

Research Article

Giorgos Stamatopoulos*

The Core of Aggregative Cooperative Games with Externalities

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Abstract: This paper analyzes cooperative games with externalities generated by aggregative normal form games. We construct the characteristic function of a coalition S for various coalition formation rules and we examine the corresponding cores. We first show that the γ -core is non-empty provided each player's payoff decreases in the sum of all players' strategies. We generalize this result by showing that if S believes that the outside players form at least $l(s) = n - s - (s - 1)$ coalitions, then S has no incentive to deviate from the grand coalition and the corresponding core is non-empty (where n is the number of players in the game and s the number of members of S). We finally consider the class of linear aggregative games (Martimort and Stole 2010). In this case, if S believes that the outsiders form at least $\hat{l}(s) = \frac{n}{s} - 1$ coalitions [where $\hat{l}(s) \leq l(s)$] a core non-emptiness result holds again.

Keywords: aggregative game, cooperative game, externalities, core

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1 Introduction

The core is the most widely used solution concept in cooperative game theory. It is the set of all allocations of the worth of the grand coalition that prevent any other coalition from forming and standing alone. To compute or even define the core, one needs to first define the characteristic function of a coalition. The characteristic function specifies the worth a group of players can attain if they act on their own, i.e., without cooperating with the players outside the coalition. For a cooperative game with externalities, namely a game where the worth of a coalition depends on the actions of the outsiders, the specification of the characteristic function requires a prediction about the behavior of the non-members, in particular their coalition structure.

*Corresponding author: **Giorgos Stamatopoulos**, Department of Economics, University of Crete, Rethymno 74100, Greece, E-mail: gstamato@uoc.gr
<http://orcid.org/0000-0003-1761-9216>

The literature has offered various such predictions or conjectures, each leading to a specific notion of core. The α and β -cores (Aumann 1967) are based on the assumption that the outsiders will try to minimize the payoff of a coalition that deviates from the society of all players. The γ -core (Chander and Tulkens 1997) is based on the premise that the outsiders will play individual best replies to the deviant coalition (i.e., the outsiders form singleton coalitions); the same approach can be followed under the additional assumption that the deviant coalition acts as a Stackelberg leader (Currarini and Marini 2003). The recursive core (Huang and Sjostrom 2003; Koczy 2007) is constructed under the assumption that the members of a coalition compute their value by looking recursively on the cores of the sub-games played among the outsiders.

Behind a cooperative game with externalities lies a normal form game where players can transfer utilities among themselves and sign binding agreements. The current paper focuses on cooperative games generated by aggregative normal form games, i.e., games where the payoff of a player depends only on his own strategy and the aggregate value of all players' strategies. Many economic models have an aggregative structure, such as common pool resource games, oligopoly models, cost sharing games, rent seeking games, etc.¹ We utilize the structure of these games in order to define and analyze various notions of core, each depending on the conjectures a deviant coalition has about the partition of the outsiders. Our goal and motivation is to provide the largest possible set of coalitional beliefs that allow the non-emptiness of the (appropriately defined) core. The paper develops gradually, starting with the case of single-valued conjectures (i.e., conjectures that focus on one partition of the outsiders) and then moving on to set-valued conjectures.

We begin with the case of γ -beliefs (Chander and Tulkens 1997), as they are often encountered in applications:² a coalition believes that should it deviate from the grand coalition its opponents will stay separate. Given these beliefs, we examine the incentive for deviation within an aggregative environment. We then generalize the analysis by determining for each coalition *a set of partitions* of the outsiders under which there is no incentive for deviation.

The paper focuses on environments with symmetric players. The results can be summarized as follows:

1. if the payoff function of a player is decreasing in the aggregate value of all players' strategies and his marginal payoff is decreasing in own strategy then the γ -core of the game is non-empty; i.e., n -player games where a

¹ The literature on aggregative games is large and expanding. See, for example, Martimort and Stole (2010) or Acemoglu and Jensen (2013) for extensive lists of references.

² We refer the reader to the literature review that follows.

- coalition with s members believes that the outsiders form $n - s$ singleton coalitions have non-empty core.
2. we introduce the additional assumption that the marginal effect of a player on another player's payoff is decreasing in the latter's own strategy; we show that the cores of games where a coalition with s members believes that the outsiders form *at least* $l(s) = \max\{n - s - (s - 1), 1\}$ coalitions are non-empty.
 3. we consider the class of linear aggregative normal form games (Martimort and Stole 2010); we show that the cores of games where a coalition with s members believes that the outsiders form *at least* $\hat{l}(s) = \max\{\frac{n}{s} - 1, 1\}$ coalitions are non-empty, where³ $\hat{l}(s) \leq l(s)$.

