#### Research Article

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# Price and Inventory Dynamics in an Oligopoly Industry: A Framework for Commodity Markets

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**Abstract:** This paper analyzes the interaction between price and inventory decisions in an oligopoly industry and its implications for the dynamics of prices. The work extends existing literature and especially the work of Hall and Rust (2007) to endogenous prices and strategic oligopoly competition. We show that the optimal decision rule is an (S,s) order policy and prices and inventory are strategic substitutes. Fixed ordering costs generate infrequent orders. Additionally, with strategic competition in prices, (S,s) inventory behavior together with demand uncertainty generates endogenous cyclical patterns in prices without any exogenous shocks. Hence, the developed model provides a promising framework for explaining dynamics of commodity markets and especially observed autocorrelation in price fluctuations.

**Keywords:** inventory dynamics, price competition, oligopoly, (*S*, *s*) policy, commodity markets

JEL classification numbers: D21, D43, E22, L81

# 1 Introduction

This paper analyzes the interaction between price and inventory decisions in an oligopoly industry and its implications for the dynamics of prices. The developed framework is able to explain important features of prices such as price dispersion observed in retail industries and autocorrelation found in commodity markets.

Cross-sectional price dispersion is a common feature in many retail markets. Since Stigler's (1961) seminal work price dispersion has usually been explained by consumer search costs. In contrast, Aguirregabiria (1999) shows

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numerically that retail inventories and lump-sum ordering cost can generate (S,s) dynamics of inventories. Under an (S,s) rule inventory moves between the target inventory level, S, and the order threshold, s, with s < S. Whenever the firm's inventory level falls below the order threshold, a new order is placed such that the target inventory level S is attained. These (S,s) order policies can explain time variability of prices of supermarket chains. However, as in the model of Aguirregabiria (1999) monopolistic competition is analyzed price dispersion between different firms cannot be observed. This paper contributes to the described work in several ways. We extend the analysis of prices and inventories with lump-sum costs to address the question how oligopolistic competition affects the dynamics. With imperfect competition dispersion of prices can actually result. Additionally, the main focus of the paper by Aguirregabiria (1999) is an empirical analysis building on a numeric simulation. Thus, the author does not aim at theoretically proving the optimality of the considered inventory decision.<sup>1</sup> Consequently, our paper is the first to formally prove the optimality of (S, s) policy with endogenous prices.

In addition to Aguirregabiria (1999) who analyze optimal inventory decisions with lump-sum costs under monopolistic competition, Hall and Rust (2007) study these dynamics under perfect competition. Their paper extends the framework of Aguirregabiria (1999) in some ways but is otherwise limited to one decision variable as prices are taken as given. Hall and Rust (2007) show that in their perfect competition model the (S,s) policy is an optimal order strategy. To the best of our knowledge, these works studying extreme cases of competition are by far the most elaborated theoretical papers investigating these decision problems.

This paper extends the literature by characterizing an equilibrium in a model of price and inventory competition in oligopoly. The paper is unique as it formally proves the optimality of (S,s) policy with endogenous prices. Additionally, the developed model does not only incorporate pricing decisions into the framework but also allows oligopolistic firms to interact strategically.

<sup>1</sup> Nevertheless, Aguirregabiria (1999) provides a theoretical proof, but this proof is not rigorously done and incomplete.

**<sup>2</sup>** Thereby Hall and Rust (2007) extend earlier work like Sethi and Cheng (1997) and Cheng and Sethi (1999) to a more general specification of the Markov process.

<sup>3</sup> There exist also some papers analyzing dynamic oligopoly with inventories without considering lump-sum ordering cost, like Kirman and Sobel (1974) or more recently Bernstein and Federgruen (2004). However, without ordering cost stationary optimal strategies result which are in essence identical to those of the corresponding static single period game. Additionally, a recent empirical analysis building on a numeric simulation by Copeland, Dunn, and Hall (2011) focusing on a durable goods market to consider inventory's role in increasing variety also studies pricing and inventory decisions of a single firm.

Besides the mentioned literature less closely related studies of oligopolistic competition exists. Dutta and Sundaram (1992) and Dutta and Rustichini (1995) analyze a discrete choice stochastic duopoly game with lump-sum costs. In these frameworks, the one abstract decision variable affecting both firms' payoffs cannot be interpreted as being related to inventory. Nevertheless, the optimality of an (S, s) policy can also be shown. More recently, Besanko and Doraszelski (2004) study decisions about prices and capacity. However, the main and important difference between inventory and capacity is that excess capacity is worthless while keeping inventory affects future competition. Hence, additional strategic effects due to kept stock are at place in a framework like ours where inventory decisions are considered. This is especially important when investigating oligopolistic competition. The developed approach also contributes to the theory and the field of estimation in recent literature like Ryan (2012) where it is assumed that firms play (S, s) investment in capacity policies.<sup>4</sup> In contrast, we prove that (S,s) strategies arise in equilibrium and derive a functional form for these policies.

The distinct characteristics of our model that is incorporating inventory and oligopoly in dynamic competition provide the most plausible framework for retail industries. Retail industries have become highly concentrated, i.e., in most categories like grocery, supermarkets, and office supplies just a handful of rivals compete locally. In the supermarket industry for example a small number of firms capture the majority of sales as supermarkets compete in tight regional oligopolies. Thus, this industry is a prime example of oligopoly. Besides, inventory costs are of major importance. Supermarkets invest in state of the art distribution systems to minimize storage and transportation costs. Hence, deciding the optimal inventory and store offer forms an important optimization problem for supermarket chains (see e.g. Beresteanu et al. 2006; Ellickson 2007; Chavas 2008).

In this work, we study the decision problem of a central store, i.e., its decision about retail prices and orders to suppliers, facing oligopolistic competition and taking into account the existence of lump-sum ordering cost. We develop a model of retail competition in which the impact of inventories on competition and prices can be evaluated. We analyze the characteristics of the optimal decision rule. We find that key factors for price fluctuations are lumpsum ordering costs and demand uncertainty. Lump-sum ordering cost generate (S,s) inventory behavior. Demand uncertainty creates a positive probability of excess demand, i.e., stockouts. The positive stockout probability has a negative effect on expected sales, which in turn creates substitutability between prices

<sup>4</sup> Moreover, Ryan (2012) assumes a functional form for the lower and upper bounds.

and inventories in the profit function such that in equilibrium prices depend negatively on the level of inventories. This results in a cyclical pattern of inventories and prices where prices decline significantly when an order is placed and consequently inventory reduction generates price increase. The pricing behavior in this model can generate cross-sectional price dispersion with cyclical patterns even without menu costs. These results are supported by empirical studies (e.g. Lach and Tsiddon 1992, 1996; Slade 1998, 1999; Pesendorfer 2002; Ellis 2009) as well as numerical simulations (e.g. Copeland, Dunn, and Hall 2011).

The developed model provides a very promising alternative for studying commodity markets. Commodity prices are extremely volatile and papers of the respective literature strand are concerned whether theory is capable of explaining the actual behavior of prices. The more recent literature on this topic (see for example Deaton and Laroque 1992, 1996; Pindyck 1994) builds on the supply and demand tradition (see e.g. Ghosh et al. 1987, for a review), but with explicit modeling of the behavior of competitive speculators who hold inventories of commodities in the expectation of making profits.<sup>5</sup> However, perfect competition and the absence of lump-sum ordering cost is always assumed in these papers. The studies are trying to explain extremely volatile prices as a result of exogenous shocks by modeling the behavior of competitive speculators holding inventories. Results are rather unsatisfying: In contrast to the models' predictions, real price fluctuations are not randomly distributed over time and this autocorrelation cannot be explained by these types of models. In addition, some probably important characteristics of commodity markets are not captured in this literature. Studies of these characteristics (e.g. Carter and MacLaren 1997; Slade and Thille 2006) find that commodity markets are best described by oligopoly instead of perfect competition. Besides, lump-sum ordering cost are realistic in some markets.<sup>6</sup> Incorporating oligopoly competition and lump sum ordering costs could be important to study the dynamics of some commodity prices. The developed model does generate time dependent patterns of prices, which is apparently in line with empirical evidence. This is in contrast to the usual hypothesis that price fluctuations are the result of exogenous shocks and therefore randomly distributed over time. In summary, the framework presented in this paper provides an explanation for autocorrelation in commodity prices, which established theory cannot provide until now. Hence, this paper makes

**<sup>5</sup>** As even estimating the models is computational demanding authors mostly use simulations. **6** For example, at London Metal Exchange orders can result in physical delivery and all contracts assume delivery.

a contribution that is useful for both the computational theory and structural estimation realms of this literature.

