical frequencies at dates $t < t'$ from any p^i to p^j does not become arbitrarily small, for t, t' sufficiently far away down the sequence. In other words, for the agents to be able to discard the assumption of Markovian prices they would need to have infinite histories and memories, so that no data can falsify that assumption if $T = \mathbb{N}$ or m is finite (on the contrary, when $T = \mathbb{Z}$ and m is infinite, the agents can compute the empirical frequency at any given date t of the transitions from any price p^i to p^j as the limit

$$\lim_{t' \to -\infty} \frac{\sum_{\tau = t'}^{t-1} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau = t'}^{t-1} \delta_\tau^i}.$$

Should this limit not exist, the Markovian assumption would then be falsified by the data in this case).

As a matter of fact, for any two consecutive terms the distance between the empirical frequencies of transitions converges to zero along the sequence, since

$$\left| \frac{\sum_{\tau=1}^{t+1} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=1}^{t+1} \delta_\tau^i} - \frac{\sum_{\tau=1}^{t} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=1}^{t} \delta_\tau^i} \right| = \frac{\delta_{t+1}^i}{\sum_{\tau=1}^{t+1} \delta_\tau^i} \cdot \left| \delta_{t+2}^j - \frac{\sum_{\tau=1}^{t} \delta_\tau^i \delta_{\tau+1}^j}{\sum_{\tau=1}^{t} \delta_\tau^i} \right| \qquad [14]$$

and

(1) either p^i is visited finitely many times and then for some t onwards $\delta_t^i = 0$, so that the distance between the empirical frequencies of transition becomes 0 from that term on, and the empirical frequency of transition from p^i to p^j becomes constant and therefore convergent,

(2) or p^i is visited countably many times and then the first factor in the right-hand side converges to zero (the numerator is bounded and the denominator is both non-decreasing and not non-increasing), while the second factor between brackets is bounded in [0, 1] (the first term is in {0, 1} and the second is in [0, 1]), so that the distance between empirical frequencies of transition from p^i to p^j converges to zero.

Thus, when $T = \mathbb{N}$ and agents have unrestricted memory, not only the agents do not have enough information to falsify the Markovian prices assumption, but also they will see vanish progressively any dependence of the probabilities of transition on earlier prices (as differences between subsequent empirical frequencies converge to zero), i.e. Markovian prices tend to be confirmed (although not proved), rather than falsified.

Of course, agents can all believe in Markovian prices while not necessarily agreeing on the specific probabilities of transition governing that process, since they have access to different bits of history when $T = \mathbb{N}$ or memory is finite. On the contrary, if memory is infinite and $T = \mathbb{Z}$, they all have to agree on the probabilities of transitions as well if the limit in eq. [13] above exists for every t; while, if memory is infinite and $T = \mathbb{N}$, they all "eventually agree", meaning that discrepancies of subsequent generations tend to vanish. In the last two cases, in which agents agree (maybe asymptotically) on the probabilities of transition, the limit of the empirical frequencies would necessarily have to be in the intersection on the unit simplex of the linear subspaces determined by the agents' first-order conditions, as proclaimed in Proposition 1 below (the proof is straightforward). In other words, if memory is infinite, the only rationally formed expectations equilibria are those for which such an intersection exists, but these equilibria are allocationally equivalent to the rational expectations (sunspot) equilibrium associated with such an intersection.

Proposition 1: *If the agents' memory m is infinite, any rationally formed expectations equilibrium of the stationary deterministic overlapping generations exchange economy* $\left(u^h, e^h\right)_{h=1}^{H}$ *is allocationally equivalent to a k-state sunspot equilibrium.*

Rationally formed expectations equilibria distinct from a rational expectations equilibrium exist in this setup, therefore, only if memory is finite. There can be many reasons why m finite is the relevant case. People tend to make forecasts based on their recent experiences, with memories of variable lengths, but certainly of finite length if only because of their actual limited recording and computing abilities. Thus the limited memory case seems to be the relevant one, while the equivalence of rationally formed expectations equilibria and rational expectations sunspot equilibria in the infinite memory case rather highlights the role played by limited knowledge in making possible rationally formed expectations equilibria distinct from rational expectations equilibria.