Our work is related to the literature on the γ -core of cooperative games. This notion of core was defined by Chander and Tulkens (1997) in the framework of a coalitional economy with environmental externalities. The authors defined the γ -characteristic function of a coalition (i.e., its payoff when the outsiders play individual best strategies) and showed that the corresponding core is non-empty under specific assumptions on the utility functions. Their result was strengthened by Helm (2001) which showed the non-emptiness of the core by proving that the induced cooperative game is balanced. The concept of γ -core has also been examined in oligopolistic markets with quantity competition. Chander (2010) proved that the cooperative oligopoly game defined by the γ -scenario is balanced and hence the corresponding core is non-empty. Non-emptiness results are obtained also in Lardon (2001) for oligopolies where firms operate under capacity constraints. The core existence results can fail though when the deviant coalition presumes for itself the role of Stackelberg leader: under this scenario, the γ -core is empty or not depending on the nature of competition among firms in the market, as shown in Currarini and Marini (2003). A similar conclusion holds for the case of economies with environmental externalities: Marini (2013) shows that in an economy with Cobb-Douglas utilities the non-emptiness of the (sequential) core depends on the relative preferences over the environmental quality and the private good.

The papers most closely related to ours are Currarini and Marini (2003) and Lekeas (2013). The first result of our paper, i.e., result (i), is connected to the work of Currarini and Marini (2003) which analyzed the non-emptiness of the γ -core for general cooperative games with externalities. Their work showed that the γ -core is non-empty under two main assumptions: (a) the underlying normal form game exhibits strategic complementarities and (b) the deviant coalition assumes for itself the role of Stackelberg leader. Interestingly, their assumptions, although

³ Note that since S has s members, the number of coalitions of the outsiders cannot exceed the number $n - s$ (and cannot be less than 1). For this range, the inequality $\hat{l}(s) \leq l(s)$ holds.

different than ours, produce a similar result. Finally, the two other results of the current paper, i.e., results (ii)–(iii), are connected to the work of Lekeas (2013). This paper analyzed a linear oligopoly market where firms compete in quantities. For each coalition of firms a set of partitions of the outsiders is found that prevent the coalition from deviating from the rest of the firms. Our work thus generalizes this analysis by considering more general cooperative games.

The paper is organized as follows. Section 2 introduces the main setting and discusses the case of γ -beliefs and the corresponding core. Section 3 analyzes more general coalitional beliefs: we first present the analysis in terms of general symmetric aggregative normal form games; we then focus on the sub-class of linear aggregative games. Section 4 offers concluding remarks.

2 Aggregative Games and γ -Beliefs

We consider a normal form game $\Gamma = \{N, (X_i, U_i)_{i \in N}\}$ where $N = \{1, 2, \dots, n\}$ is the set of players; $X_i \subset \mathbb{R}$ is player i 's strategy set; and $U_i : X \rightarrow \mathbb{R}$ is i 's payoff function, where X is the cartesian product of the individual strategy sets. We make the standard assumptions that X_i is compact and $U_i(x_1, x_2, \dots, x_n)$ is concave in argument x_i and continuous in all arguments jointly. An aggregative normal form game arises when the payoff of a player can be expressed as a function of two elements only: his own strategy and an aggregate of the strategies of all players. We take this aggregate to be simply the sum. Thus Γ is an aggregative game if for each player i there is a function $u_i : X_i \times Y \rightarrow \mathbb{R}$ such that $U_i(x_1, x_2, \dots, x_n) = u_i\left(x_i, \sum_{k \in N} x_k\right)$, where $Y \subset \mathbb{R}$.

We will focus on symmetric aggregative games. Hence, throughout the paper the following condition holds:⁴

$$AO \quad X_i = X_j; \text{ and } u_i\left(x_i, \sum_{k \in N} x_k\right) = u_j\left(x_j, \sum_{k \in N} x_k\right) \text{ if } x_i = x_j, \text{ for any } i, j.$$

We consider situations where players can form coalitions and sign binding contracts. A partition of set N into disjoint subsets (coalitions) is given by $\pi = \{S_1, S_2, \dots, S_l\}$. The strategy set of coalition $S_i \in \pi$ is given by the set $\times_{i \in S_i} X_i$; and its payoff function by

$$u_{S_i}(x_1, x_2, \dots, x_n) = \sum_{i \in S_i} u_i\left(x_i, \sum_{k \in N} x_k\right)$$

⁴ Since AO holds throughout the paper, we shall not refer to it when stating our results.

We denote by⁵ Γ_π the normal form game that arises under partition π . The equilibrium outcome of Γ_π is denoted by⁶ $(x_1^\pi, x_2^\pi, \dots, x_n^\pi)$.