The rest of the paper is organized as follows. Section 2 introduces the model and shows important characteristics of firms' expected sales. Section 3 characterizes the optimal decision rules. Section 4 concludes while the Appendix contains the proofs of the results stated in the text.

## 2 The Model

Consider an oligopoly market where risk neutral firms, indexed by  $i \in \{1, 2, ..., N\}$ , sell differentiated storable products. Each firm sells a variety of the product. Firms compete in prices and they have uncertainty about temporary demand shocks. In the short run, firms cannot respond to these temporary shocks neither by changing prices nor by increasing supply, in case of excess demand. Firms do not face any delivery lags and cannot backlog unfilled orders. Thus, whenever demand exceeds quantity on hand, the residual unfilled demand is lost. Therefore, the quantity sold by firm i at period t is the minimum of supply and demand:

$$y_{it} = \min\{s_{it} + q_{it}, d_{it}\},$$
 [1]

where  $y_{it}$  is the quantity sold;  $s_{it}$  is the level of inventories at the beginning of period t;  $q_{it}$  represents new orders to suppliers during period t; and  $d_{it}$  is consumers' demand. Thus, the timing of events in every period is as follows: first, firms place orders that are fulfilled and set prices. Afterwards demand is realized and firms sell the minimum of demand and inventory on hand. Every period t a firm knows the levels of inventories of all the firms in the market, i.e., the vector  $\mathbf{s}_t = \{s_{1t}, s_{2t}, ..., s_{Nt}\}^7$  Given this information, the firm decides on prices and new orders  $(p_{it}, q_{it})$  to maximize its expected value  $E_t(\sum_{r=1}^{\infty} \beta^r \prod_{i,t+r})$ , where  $\beta \in (0,1)$  is the discount factor and  $\Pi_{it}$  is the current profit of firm i at

A firm's current profit is equal to revenue minus ordering cost and inventory holding cost:

$$\Pi_{it} = p_{it} y_{it} - c_i q_{it} - k_i I\{q_{it} > 0\} - h_i s_{it},$$
 [2]

<sup>7</sup> This is a very reasonable assumption as firms can observe prices and are therefore able to learn and deduce stock levels.

where  $c_i$  is the unit ordering cost;  $k_i$  is the fixed or lump-sum ordering cost; and  $h_i$  is the inventory holding cost. The transition rule of inventories, i.e., state variables, is:

$$s_{it+1} = s_{it} + q_{it} - y_{it} = \max\{0, s_{it} + q_{it} - d_{it}\}.$$
 [3]

Firms have uncertainty about current demand. The demand of product i at period t is

$$d_{it} = \exp\{\varepsilon_{it}\}d_{it}^e$$
.

Here,  $\varepsilon_{it}$  is a temporary and idiosyncratic demand shock that is independently and identically distributed over time with cumulative distribution function  $F(\cdot)$  that is continuously differentiable on the Lebesgue measure. These shocks are unknown to firms when they decide prices and orders. Furthermore,  $d^e_{it}$  is the expected demand that depends on the endogenous prices and the exogenous qualities of all products. The expected demand  $d^e_{it}$  is a function of the prices of all firms such that it is strictly decreasing in the own price, strictly increasing in the prices of competitors, and the revenue function  $p_i d^e_i$  is strictly concave in  $p_i$ . By definition of expected demand, we have that  $E(\exp\{\varepsilon_{it}\})=1$ . For technical reasons it is useful to assume that  $F(\cdot)$  is such that the respective hazard rate  $h(\cdot)=\frac{f(\cdot)}{1-F(\cdot)}$  is smaller than one. For examples and numerical exercises it may be useful to consider a logit demand model for the expected demand:

$$d_{it}^{e} = \frac{\exp\{w_{i} - \alpha p_{it}\}}{1 + \sum_{i=1}^{N} \exp\{w_{i} - \alpha p_{it}\}},$$
 [4]

where  $\{w_i: i=1, 2, ..., N\}$  are exogenous parameters that represent product qualities, and  $\alpha$  is a parameter that represents the marginal utility of income. The logit demand model is convenient for the derivation and illustration of some future results, but it can be relaxed for all our results.

# 2.1 Implications of Demand Uncertainty for Expected Sales

As a firm does not know the temporary demand shock  $\varepsilon_{it}$ , it does not know actual sales  $y_{it}$ . Expected profits are  $\Pi_{it}^e = p_{it} \ y_{it}^e - c_i \ q_{it} - k_i \ I\{q_{it} > 0\} - h_i s_{it}$ , where

<sup>8</sup> This assumption is especially helpful for proving Lemma 2, although it is only a sufficient but not necessary condition.

**<sup>9</sup>** See Aguirregabiria (2007) for a derivation of this demand model from a model of consumer behavior under possible excess demand.

 $y_{it}^e$  represents expected sales, i.e.,  $y_{it}^e = E[\min\{d_{it}, s_{it} + q_{it}\}]$ . Demand uncertainty has important implications for the relationship between prices and inventories.

**Lemma 1** Expected sales  $y_{it}^e$  are equal to expected demand  $d_{it}^e$  times a function  $\lambda\left(\frac{s_{it}+q_{it}}{d^e}\right)$ , i.e.,

$$y_{it}^e = d_{it}^e \lambda \left( \frac{s_{it} + q_{it}}{d_{it}^e} \right).$$
 [5]

The function  $\lambda(x)$  is defined as  $\lceil \min\{x, \exp(\varepsilon)\} dF(\varepsilon) \rceil$  and it has the following properties:

- (i) It is continuously differentiable:
- (ii) it is strictly increasing;
- (iii)  $\lambda(0) = 0$ ;
- (iv)  $\lambda(\infty) = E(\exp(\varepsilon)) = 1$ ; and
- (v) for x > 0,  $\lambda'(x) = \int_{-\infty}^{\ln(x)} dF(\varepsilon) = 1 F(\ln(x)) \in (0, 1)$ .

#### **Proof** See Appendix 5.1.

In case of a very small (close to zero) supply-to-expected-demand-ratio  $\frac{s_{it}+q_{it}}{d^e}$ stockout probability is very large such that expected sales are much lower than expected demand (approaching zero). On the other hand, a high ratio (approaching infinity) yields low probability for stockouts such that expected sales are almost equal to expected demand. The higher the supply-to-expecteddemand-ratio the lower gets the probability of stockout and the more do expected sales converge to expected demand. This is formalized in properties (ii) - (iv). From property (v) yielding  $\lambda''(x) < 0$  it is now clear that the gain of a higher supply-to-expected-demand-ratio for expected sales is higher the lower the ratio. For low ratios the gain is almost equal to the increase of stock as one unit more in stock in essence is a unit more sold. For high ratios the probability of selling an additional unit in stock decreases to zero.

Therefore, variability over time in the supply-to-expected-demand-ratio can generate significant fluctuations in expected sales and thus in optimal prices.

