Contributions

Yasushi Asako*

One-Sided Games in a War of Attrition

Abstract: This study develops a war-of-attrition model with the asymmetric feature that one player can be defeated by the other but not vice versa; that is, only one player has an exogenous probability of being forced to capitulate. With complete information, the equilibria are almost identical to the canonical war-of-attrition model. On the other hand, with incomplete information on a player's robustness, a war where both players fight for some duration emerges. Moreover, a player who is never defeated may capitulate in equilibrium, and this player will give in earlier if the other player's fighting costs are greater.

Keywords: war, attrition, Bayesian learning, asymmetric robustness
JEL Classification: C72, D82, D83

DOI 10.1515/bejte-2014-0078

1 Introduction

This study expands upon Maynard Smith's (1974) war-of-attrition model by introducing a new feature where one player can be defeated by the other but not vice versa. To be precise, only one player has an exogenous probability of being forced to capitulate and never fighting again (i.e. of being *defeated*).

The war-of-attrition model has been applied to many topics, such as price wars and exits (Kreps and Wilson, 1982; Ghemawat and Nalebuff, 1985; Fudenberg and Tirole, 1986), patent races (Fudenberg et al. 1983), public goods provisions (Bliss and Nalebuff, 1984), labor strikes (Kennan and Wilson, 1989), and real wars (Langlois and Langlois, 2009). However, these studies fail to account for a situation in which only one player may be defeated.

***Corresponding author: Yasushi Asako,** School of Political Science and Economics, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan, E-mail: yasushi.asako@aoni.waseda.jp

A fitting example is a war against terrorism. In this conflict, only the terrorist group faces the possibility of being defeated because the targeted state has a far stronger military and substantially more resources. However, despite the possibility of defeat, the terrorist group may still decide to attack, which, in turn, may lead the targeted state to compromise with them.[1] Another possible example is a price war (or patent race) between a large firm and a small store. In such a competition, the small store faces the possibility that financial institutions may not lend them additional money, whereas a large firm usually has many channels for funding.

This study analyzes two-player models with both complete and incomplete information. Both players are at war, and their strategic variable is the timing of their capitulation. The war continues until one of the players either concedes or is defeated. Suppose that player 2 may be defeated by player 1, but not vice versa. With complete information, the equilibria are almost identical to the canonical war-of-attrition model: either player gives in immediately or a war endures as long as the players choose mixed strategies. Unlike in the standard model, there is a unique equilibrium where player 2 immediately surrenders when player 1 has a sufficiently greater benefit and lower cost.

In the incomplete information model, player 2 knows his/her robustness, but player 1 does not. A class of equilibria then emerge where both players fight for some duration, so an enduring war that lasts for an indeterminate amount of time can emerge. Our innovation lies in player 1's Bayesian learning of his/her adversary's robustness. As the war wages on, player 1 updates his/her belief regarding player 2's robustness. A prolonged war, thus, indicates to player 1 that player 2 is harder to defeat than originally anticipated. Thus, player 1 prefers to fight in the early periods but gives in when the war is prolonged, and player 2 has an incentive to wait until player 1 gives in. Thus, even in one-sided games, an invincible player may concede in equilibrium. Moreover, player 1 may capitulate earlier if player 2's fighting cost is greater and benefit from winning is lower. This is because, under these circumstances, a weaker player 2 would avoid fighting; therefore, by doing so any way, player 1 would be led to believe that he/she was fighting against a stronger opponent.

---

1 Attrition is a major strategy for terrorist groups. For example, the Irish Republican Army (IRA) explicitly included attrition among its primary strategies in its manual, the *IRA Green Book*, which states: "[a] war of attrition against enemy personnel… is aimed at causing as many casualties as possible so as to create a demand from their people at home for their withdrawal (O'Brien, 1999)." Kydd and Walter (2006) also recognize attrition as one of the four strategies deployed by terrorists to influence their target state's policies.

1.1 Related Literature

The war-of-attrition model with complete information was generalized by Bishop and Cannings (1978) and Hendricks et al. (1988). Various versions of the model with incomplete information were developed by Bishop et al. (1978), Riley (1980), Milgrom and Weber (1985), Nalebuff and Riley (1985), Ponsati and Sákovics (1995), Bulow and Klemperer (1999), and Hörner and Sahuguet (2011), just to name a few. However, these studies do not consider the possibility that a player could be defeated, and thus, only deem the wars over when one player concedes.

Some studies suppose that the war has an exogenous (and possibly random) end period (Ordover and Rubinstein, 1986; Kim and Xu Lee, 2014). In this scenario, though, both players may be able to obtain positive benefits at the end of the war, which does not imply that one of the players is defeated. The possibility of defeat is explored by Langlois and Langlois (2009). In their model, players' resources decrease over time, and if a player's resources reach zero, they are defeated; consequently, both players can be defeated. Thus, to the best of my knowledge, this study is the first that analyzes one-sided games in a war of attrition where one player can be defeated by the other, but not vice versa.

On the other hand, some studies analyze wars of attrition between asymmetric players who have different benefits, costs, or discount factors (Kambe, 1999; Abreu and Gul, 2000; Myatt, 2005). One can infer that a player who has a higher benefit, lower cost, or higher discount factor is stronger than the other since such a player has a higher incentive to fight. My model provides a different description of asymmetric robustness; that is, only one player can be defeated by the other.

The rest of the paper proceeds as follows: Section 2 develops a model with complete information. Section 3 further extends the model to include player 1's asymmetric information regarding player 2's robustness, and Section 5 concludes.

2 Complete Information

2.1 Settings

The game involves two players, 1 and 2, who are at war. The model assumes that time is continuous, $t \in 0, \infty]$. Players 1 and 2 strategically choose times T_1 and T_2, respectively, to settle the war. Similar to a standard war-of-attrition model, these strategic decisions are made simultaneously at the beginning of the game. Therefore, the equilibrium concepts are a Nash equilibrium with complete information and a Bayesian–Nash equilibrium with incomplete information.