We are interested in the formation of the grand coalition. Denote the resulting partition by $\pi^* = \{N\}$. The objective function of the grand coalition is

$$u_N(x_1, x_2, \dots, x_n) = \sum_{i \in N} u_i \left(x_i, \sum_{k \in N} x_k \right)$$

Denote by $(x_1^{\pi^*}, x_2^{\pi^*}, \dots, x_n^{\pi^*})$ the strategy profile that maximizes the above sum. Then the worth of the grand coalition is

$$v(N) = \sum_{i \in N} u_i \left(x_i^{\pi^*}, \sum_{k \in N} x_k^{\pi^*} \right)$$

The formation of the grand coalition is potentially blocked by the formation of smaller coalitions. Let $S \subset N$ be such a coalition, with $|S| = s$ members. Denote by $N \setminus S$ the set of all non-members of S . The payoff of S depends on how the $n - s$ outsiders partition off into coalitions. Let Π_{-S} be the set of all partitions that the outsiders can form. In this section we focus on a specific member of Π_{-S} , namely the partition that corresponds to the γ -scenario: S believes that, if it deviates from the grand coalition, the outside players will stay separate. The γ -scenario was introduced by Chander and Tulkens (1997) for economies with environmental externalities. Under the γ -approach, if S deviates from the grand coalition, the normal form game $\Gamma_{\pi'_S}$ is to be played, where⁷ $\pi'_S = \{S, \pi'_{-S}\}$ and π'_{-S} is the set of singleton coalitions of all players outside S .

Denote by $(x_1^{\pi'_S}, x_2^{\pi'_S}, \dots, x_n^{\pi'_S})$ the equilibrium choices in $\Gamma_{\pi'_S}$. The worth of coalition S then is

$$v_\gamma(S) = \sum_{i \in S} u_i \left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right)$$

The resulting cooperative game is denoted by (N, v_γ) . An allocation is a vector $(w_1, w_2, \dots, w_n)$ satisfying $\sum_{k \in N} w_k = v(N)$. The γ -core is the set of all allocations that no coalition S can block given the γ -scenario.

We will determine conditions for non-emptiness of the γ -core under the aggregative normal form structure. In some parts of the paper we will use the following:

⁵ The normal form game Γ corresponds to the case where $\pi = \{\{1\}, \{2\}, \dots, \{n\}\}$.

⁶ We express this outcome in terms of individual players' strategies in order to economize on the notation.

⁷ We use subscript S to keep track of the fact that we analyze the game from the viewpoint of deviant coalition S .

$A1 u_i \left(x_i, \sum_{k \in N} x_k \right)$ is continuously differentiable in x_i and in $\sum_{k \in N} x_k$.

$A2 u_i \left(x_i, \sum_{k \in N} x_k \right)$ is decreasing in $\sum_{k \in N} x_k$.

Numerous economic models satisfy condition A2, such as rent seeking games, oligopoly games, common pool resource games, cost or surplus sharing games, etc. Consider, for example, a market where firms compete in quantities. The set of firms is N . The firms produce a homogeneous product. Firm i produces quantity q_i . The market price is given by the price function $p \left(\sum_{k \in N} q_k \right)$, where $p' \left(\sum_{k \in N} q_k \right) < 0$. The cost of firm i is $C(q_i)$. Its payoff is

$$u_i \left(q_i, \sum_{k \in N} q_k \right) = p \left(\sum_{k \in N} q_k \right) q_i - C(q_i)$$

Condition A2 holds since the price function is decreasing.

Consider now a cost-sharing game. There is a set N of agents which produce an output of magnitude Y . The total cost of producing Y units is given by the increasing function $C(Y)$. The payoff v of an agent is increasing in his consumption of output and decreasing in his cost contribution. Agent i consumes y_i units. Consumption by all agents exhausts total output, i.e., $\sum_{k \in N} y_k = Y$. Assuming that agent i 's share of the cost is δ_i , the utility of i is written as

$$u_i \left(y_i, \sum_{k \in N} y_k \right) = v \left(y_i, \delta_i \cdot C \left(\sum_{k \in N} y_k \right) \right)$$

Given that the cost function is increasing, condition A2 holds.

Finally, let us look on a game with multilateral environmental externalities. Consider a set N of agents whose production activities generate negative externalities affecting one another. Agent i produces a private good at quantity x_i using the quantity e_i of an input. The production process is described by $x_i = f(e_i)$, where $f(e_i)$ is an increasing production function. The utility function of agent i is

$$u_i \left(e_i, \sum_{k \in N} e_k \right) = f(e_i) - d \left(\sum_{k \in N} e_k \right)$$

where $d(\cdot)$ is the damage function, which describes the loss of utility caused by the aggregate use of the input. If we assume that the damage function is increasing in the aggregate value of the externality then A2 is satisfied once more.

Let us now return to our general framework. In addition to our previous assumptions we shall also assume that the optimal strategies of the players in $\Gamma_{\pi'_S}$ always are in the interior of the corresponding strategy sets.

A3 $x_i^{\pi'_S} \in \text{int } X_i$, for all $i \in N$, where int denotes the interior of a set.

We begin with two preliminary results (Lemmas 1 and 2) which hold for any deviant coalition S .

Lemma 1 Assume A1-A3 hold and that $\partial u_i(x_i, \sum_{k \in N} x_k) / \partial x_i$ is decreasing in the first argument x_i . Let $i \in S$ and $j \notin S$. Then $x_j^{\pi'_S} \geq x_i^{\pi'_S}$.