## 2.2 Markov Perfect Equilibrium

The model has a Markov structure and we assume that firms play Markov strategies. That is, a firm's strategy depends only on payoff relevant state variables, which in this model is the vector of inventories  $\mathbf{s}_t$ . Therefore, a strategy for firm i is a function  $\sigma_i(\mathbf{s}_t)$  from the space of the vector of inventories,

 $\mathbb{R}^N_+$ , into the space of the decision variables  $(p_{it},q_{it})$ ,  $\mathbb{R}^2_+$ , i.e.,  $\sigma_i(\mathbf{s}_t)$  is a function from  $\mathbb{R}^N_+$  into  $\mathbb{R}^2_+$ . Let  $\sigma \equiv \{\sigma_i\colon i=1,2,...,N\}$  be a set of strategy functions, one for each firm. Suppose that firm i considers the rest of the firms to behave according to their respective strategies in  $\sigma$ . Under this condition, other firms' inventories,  $\mathbf{s}_{-it}$ , follow a Markov transition probability function  $F^\sigma_{\mathbf{s}_{-i}}(\mathbf{s}_{-it+1}|\mathbf{s}_{-it})$ . Note that this transition probability function depends on the other firms' strategies in  $\sigma$ . Taking  $F^\sigma_{\mathbf{s}_{-i}}$  as given, firm i's decision problem can be represented using the Bellman equation:

$$\begin{split} V_{i}^{\sigma}(\mathbf{s}_{t}) &= \max_{\{p_{i}, q_{i}\}} \Big\{ \Pi_{i}^{\sigma}(p_{i}, s_{it} + q_{i}) \\ &+ \beta \int V_{i}^{\sigma}(s_{i, t+1}, \mathbf{s}_{-it+1}) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+1}|\mathbf{s}_{-it}) \Big\}. \end{split}$$
 [6]

The (expected) profit function is continuously differentiable and the standard regularity conditions apply such that the value function  $V_i^{\sigma}$  is uniquely determined as the fixed point of a contraction mapping. Note that this value function is conditional to the other firms' strategies. A Markov perfect equilibrium (MPE) is a set of equilibrium strategies  $\sigma$  such that for every firm i and for every vector  $\mathbf{s}_t \in \mathbb{R}^N_+$  we have that

$$\begin{split} \sigma_{i}(\mathbf{s}_{t}) &= \arg\max_{\{p_{i},\,q_{i}\}} \Big\{ \Pi_{i}^{\sigma}(p_{i},s_{it}+q_{i}) \\ &+ \beta \left[ V_{i}^{\sigma}(s_{i,\,t+1},\mathbf{s}_{-it+1}) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+1}|\mathbf{s}_{-it}) \right\}. \end{split}$$
 [7]

# 3 Optimal Decision Rule

Let us now characterize the optimal decision rule for a firm in this game of oligopolistic competition.

In this section we will show that the (S,s) rule, where a new order is placed whenever the inventory level falls below the order threshold s such that the target level S is attained, is indeed the best response not only to an (S,s) rule but to any given strategy of the opponents. This, of course, implies that the equilibrium resulting from (S,s) strategies by all players is a MPE.

In order to represent the optimal decision rule of the oligopolists, it is convenient to represent the decision problem in terms of the variables  $p_{it}$  and  $z_{it} \equiv s_{it} + q_{it}$ . The variable  $z_{it}$  represents the total supply of the product during period t. It is also useful to define the following "value" function which is independent of the firm's own current inventory, i.e., the only state variable

the firm can influence (however, it is not independent of the current state per se), and taking the other firms' strategies in  $\sigma$  and so  $F_s^{\sigma}$ , as given:

$$Q_{i}^{\sigma}(z_{it}, p_{it}; \mathbf{s}_{-it}) = -cz_{it} + p_{it} \int \min \left\{ z_{it}; e^{\varepsilon_{it}} d_{it}^{e\sigma}(p_{it}) \right\} dF(\varepsilon_{it})$$

$$+ \beta \int V_{i}^{\sigma} \left( \max \left\{ 0; z_{it} - e^{\varepsilon_{it}} d_{it}^{e\sigma}(p_{it}) \right\}; \mathbf{s}_{-it+1} \right) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+1} | \mathbf{s}_{-it})$$
[8]

such that

$$V_i^{\sigma}(\mathbf{s}_t) = \max_{\{p_i, q_i\}} \big\{ Q_i^{\sigma}(s_{it} + q_i, p_i; \, \mathbf{s}_{-it}) - (h_i - c_i) s_{it} - k_i I_{\{q_i > 0\}} \big\}.$$

Given the function  $Q_i^{\sigma}$ , it is clear that an oligopolist chooses  $(z_{it}, p_{it})$  as a best response to the other firms' strategies in  $\sigma$ , i.e., other firms order and pricing decisions, to maximize  $Q_i^{\sigma}(z_{it}, p_{it}; \mathbf{s}_{-it}) - kl\{z_{it} > s_{it}\}$ . Making use of this "value" function  $Q_i^{\sigma}$  we can derive important characteristics of competition in prices and inventories:

### **Lemma 2** The function $Q_i^{\sigma}$ is such that:

- (i)  $Q_i^{\sigma}$  is strictly concave in prices, i.e.,  $\partial^2 Q_i^{\sigma}(z_i, p_i)/\partial p_i \partial p_i < 0$ .
- (ii) Prices and total supply are strategic substitutes, i.e.,  $\partial^2 Q_i^{\sigma}(z_i, p_i)/\partial p_i \partial z_i \leq 0$ .

#### **Proof** See Appendix 5.2.

The positive stockout probability has a negative effect on expected sales which in turn creates substitutability between prices and inventories in the profit function. This is the case as with low inventory optimal expected demand (under given demand uncertainty) is low and thus the optimal price is high.

Using  $\sigma_p^{\sigma}(\mathbf{s})$  and  $\sigma_z^{\sigma}(\mathbf{s})$  to represent the optimal response rules for p and z, respectively, we have

$$\{\sigma_{iz}^{\sigma}(\mathbf{s}), \sigma_{ip}^{\sigma}(\mathbf{s})\} = \arg\max_{\{z_i \geq s_i, p_i \geq 0\}} \left\{Q_i^{\sigma}(z_i, p_i; \mathbf{s}_{-it}) - kI\{z_i > s_i\}\right\}.$$

We define the optimal price as a function of current supply:

$$\bar{p}_i^{\sigma}(z_i; \mathbf{s}_{-i}) \equiv \arg\max_{\{p_i\}} Q_i^{\sigma}(z_i, p_i; \mathbf{s}_{-i}).$$
 [9]

Since  $Q_i^{\sigma}$  is continuously differentiable and strictly concave in prices,  $\bar{p}_i^{\sigma}(z; \mathbf{s}_{-i})$  is implicitly defined by the first order condition  $\frac{\partial Q_i^{\sigma}(z_i, \bar{p}_i; \mathbf{s}_{-i})}{\partial \bar{p}_i} = 0$ .

It is now possible to show that the best response to any strategy is an (S,s) rule:

**Proposition 1** Firm i considers the rest of the firms to behave according to their respective strategies in  $\sigma$ . Taking  $F_{\mathbf{s}_{-i}}^{\sigma}$  as given, let firm i's best response rule for total supply and prices be  $\sigma_{iz}^{\sigma}(\mathbf{s})$  and  $\sigma_{ip}^{\sigma}(\mathbf{s})$ , respectively. These functions are such that:

- 1.  $\sigma_{ip}^{\sigma}(\mathbf{s}) = \bar{p}_{i}^{\sigma}(\sigma_{iz}(\mathbf{s}); \mathbf{s}_{-i})$ , where  $\bar{p}_{i}^{\sigma}(z_{i}; \mathbf{s}_{-i})$  is continuous and strictly decreasing in  $z_{i}$ ; and
- 2.  $\sigma_{iz}^{\sigma}(\mathbf{s})$  has the following form:

$$\sigma_{iz}^{\sigma}(\mathbf{s}) = \begin{cases} s_i^{\star \sigma}(\mathbf{s}_{-i}) & \text{if} \quad s_{it} \leq \underline{s}_i^{\sigma}(\mathbf{s}_{-i}) \\ s_{it} & \text{if} \quad s_{it} > \underline{s}_i^{\sigma}(\mathbf{s}_{-i}), \end{cases}$$
[10]

where  $s_i^{\star\sigma}$  and  $\underline{s}_i^{\sigma}$  are scalars, with  $s_i^{\star\sigma} > \underline{s}_i^{\sigma} \ \forall \mathbf{s}_{-i}$ , and the following definitions:

$$\boldsymbol{s}_{i}^{\star\sigma}(\mathbf{s}_{-i}) \equiv \arg\max_{\{z_{i}\}} Q_{i}^{\sigma}(z_{i}, \bar{p}_{i}(z_{i}); \mathbf{s}_{-i}), \tag{11}$$

$$\underline{s}_{i}^{\sigma}(\mathbf{s}_{-i}) \equiv \inf\{s_{i}|Q_{i}^{\sigma}(s_{i}^{\star\sigma}, \bar{p}_{i}(s_{i}^{\star\sigma}); \mathbf{s}_{-i}) - k \leq Q_{i}^{\sigma}(s_{i}, \bar{p}_{i}(s_{i}); \mathbf{s}_{-i})\}.$$
 [12]