Player 2 faces a risk of defeat, where he/she is not strong enough to overcome player 1. Player 2 is defeated when $t = \tau$, where $\tau \in [0, \infty)$ is a random variable with the cumulative distribution function $F(\tau) \equiv 1 - \exp(-r\tau)$.[2] The parameter $r \in (0, 1)$ denotes player 2's robustness in fighting. Because the expected timing of player 2's defeat is $1/r$, a larger r implies that player 2 is more likely to be defeated early.

If player 1 concedes before player 2 gives in or is defeated ($T_1 < \min\{\tau, T_2\}$), player 2 wins a one-shot benefit, $b_2 > 0$, at $t = T_1$, while player 1 gains nothing. By contrast, if player 2 gives in or is defeated before or at the same time as player 1's concession ($T_1 \geq \min\{\tau, T_2\}$), player 1 secures benefit b_1 at $t = \min\{\tau, T_2\}$, while player 2 gains nothing.[3] The war inflicts costs c_1 and c_2 on player 1 and player 2, respectively, per unit of time.

The players' expected payoffs at the game's onset can be obtained as follows:

$$\begin{aligned} U_1(T_1, T_2) &\equiv I(T_1 < T_2)F(T_1|r)b_1 + I(T_1 \geq T_2)b_1 \\ &\quad - c_1\left(\int_0^{\min\{T_1,T_2\}} \tau dF(\tau|r) + \int_{\min\{T_1,T_2\}}^{\infty} \min\{T_1, T_2\} dF(\tau|r)\right) \\ &= I(T_1 < T_2)(1 - \exp(-rT_1))b_1 + I(T_1 \geq T_2)b_1 - \frac{c_1}{r} \\ &\quad (1 - \exp(-r\min\{T_1, T_2\})) \\ U_2(T_1, T_2) &\equiv I(T_1 < T_2)\big(1 - F\big(T_2|r^i\big)\big)b_2 \\ &\quad - c_2\left(\int_0^{\min\{T_1,T_2\}} \tau dF(\tau|r) + \int_{\min\{T_1,T_2\}}^{\infty} \min\{T_1, T_2\} dF(\tau|r)\right) \\ &= I(T_1 < T_2)\exp(-rT_1)b_2 - \frac{c_2}{r}(1 - \exp(-r\min\{T_1, T_2\})), \end{aligned}$$

where $I(\cdot)$ is an indicator that equals 1 if its condition holds and 0 otherwise.

Following the standard war-of-attrition model, we assume that the players are risk-neutral and that there is no time discounting. Even if risk aversion and time discounting were to be introduced, the main implications of our model would not change, though the duration of the war would be shorter.

2 The exponential distribution greatly simplifies the players' equilibrium strategies, so this study employs it.

3 The assumption regarding payoffs when $T_1 = T_2$ is not problematic because $T_1 \neq T_2$ in equilibrium, which I shall show later. In addition, the probability that $T_1 = \tau$ is zero and can be ignored.

2.2 Equilibrium

The following proposition summarizes the Nash equilibria of the model with complete information.[4]

Proposition 1 *The game has Nash equilibria with the following properties*:

(i) *If* $b_1r > c_1$, player 2 *immediately gives in* ($T_2 = 0$).

(ii) *If* $b_1r < c_1$, *the following three types of equilibria emerge*:

(a) *Player 1 immediately gives in* ($T_1 = 0$).

(b) *Player 2 immediately gives in* ($T_2 = 0$).

(c) *Both players back down probabilistically such that*

$$\Pr(T_1 < t) = 1 - \exp\left(-\left(\frac{c_2}{b_2} + r\right)t\right),$$
$$\Pr(T_2 < t) = 1 - \exp\left(\left(r - \frac{c_1}{b_1}\right)t\right).$$

Proof. See Appendix A.1. ■

Player 1's rational choice of T_1 depends on the relative sizes of the marginal benefit (b_1r) and the marginal cost (c_1) of extending the war. The marginal benefit and cost are independent of T_1 because the exponential distribution is memoryless. That is, $Pr(\tau > t + \Delta t | \tau > t) = \Pr(\tau > \Delta t)$. If b_1r is sufficiently low and c_1 is sufficiently high (Proposition 1 (ii)), similar to the canonical war-of-attrition model, both players will receive a negative payoff for extending the fight. Thus, the equilibria resemble those of the standard war-of-attrition model. In the pure-strategy equilibria (a, b), the game immediately ends. It is only in the mixed-strategy equilibrium (c) that the war can last for an indeterminate length of time. On the other hand, if b_1r is sufficiently high and c_1 is sufficiently low (Proposition 1 (i)), there exists a unique type of equilibrium where player 2 concedes immediately. This is because player 1 has a positive payoff for extending the fight (because of high b_1r and low c_1), so he/she has an incentive to wait until player 2 is defeated. Thus, as player 2 has no hope of obtaining b_2, he/she will give in at the beginning of the game.

These results suggest that the war can only be maintained in the mixed-strategy equilibrium in which player 1 has a higher probability of conceding

4 It is also possible that $b_1r = c_1$, while player 1 is indifferent to T_1. To simplify the analysis, I disregard this case. Put simply, in equilibrium, player 1 chooses any T_1, and if this T_1 is sufficiently high, player 2 immediately surrenders. If T_1 is sufficiently low, player 2 chooses $T_2 > T_1$.

earlier when $b_1 = b_2$ and $c_1 = c_2$.[5] This seems unrealistic as it requires an invincible player to concede more quickly than a vulnerable one. A similar problem can be found in the standard war-of-attrition model with asymmetric benefits ($b_1 \neq b_2$) and/or costs ($c_1 \neq c_2$): The player with a higher b_i and a lower c_i has a higher probability of conceding earlier.[6] Kornhauser et al. (1989) assert that "the weaker player … should concede immediately." Therefore, even though the conflict is maintained in a mixed-strategy equilibrium, it is difficult to justify these strategies.[7] Proposition 1 (i) shows the unique type of equilibria proposed by Kornhauser et al. (1989) in which the weak player concedes immediately. However, this cannot explain why such wars occur.