Proof Given A3 $x_i^{\pi'_S}$ and $x_j^{\pi'_S}$ satisfy respectively

$$\frac{\partial u_i(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S})}{\partial x_i} + \sum_{\substack{r \in S \\ r \neq i}} \frac{\partial u_r(x_r^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S})}{\partial x_i} = 0 \quad [1]$$

$$\frac{\partial u_j(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S})}{\partial x_j} = 0 \quad [2]$$

Assumption A2 implies that each term in the sum of the derivatives in eq. [1] is negative. Hence by eq. [1] we have that

$$\frac{\partial u_i(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S})}{\partial x_i} > 0 \quad [3]$$

By assumption, the function $\partial u_i(x_i, \sum_{k \in N} x_k) / \partial x_i$ is decreasing in the first argument x_i , for any fixed $\sum_{k \in N} x_k$. Hence if $x_i^{\pi'_S} > x_j^{\pi'_S}$ then by eq. [3] we would have

$$\frac{\partial u_i(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S})}{\partial x_i} > 0$$

and hence by symmetry

$$\frac{\partial u_j(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S})}{\partial x_j} > 0$$

which violates eq. [2]. We conclude that $x_j^{\pi'_S} \geq x_i^{\pi'_S}$. ■

Let us identify a well-known environment where the condition that $\partial u_i(x_i, \sum_{k \in N} x_k) / \partial x_i$ decreases in the first argument is met. Consider the n -firm

market described in page 5. For simplicity let $\sum_{k \in N} q_k = Q$. Then the profit of firm i is

$$u_i(q_i, Q) = p(Q)q_i - C(q_i)$$

and thus

$$\frac{\partial u_i(q_i, Q)}{\partial q_i} = p(Q) + p'(Q)q_i - C'(q_i)$$

The condition that $\partial u_i(q_i, Q)/\partial q_i$ decreases in the first argument q_i is met if $p'(Q) - C''(q_i) \leq 0$, which is the condition that results into a quasi-competitive oligopoly market, i.e., a market where industry output (price) increases (decreases) in the number of firms (see Amir and Lambson 2000).

Lemma 2 Assume the conditions of Lemma 1 hold. Let $i \in S$ and $j \notin S$. Then

$$u_j\left(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S}\right) \geq u_i\left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S}\right).$$

Proof We have that

$$\begin{aligned} u_j\left(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S}\right) &= u_j\left(x_j^{\pi'_S}, \sum_{\substack{k \in N \\ k \neq j}} x_k^{\pi'_S} + x_j^{\pi'_S}\right) \geq u_j\left(x_i^{\pi'_S}, \sum_{\substack{k \in N \\ k \neq j}} x_k^{\pi'_S} + x_i^{\pi'_S}\right) \\ &\quad [\text{because } x_j^{\pi'_S} \text{ optimal for } j] \\ &= u_j\left(x_i^{\pi'_S}, \sum_{\substack{k \in N \\ k \neq j, k \neq i}} x_k^{\pi'_S} + x_i^{\pi'_S} + x_i^{\pi'_S}\right) \\ &\geq u_j\left(x_i^{\pi'_S}, \sum_{\substack{k \in N \\ k \neq j, k \neq i}} x_k^{\pi'_S} + x_j^{\pi'_S} + x_i^{\pi'_S}\right) [\text{because of A2 and } x_j^{\pi'_S} \geq x_i^{\pi'_S}] \\ &= u_i\left(x_i^{\pi'_S}, \sum_{\substack{k \in N \\ k \neq j, k \neq i}} x_k^{\pi'_S} + x_j^{\pi'_S} + x_i^{\pi'_S}\right) [\text{because of A0}] \\ &= u_i\left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S}\right) \end{aligned}$$

So the result is proved. ■

Proposition 1 Assume the conditions of Lemmas 1–2 hold. Then the core of (N, v_γ) is non-empty.

Proof Given symmetry, the γ -core is non-empty if for each S

$$\frac{v(N)}{n} \geq \frac{v_\gamma(S)}{s} \quad [4]$$

where s is the number of players in S . By symmetry all members of S select the same strategy; likewise, all players outside S select the same strategy too. Hence we can write

$$\sum_{i \in S} u_i \left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right) = s u_i \left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right) \quad [5]$$

$$\sum_{j \in N \setminus S} u_j \left(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right) = (n-s) u_j \left(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right) \quad [6]$$

We then have

$$\begin{aligned} \frac{v(N)}{n} &= \frac{\sum_{r \in N} u_r \left(x_r^{\pi^*}, \sum_{k \in N} x_k^{\pi^*} \right)}{n} \geq \frac{\sum_{r \in N} u_r \left(x_r^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right)}{n} \\ &= \frac{s u_i \left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right) + (n-s) u_j \left(x_j^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right)}{n} \\ &\quad [\text{by eqs [5] and [6], where } i \in S, j \notin S] \\ &\geq \frac{(s+n-s) u_i \left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right)}{n} \quad [\text{by Lemma 2}] \\ &= \frac{s u_i \left(x_i^{\pi'_S}, \sum_{k \in N} x_k^{\pi'_S} \right)}{s} = \frac{v_\gamma(S)}{s} \end{aligned}$$

So eq. [4] holds. ■

If S deviates from the grand coalition then each member of S selects a lower strategy than each non-member, as Lemma 1 shows. This is driven (to a large extend) by assumption A2: acting in cooperative way, a member of S selects his strategy by internalizing the negative impact of his choice on the other members. As a result, the per-member payoff in S is relatively low (as a matter of fact, it is lower than the payoff of a non-member), which prevents S from deviating.