#### **Proof** See Appendix 5.3.

The proposition shows that consideration of oligopolistic competition does not affect the optimality of (S,s) inventory rules. Fixed ordering costs generate infrequent orders. The upper band  $s_i^{\star\sigma}$  is defined as the optimal order quantity when the firm has no inventory on hand, i.e., the optimal inventory level. The lower band  $\underline{s}_i^{\sigma}$  is the smallest value of inventory such that the desired order quantity is zero. This inventory level at which a new order is placed is decreasing in the lump-sum ordering costs k as  $Q_i^{\sigma}(s_i, \bar{p}(s_i); \mathbf{s}_{-i})$  is a strictly k-concave function. Thus, the higher the lump-sum ordering costs the less often are orders placed.

The resulting order policy might appear to be a very natural and intuitive strategy. However, as shown in the appendix the value function is not concave such that in principle a much more complex decision rule could be optimal. Besides, oligopolistic competition assures that no additional assumption on prices like the "no expected loss condition" of Hall and Rust (2007) is necessary for the optimal trading strategy to be of the (S, s) form. <sup>11</sup> With endogenous

**<sup>10</sup>** However, as thresholds depend on the competitors' inventories, we have an  $(S(\mathbf{s}_{-i}), s(\mathbf{s}_{-i}))$  decision rule.

<sup>11</sup> The "no expected loss condition" requires that the exogenous nonconstant retail price exceeds a certain (endogenous) nonconstant threshold any time.

prices, the optimality of the (S,s) rule is not limited but always fulfilled. Additionally, our model of oligopolistic competition with prices as decision variables of the firms allows for analyzing the dynamics of prices. The resulting (S,s) inventory behavior together with demand uncertainty generates cyclical patterns in prices. 12 The optimal price is a strictly decreasing function of a firm's inventory on hand  $z_i$  as the positive probability of stockouts creates strategic substitutability between prices and inventories. Thus, the price increases between two orders when the stock level decreases and it drops down when new orders are placed. This is the case as with low inventories the optimal expected demand is lower and hence the optimal price is higher. When the level of inventories decreases between two orders, the probability of stockout increases and so expected sales decrease and become more inelastic with respect to the price. Thus, the optimal price increases between two orders, and decreases when the elasticity of sales goes up as the result of positive orders.

The largest price increase occurs just after a positive order and the increments tend to be smaller when we approach to the next positive order. The reason for this behavior is that the cyclical path of prices generates a cyclical behavior in sales. The largest sales and, consequently, the largest stock reductions and price increases, occur just after a positive order.

# 4 Conclusion

We have shown that the best response not only to (S,s) strategies but to any strategy is an (S,s) rule. This result extends earlier findings of models without price competition (Hall and Rust 2007) and models without strategic competition (Aguirregabiria 1999) where fixed ordering costs generate infrequent orders. Thus, the (S,s) policy might appear to be a very robust strategy. However, it is not hard to change assumptions in ways that destroy its optimality. Additionally, with strategic competition in prices (S, s) inventory behavior together with demand uncertainty generates cyclical pattern in prices.

The model developed in this paper provides a very promising alternative for studying commodity markets. Observed autocorrelation in prices cannot not be

<sup>12</sup> Pricing behavior in this model can also generate cross-sectional price dispersion as a result of non-synchronized cyclical patterns. The magnitude of this price dispersion will depend on the magnitude of lump-sum ordering costs, the sensitivity of the price elasticity of sales to changes in the probability of stockout, and the degree of correlation between the demand shocks at individual firms.

explained by existing models of competitive speculators with the usual hypothesis that price fluctuations are the result of exogenous shocks and therefore randomly distributed over time. Incorporating oligopoly competition and lump sum ordering costs could be important to study the dynamics of commodity prices. The developed model accounts for important characteristics of commodity markets and thereby generates time dependent price fluctuations that are in principle in line with empirical evidence. Making use of the presented model, it should now be possible to relate findings to commodity price dynamics and show that lump-sum ordering cost and oligopoly competition can be important to explain extremely volatile prices and especially time dependencies in price fluctuations.

However, due to the relatively high complexity of the framework further research requires numerical experiments. By this means, other topics like precise reactions of firms on competitors' orders provide scope for interesting studies. This important work is left for future research. Nevertheless, the results derived in this paper – in particular showing that (S,s) inventory policies arise in equilibrium – allow to restrict attention to according policies instead of having to consider all possible kinds of behavior. This can make it easier to compute equilibria to this model as well as to estimate these models.

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# A Appendix

## A.1 Expected Sales: Proof of Lemma 1

**Proof**: For notational simplicity, we omit here the firm and time subindexes. By definition, expected sales  $y^e$  are:

$$y^e = \int \min\{s+q, d^e \exp(\varepsilon)\} dF(\varepsilon) = d^e \lambda \left(\frac{s+q}{d^e}\right)$$

where  $\lambda(x)$  is defined as  $\int \min\{x, \exp(\varepsilon)\} dF(\varepsilon)$ . The function  $\lambda(x)$  has the following properties:

$$\lim_{x\to 0} \lambda(x) = \int \min\{0, \exp(\varepsilon)\} dF(\varepsilon) = 0.$$

Also,

$$\lim_{x\to\infty}\lambda(x)=\lim_{x\to\infty}\int\min\{x,\,\exp(\varepsilon)\})dF(\varepsilon)=\int\exp(\varepsilon)dF(\varepsilon)=1.$$

Finally,

$$\lambda'(x) = \int I\{x < \exp(\varepsilon)\} dF(\varepsilon) = 1 - F(\ln x).$$

#### A.2 The "Value" Function: Proof of Lemma 2

**Proof:** We use backwards induction and first show that the properties of Lemma 2 hold for the finite horizon problem with time horizon equal to T.

Let us consider  $Q_{iT}^{\sigma}(\cdot)$  to represent the profit function in the last period, i.e.,

$$\begin{split} Q_{iT}^{\sigma}(z_i,p_i;\,\mathbf{s}_{-i}) &= -cz_i + p_i y_i^{e\sigma}(z_i,p_i) \\ &= -cz_i + p_i d_i^{e\sigma}(p_i) \lambda \left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \\ &= -cz_i + p_i \int \min \left\{z_i;\,e^{\varepsilon_i} d_i^{e\sigma}(p_i)\right\} dF(\varepsilon_i). \end{split}$$

Therefore,

$$\frac{\partial Q_{iT}^{\sigma}(\cdot)}{\partial p_i} = y_i^{e\sigma}(z_i, p_i) + p_i \frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i},$$

and

$$\frac{\partial^2 Q_{iT}(\cdot)}{\partial p_i^2} = 2 \frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} + p_i \frac{\partial^2 y_i^{e\sigma}(z_i, p_i)}{\partial p_i^2}.$$
 [13]

Given that  $y_i^{e\sigma}(z_i,p_i)=d_i^{e\sigma}(p_i)\lambda\Big(\frac{z_i}{d_i^{e\sigma}(p_i)}\Big)$ , we have that

$$\frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} = \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} F(\ln z_i - \ln d_i^{e\sigma}(p_i)),$$

and

$$\frac{\partial^2 y_i^{e\sigma}(z_i,p_i)}{\partial p_i^2} = \frac{\partial^2 d_i^{e\sigma}(p_i)}{\partial p_i^2} F(\ln z_i - \ln d_i^{e\sigma}(p_i)) - \left(\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}\right)^2 \frac{f(\ln z_i - \ln d_i^{e\sigma}(p_i))}{d_i^{e\sigma}(p_i)}.$$