The next proposition establishes the main result of the paper, namely that any deterministic stationary overlapping generations economy with sunspot equilibria can be perturbed robustly in order to produce rationally formed expectations equilibria that no sunspot equilibrium can match.

Proposition 2: *Arbitrarily close*[9] *to every deterministic stationary overlapping generations economy (with at least two agents of a given type) with a k-state stationary sunspot equilibrium, there exists an economy with finite-memory rationally formed expectations equilibria distinct from any rational expectations equilibrium.*

9 In the topology of C^1-convergence over compacta in the space of utility functions.

Proof: Let $\left(u^h, e^h\right)_{h\in H}$ be the utility and endowments *types* of the members of the representative generation of a stationary overlapping generations economy, with at least one type of agents h_0 with two agents or more. Let $\left\{p^i, \left(\bar{c}_1^{hi}, \bar{c}_2^{hi}\right)_{h\in H}\right\}_{i=1}^k$ be the contingent prices and consumptions of a k-state stationary sunspot equilibrium of the economy driven by a Markov chain with probabilities of transition $\left(\pi^{ij}\right)_{i,j=1}^k$.

Consider a new economy with a representative generation consisting of replacing in H just one agent of type h_0 by an agent h_1 with the same endowments and consumptions as agent h_0 – the new allocation of the new economy is therefore feasible – and a utility function u^{h_1} with gradients at the consumption bundles $(\bar{c}_1^{h_1 i}, \bar{c}_2^{h_1 j})_{i,j=1}^k$ such that, for some $i = 1, ..., k$, the vector of products $A_{u^h}^{ij} \equiv D_1 u^h(\bar{c}_1^{hi}, \bar{c}_2^{hj})\left(\bar{c}_1^{hi} - e_1^h\right) + D_2 u^h(\bar{c}_1^{hi}, \bar{c}_2^{hj})(\bar{c}_2^{hj} - e_2^h)$ – where $D_i u^h$ stands for the partial derivative of u^h with respect to c_i – of the gradient of utility and the trade implied by the planned consumptions for each possible transition starting from i for agent h_1

$$A_{u^{h_1}}^i = \left(A_{u^{h_1}}^{i1}, \ldots, A_{u^{h_1}}^{ik}\right) \tag{15}$$

is distinct from, but in the span of, $A_{u^{h_0}}^i$ and $\mathbf{1} = (1, ..., 1)$, i.e.

$$A_{u^{h_1}}^i = \alpha A_{u^{h_0}}^i + \beta \mathbf{1} \tag{16}$$

for some α and some $\beta \neq 0$.[10] Then the system

$$\begin{aligned} \pi^{i1} A_{u^{h_0}}^{i1} + \cdots + \pi^{ik} A_{u^{h_0}}^{ik} &= 0 \\ \pi^{i1} A_{u^{h_1}}^{i1} + \cdots + \pi^{ik} A_{u^{h_1}}^{ik} &= 0 \end{aligned} \tag{17}$$

has no solution in the probabilities $\pi^{i1}, \ldots, \pi^{ik}$. Indeed, should there be one, using eq. [16] above, the second equation in [17] can be written equivalently as

$$\alpha\left(\pi^{i1} A_{u^{h_0}}^{i1} + \cdots + \pi^{ik} A_{u^{h_0}}^{ik}\right) + \beta\left(\pi^{i1} + \cdots + \pi^{ik}\right) = 0 \tag{18}$$

but from the first equation in [17] and $\beta \neq 0$, then it should hold

$$\pi^{i1} + \cdots + \pi^{ik} = 0 \tag{19}$$

which cannot be since these probabilities should add up to 1. Since a solution to eq. [17] is needed for the given allocation to be that of a sunspot equilibrium,

10 Note that, since $\sum_{j=1}^k \pi^{ij} A_{u^{h_0}}^{ij} = 0$, the vector $A_{u^{h_0}}^i$ cannot be collinear to $\mathbf{1}$. Moreover there is a 1-dimensional manifold of *directions* that the vector $A_{u^{h_1}}^i$ can take while satisfying condition [16].

this establishes that the prices and consumptions $\left\{p^i, \left(\bar{c}_1^{hi}, \bar{c}_2^{hi}\right)_{h\in H}\right\}_{i=1}^{k}$, with $\left(\bar{c}_1^{h_1 i}, \bar{c}_2^{h_1 j}\right)_{i,j=1}^{k} = \left(\bar{c}_1^{h_0 i}, \bar{c}_2^{h_0 j}\right)_{i,j=1}^{k}$, are *not* those of a sunspot equilibrium of the new, arbitrarily close economy.