Notice also that assumption A2 means there are positive externalities from coalition formation. The merging of, say, S with $S_k = \{k\}$ would induce each member of the new coalition $S \cup S_k$ to select a strategy by internalizing his negative impact on more players. Hence compared to the pre-merging situation, each member of the new coalition would select a lower strategy. Then, this would benefit the players outside $S \cup S_k$ (by A2 again).

3 Aggregative Games and Set-Valued Beliefs

We generalize the analysis of the previous section by allowing a deviant coalition to have set-valued beliefs over the coalitional behavior of the outsiders. In particular, for each coalition S we will determine a collection $\mathcal{T}_{-S} \subseteq \Pi_{-S}$ with the property that if S believes that a partition $\pi_{-S} \in \mathcal{T}_{-S}$ will form, then it has no incentive to break-off from the grand coalition.

As in the previous section, consider a candidate deviant coalition S with s members. Define the following number

$$l(s) = \begin{cases} 1, & \text{if } \frac{n}{2} \leq s \leq n-1 \\ n-s-(s-1), & \text{if } 1 \leq s < \frac{n}{2} \end{cases} \quad [7]$$

Remark 1 Given coalition S , consider a partition of the outsiders with l coalitions, $\pi_{-S} = \{S_1, S_2, \dots, S_l\}$. Assume $l \geq l(s)$. Then there is no $S_j \in \pi_{-S}$ with $|S_j| > s$.

Proof Clearly for $s \geq n/2$ the above holds immediately. Let next $s < n/2$. To show the validity of the Remark for this case, it suffices to consider the case where $l-1$ members of π_{-S} have one player each and one member of π_{-S} has $s^* > s$ players and show that such a partition cannot occur.

Indeed, the total number of players in the suggested partition is $l-1+s^*$. Since $l \geq l(s)$ and $s^* > s$, we have that $l-1+s^* > (n-s)-(s-1)-1+s = n-s$, i.e., the number of players in the suggested partition exceeds the number of outsiders. ■

Given S , let Π_{-S}^l be the set of all partitions of the outsiders that have at least l coalitions, where $l \in \{1, 2, \dots, n-s\}$. I.e.,

$$\Pi_{-S}^l = \{\pi_{-S} \in \Pi_{-S} : |\pi_{-S}| \geq l\}$$

Consider a partition $\pi_{-S} \in \Pi_{-S}^{l(s)}$, where $l(s)$ is defined in eq. [7], and let $\pi_S = \{S, \pi_{-S}\}$. Denote by Γ_{π_S} the corresponding normal form game; and denote by $(x_1^{\pi_S}, x_2^{\pi_S}, \dots, x_n^{\pi_S})$ the equilibrium choices in Γ_{π_S} .

Lemma 3 below extends Lemma 1 in the following sense: it shows that the equilibrium strategy of a player in S under Γ_{π_S} is not higher than the equilibrium strategy of any player not in S , irrespective of where this other player belongs to. To show this, we need an assumption analogue of A3.

A4 $x_i^{\pi_S} \in \text{int } X_i$, for all $i \in N$.

Lemma 3 Assume A1–A2 and A4 hold and that

- (I) $\partial u_i(x_i, \sum_{k \in N} x_k) / \partial x_i$ is decreasing in the first argument x_i
 (II) $\partial u_i(x_i, \sum_{k \in N} x_k) / \partial x_r$ is decreasing in the first argument x_i , for any $r \neq i$

Consider a partition $\pi_{-S} \in \Pi_{-S}^{l(s)}$ and take any $S_j \in \pi_{-S}$. Let $i \in S$ and $j \in S_j$. Then $x_j^{\pi_S} \geq x_i^{\pi_S}$.

Proof Take a coalition $S_j \in \pi_{-S}$. Let $|S_j| = s_j$. Since $\pi_{-S} \in \Pi_{-S}^{l(s)}$, we have $s \geq s_j$. Let $\bar{S}_j$ be a subset of S with s_j members (given symmetry we do not need to fully specify who these members are).