Inserting these expressions in eq. [13], we get:

$$\begin{split} \frac{\partial^2 Q^{\sigma}_{iT}(\cdot)}{\partial p_i^2} &= 2\frac{\partial d^{e\sigma}_i(p_i)}{\partial p_i} F(\ln z_i - \ln d^{e\sigma}_i(p_i)) \\ &+ p_i \bigg( \frac{\partial^2 d^{e\sigma}_i(p_i)}{\partial p_i^2} F\big(\ln z_i - \ln d^{e\sigma}_i(p_i)\big) \\ &- \bigg( \frac{\partial d^{e\sigma}_i(p_i)}{\partial p_i} \bigg)^2 \frac{f(\ln z_i - \ln d^{e\sigma}_i(p_i))}{d^{e\sigma}_i(p_i)} \bigg) \\ &= F(\ln z_i - \ln d^{e\sigma}_i(p_i)) \bigg( 2\frac{\partial d^{e\sigma}_i(p_i)}{\partial p_i} + p_i \bigg( \frac{\partial^2 d^{e\sigma}_i(p_i)}{\partial p_i^2} \bigg) \bigg) \\ &- \bigg( \frac{\partial d^{e\sigma}_i(p_i)}{\partial p_i} \bigg)^2 \frac{f(\ln z_i - \ln d^{e\sigma}_i(p_i))}{d^{e\sigma}_i(p_i)} . \end{split}$$

The first term is negative because  $\left(2\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} + p_i\left(\frac{\partial^2 d_i^{e\sigma}(p_i)}{\partial p_i^2}\right)\right)$  is just the second derivative of the function  $p_i d_i^{e\sigma}(p_i)$ , that is strictly concave by assumption. It is clear that the second term is also negative. Therefore,  $\frac{\partial^2 Q_{iT}(\cdot)}{\partial p_i^2} < 0$ .

Furthermore, since  $\frac{\partial \mathcal{Q}_{iT}^{er}(\cdot)}{\partial p_i} = y_i^{e\sigma}(z_i, p_i) + p_i \frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i}$ , we have that

$$\frac{\partial^2 Q_{iT}^{\sigma}(\cdot)}{\partial p_i \partial z_i} = \frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} + p_i \frac{\partial^2 y_i^{e\sigma}(z_i, p_i)}{\partial p_i \partial z_i}.$$
 [14]

As we have shown above,  $\frac{\partial y_i^{e\sigma}(z_i,p_i)}{\partial z_i} = \lambda' \left( \frac{z_i}{d_i^{e\sigma}(p_i)} \right) = 1 - F \left( \ln z_i - \ln d_i^{e\sigma}(p_i) \right)$ . We have also shown that  $\frac{\partial y_i^{e\sigma}(z_i,p_i)}{\partial p_i} = \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} F \left( \ln z_i - \ln d_i^{e\sigma}(p_i) \right)$ , and therefore

$$\frac{\partial^2 y_i^{e\sigma}(z_i,p_i)}{\partial p_i \partial z_i} = \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} \frac{f(\ln z_i - \ln d_i^{e\sigma}(p_i))}{z_i}.$$

Inserting these expressions into the eq. [14], we get:

$$\frac{\partial^2 Q_{iT}^{\sigma}(\cdot)}{\partial p_i \partial z_i} = 1 - F(\ln z_i - \ln d_i^{e\sigma}(p_i)) + \frac{p_i}{z_i} \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} f(\ln z_i - \ln d_i^{e\sigma}(p_i)).$$

With  $\eta_d(p_i) \equiv -\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} \frac{p_i}{d_i^{e\sigma}(p_i)} > 0$  as the elasticity of expected demand, and  $\eta_\lambda \left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \equiv -\lambda' \left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \frac{z_i}{\lambda(\cdot)d_i^{e\sigma}(p_i)} < 0$  as the elasticity of the  $\lambda(\cdot)$ -function the above expression can be written as

$$\frac{\partial^{2} Q_{iT}^{\sigma}(\cdot)}{\partial p_{i} \partial z_{i}} = \lambda'(\cdot) (1 - \eta_{d}(\cdot) (1 - \eta_{\lambda}(\cdot))) + \lambda(\cdot) \eta_{d}(\cdot) \eta'_{\lambda}(\cdot)$$
[15]

with

$$\eta_{\lambda}^{'}(\cdot) = -\frac{\frac{z_{i}}{d_{i}^{eor}(p_{i})}\lambda^{\prime}(\cdot)^{2} + \lambda(\cdot)(\lambda^{\prime}(\cdot) + \frac{z_{i}}{d_{i}^{eor}(p_{i})}\lambda^{\prime\prime}(\cdot))}{\lambda(\cdot)^{2}}.$$

The term  $\eta_{\lambda}^{'}(\cdot)$  is negative as  $\lambda'(\cdot) + \frac{z_i}{d_i^{e_i}(p_i)}\lambda''(\cdot) = 1 - F(\cdot) - f(\cdot)$  is positive for  $1 - F(\cdot) > f(\cdot)$  which is fulfilled by assumption. Thus, the second term of eq. [15] is negative.

Now, let's particularize expression [15] at  $(z, \bar{p}_T(z))$ . We can write

$$\frac{\partial Q_{iT}^{\sigma}(\cdot)}{\partial p_i} = y^{e\sigma}(z_i, p_i)(1 - \eta_d(\cdot)(1 - \eta_\lambda(\cdot)))$$

such that  $1-\eta_d(\cdot)(1-\eta_\lambda(\cdot))$  can never be positive at the optimal decision and therefore  $\frac{\partial^2 Q^\sigma_{iT}(\cdot)}{\partial p_i \partial z_i} < 0$  holds.

We will now show that if  $Q_i^{\sigma}(z_{it+1}, p_{it+1}; \cdot)$  is strictly concave in prices and prices and supply are strategic substitutes in t+1, then  $Q_i^{\sigma}(z_{it}, p_{it}; \cdot)$  is strictly concave in prices and prices and supply are strategic substitutes in t as well.

We make use of the fact that the profit function is bounded from above. More specifically,

$$\max_{s_i \geq 0} \max_{\{z_i \geq s_i, p_i \geq 0\}} \left\{ p_i d_i^{e\sigma}(p_i) \lambda \left( \frac{z_i}{d_i^{e\sigma}(p_i)} \right) - c_i z_i - k_i I_{\{z_i > s_i\}} \right\}$$

is smaller than some constant  $\tau < \infty$ . This property guarantees that for any values of  $z_i$  and  $p_i$ 

$$Q_i^{\sigma}(z_i, p_i) = \lim_{T \to \infty} Q_{iT}^{\sigma}(z_i, p_i).$$

Thus, as in t+1 the "value" function given as

$$\begin{split} Q_i^{\sigma}(z_{it+1},p_{it+1};\cdot) &\equiv -cz_{it+1} + p_{it+1} \int \min \left\{ z_{it+1}; \, e^{\varepsilon_{it+1}} d_{it+1}^{e\sigma}(p_{it+1}) \right\} dF(\varepsilon_{it+1}) \\ &+ \beta \int V_i^{\sigma} \left( \max \left\{ 0; \, z_{it+1} - e^{\varepsilon_{it+1}} d_{it+1}^{e\sigma}(p_{it+1}) \right\}; \cdot \right) \\ &dF(\varepsilon_{it+1}) dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+2} | \mathbf{s}_{-it+1}) \end{split}$$

is strictly concave in prices and prices and supply are strategic substitutes, so is the function in t. This completes the proof.

## A.3 Optimal Decision Rule: Proof of Proposition 1

Following Scarf (1960), the key to proving that the optimal strategy is of the (S, s) form is to show that the value function V is k-concave. Our proof exploits several properties of k-concave functions.