They are, nevertheless, the allocation and prices of a rationally formed expectations equilibrium of an arbitrarily close economy. In effect, for all $h \in H$, all $t \in T$, and all $\delta \in \Delta$, there exists $\left(\pi_{t\delta}^{hij}\right)_{i,j=1}^{k}$ solution to

$$\begin{aligned} &\max_{\pi^{ij}} \prod_{i,j=1}^{k} \left(\pi^{ij}\right)^{\sum_{\tau=1}^{m} \delta_{t-\tau}^{i} \delta_{t-\tau+1}^{j}} \\ &\text{s.t. } \forall i,\ \pi^{i\cdot} \in S^{k-1} \\ &\qquad \forall i, \left(\bar{c}_1^{hi}, \left\{\bar{c}_2^{hj}\right\}_j\right) = \arg\max \sum_j \pi^{ij} u^h\left(c_1^i, c_2^j\right) \\ &\qquad\qquad\qquad\qquad \text{s.t. } p^i\left(c_1^i - e_1^h\right) + p^j\left(c_2^j - e_2^h\right) = 0, \forall j \end{aligned} \qquad [20]$$

– where $\delta_{t-\tau}^{i} = 0$ for $\tau \geq t$ if $T = \mathrm{N}$ – since the objective function is continuous, and the constrained set is non-empty and compact. The same is true for any $\tilde{u}^{h_1}$ close enough to u^{h_1} in the topology of C^1-convergence over compacta, i.e. with $A_{\tilde{u}^{h_1}}^{i}$ not necessarily in the span of $A_{u^{h_0}}^{i}$ and $\mathbf{1}$.

Finally, since m is finite, the remembered empirical frequencies of the transitions do not falsify the agents' beliefs.

Q.E.D.

4 Epistemic Status of Rationally Formed Expectations Equilibria

This section intends to compare the epistemic status of rational expectations equilibria and rationally formed expectations equilbria in the previous sections, where the economy considered is a sequential markets economy in which an objective process may be driving prices (or even fundamentals). In such a setup rational expectations require therefore all agents to hold expectations that follow from a common belief on a single process that happens to be the actual process driving prices or fundamentals. Nonetheless, the comparison itself is done, for the sake of simplicity, in a setup that abstracts from the demographics and infinities of the actual overlapping generations economies of Sections 2 and 3. An additional assumption of a common prior is crucial to make the connection with the equilibria studied in Sections 2 and 3.

Specifically, in order to compare the epistemic requirements of a rationally formed expectations equilibrium with those of rational expectations equilibria, I will discuss it here within a model specifying an interactive system of beliefs and higher order beliefs. For the sake of clarity, this will be done without specifying the cardinality of agents and goods, as well as the demographics of the economy. Thus the case where the number of states, goods and agents is not finite, and agents may be endowed with, and have preferences on, only a few goods (of which the overlapping generations setup is an instance) is therefore comprised in the following discussion. Obviously, sums would stand for the adequate aggregations over measures when infinities are involved. As it will be seen below, the equilibrium notions do not depend conceptually on these details, while there is a clear notational advantage in overlooking them at this stage.

A definition of a rational expectations equilibrium in an implicitly sequential markets setup is next first, followed by a discussion of its epistemic implicit assumptions.