Note that for $i \in S$, $x_i^{\pi_S}$ satisfies⁸

$$\frac{\partial u_i \left(x_i^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} + \sum_{\substack{r \in \bar{S}_j \\ r \neq i}} \frac{\partial u_r \left(x_r^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} + \sum_{r \in S \setminus \bar{S}_j} \frac{\partial u_r \left(x_r^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} = 0 \quad [8]$$

On the other hand, for $j \in S_j$, $x_j^{\pi_S}$ satisfies

$$\frac{\partial u_j \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_j} + \sum_{\substack{m \in S_j \\ m \neq j}} \frac{\partial u_m \left(x_m^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_j} = 0 \quad [9]$$

Assumption A2 implies that each term in the second sum of the derivatives in eq. [8] is negative. Hence the sum of the first two parts in eq. [8] is positive, i.e.,

$$\frac{\partial u_i \left(x_i^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} + \sum_{\substack{r \in \bar{S}_j \\ r \neq i}} \frac{\partial u_r \left(x_r^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} > 0 \quad [10]$$

Note that $x_i^{\pi_S} = x_r^{\pi_S}$ for $r \in S$ and $x_j^{\pi_S} = x_m^{\pi_S}$ for $m \in S_j$. If $x_i^{\pi_S} > x_j^{\pi_S}$ then conditions (I) and (II) of the current Lemma and eq. [10] would imply that

$$\frac{\partial u_i \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} + \sum_{\substack{r \in \bar{S}_j \\ r \neq i}} \frac{\partial u_r \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} > 0 \quad [11]$$

Using symmetry, the above is equivalent to

⁸ W.l.o.g. assume $i \in \bar{S}_j$.

$$\frac{\partial u_i \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} + (s_j - 1) \frac{\partial u_r \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_i} > 0 \quad [12]$$

By symmetry again, we can use eq. [12] and write

$$\frac{\partial u_j \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_j} + (s_j - 1) \frac{\partial u_m \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{\partial x_j} > 0 \quad [13]$$

which violates eq. [9]. We conclude that $x_j^{\pi_S} \geq x_i^{\pi_S}$. Since S_j is an arbitrary member of π_{-S} , a similar inequality holds when we compare the strategy of a player in S and the strategy of a player in any $S_t \in \pi_{-S}$. ■

To give an example that satisfies condition (II) of Lemma 3, we return to the market described after Lemma 1. We have $\partial u_i(q_i, Q)/\partial q_r = p'(Q)q_i$, for any $r \neq i$. This function is decreasing in the argument q_i , for any Q , since $p'(Q) < 0$.

Lemma 4 Assume the conditions of Lemma 3 hold. Consider a partition $\pi_{-S} \in \Pi_{-S}^{l(s)}$ and take any $S_j \in \pi_{-S}$. Let $i \in S$ and $j \in S_j$. Then $u_j \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right) \geq u_i \left(x_i^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)$.

Proof Take a coalition $S_j \in \pi_{-S}$. Let $|S_j| = s_j$. Notice that

$$\sum_{j \in S_j} u_j \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right) = \sum_{j \in S_j} u_j \left(x_j^{\pi_S}, s_j x_j^{\pi_S} + \sum_{\substack{k \in N \\ k \notin S_j}} x_k^{\pi_S} \right) \geq \sum_{j \in S_j} u_j \left(x_i^{\pi_S}, s_j x_i^{\pi_S} + \sum_{\substack{k \in N \\ k \notin S_j}} x_k^{\pi_S} \right)$$

where the above is due to symmetry and to the fact that the payoff of coalition S_j is maximized when its (symmetric) members select $x_j^{\pi_S}$. Symmetry, Lemma 3 and A2 then imply that

$$\begin{aligned} u_j \left(x_j^{\pi_S}, s_j x_j^{\pi_S} + \sum_{\substack{k \in N \\ k \notin S_j}} x_k^{\pi_S} \right) &\geq u_j \left(x_i^{\pi_S}, s_j x_i^{\pi_S} + \sum_{\substack{k \in N \\ k \notin S_j}} x_k^{\pi_S} \right) \geq u_j \left(x_i^{\pi_S}, s_j x_j^{\pi_S} + \sum_{\substack{k \in N \\ k \notin S_j}} x_k^{\pi_S} \right) \\ &= u_i \left(x_i^{\pi_S}, s_j x_j^{\pi_S} + \sum_{\substack{k \in N \\ k \notin S_j}} x_k^{\pi_S} \right) \end{aligned}$$

as claimed. Since S_j is chosen arbitrarily from π_{-S} we can derive a similar conclusion regarding the comparison between S and any other $S_t \in \pi_{-S}$.

Denote by $v_{\pi_S}(S)$ the worth of S when the outsiders form partition π_{-S} and thus $\pi_S = \{S, \pi_{-S}\}$.

Lemma 5 Assume the conditions of Lemmas 3 and 4 hold. Then $\frac{v(N)}{n} \geq \frac{v_{\pi_S}(S)}{s}$, for all $\pi_S = \{S, \pi_{-S}\}$ s.t. $\pi_{-S} \in \Pi_{-S}^{l(s)}$.