3 Asymmetric Information

3.1 Settings

Next, I introduce incomplete information about player 2's robustness into the model. Suppose that player 1 is uncertain about player 2's robustness, r, while player 2 may know it. Although player 1 does not know the true value of r, he/she knows that player 2 is either a strong (S) type with $r = r_S$ or a weak (W) type with $r = r_W > r_S$, and that this characteristic is distributed according to prior probabilities $Pr(r_S) \in (0, 1)$ and $Pr(r_W) = 1 - \Pr(r_S)$. If Player 2 has r_W, he/she is more likely to be defeated early.

In order to rule out uninteresting cases that resemble that with complete information (Proposition 1), I impose the following restrictions:

Assumption 1 $r_S < c_1/b_1 < \Pr(r_S)r_S + \Pr(r_W)r_W$.

If $r_S > c_1/b_1$, the equilibria are identical to Proposition 1 (i); furthermore, if $\Pr(r_S)r_S + \Pr(r_W)r_W < c_1/b_1$, the equilibria are identical to Proposition 1 (ii).[8]

5 This is because a player's mixed strategy must make his/her opponent indifferent between fighting and giving in. Since player 2 has a lower expected payoff than player 1 when $b_1 = b_2$ and $c_1 = c_2$ (because of the probability of defeat), player 1 must have a higher probability of giving in than player 2.

6 My model has the same implication when $r = 0$.

7 Kambe (1999), Abreu and Gul (2000), and Myatt (2005) present reasonable equilibria for the model with asymmetric players by introducing a small probability that a player never concedes.

8 I also disregard the cases where player 1 may be indifferent between fighting and not fighting, that is, $r_S = c_1/b_1$ and $\Pr(r_S)r_S + \Pr(r_W)r_W = c_1/b_1$.

3.2 Equilibrium with an Ongoing War

This section shows an equilibrium where player 1 chooses a pure strategy to fight until a certain period (i.e. $T_1 \in (0, \infty)$), because these novel equilibria do not appear in the standard war-of-attrition model or the model with complete information. The other equilibria will be discussed in Section 3.3.

When player 1 chooses a pure strategy, $T_1 \in (0, \infty)$, player 2 (who is some type-$R \in \{S, W\}$) may adopt a mixed strategy between fighting on ($T_2 > T_1$) and giving in immediately ($T_2 = 0$); namely, with probability $\sigma_R \in [0, 1]$, type R intends to fight until player 1 gives in ($T_2 > T_1$), and with probability $1 - \sigma_R$, type R immediately gives in ($T_2 = 0$). I focus only on $T_2 = 0$ and $T_2 > T_1$, because any $T_2 \in (0, T_1]$, which is costly but never wins b_2, is strictly dominated by $T_2 = 0$. I define σ_R as each type of R's mixed strategy and Σ as the set of the two types of mixed strategies ($\Sigma \equiv (\sigma_S, \sigma_W)$).

3.2.1 Player 1's Incentive to Fight

Player 1's expected payoff at the game's onset is expressed as follows:

$$V_1(T_1, \Sigma) \equiv \sum_{r \in \{r_S, r_W\}} \Pr(r|\Sigma)\left((1 - \exp(-rT_1))b_1 - \frac{c_1}{r}(1 - \exp(-rT_1))\right). \qquad [1]$$

Player 1's rational decision (not) to give in is based on his/her estimate of player 2's robustness, r, in each period.[9] By Bayes' rule, player 1's belief regarding the group's type of weakness in period t can be shown as follows:

$$\Pr(r_W|t, \Sigma) \equiv \frac{\Pr(r_W)\sigma_W \exp(-r_W t)}{\Pr(r_S)\sigma_S \exp(-r_S t) + \Pr(r_W)\sigma_W \exp(-r_W t)},$$

which decreases with t. This formula suggests that longer periods of fighting drive player 1 to revise his/her estimate of player 2's robustness (or lower the expected value of r), expressed as

$$E(r|t, \Sigma) \equiv 1 - \Pr(r_W|t, \Sigma)]r_S + \Pr(r_W|t, \Sigma)r_W.$$

The following lemma summarizes player 1's incentive to fight.

9 Even though player 1 decides T_1 at period 0, he/she predicts future, revised beliefs and chooses based upon them. This assumption is not equivalent to sequential rationality because I do not assume rationality in each period, and player 1 can know these revised beliefs in period 0.

Lemma 1 *Suppose Assumption 1 holds. If* $c_1/b_1 < E(r|t=0,\Sigma)$, *player 1 has an incentive to fight without conceding, at least until* $t = T_1^*(\Sigma) \in (0,\infty)$ *such that* $E\big(r|T_1^*(\Sigma),\Sigma\big)b_1 = c_1$ *or*

$$T_1^*(\Sigma) \equiv \ln\left(\frac{\Pr(r_W)\sigma_W}{\Pr(r_S)\sigma_S}\frac{r_W - \frac{c_1}{b_1}}{\frac{c_1}{b_1} - r_S}\right)^{\frac{1}{r_W - r_S}}. \qquad [2]$$

If $c_1/b_1 \geq E(r|t=0,\Sigma)$, $T_1^*(\Sigma) = 0$.

Proof. Suppose $c_1/b_1 < E(r|t=0,\Sigma)$. As in Proposition 1, the relative sizes of the marginal benefit, $E(r|t,\Sigma)b_1$, and marginal cost, c_1, of extending the fight (both of which can be derived from eq. [1]) determine the timing, T_1. The timing, $T_1^*(\Sigma)$ can be derived from the condition that $E(r|T_1,\Sigma)b_1 = c_1$, which yields $T_1 = T_1^*(\Sigma)$ in eq. [2]. By Assumption 1, $T_1^*(\Sigma)$ is positive and finite when $c_1/b_1 < E(r|t=0,\Sigma)$. If $c_1/b_1 \geq E(r|t=0,\Sigma)$, then $T_1^*(\Sigma)$ in eq. [2] is negative, so player 1 does not have an incentive to fight at period 0. ■

If the marginal benefit of extending the fight ($E(r|T_1,\Sigma)b_1$) is greater than the marginal cost (c_1), player 1 has an incentive to fight and never concede. As it is gradually revealed that player 2 is strong, the marginal benefit decreases while the marginal cost remains unchanged. Because of Assumption 1, the marginal cost is greater than the marginal benefit when it is known that player 2 is strong. Thus, player 1 will fight at least until $T_1^*(\Sigma)$ because the marginal benefit will be greater than the marginal cost until that point.