Definition 1: *Given an objective probability distribution* π *over a set of states of nature* S *and an economy* $\left\{\left(u_s^h, e_s^h\right)_{s\in S}, I^h\right\}_{h\in H}$ *(where, for all* $h \in H$*,* I^h *is a partition of* S*)*[11] *a* **rational expectations equilibrium** *is a set of contingent consumptions, beliefs, and prices* $\left\{\left(c_s^h, \pi_s^h\right)_{h\in H, p_s}\right\}_{s\in S}$ *such that*

(a.1) *for all* $s \in S$,

$$\sum_{h\in H}\left(c_s^h - e_s^h\right) = 0 \tag{21}$$

(a.2) *for all* $h \in H$ *and all* $s \in S$,

$$\begin{gathered}\left(c_{s'}^h\right)_{s'\in F_s^h} \in \arg\max_{(c_{s'})_{s'\in F_s^h}} \sum_{s'\in F_s^h} \pi_{s'}^h u_{s'}^h(c_{s'}) \\ p_{s'}\left(c_{s'} - e_{s'}^h\right) \le 0, \forall s' \in F_s^h\end{gathered} \tag{22}$$

c is F^h-measurable

where $F^h = I^h \vee p^{-1}(p)$ *is the partition of* S *such that* $F_s^h = I_s^h \cap p^{-1}(p_s)$, $\forall s \in S$, *and*

11 The endowments e_s^h are measurable with respect to I^h, so that agents know their endowments. In what follows I_s^h denotes the element of the partition I^h containing (state) s. Also, for all s and all h, u_s^h has the usual properties.

$$\pi^h_{s'} = 0 \ if \ s' \notin F^h_s \, and \ \pi^h_{s'} = \frac{\pi_{s'}}{\sum_{s'' \in F^h_s} \pi_{s''}} \ otherwise$$

Note that, at a rational expectations equilibrium, the resulting $\left(c^h_s\right)_{s \in S}$ is required to be $[I^h \vee p^{-1}(p)]$-measurable, where $p^{-1}(p_s)$ denotes the partition $\{p^{-1}(p_s)\}_{s \in S}$ of S induced by prices, and $p \in \left(\mathbb{R}^L_{++}\right)^S$ stands for the function assigning p_s to s. (Indeed, h's choice in state s in eq. [22] must be contingent to the information available to the agent, so that $c^h_{s'}$ has to be the same for all s' in the element I^h_s of his private information partition I^h of S that contains s, and in the element $p^{-1}(p_s)$ of the partition induced by prices containing the price p_s observed at s, i.e. for all s' in $I^h_s \cap p^{-1}(p_s)$. As a consequence, h's choice $\left(c^h_s\right)_{s \in S}$ needs to be, as a function of s, measurable with respect to the join $I^h \vee p^{-1}(p)$ of h's private information and the partition induced by prices). Thus condition (a.2) in Definition 1 prevents agents that are (possibly asymmetrically) uncertain about the state of nature s from conditioning on things they cannot see, either directly through their information partition I^h or by being revealed by prices.

Implicitly in the previous definition each agent h obviously knows at least $\left\{\pi^h_s, u^h_s, e^h_s, p_s\right\}_{s \in S}$ and I^h – otherwise his choice could not be modeled as in (a.2) above – other than this, the agents do not need to have any further knowledge at a rational expectations equilibrium as defined above. In particular, no common knowledge of anything is needed to sustain a rational expectations equilibrium (some common knowledge has nonetheless been required to address some strong features of the definition above, like the need of agents to know the entire price function $(p_s)_{s \in S}$ and the generic full revelation of prices – see McAllister (1990) – but as far as the epistemic requirements of rational expectations equilibria as defined above is concerned, nothing more than $\left\{\pi^h_s, u^h_s, e^h_s, p_s\right\}_{s \in S}$ and I^h for each agent h is required).[12] Notwithstanding, at a rational expectations equilibrium each agent h knows implicitly more than just $\left\{\pi^h_s, u^h_s, e^h_s, p_s\right\}_{s \in S}$ and I^h: he actually knows that *and what he can deduce from*

12 This has a parallel in the epistemic conditions for a Nash equilibrium characterized in Aumann and Brandenburger (1995). In effect, as the authors point out there, Nash equilibria – understood, as usual, as profiles of randomizations over pure strategies – require (besides the agents' rationality) only the knowledge by the agents of their own payoffs and their *mutual* knowledge of each other's strategies. Interestingly enough, again no common knowledge of anything is actually required (common knowledge is only required to make sense of the interpretation of Nash equilibria as profiles of commonly held *conjectures* about each player's action, and this only when there are at least three players). In the current context, this amounts to the agents knowledge of the elements determining (and constraining) their payoffs, i.e. $\left\{\pi^h_s, u^h_s, e^h_s, p_s\right\}_{s \in S}$ (the assumed price-taking behavior voiding of content in this case the requirement of mutual knowledge of each others' decisions).