Proof Let $\pi_S = \{S, \pi_{-S}\}$, for $\pi_{-S} \in \Pi_{-S}^{l(s)}$. We then have

$$\begin{aligned}
 \frac{v(N)}{n} &= \frac{\sum_{r \in N} u_r \left(x_r^{\pi^*}, \sum_{k \in N} x_k^{\pi^*} \right)}{n} \geq \frac{\sum_{r \in N} u_r \left(x_r^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{n} \\
 &= \frac{su_i \left(x_i^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right) + \sum_{S_j \in \pi_{-S}} |S_j| u_j \left(x_j^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{n} \\
 &\geq \frac{(s + n - s) u_i \left(x_i^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{n} \quad [\text{by Lemma 4 and } \sum_{S_j \in \pi_{-S}} |S_j| = n - s] \\
 &= \frac{su_i \left(x_i^{\pi_S}, \sum_{k \in N} x_k^{\pi_S} \right)}{s} = \frac{v_{\pi_S}(S)}{s}
 \end{aligned}$$

Notice by eq. [7] that if a candidate deviant coalition S has at least $n/2$ members then $\Pi_{-S}^{l(s)} = \Pi_{-S}$. Given this observation, define

$$\mathcal{T}_{-S} = \begin{cases} \Pi_{-S}, & \text{if } \frac{n}{2} \leq |S| = s \leq n - 1 \\ \Pi_{-S}^{l(s)}, & \text{if } 1 \leq |S| = s < \frac{n}{2} \end{cases} \quad [14]$$

Denote by $(N, v_{\pi_S}(S))$ a cooperative game where the payoff of coalition S is computed under a certain partition $\pi_S = \{S, \pi_{-S}\}$. Define the set of games

$$\mathcal{U}^T = \{(N, v_{\pi_S}(S)) : \text{for each } S \subset N, \pi_S = \{S, \pi_{-S}\} \text{ with } \pi_{-S} \in \mathcal{T}_{-S}\}$$

The above analysis implies the following result.

Proposition 2 Assume the conditions of Lemmas 3 and 4 hold. Then each game in $\mathcal{U}^T$ has non-empty core.

Recall that under the γ -scenario a coalition with s members believes the opponents will form $n - s$ coalitions. For singleton coalitions, $l(1) = n - 1$, i.e., as in the γ -core scenario. For $s > 1$, we have that $n - s > l(s)$. Hence for $s > 1$, a coalition does not deviate from the grand coalition not only when the outsiders form $n - s$ singleton coalitions, but also when they form fewer coalitions.

In conclusion, we can informally re-state Proposition 2 as follows: Cooperative games that satisfy the conditions of Lemmas 3–4 and in which

- (i) singleton coalitions believe outsiders form $n - 1$ coalitions
- (ii) coalitions with size $s \in \{2, \dots, \frac{n}{2} - 1\}$ believe outsiders form at least $l(s)$ coalitions, have non-empty core, irrespective of the beliefs of coalitions with size $s \geq n/2$.

The results in this section are an extension of the previous section's results and so is their explanation. As long as the players outside S form at least $l(s)$ coalitions then each of these coalitions has fewer members than S . This implies that the internalization of the negative intra-coalitional externalities is more intense in S than in any coalition of the outsiders (as in the case of γ -scenario studied in the previous section). As a result, the strategy and the payoff per member of S are relatively low (as Lemmas 3 and 4 show) making its deviation undesirable.

Note, finally, that in the particular cases where S has at least $n/2$ members, it becomes the largest coalition irrespective of the partition of the outsiders. Hence, in these cases the above implications (intense internalization of externalities and thus low strategy and low payoff per member of S) hold no matter how the outsiders split into coalitions. Hence S never deviates.

3.1 An Oligopoly Example

To illustrate the above via an example, consider an oligopoly market where firms compete in quantities and produce a homogeneous product. The set of firms is $N = \{1, 2, \dots, n\}$. The cost function of firm i is $C(q_i) = cq_i + q_i^2/2$, where q_i is the quantity of firm i and $c > 0$. The inverse demand function is $p = a - Q$, where p is the price, $Q = \sum_{k \in N} q_k$ and $a > c$. The payoff function of individual firm i is

$$u_i(q_i, Q) = (a - Q - c)q_i - q_i^2/2$$

Consider a candidate deviant coalition S and a partition $\pi_S = \{S, \pi_{-S}\}$, where $\pi_{-S} = \{S_1, S_2, \dots, S_l\}$. Assume $|S| = s$ and $|S_j| = s_j$, for $S_j \in \pi_{-S}$. The objective function of S in the resulting game Γ_{π_S} is

$$u_{S_j}(q_1, q_2, \dots, q_n) = \sum_{i \in S} ((a - Q - c)q_j - q_j^2/2) \quad [15]$$

The objective function of coalition $S_j \in \pi_{-S}$ is

$$u_{S_j}(q_1, q_2, \dots, q_n) = \sum_{j \in S_j} ((a - Q - c)q_j - q_j^2/2)$$

The maximization problems are

$$\max_{\{(q_i)_{i \in S}\}} u_S(q_1, q_2, \dots, q_n)$$

and

$$\max_{\{(q_j)_{j \in S_j}\}} u_{S_j}(q_1, q_2, \dots, q_n), \quad \text{for } S_j \in \pi_{-S}$$

In Appendix A we calculate the solution of these problems and we show that the worth of S is

$$\frac{v_{\pi_S}(S)}{s} = (a - c)^2 \frac{(2s + 1)}{2\Delta^2(1 + s)^2} \quad [16]$$

where $\Delta = 1 + \frac{s}{1+s} + \sum_{j=1}^l \frac{s_j}{1+s_j}$.