Hereafter, suppose that player 1 concedes at $T_1^*(\Sigma)$. Lemma 1 is related to Proposition 1 in that this strategy corresponds to the shift of the equilibrium from the case in Proposition 1-(i) to the case in Proposition 1-(ii-a). That is, when the marginal benefit falls beneath the marginal cost, player 1 will concede. This is because, as player 1 discovers that player 2 is strong, he/she becomes less confident about quickly defeating his/her adversary and more weary of the war. Consequently, player 1 will decide to give in.

Note that there is still a possibility that players may choose mixed strategies after $T_1^*(\Sigma)$ (as in Proposition 1-(ii-c)). However, if the players choose such strategies, player 2 will be indifferent between fighting and giving in after $T_1^*(\Sigma)$. This means that player 2's expected utility at time $T_1^*(\Sigma)$ (and thereafter) is zero. In this case, player 2 has no incentive to fight until $T_1^*(\Sigma)$ because, at period 0, player 2's expected payoff for fighting is the same as the expected cost until that point (thus negative).

Moreover, there is a possibility that player 1 will choose a pure strategy, T_1, such that $T_1 > T_1^*(\Sigma)$ (Proposition 1-(ii-b)). However, player 1 has no reason to choose $T_1 > T_1^*(\Sigma)$ such that $T_2 > T_1$ because fighting in $(T_1^*(\Sigma), T_1]$ has a negative expected payoff, so player 1 has an incentive to choose $T_1^*(\Sigma)$ instead. Furthermore, since any $T_2 \in (0, T_1]$ is strictly dominated by $T_2 = 0$ for player 2, $T_1 > T_1^*(\Sigma)$ can be the equilibrium only if $T_2 = 0$. I will discuss this equilibrium in Section 3.3.

3.2.2 Equilibrium

When player 1 chooses a pure strategy, T_1, each type of R's payoff at the beginning of the game for continuing to fight ($T_2 > T_1$) is as follows:

$$V_R(T_1) \equiv \exp(-r_R T_1) b_2 - \frac{c_2}{r_R}(1 - \exp(-r_R T_1)). \quad [3]$$

Therefore, player 2, no matter his/her type, will be willing to wage war if $V_R(T_1) \geq 0$. Player 2's rational strategies comprise the following relationships between the two types.

Lemma 2 *Suppose that player 1 chooses a pure strategy, $T_1 \in (0, \infty)$. Then, (i) if a weak type has a non-negative expected payoff for fighting ($T_2 > T_1$), then a strong type will have a positive expected payoff, so if $\sigma_W > 0$ in equilibrium, $\sigma_S = 1$. (ii) If a strong type has a non-positive expected payoff for fighting, then a weak type will have a negative payoff, so if $\sigma_S < 1$ in equilibrium, $\sigma_W = 0$.*

Proof. See Appendix A.2. ■

Lemma 2 assures that a strong type of player 2 will fight just as often as a weak type. Therefore, there are three possible equilibria (except $\Sigma = (0, 0)$):
- $\Sigma^{\mathrm{I}} \equiv (1, 1)$,
- $\Sigma^{\mathrm{II}} \equiv (1, \sigma_W)$ with $\sigma_W \in (0, 1)$,
- $\Sigma^{\mathrm{III}} \equiv (\sigma_S, 0)$ with $\sigma_S \in (0, 1]$.

However, Σ^{III} is not an equilibrium. If only the strong type fights, the weak type will obtain a positive expected payoff by choosing $\sigma_W = 1$, because, from Assumption 1 and Lemma 1, $T_1^*(\Sigma^{\mathrm{III}}) = 0$. As such, the following proposition is obtained.

Proposition 2 *Suppose Assumption 1 holds. Then, the following two types of equilibria exist*:

(i) **Equilibrium I (Pooling equilibrium)**: If and only if $V_W(T_1^*(\Sigma^{\mathrm{I}})) \geq 0$, will player 1 fight until $t = T_1^*(\Sigma^{\mathrm{I}})$ (as defined in Lemma 1) and player 2 will fight continually ($T_2 > T_1^*(\Sigma^{\mathrm{I}}) + 1$) regardless of his/her type.

(ii) **Equilibrium II (Semi-separating equilibrium)**: (a) If and only if $V_W(T_1^*(\Sigma^{\mathrm{I}})) < 0$, will player 1 choose $T_1 = T_1^*(\hat{\Sigma}^{\mathrm{II}})$ and player 2 will choose $\hat{\Sigma}^{\mathrm{II}} \equiv (1, \hat{\sigma}_W)$, which satisfies $V_W(T_1^*(\hat{\Sigma}^{\mathrm{II}})) = 0$, or

$$b_2 \exp\left(-r_W T_1^*(\hat{\Sigma}^{II})\right) = c_2 \left(\frac{1}{r_W} - \frac{1}{r_W} \exp\left(-r_W T_1^*(\hat{\Sigma}^{II})\right)\right). \quad [4]$$

(b) Additionally, $\hat{\sigma}_W$ is uniquely determined in equilibrium.[10]

Proof. See Appendix A.3. ■

First, if $V_R(T_1^*(\Sigma^{\mathrm{I}})) \geq 0$ for both types, player 2 will be willing to fight regardless of his/her robustness, r. Namely, $T_2 > T_1^*(\Sigma^{\mathrm{I}}) + 1$ so that player 2's strategy does not reveal his/her robustness, r, to player 1.[11] The Bayesian–Nash equilibrium must thus be a pooling equilibrium, where player 1 adopts the same strategy, $T_1^*(\Sigma^{\mathrm{I}})$, for both types.[12]

On the other hand, if $V_R(T_1^*(\Sigma^{\mathrm{I}})) < 0$, a semi-separating equilibrium with Σ^{II} exists. In order for the weak type to randomize his/her strategy in equilibrium, he/she must be indifferent between fighting and not fighting, or he/she will choose $\hat{\sigma}_W$ such that eq. [4] holds. As σ_W decreases, $T_1^*(\Sigma^{\mathrm{II}})$ decreases (to zero), and $V_W(T_1^*(\Sigma^{\mathrm{II}}))$ increases (to be positive). Thus, there exists $\hat{\sigma}_W$. Note that a weak player 2 never chooses σ_W such that $c_1/b_1 \geq E(r|t = 0, \Sigma)$. If he/she did, player 1 would concede immediately, and a weak player 2 would certainly fight.