A real-valued function f(s) is a k-concave function if and only if for every  $s_0$  and  $s_1$  such that  $s_0 \le s_1$  and every scalar  $\delta \in (0,1)$ :

$$\delta f(s_0) + (1 - \delta)f(s_1) \le (1 - \delta)k + f(\delta s_0 + (1 - \delta)s_1).$$
 [16]

Consider the following properties of k – concave functions:

- (i) If *f* is strictly *k*-concave it has a unique global maximum.
- (ii) If f is strictly k-concave, and  $s^*$  is the global maximum, then the equation  $f(z) = f(s^*) k$  has two solutions,  $s^L$  and  $s^H$  with  $s^L < s^H$ . Furthermore,  $f(s) > f(s^*) k$  if and only if  $s \in (s^L, s^H)$ .
- (iii) If f(x, y) is k-concave in x for any value of y, and k-concave in y for any value of x, and  $y^*(x) \equiv \arg\max_y f(x, y)$ , then  $g(x) \equiv f(x, y^*(x))$  is k-concave.
- (iv) If  $f_1(\cdot)$  is  $k_1$ -concave,  $f_2(\cdot)$  is  $k_2$ -concave, and  $\alpha_1$ ,  $\alpha_2$  are two positive scalars, then  $\alpha_1 f_1 + \alpha_2 f_2$  is  $(\alpha_1 k_1 + \alpha_2 k_2)$ -concave.

**Proof:** Suppose that  $Q_i^{\sigma}$  is strictly k-concave in  $z_i$  for any value of  $p_i$  and strictly k-concave in  $p_i$  for any value of  $z_i$  for all values of  $\mathbf{s}_{-it}$ .

The optimal price decision can be written as

$$\sigma_{ip}^{\sigma}(\mathbf{s}) \equiv \bar{p}_{i}^{\sigma}(z_{i}; \mathbf{s}_{-i}).$$

That means, giving the optimal pricing function  $\bar{p}_i^{\sigma}(z; \mathbf{s}_{-i})$  the firm chooses inventory level  $\sigma_{iz}^{\sigma}(\mathbf{s})$  which results in pricing  $\sigma_{ip}^{\sigma}(\mathbf{s})$  as a function of the preorder inventory.

As  $Q_i^{\sigma}(\cdot)$  is strictly k-concave,  $s_i^{\star\sigma}(\mathbf{s}_{-i})$  and  $\bar{p}_i^{\sigma}(s_i^{\star\sigma}(\mathbf{s}_{-i}),\mathbf{s}_{-i})$  are unique and  $\bar{p}_i^{\sigma}(\cdot,\cdot)$  is a real function. Furthermore,  $Q_i^{\sigma}(z_i,\bar{p}_i^{\sigma}(z_i);\mathbf{s}_{-i})$  is also strictly k-concave. By definition of  $\sigma_{iz}^{\sigma}(\mathbf{s})$ ,  $s_i^{\star\sigma}(\mathbf{s}_{-i})$ , and  $\bar{p}_i^{\sigma}(s_i^{\star\sigma}(\cdot),\cdot)$ , it is clear that

$$\sigma_{iz}^{\sigma}(\mathbf{s}) = \begin{cases} s_i^{\star\sigma}(\mathbf{s}_{-i}) & \text{if} \quad Q_i^{\sigma}(s_i^{\star\sigma}, \bar{p}_i^{\sigma}(s_i^{\star\sigma}); \cdot) - k > Q_i^{\sigma}(s_i, \bar{p}_i^{\sigma}(s_i); \cdot) \\ s_i & \text{if} \quad Q_i^{\sigma}(s_i^{\star\sigma}, \bar{p}_i^{\sigma}(s_i^{\star\sigma}); \cdot) - k \leq Q_i^{\sigma}(s_i, \bar{p}_i^{\sigma}(s_i); \cdot). \end{cases}$$

Due to the *k*-concavity of  $Q_i^{\sigma}(z_i, \bar{p}_i^{\sigma}(z_i); \cdot)$  the equation  $Q_i^{\sigma}(s_i^{\star \sigma}, \bar{p}_i^{\sigma}(s_i^{\star \sigma}); \cdot) - k = Q_i^{\sigma}(s_i, \bar{p}_i^{\sigma}(s_i); \cdot)$  has only two solutions.

Let these two solutions be  $s_i^L(\cdot)$  and  $s_i^H(\cdot)$ , where  $s_i^L(\cdot) \le s_i^{\star\sigma}(\cdot) \le s_i^H(\cdot)$ . Then, k-concavity implies

$$Q_i^{\sigma}(s_i^{\star\sigma},\bar{p}_i^{\sigma}(s_i^{\star\sigma});\cdot)-k\leq Q_i^{\sigma}(s_i,\bar{p}_i^{\sigma}(s_i);\cdot) \Leftrightarrow s_i^L(\cdot)\leq s_i(\cdot)\leq s_i^H(\cdot).$$

It is clear that the conditions  $s_i > s_i^H$  and  $s_i \le s_i^H$  do not play any role because the stock level is always lower or equal to  $s_i^{\star \sigma}$ . With  $\underline{s}_i^{\sigma}$  as the smaller of the two solutions by definition we can write the optimal decision as

$$\sigma_{iz}^{\sigma} = \begin{cases} s_i^{\star \sigma} & \text{if} \quad s_i \leq \underline{s}_i^{\sigma}, \\ s_i & \text{if} \quad s_i \geq \underline{s}_i^{\sigma}. \end{cases}$$

The according optimal pricing decision for the inventory before ordering is

$$\sigma_{ip}^{\sigma}(\mathbf{s}) = \bar{p}^{\sigma}(\sigma_{iz}(\mathbf{s})) = \begin{cases} \bar{p}^{\sigma}(s_i^{\star\sigma}) & \text{if} \quad s_i \leq \underline{s}_i^{\sigma}, \\ \bar{p}^{\sigma}(s_i) & \text{otherwise.} \end{cases}$$

It further remains to show that  $Q_i^{\sigma}$  is indeed *k*-concave.

We proceed in three steps:

- If  $V_i^{\sigma}(\mathbf{s})$  is strictly k-concave in  $s_i$ , then  $Q_i^{\sigma}(\cdot)$  is strictly k-concave in  $z_i$  for any value of  $p_i$ .
- If  $V_i^{\sigma}(\mathbf{s})$  is strictly *k*-concave in  $s_i$ , then  $Q_i^{\sigma}(\cdot)$  is strictly *k*-concave in  $p_i$  for (b) any value of  $z_i$ .
- $V_i^{\sigma}(\mathbf{s})$  is strictly *k*-concave in  $s_i$ . (c)
- We will now show that if  $V_i^{\sigma}(\mathbf{s})$  is strictly *k*-concave in  $s_i$ , then  $Q_i^{\sigma}(z_i, p_i; \mathbf{s}_{-i})$ is strictly k-concave in  $z_i$  for any value of  $p_i$ .