that. Indeed, firstly an agent h is able to tell, at any given state s, whether an event $E \subset S$ has happened or not, based on I^h, if, and only if, he assigns a probability to E conditional to I_s^h of either 1 or 0 respectively, i.e. if, and only if, $I_s^h \subset E$ or $I_s^h \subset E^C$ respectively, where $E^C = S \backslash E$ (more generally, he attaches at I_s^h a probability $P(E|I_s^h)$ to any event E). Nevertheless, the knowledge of the equilibrium prices $(p_s)_{s\in S}$ allows him to tell as well whether an event has happened or not based also on the partition $\{p^{-1}(p_s)\}_{s\in S}$ induced by prices. Of course this means that at a rational expectations equilibrium h is able to tell whether E has happened or not at state s if, and only if, $I_s^h \cap p^{-1}(p_s) \subset E$ or $I_s^h \cap p^{-1}(p_s) \subset E^C$ respectively (more generally, he attaches at I_s^h and p_s a probability $P(E|I_s^h \cap p^{-1}(p_s))$ to event E), which allows him to notice (and hence condition on) a bigger set of events than with I^h alone.

As the definition above makes clear, a defining feature of a rational expectations equilibrium is that agents are supposed to share the same prior π over the states of nature. At a rationally formed expectations equilibrium this requirement is dropped instead, and just a rational use of the available information (I^h and p for each agent h) is required. A formal definition on a rationally formed expectations equilibrium is next, after which we discuss how its implicit epistemic assumptions compare to those of the rational expectations equilibria.

Definition 2: *Given a set of states of nature S and an economy $\left\{(u_s^h, e_s^h)_{s\in S}, I^h\right\}_{h\in H}$ (where, for all $h \in H$, I^h is a partition of S)*[13] *a* **rationally formed expectations equilibrium** *is a set of contingent consumptions, beliefs, and prices $\left\{(c_s^h, \pi_s^h)_{h\in H}, p_s\right\}_{s\in S}$ such that*

(a.1) *for all $s \in S$,*

$$\sum_{h\in H} (c_s^h - e_s^h) = 0 \tag{23}$$

(a.2) *for all $h \in H$ and all $s \in S$,*

$$\begin{gathered} (c_{s'}^h)_{s'\in F_s^h} \in \arg\max_{(c_{s'})_{s'\in F_s^h}} \sum_{s'\in F_s^h} \pi_{s'}^h u_{s'}^h(c_{s'}) \\ p_{s'}(c_{s'} - e_{s'}^h) \le 0, \forall s' \in F_s^h \end{gathered} \tag{24}$$

where $F_s^h = I_s^h \cap p^{-1}(p_s)$

(a.3) *for all $h \in H$, all $s \in S$, and all π such that, for all $\tilde{s} \in S$,*

13 The endowments e_s^h are measurable with respect to I^h, so that agents know their endowments.

$$\left(c_{s'}^h\right)_{s'\in F_{\tilde{s}}^h} \in \arg\max_{(c_{s'})_{s'\in F_{\tilde{s}}^h}} \sum_{s'\in F_{\tilde{s}}^h} \pi_{s'} u_{s'}^h(c_{s'})$$
$$p_{s'}\left(c_{s'} - e_{s'}^h\right) \le 0, \forall s' \in F_{\tilde{s}}^h \qquad [25]$$

it holds that

$$\sum_{s'\in F_s^h} \pi_{s'} \le \sum_{s'\in F_s^h} \pi_{s'}^h \qquad [26]$$

As opposed to the previous Definition 1 of a rational expectations equilibrium, the agents are not required to hold the same prior anymore, but still a condition (a.3) imposes a consistency that prevents agents to hold arbitrary priors, namely that no other beliefs rationalizing the agents choices attach a higher likelihood to the observed event.