In these equilibria, player 1 and a strong player 2 fight for a certain period. Thus, the war may continue until $T_1^*(\Sigma)$ (unless player 2 is defeated). Moreover, Proposition 2 implies that there are equilibria where even a weak player 2 will

10 My interpretation of the weak type's mixed strategy in equilibrium is similar to that of Harsanyi (1973), according to whom a mixed strategy can be "purified" by incorporating uncertainty about the player's preference. In my model, the weak type may be further divided into two subcategories depending upon his/her choice of pure strategies: a moderate type (who emerges with probability $\Pr(r_W)\hat{\sigma}_W$) and a very weak type (with probability $\Pr(r_W)(1 - \hat{\sigma}_W)$).

11 For player 1 to back down at $t = T_1^*(\Sigma)$, player 2 must be willing to fight as long as $T_2 > T_1^*(\Sigma) + E\big(r|T_1^*(\Sigma), \Sigma\big)(b_1/c_1)$, for which $E\big(r|T_1^*(\Sigma), \Sigma\big)(b_1/c_1) = 1$ (Lemma 1). Player 1 will continue to fight even after $t = T_1^*(\Sigma)$ if player 2 gives in quickly.

12 A similar equilibrium emerges even if player 2 does not know the true value of his/her own robustness, r (but knows the prior probability distribution). Without being informed of the value of r, player 2 is willing to fight ($T_2 > T_1 + 1$) if $\sum_{r_R \in \{r_S, r_W\}} \Pr(r_R)$ $[\exp(-r_R T_1) b_2 - (c_2/r_R)(1 - \exp(-r_R T_1))] \geq 0$ and to immediately give in ($T_2 = 0$) otherwise.

attempt to influence player 1's decision. In my model, although player 2 has no chance to defeat player 1, even a weak type may fight if he/she anticipates that player 1 will back down (and a strong type will definitely fight). Indeed, player 1 may concede in equilibrium even though he/she can never be defeated and can beat player 2.

My equilibrium results imply the following.

Proposition 3 *(i) Player 1 capitulates earlier if the cost of fighting is higher for player 2 and the benefit lower; that is,* $T_1^*(\hat{\Sigma}^{II})$ *is smaller for a larger* c_2/b_2 *in a semi-separating equilibrium. (ii) In a pooling equilibrium,* c_2/b_2 *does not affect* $T_1^*(\Sigma^I)$.

Proof. (i) As c_2/b_2 increases, $T_1^*(\hat{\Sigma}^{II})$ must maintain equality [4]. (ii) The timing of player 1's capitulation, $T_1^*(\Sigma^I)$, is determined by the condition $E\left(r|T_1^*(\Sigma),\Sigma\right)b_1 = c_1$ (Lemma 1), which is unaffected by the change in c_2/b_2 as long as player 2 still has an incentive to fight ($V_W(T_1^*(\Sigma^I)) \geq 0$). ■

Contrary to what one might think, Proposition 3 implies that as player 2's fighting costs rise and the benefit falls, player 1 may concede *earlier* rather than later. If fighting poses a heavy burden, a larger fraction of the weak types will avoid fighting (or $\hat{\sigma}_W$ will fall), and as a result, player 1 is more likely to confront strong types, making it more difficult for player 1 to defeat his/her opponent.

3.3 Other Equilibria

Proposition 4 *Suppose Assumption 1 holds. The following two types of equilibria also exist.*

(i) **Equilibrium III**: Player 2 immediately gives in ($T_2 = 0$) regardless of his/her type ($\Sigma = (0,0)$), and player 1 fights until $t = T_1$ such that $V_S(T_1) < 0$.

(ii) **Equilibrium IV**: A weak player 2 immediately gives in. Player 1 and a strong player 2 back down probabilistically such that

$$\Pr(T_1 < t) = 1 - \exp\left(-\left(\frac{c_2}{b_2} + r_S\right)t\right),$$

$$\Pr(T_2 < t) = 1 - \exp\left(\left(r_S - \frac{c_1}{b_1}\right)t\right).$$

Proof. (i) If player 1 fights long enough (such that $V_R(T_1) < 0$ for $R = S$ and W), player 2 will immediately give in ($T_2 = 0$). (ii) Suppose that a weak player 2 does not fight, but rather player 1 and a strong player 2 choose mixed strategies. For a strategy profile to form a mixed-strategy equilibrium, both players must be indifferent between fighting and not fighting in each period. Thus, their mixed strategies are derived using the same reasoning as in Proposition 1 (ii-c). According to Lemma 2, when a strong type has a payoff of zero, a weak type has a negative payoff,[13] and thus will not fight. ■

These equilibria are the same as those in the model with complete information (Proposition 1 (ii-b) and (ii-c)) and the standard war-of-attrition model.

Corollary 1 *Suppose Assumption 1 holds. There does not exist any equilibrium other than Equilibria I, II, III, and IV.*

Proof. See Appendix A.4. ■

One important difference between my model and the standard war-of-attrition model is that there is no equilibrium in which player 1 concedes immediately. Both strong and weak types of player 2 would have an incentive to fight because they could get b_2 immediately. However, under Assumption 1, if both types fight, player 1 also has an incentive to fight ($T_1^*(\Sigma^{\mathrm{I}}) > 0$).