By the first part of the proof, there exist  $s_i^{\star\sigma}$  and  $\underline{s}_i^{\sigma}$  satisfying  $0 \le \underline{s}_i^{\sigma} \le s_i^{\star\sigma}$  for which  $V_i^{\sigma}$  can be represented as

$$V_{i}^{\sigma}(\mathbf{s}) = V(Q)_{i}^{\sigma}(\mathbf{s}, \sigma_{ip}^{\sigma}(\mathbf{s})) = \begin{cases} Q_{i}^{\sigma}(s_{i}^{\star\sigma}, \bar{p}_{i}(s_{i}^{\star\sigma}); \mathbf{s}_{-i}) + cs_{i} - h(s_{i}) - k & \text{if} \quad s_{i} \in [0, \underline{s}_{i}^{\sigma}), \\ Q_{i}^{\sigma}(s_{i}, \sigma_{p}(s_{i}); \mathbf{s}_{-i}) + cs_{i} - h(s_{i}) & \text{if} \quad s \geq \underline{s}. \end{cases}$$

$$[17]$$

 $V_i^{\sigma}(\mathbf{s})$  can be extended to be a function defined on  $\mathbb{R} \times \mathbb{R}^{n-1}$ :

$$V_i^{\sigma}(\mathbf{s}) = \left\{ egin{array}{ll} V_i^{\sigma}(0,\mathbf{s}_{-i}) + cs_i & ext{if } s_i \leq 0, \\ V_i^{\sigma}(\mathbf{s}) & ext{else,} \end{array} 
ight.$$

which is needed as the proof of (c) implies that  $V_i$  is k-concave in  $s_i$  over  $\mathbb{R}$ . We can write  $Q_i^{\sigma}$  as

$$Q_i^{\sigma}(\cdot) = Q_i^{\sigma R}(\cdot) + \beta Q_i^{\sigma V}(\cdot),$$

where

$$\begin{split} Q_{i}^{\sigma R}(\cdot) &\equiv -cz_{i} + p_{i} \int \min \left\{ z_{i}; \, e^{\varepsilon_{i}} d_{i}^{e\sigma}(p_{i}) \right\} dF(\varepsilon_{i}) \\ &= -cz_{i} + p_{i} d_{i}^{e\sigma}(p_{i}) \lambda \left( \frac{z_{i}}{d_{i}^{e\sigma}(p_{i})} \right) \end{split}$$

and

$$Q_i^{\sigma V}(\cdot) \equiv \int V_i^{\sigma} \left( \max \left\{ 0; \, z_{it} - e^{\varepsilon_{it}} d_{it}^{e\sigma}(p_{it}) \right\}; \, \mathbf{s}_{-it+1} \right) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+1} | \mathbf{s}_{-it}).$$

Let us now consider the function  $\int V_i^{\sigma} \left(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot\right) dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^{\sigma}$ . Since each  $V_i^{\sigma}(\cdot)$  is k-concave in  $s_i$  over  $\mathbb{R}$ , and since positive linear combinations of pointwise limits of k-concave functions are k-concave, it follows that  $\int V_i^{\sigma} \left(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot\right) dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^{\sigma}$  is k-concave in  $s_i$  on  $\mathbb{R}$ . With  $\bar{\varepsilon}_i(\cdot)$  as the value of  $\varepsilon_i$  for which demand is equal to supply  $z_i$ , i.e.  $z_i = \exp(\bar{\varepsilon}_i(z_i)) d^{e\sigma}(\cdot)$ , we have

$$\begin{split} &\int V_{i}^{\sigma}\left(s_{i}-e^{\varepsilon_{i}}d_{i}^{e\sigma}(\cdot);\cdot\right)dF(\varepsilon_{i})dF_{\mathbf{s}_{-i}}^{\sigma} \\ &=\int_{-\infty}^{\bar{\varepsilon}_{i}(s_{i})}V_{i}^{\sigma}\left(s_{i}-e^{\varepsilon_{i}}d_{i}^{e\sigma}(\cdot);\cdot\right)dF(\varepsilon_{i})dF_{\mathbf{s}_{-i}}^{\sigma} \\ &+\int_{\bar{\varepsilon}_{i}(s_{i})}^{\infty}V_{i}^{\sigma}\left(s_{i}-e^{\varepsilon_{i}}d_{i}^{e\sigma}(\cdot);\cdot\right)dF(\varepsilon_{i})dF_{\mathbf{s}_{-i}}^{\sigma} \\ &=\int_{-\infty}^{\bar{\varepsilon}_{i}(s_{i})}V_{i}^{\sigma}\left(s_{i}-e^{\varepsilon_{i}}d_{i}^{e\sigma}(\cdot);\cdot\right)dF(\varepsilon_{i})dF_{\mathbf{s}_{-i}}^{\sigma} +V_{i}^{\sigma}(0;\cdot)\int_{\bar{\varepsilon}_{i}(s_{i})}^{\infty}dF(\varepsilon_{i})dF_{\mathbf{s}_{-i}}^{\sigma} \\ &+c\int_{\bar{\varepsilon}_{i}(s_{i})}^{\infty}\left(s_{i}-e^{\varepsilon_{i}}d_{i}^{e\sigma}(\cdot)\right)dF(\varepsilon_{i}) \\ &=Q_{i}^{\sigma V}(s_{i},p_{i},\cdot)+c\int_{\bar{\varepsilon}_{i}(s_{i})}^{\infty}\left(s_{i}-e^{\varepsilon_{i}}d_{i}^{e\sigma}(\cdot)\right)dF(\varepsilon_{i}). \end{split}$$

Using the definition of  $Q_i^{\sigma}$ , we have

$$\begin{split} Q_{i}^{\sigma}(\cdot) &= Q_{i}^{\sigma R}(\cdot) \ + \ \beta Q_{i}^{\sigma V}(\cdot) \\ &= p_{i} \int \min \big\{ z_{i}; \ e^{\varepsilon_{i}} d_{i}^{e\sigma}(p_{i}) \big\} dF(\varepsilon_{i}) - cz_{i} \\ &+ \beta \int_{-\infty}^{\bar{\varepsilon}_{i}(z_{i})} V_{i}^{\sigma} \big( z_{i} - e^{\varepsilon_{i}} d_{i}^{e\sigma}(\cdot); \cdot \big) dF(\varepsilon_{i}) dF_{\mathbf{s}_{-i}}^{\sigma} \\ &+ \beta \int_{\bar{\varepsilon}_{i}(z_{i})}^{\infty} V_{i}^{\sigma} \big( z_{i} - e^{\varepsilon_{i}} d_{i}^{e\sigma}(\cdot); \cdot \big) dF(\varepsilon_{i}) dF_{\mathbf{s}_{-i}}^{\sigma} \\ &- \beta c \int_{\bar{\varepsilon}_{i}(z_{i})}^{\infty} \big( z_{i} - e^{\varepsilon_{i}} d_{i}^{e\sigma}(\cdot) \big) dF(\varepsilon_{i}). \end{split}$$

The sum of the third and fourth terms in the last equation is k-concave since  $\int V_i^{\sigma}(s_i - e^{\varepsilon_i}d_i^{e\sigma}(\cdot);\cdot)dF(\varepsilon_i)dF_{s_i}^{\sigma}$  is *k*-concave. Since  $cz_i$  is a linear and hence convex function of  $z_i$ , a sufficient condition for the k-concavity of  $Q_i^{\sigma}(\cdot)$  is that the function

$$p_i d_i^{e\sigma}(p_i) \lambda \left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) - \beta c \int_{\bar{\varepsilon}_i(z_i)}^{\infty} \left(z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot)\right) dF(\varepsilon_i)$$

is concave in  $z_i$ . The function is continuously differentiable in  $z_i$  with second derivatives

$$(p_i - \beta c)(1 - F(\ln z_i - \ln d_i^{e\sigma}(\cdot))).$$

As  $F(\cdot)$  < 1, this expression is non-positive and hence  $Q_i^{\sigma}$  is k-concave as long as  $p_i \ge \beta c_i$ . (Obviously, a weaker condition for that result exists.)

For proving that  $Q_i^{\sigma}$  is indeed *k*-concave we need to show that  $\sigma_{iv}^{\sigma}(\mathbf{s}) - \beta c \ge 0$ holds. Recall

$$\begin{split} Q_{i}^{\sigma}(z_{i},p_{i};\,\mathbf{s}_{-i}) &\equiv -cz_{i} + p_{i}d_{i}^{e\sigma}(p_{i})\lambda\left(\frac{z_{i}}{d_{i}^{e\sigma}(p_{i})}\right) \\ &+ \beta\int V_{i}^{\sigma}\left(\max\left\{0;\,z_{i} - e^{\varepsilon_{i}}d_{i}^{e\sigma}(p_{i})\right\};\,\mathbf{s}_{-it+1}\right)dF(\varepsilon_{i})dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+1}|\mathbf{s}_{-i}) \end{split}$$

and

$$V_i^{\sigma}(\mathbf{s}_t) = \max_{\{p_i,\,q_i\}} \big\{ Q_i^{\sigma}(s_{it} + q_i,p_i;\,\mathbf{s}_{-it}) - (h_i - c_i)s_{it} - k_i I_{\{q_i > 0\}} \big\}.$$

where the expected sales  $d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d^{e\sigma}(p_i)}\right)$  are always smaller than or equal to total supply  $z_i$ . Let's suppose to the contrary that there is an optimal price  $\sigma_{ip}^{\sigma} < \beta c < c$ . In that case  $-cz_i + p_i d_i^{e\sigma}(p_i) \lambda \left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right)$  would be negative. Thus, without a new order the current value  $V_i^{\sigma}(\mathbf{s}_t)$  would be smaller than the expected value  $V_i^{\sigma}(\mathbf{s}_{t+1})$  after selling the goods at price  $\sigma_p(s_t)$  although the inventory is larger, i.e.,  $s_{it} > s_{it+1}$ . This cannot be the case in equilibrium. The same is true in the case with ordering. Ordering goods and simultaneously selling them for a price lower than the purchase price cannot be an optimal strategy. Thus, the optimal price  $\sigma_{iv}^{\sigma}$  is always greater c.