It is worth noting that, firstly, the constructive argument used to establish Proposition 2 allows to establish in this setup too the existence of rationally formed expectations equilibria distinct form rational expectations equilibria for economies arbitrarily close to an economy with a rational expectations equilibrium (see Radner [1979] and Spear and Srivastava [1986]), as stated in Proposition 3 below, whose proof is analogous to that of Proposition 2.

Proposition 3: *Arbitrarily close*[14] *to every economy with a rational expectations equilibrium there exists an economy* $\left\{\left(u_s^h, e_s^h\right)_{s\in S}, I^h\right\}_{h\in H}$ *with rationally formed expectations equilibria distinct from any rational expectations equilibrium.*

Secondly, note that from the definitions neither rational expectations implies rationally formed expectations, nor conversely: since (a.1) and (a.2) are the same for both equilibrium concepts but neither the common prior implies (a.3) nor the other way round, then none of the two equilibrium notions is a particular case of the other. In effect, for a rationally formed expectations equilibrium to be a rational expectations equilibrium all the agents would have to hold a common prior on the state of the world, which need not be the case. Also for a rational expectations equilibrium to be a rationally formed expectations equilibrium the condition (a.3) above needs to be satisfied, which again needs not be the case for any given rational expectations equilibrium.

Nonetheless, even though none of the two equilibrium notions is a particular case of the other, still according to the definitions above the two share the same epistemic requirements, since what is implied about the agents' knowledge by the definition of a rationally formed expectations equilibrium is the

14 In the topology of C^1-convergence over compacta in the space of utility functions.

same as in a rational expectations equilibrium. Indeed, what the agents are supposed to know at a rationally formed expectations equilibrium, as well as what they can deduce from that knowledge, is – as in the case of a rational expectations equilibrium – determined only by their information partitions I^h, for each h, and the partition $\{p^{-1}(p_s)\}_{s\in S}$ induced by prices, which is the information about the state of nature conveyed by prices. Neither the fact that in a rationally formed expectations equilibrium the agents may hold different priors about the state of nature, nor its condition (a.3) adds anything that is not already implicit in the knowledge by each agent h of $\{\pi_s^h, u_s^h, e_s^h, p_s\}_{s\in S}$ and I^h and what is implied by this.

In order to see that, on top of common knowledge not being necessary to sustain neither a rational expectations equilibrium nor a rationally formed expectations equilibrium, the two concepts are actually more stringent than common knowledge of rationality and market clearing,[15] consider each agent to be of one among several types that, while having no impact on the fundamentals, can nonetheless be relevant for the equilibrium, if only because the agents may believe that opponents of different types may behave differently. Making this possibility explicit calls for a system of beliefs and high order beliefs about the other agents beliefs in which the consistency with common knowledge of rationality and of market clearing can be addressed. This, of course, leaves room for the agents to hold different beliefs about prices in different states of the world (which now include the profile of agents' types $(t^h)_{h\in H}$ alongside the state of nature s) thus necessarily departing from the rational expectations equilibrium notion. Introducing the adequate notation, it is straightforward to see that any rational expectations equilibrium is consistent with (although does not require) common knowledge of rationality and market clearing. The fact that the converse is not true is precisely what has been established for finite exchange economies with asymmetric information in Ben-Porath and Heifetz (2010).

Similarly, the Definition 2 of a rationally formed expectations equilibrium can be extended to include a system of interactive beliefs guaranteeing common knowledge of rationality, market clearing, and belief formation rationality. Since any rationally formed expectations equilibrium is straightforwardly consistent with (although does not require) the conditions for common knowledge of rationality, market clearing, and belief formation rationality, and, according to Proposition 2, there are rationally formed expectations equilibria distinct from

15 For rational expectations equilibria this has been established in Ben-Porath and Heifetz (2010) for finite exchange economies with asymmetric information.

any rational expectations equilibrium, adding common knowledge of belief formation rationality to that of rationality and market clearing is still not enough to guarantee rational expectations equilibrium outcomes.