Note that these equilibria (in Propositions 2 and 4) satisfy the conditions of a perfect Bayesian equilibrium. This is because the model assumes continuous and infinite periods, thus, all periods are identical except in regards to the revised beliefs, which are the same as those in the Bayesian–Nash equilibrium.[14] I employ Bayesian–Nash equilibria to facilitate comparisons between my model and the standard war-of-attrition model, which uses them as well.

---

13 To be precise, a strong player 2 chooses a mixed strategy when eq. [5] in Appendix A.1 is satisfied with $r = r_S$. The left-hand side of eq. [5] is the marginal benefit of extending the fight for player 2, so a weak player 2 has a lower marginal benefit than a strong one since $r_W > r_S$. Thus, when a strong player 2 chooses a mixed strategy (i.e. eq. [5] holds with $r = r_S$), a weak player 2 does not have an incentive to fight.

14 Put simply, if player 1 chooses to fight, and player 2 chooses to concede in every period, the result is identical to that of Equilibrium III. The marginal change in the mixed strategies' probabilities in Equilibrium IV does not depend on t, so Equilibrium IV is also a perfect Bayesian equilibrium. Moreover, in Equilibria I and II, player 1 has an incentive to fight in every period before $T_1^*(\Sigma)$ (but not after) based on his revised beliefs; thus, they are also perfect Bayesian equilibria.

3.4 Differences from the Standard Model

There are two significant differences between my model's implications and those of the classical war-of-attrition model. First, my model allows for a unique equilibrium in which player 2, who can be defeated by player 1, concedes immediately. This occurs when the probability of defeat is sufficiently high ($r > c_1/b_1$ in the complete-information model and $r_S > c_1/b_1$ in the asymmetric-information model). Moreover, when Assumption 1 holds in the asymmetric-information model, there is no equilibrium in which player 1 (who is invincible) concedes immediately. These equilibria confirm the suggestion of Kornhauser et al. (1989) who argued that a weak player (i.e. a vulnerable player in my model) should concede immediately.

Second, and more importantly, an ongoing war can occur in equilibrium. In Equilibria I and II, player 1 and player 2 (strong types and some weak types) choose to fight for a certain period ($T_1^*(\Sigma)$). During these periods, the war will end if and only if player 2 is defeated. These equilibria simply show (i) why conflicts between players who have asymmetric power (such as a war against terrorism) occur, (ii) why a vulnerable player (such as a terrorist group) decides to attack even though it faces the risk of defeat, and (iii) why an invincible player (such as the targeted state) decides to compromise in a war. To my knowledge, such an equilibrium has not been found in any past extensions of the classical war-of-attrition model.

4 Conclusion

In this article, I present a war-of-attrition model with a one-sided game: one player (player 2) can be defeated by the other (player 1) but not vice versa. The model with complete information has the same implications as the standard war-of-attrition model if the probability that player 1 defeats player 2 is sufficiently low: the war will either (i) end immediately because one of the players gives in at the outset, or (ii) endure so long as the players capitulate probabilistically. However, if the probability that player 1 defeats player 2 is sufficiently high, there exists a unique type of equilibria where player 2 immediately surrenders.

On the other hand, in the model where player 1 is uncertain about player 2's robustness, there exists an equilibrium in which players fight for a period of time. In this model, player 1 would prefer to fight in the early periods but not in the long run, because he/she would start to believe that player 2 was strong and difficult to defeat. Thus, even though player 1 can never be defeated, he/she will give in. Player 2 will expect this, so even if he/she is weak, he/she may prefer to fight and may even benefit from player 1's concession. In addition, player 1 may capitulate earlier if player 2's fighting costs are higher and benefit from winning lower. This is because a weak player 2 tends to avoid wars with high costs and low benefits, implying that, once the fighting has begun, player 1 has a higher probability of facing a strong opponent, to whom he/she would prefer to concede.

I believe that there are many potential applications for this model (as mentioned in the introduction). However, the implications described above must be investigated in greater detail in order to pursue these applications. It may be useful for future studies to endogenize the probability of player 2's defeat. This paper assumes an exogenous and identical probability in every period; however, it is possible that this probability will change overtime or that player 2 (or other players) may be able to increase or decrease it.

Acknowledgments: The author is deeply grateful to two anonymous referees, Scott Gehlbach, Andrew Kydd, Keisuke Nakao, and William Sandholm for their helpful discussions. The author would also like to acknowledge comments from Swati Dhingra, Ching-Yang Lin, John Morrow, and Mian Zhu. This paper was presented at Keio University, Osaka University, Sophia University, the University of Wisconsin-Madison, the American Political Science Association, and the Workshop on the Frontiers of Statistical Analysis and Formal Theory of Political Science.

Appendix

A Proofs

A.1 Proposition 1

(i) The marginal benefit of an infinitesimal extension of the war is:

$$\frac{\frac{d}{dT_1} b_1(1-\exp(-rT_1))}{\exp(-rT_1)} = b_1 r,$$

On the other hand, the marginal cost is

$$\frac{\frac{d}{dT_1} c_1\left(\frac{1}{r}-\frac{1}{r}\exp(-rT_1)\right)}{\exp(-rT_1)} = c_1.$$

If $b_1 r > c_1$, player 1 is willing to fight until player 2 gives in or is defeated ($T_1 > T_2$). In the absence of the possibility of winning, player 2 is unwilling to fight ($T_2 = 0$).