(b) We will show that if  $V_i^{\sigma}(\mathbf{s})$  is strictly *k*-concave in  $s_i$ , then  $Q_i^{\sigma}(\cdot)$  is strictly *k*-concave in  $p_i$  for any value of  $z_i$ .

We can represent the function  $Q_i^{\sigma R}(\cdot)$  as  $-cz_i + p_i y^{\sigma e}(z_i, p_i; \mathbf{s}_{-i})$ , where  $y^{\sigma e}(\cdot)$  is the expected sales function. The function  $Q_i^{\sigma R}(\cdot)$  is the same as the function  $Q_i^{\sigma}$  at the last period  $Q_{iT}^{\sigma}$ . We have shown in the proof of Lemma 2 that this function is convex.

Therefore,  $\frac{\partial^2 Q_i^{\sigma R}(\cdot)}{\partial p_i^2} < 0$ .

An argumentation analogous to part (a) yields a similar sufficient condition for the *k*-concavity of  $Q_i^{\sigma}(\cdot)$  in  $p_i$ , namely that the function

$$p_i d_i^{e\sigma}(p_i) \lambda \left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) - \beta c \int_{\bar{\varepsilon}_i(z_i)}^{\infty} \left(z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot)\right) dF(\varepsilon_i)$$

is concave in  $p_i$ . The function is continuously differentiable in  $p_i$  with a second derivative that is negative. Therefore,  $Q_i^{\sigma}(\cdot)$  is k-concave in  $p_i$ .

(c) Finally, we show that  $V_i^{\sigma}(\mathbf{s})$  is strictly *k*-concave in  $s_i$ .

Like in proof of Lemma 2 we make use of the fact that the profit function is bounded from above. This property guarantees that for any value of  $s_i$ 

$$V_i^{\sigma}(s_i;\cdot) = \lim_{T\to\infty} V_{iT}^{\sigma}(s_i;\cdot)$$

with  $V_{iT}^{\sigma}(s_i)$  as the value function for the finite horizon problem with time horizon equal to T. We prove k-concavity by induction.

For T=1 we have  $Q_{i1}^{\sigma}(\cdot)$  is strictly concave in  $z_i$  and  $p_i$  due to (a) and (b). Using the result of the first part of the proof, the optimal decision for this one-period problem has the form of eqs [9] and [10]. Hence, the value function of this one period problem is

$$\begin{split} V_{i1}^{\sigma}(s_i,\cdot) &= I(s_i < \underline{s}_{i1}^{\sigma}) \left( Q_{i1}^{\sigma}(s_{i1}^{\star\sigma}, \bar{p}_{i1}^{\sigma}(s_{i1}^{\star\sigma})) - k \right) \\ &+ I(s_i \ge \underline{s}_{i1}^{\sigma}) Q_{i1}^{\sigma}(s_i, \bar{p}_{i1}^{\sigma}(s_i, \cdot)) - (h_i - c_i) s_i. \end{split}$$

With  $Q_{i1}^{\sigma}(\cdot)$  being concave, it is simple to verify that  $V_{i1}^{\sigma}(s_i,\cdot)$  fulfills the definition of strict k-concavity.

Assume now that for any  $t \ge 1$ ,  $V_{it}^{\sigma}(s_i, \cdot)$  is strictly k-concave. Then,

$$\begin{split} Q_{it+1}^{\sigma}(z_i,\bar{p}_{it+1}^{\sigma}(\cdot);\mathbf{s}_{-it+1}) &= -cz_i + \bar{p}_{it+1}^{\sigma}(\cdot)d_i^{e\sigma}(\bar{p}_{it+1}^{\sigma}(\cdot))\lambda\bigg(\frac{z_i}{d_i^{e\sigma}(\bar{p}_{it+1}^{\sigma}(\cdot))}\bigg) \\ &+ \beta\int V_i^{\sigma}\left(\max\left\{0;z_i - e^{\varepsilon_i}d_i^{e\sigma}(\bar{p}_{it+1}^{\sigma}(\cdot))\right\};\cdot\right) \\ &dF(\varepsilon_i)dF_{\mathbf{s}_{-i}}^{\sigma}(\mathbf{s}_{-it+2}|\mathbf{s}_{-it+1}). \end{split}$$

As  $\bar{p}_{it+1}^{\sigma}(\cdot)d_{i}^{e\sigma}(\bar{p}_{it+1}^{\sigma}(\cdot))\lambda\left(\frac{z_{i}}{d_{i}^{e\sigma}(\bar{p}_{it+1}^{\sigma}(\cdot))}\right)-cz_{i}$  is again strictly concave and  $V_{it}^{\sigma}(s_{i},\cdot)$  is strictly k-concave, due to property (iv) of k-concave functions,  $Q_{it+1}^{\sigma}(z_{i},\bar{p}_{it+1}^{\sigma}(\cdot);\cdot)$  is also strictly k-concave. Hence, the optimal decision has again the form of eqs [9] and [10] and the value function of this finite-horizon problem is

$$\begin{split} V_{it+1}^{\sigma}(s_{i},\cdot) &= I(s_{i} < \underline{s}_{it+1}^{\sigma}) \left( Q_{it+1}^{\sigma}(s_{it+1}^{\star\sigma}, \bar{p}_{it+1}^{\sigma}(s_{it+1}^{\star\sigma})) - k \right) \\ &+ I(s_{i} \ge \underline{s}_{it+1}^{\sigma}) Q_{it+1}^{\sigma}(s_{i}, \bar{p}_{it+1}^{\sigma}(s_{i}, \cdot)) - (h_{i} - c_{i}) s_{i}. \end{split}$$

Similar to  $V_{i1}^{\sigma}(s_i,\cdot)$ , this value function is strictly k-concave which completes the proof by induction. Therefore,  $V_i^{\sigma}(s_i;\cdot) = \lim_{T\to\infty} V_{iT}^{\sigma}(s_i;\cdot)$  is strictly *k*-concave.

This completes the proof of the optimality of the described ordering strategy.

**Properties of the optimal price:** We complete the proof of Proposition 1 by showing that  $\bar{p}(\cdot)$  is a continuous and strictly decreasing function.

The function  $\bar{p}_i^{\sigma}$  is the value of  $p_i$  that maximizes  $Q_i^{\sigma}$  in  $p_i$  for a given  $z_i$ . Since  $Q_i^{\sigma}$  is continuously differentiable and strictly concave in prices,  $\bar{p}_i^{\sigma}(z; \mathbf{s}_{-i})$  is implicitly defined by the first order condition  $\frac{\partial Q_i^{\sigma}(z_i, \bar{p}; \mathbf{s}_{-i})}{\partial p_i} = 0$ . By the implicit function theorem, we have that  $\frac{d\bar{p}_i(z_i)^{\sigma}}{dz_i} = -\frac{\partial^2 Q_i^{\sigma}(z_i,\bar{p}_i)/\partial p_i\partial z_i}{\partial^2 Q_i^{\sigma}(z_i,\bar{p}_i)/\partial p_i\partial p_i}$ , that by Lemma 2 is negative.

This completes the proof.

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