Finally, it is important to realize also that deterministic environments do not necessarily imply deterministic equilibrium allocations and prices. Indeed, in the definition of a rational expectations equilibrium provided, the fundamentals u_s^h and e_s^h might actually not depend on the state of the world s (which is then a sunspot) and the economy might still have non-deterministic rational expectations equilibria, i.e. sunspot equilibria. This is a well-known fact that follows from the sufficient characterization by Cass and Shell (1983) of the conditions under which sunspots *do not* matter (basically those of a finite, convex, complete markets Arrow-Debreu economy), which essentially opened the path towards establishing subsequently that sunspots *do* matter in almost any other setup. That the same can be said about rationally formed expectations equilibria is what this paper establishes. Note once more that the epistemic status of both rational expectations and rationally formed expectations equilibria, being the same, has no import on this fact.

5 Discussion

Firstly, given that Proposition 2 establishes that rationally formed expectations equilibria can account for more fluctuations than rational expectations equilibria, one would like to have an idea of where do the limits of this expansion lay or, at least, whether the proposed equilibrium notion does not go too far as to be able to rationalize *any* fluctuations as an equilibrium phenomenon. In order to see that not anything can be made into a rationally formed expectations equilibrium, consider a feasible allocation of consumptions c_1^i, c_2^i, for all $i = 1, \ldots, k$, such that for some agent and some price p^i, it holds that all his trades contingent to any price p^j he may face in his second period of life imply a higher marginal rate of substitution of future for present consumption than the corresponding implicit relative price, i.e.

$$A^{ij} \equiv D_1 u\left(c_1^i, c_2^j\right)\left(c_1^i - e_1\right) + D_2 u\left(c_1^i, c_2^j\right)\left(c_2^j - e_2\right) < 0 \qquad [27]$$

for all $j = 1, \ldots, k$. For this to happen, it suffices – in the case the marginal rate of substitution is smaller than 1 at the endowments point – that c_1^i is small enough whenever the solutions are guaranteed to be always interior. Then the set of expectations consistent with the agent's choice of c_1^i when facing p^i in his first period of life is empty (the first-order conditions of eq. [11] in the definition cannot be satisfied, since the associated hyperplane does not intersect the unit simplex, its normal direction being in the strictly positive orthant). As a consequence, no fluctuations between the feasible allocation of such consumption levels c_1^i, c_2^i, for all $i=1,...,k$, can result from a rationally formed expectations equilibrium.

Finally, the rationality condition considered here on the formation of expectations seems reminiscent of the one underlying the rational beliefs equilibrium concept of Kurz (1994a, 1994b). Nonetheless, rationally formed expectations differ essentially from Kurz's rational beliefs. The two concepts only share the idea that the rationality of expectations or "beliefs should be defined relative to what is learnable from the data" (Kurz (1994b), p. 879). Otherwise, Kurz (1994b) requires the agents to believe that prices are driven by a process whose *long term* behavior coincides with that of the true process. Leaving aside the problem posed by the ad hoc character of such a true process in the pure extrinsic uncertainty case, in order to infer such long term behavior Kurz (1994b) assumes that the agents have access to infinitely long histories of past prices, a formidable feat that the rationally formed expectations equilibrium deliberately avoids to assume.

Also Hommes (1998) and Hommes and Sorger (1998) introduce in a different setup an equilibrium notion, the consistent expectations equilibrium, which imposes as well a condition of consistency with available data, namely the zero (limit of) autocorrelations of errors made in past forecasts based on history, so that they cannot be distinguished from white noise. Note however that in a consistent expectations equilibrium agents try to forecast a relevant variable, say a price, while in the setup considered here they try to forecast the *probability distribution* of that variable. Also the consistent expectations equilibrium notion makes implicitly the counterfactual assumption, as in Kurz (1994a, 1994b), that infinitely many records of past realizations of this variable are always available and agents have an infinite memory and computation ability allowing to process them, otherwise finite sample autocorrelations of a given number of lags will always be typically nonzero.

Acknowledgements: The author wants to thank very insightful and useful remarks from Edward J. Green and two anonymous referees on a previous version of this paper.

Funding: Research funding from the research grants MIS-Ulysse 6007.09 and PDR T.0044.13 of the Belgian F.R.S-FNRS is gratefully acknowledged.

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