(ii) If $b_1 r < c_1$, a player's rational strategy depends on how quickly the opponent gives in. (ii-*a*) If player 2 is willing to fight long enough, player 1 will immediately give in ($T_1 = 0$). (ii-*b*) If player 1 is willing to fight long enough, player 2 will immediately give in ($T_2 = 0$). (ii-*c*) For a strategy profile to form a mixed-strategy equilibrium, both players must be indifferent between fighting and giving in. It suffices that

$$\left(\frac{\frac{d}{dt}\Pr(T_2 < t)}{1 - \Pr(T_2 < t)} + r\right) b_1 = c_1 \quad \text{for player 1}$$

$$\left(\frac{\frac{d}{dt}\Pr(T_1 < t)}{1 - \Pr(T_1 < t)} - r\right) b_2 = c_2 \quad \text{for player 2.} \qquad [5]$$

The mixed strategies are derived from the two differential equations. ■

A.2 Lemma 2

The proof focuses on the sign of $V_R(T_1)$, because the sign determines type R's rational behavior. Put formally, if $V_R(T_1) < 0$, $\sigma_R = 0$; if $V_R(T_1) = 0$, $\sigma_R \in [0,1]$; and if $V_R(T_1) > 0$, $\sigma_R = 1$ (in addition, $\sigma_R \in (0,1)$ only if $V_R(T_1) = 0$). I thus examine how the sign of $V_R(T_1)$ changes with r_R (recall that $r_S < r_W$). For each R, I define

$$\Phi_R \equiv \exp(r_R T_1) V_R(T_1), \qquad [6]$$

whose sign coincides with that of $V_R(T_1)$, because $\exp(r_R T_1) > 0$.[15] Its derivative with respect to r_R is

$$\frac{\partial \Phi_R}{\partial r_R} = \frac{c_2}{(r_R)^2}(-\exp(-r_R T_1) + (1 - r_R T_1))\exp(r_R T_1),$$

which is negative unless $r_R T_1 = 0$. The negativity of $\partial \Phi_R / \partial r_R$ indicates that (i) if $\sigma_W > 0$, $\sigma_S = 1$ and (ii) if $\sigma_S < 1$, $\sigma_W = 0$. ■

15 I introduce Φ_R to simplify the analysis. The change in r_R has complex effects in that $\partial V_R(T_1)/\partial r_R$ can be positive, zero, or negative depending on T_1. This is because as r_R increases, the benefit (i.e. the first term on the right-hand side of eq. [3]) decreases, whereas the cost (the second term) may decrease. I focus only on how the *sign*, not the value, of $V_R(T_1)$ changes.

A.3 Proposition 2

(i) By Lemma 1, $T_1 = T_1^*(\Sigma^{\mathrm{I}})$ for player 1. For type R, any T_2 that is greater than T_1 ($T_2 > T_1$) is incentive compatible. If $V_W(T_1^*(\Sigma^{\mathrm{I}})) < 0$, the weak type has an incentive to deviate by choosing $\sigma_W = 0$.

(ii) (*a*) By Lemma 1, $T_1 = T_1^*(\hat{\Sigma}^{\mathrm{II}})$ for player 1. According Lemma 2, $\sigma_S = 1$ because $\hat{\sigma}_W > 0$ and the weak type is indifferent because Equality [4] is satisfied. If $V_W(T_1^*(\Sigma^{\mathrm{I}})) \geq 0$, there is no $\hat{\sigma}_W \in (0, 1)$ that satisfies equality [4] because $V_W(T_1^*(\Sigma^{\mathrm{II}})) > V_W(T_1^*(\Sigma^{\mathrm{I}})) \geq 0$ for all Σ^{II}. (*b*) First, if $\sigma_W = 0$, the expected payoff for the weak type is positive because $T_1^*(\Sigma^{\mathrm{III}}) = 0$. Second, if $\sigma_W = 1$, the expected payoff is negative because $V_W(T_1^*(\Sigma^{\mathrm{I}})) < 0$. Third, as σ_W increases, $T_1^*(\Sigma^{\mathrm{II}})$ from eq. [2] increases. As $T_1^*(\Sigma^{\mathrm{II}})$ increases, $V_W(T_1^*(\Sigma^{\mathrm{II}}))$ continuously and strictly decreases. Thus, $\hat{\sigma}_W$ is uniquely determined. ■

A.4 Corollary 1

In Section 3.2, I showed that, aside from Equilibria I, II, and III, there are no other equilibria in which player 1 will choose a pure strategy, T_1. Thus, suppose that player 1 chooses a mixed strategy, in which case Lemma 2 still holds. This is because player 2's expected utility is the sum of all the utilities ($V_R(T_1)$ in (3)) of player 1's pure strategies, T_1 that could result from his/her mixed strategy, weighted by the probability that he/she fights until T_1. Thus, (i) a weak player 2 never fights longer than a strong player 2, and (ii) when a strong player 2 chooses a mixed strategy (i.e. he/she is indifferent between fighting and not fighting in *t*), a weak player 2 will not fight (in *t*). I denote $\overline{T}_2$ as the longest amount of time that the strong type is willing to fight in his/her (pure or mixed) strategy.

First, player 1 never choose a discrete mixed strategy before $\overline{T}_2$.[16]

Lemma 3 *Player 1 never choose a mixed strategy such that there is a positive probability that he/she will give in at t' or t'', where $t' < t''$ and $t' < \overline{T}_2$, but not in (t', t'').*

Proof. Suppose that $t' < t'' < \overline{T}_2$. First, if the probability that player 2 concedes at $[t', t'']$ is not positive, and a weak player 2 chooses to fight in $[t', t'']$, then the

16 The one exception to this is the case in which player 1 chooses a discrete mixed strategy: When $T_2 = 0$, regardless of the type, player 1 chooses a mixed strategy in equilibrium from some sufficiently long periods (Equilibrium III).

marginal benefit of extending the fight at t'' will be lower than the one at t' (and the marginal cost of fighting, c, will not change) since $E(r|t, \Sigma)$ decreases over $[t', t'']$. Thus, player 1 cannot be indifferent between t' and t'', so it cannot be an equilibrium. If a weak player 2 does not fight during $[t', t'']$, player 1 will have a negative expected payoff for fighting in that period (since the marginal benefit is lower than the marginal cost against a strong type), so it is profitable to deviate from t'' to t'.

Second, if the probability that player 2 concedes is positive in $[t', t'']$, player 2 will prefer to give in at $t' + \varepsilon$ where ε is close to zero as opposed to giving in at $(t' + \varepsilon, t'']$ because player 1 will never concede in (t', t''). Third, since ε is close to zero, player 1 will prefer to give in at $t' + 2\varepsilon$ rather than t'. Thus, it is not an equilibrium.

Suppose that $t' < \overline{T}_2 \leq t''$. Player 2 prefers giving in at $t' + \varepsilon$ to giving in at $(t' + \varepsilon, t'']$ because player 1 never concedes in (t', t''). Thus, $\overline{T}_2 = t' + \varepsilon$ when ε is close to zero. Since ε is close to zero, player 1 will prefer giving in at $t' + 2\varepsilon$ to giving in at t'. Thus, it is not equilibrium.

The model considers time to be continuous, so there is such an ε for all of player 1's discrete mixed strategies. Thus, player 1 never chooses such a strategy in equilibrium. ■

Suppose player 1 chooses a continuous mixed strategy. To make player 1 indifferent between fighting and not fighting in any period, player 2 must also choose a continuous mixed strategy. Note that player 1 not giving in until $t > 0$, then choosing a mixed strategy after t is not an equilibrium. If player 1 chooses such strategy, player 2 will be indifferent between fighting and not fighting after t, which means that player 2's expected utility at (and after) t is zero. In this case, there is no incentive for player 2 to fight until t since player 2's expected payoff is negative at period zero. Thus, both players need to choose continuous mixed strategies from period zero onward to be in equilibrium.

As described in Lemma 2, when the weak type chooses a mixed strategy or fights continually, the strong type has an incentive to fight continually. Thus, there are two possible cases.

1. A strong type and player 1 choose a mixed strategy and a weak type concedes at period 0 (Equilibrium IV).
2. A strong type fights continually ($\overline{T}_2 = \infty$) and a weak type and player 1 choose a mixed strategy.

Case 2 is not an equilibrium. Under Assumption 1, player 1 has an incentive fight continually (i.e. the marginal benefit of extending the fight is always greater than the marginal cost) against the weak type. On the other hand, player 1's marginal benefit of fighting against the strong type is lower than the marginal cost. Thus, regardless of the weak type's mixed strategy, player 1's marginal benefit decreases over time because $E(r|t,\Sigma)$ decreases over time. This means that player 1 cannot be indifferent between fighting and not fighting in each period. ■

References

Abreu, D., and F. Gul. 2000. "Bargaining and Reputation." *Econometrica* 68 (1):85–117.

Bishop, T., and C. Cannings. 1978. "A Generalized War of Attrition." *Journal of Theoretical Biology* 70 (1):85–124.

Bishop, T., C. Cannings, and J. Maynard Smith. 1978. "The War of Attrition with Random Rewards." *Journal of Theoretical Biology* 74 (3):377–88.

Bliss, C., and B. Nalebuff. 1984. "Dragon-Slaying and Ballroom Dancing: The Private Supply of a Public Good." *Journal of Public Economics* 25 (1–2):1–12.

Bulow, J., and P. Klemperer. 1999. "The Generalized War of Attrition." *American Economic Review* 89 (1):175–81.

Fudenberg, D., and J. Tirole. 1986. "A Theory of Exit in Duopoly." *Econometrica* 54 (4):943–60.

Fudenberg, D., R. Gilbert, J. Stiglitz, and J. Tirole. 1983. "Preemption, Leapfrogging and Competition in Patent Races." *European Economic Review* 22 (1):3–31.

Ghemawat, P., and B. Nalebuff. 1985. "Exit." *Rand Journal of Economics* 16 (2):184–88.

Harsanyi, J. C. 1973. "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points." *International Journal of Game Theory* 2 (1):1–23.

Hendricks, K., A. Weiss, and C. Wilson. 1988. "The War of Attrition in Continuous Time with Complete Information." *International Economic Review* 29 (4):663–71.

Hörner, J., and N. Sahuguet. 2011. "A War of Attrition with Endogenous Effort Levels." *Economic Theory* 47 (1):1–27.

Kambe, S. 1999. "Bargaining with Imperfect Information." *Games and Economic Behavior* 28 (2):217–37.

Kennan, J., and R. Wilson. 1989. "Strategic Bargaining Models and Interpretation of Strike Data." *Journal of Applied Econometrics* 4 (S1):S87–130.

Kim, K., and F. Z. Xu Lee. 2014. "Information Acquisition in a War of Attrition." *American Economic Journal: Microeconomics* 6 (2):37–78.

Kornhauser, L., A. Rubinstein, and C. Wilson. 1989. "Reputation and Patience in the War of Attrition." *Economica* 56 (221):15–24.

Kreps, D. M., and R. Wilson. 1982. "Reputation and Imperfect Information." *Journal of Economic Theory* 27 (2):253–79.

Kydd, A., and B. F. Walter. 2006. "The Strategies of Terrorism." *International Security* 31 (1):49–80.

Langlois, C. C., and J.-P. P. Langlois. 2009. "Does Attrition Behavior Help Explain the Duration of Interstate Wars? A Game Theoretic and Empirical Analysis." *International Studies Quarterly* 53 (4):1051–73.

Maynard Smith, J. 1974. "The Theory of Games and the Evolution of Animal Conflicts." *Journal of Theoretical Biology* 47:209–21.

Milgrom, P. R., and R. J. Weber. 1985. “Distributional Strategies for Games with Incomplete Information.” *Mathematics of Operations Research* 10 (4):619–25.
Myatt, D. P. 2005. “Instant Exit from the Asymmetric War of Attrition.” Discussion Paper Series No. 160, Department of Economics, University of Oxford.
Nalebuff, B., and J. Riley. 1985. “Asymmetric Equilibria in the War of Attrition.” *Journal of Theoretical Biology* 113 (3):517–27.
O’Brien, B. 1999. *The Long War: The IRA and Sinn Fein*. Syracuse: Syracuse University Press.
Ordover, J. A., and A. Rubinstein. 1986. “A Sequential Concession Game with Asymmetric Information.” *Quarterly Journal of Economics* 101 (4):879–88.
Ponsati, C., and J. Sákovics. 1995. “The War of Attrition with Incomplete Information.” *Mathematical Social Sciences* 29 (3):239–54.
Riley, J. G. 1980. “Strong Evolutionary Equilibrium and the War of Attrition.” *Journal of Theoretical Biology* 82 (3):383–400.