Finally, the objective function of the grand coalition is

$$u_N(q_1, q_2, \dots, q_n) = \sum_{r \in N} ((a - Q - c)q_r - q_r^2/2)$$

The maximization problem the grand coalition faces is

$$\max_{\{(q_r)_{r \in N}\}} u_N(q_1, q_2, \dots, q_n)$$

By straightforward calculations we get

$$\frac{v(N)}{n} = \frac{(a - c)^2}{2(2n + 1)}$$

Define the function

$$\Delta_{-S}(s_1, s_2, \dots, s_l) \equiv \sum_{j=1}^l \frac{s_j}{1 + s_j}$$

where s_j is defined above. Given that the players outside S form l coalitions, the payoff of S is maximized when Δ is minimum, i.e., when $\Delta_{-S}(s_1, s_2, \dots, s_l)$ is minimum.

Recall that $\sum_{j=1}^l s_j = n - s$ (there are $n - s$ outside players). Then the function $\Delta_{-S}(s_1, s_2, \dots, s_l)$ is minimized over $\{s_1, s_2, \dots, s_l\}$ if $s_j = 1$ for all j but one; say $s_1 = s_2 = \dots = s_{l-1} = 1$ and $s_l = n - s - (l - 1)$. Thus given that the players outside S form l coalitions, the payoff of S is maximized when $l - 1$ coalitions have one member each and one coalition includes all the remaining players. Denote the resulting partition of the outsiders by $\bar{\pi}_{-S}$ and let $\bar{\pi}_S = \{S, \bar{\pi}_{-S}\}$. Then the value of $\Delta_{-S}(s_1, s_2, \dots, s_l)$ computed at $(s_1, s_2, \dots, s_l) = (1, 1, \dots, n - s - (l - 1))$ is

$$\Delta_{-S}(1, 1, \dots, n - s - (l - 1)) = \frac{l - 1}{2} + \frac{n - s - (l - 1)}{n - s - (l - 1) + 1}$$

Denote the corresponding value of Δ by Δ_{min} , i.e.,

$$\begin{aligned}\Delta_{\min} &= 1 + \frac{s}{1+s} + \Delta_{-S}(1, 1, \dots, n-s-(l-1)) \\ &= \frac{1+2s}{s} + \frac{l-1}{2} + \frac{n-s-(l-1)}{n-s-(l-1)+1}\end{aligned}$$

Hence if the players outside S form l coalitions, the payoff of S is maximized if the partition $\bar{\pi}_S$ forms and

$$\frac{v_{\bar{\pi}_S}(S)}{s} = (a-c)^2 \frac{(2s+1)}{2\Delta_{\min}^2(1+s)^2} \quad [17]$$

To continue, and in accordance to our previous analysis, we can discriminate between two cases: $s \geq n/2$ and $s < n/2$.

Case A: $s \geq n/2$. Recall by Proposition 2 that in this case coalition S does not deviate irrespective of the partition of the outsiders. Indeed, notice that $\Delta_{\min}$ is increasing in l and hence $v_{\bar{\pi}_S}(S)$ is decreasing in l . Hence the worth of S is maximized when $l = 1$. Plugging $l = 1$ in eq. [17] gives

$$\frac{v_{\bar{\pi}_S}(S)}{s} = (a-c)^2 \frac{(2s+1)(n-s+1)^2}{2((1+s)(n-s) + (1+2s)(n-s+1))^2} \quad [18]$$

Using eq. [18] it is straightforward to show that

$$\frac{v(N)}{n} \geq \frac{v_{\bar{\pi}_S}(S)}{s} \Leftrightarrow \zeta(n, s) \geq 0 \quad [19]$$

where $\zeta(n, s) = -2(2s+1)n^2 + (13s^2 + 4s - 1)n - 9s^3 + 2s^2 + 3s$. Recall we are in the case where $s \geq n/2$ (and also $s < n$). In this range the function $\zeta(n, s)$ is positive, as it can be easily checked. Since the worth of S is maximized at eq. [18] [for $l = 1$] and for this value S does not deviate, we conclude that S never deviates.

Case B: $s < n/2$. In this case Proposition 2 predicts that S does not deviate if the outsiders form at least $l(s)$ coalitions, where $l(s) = n-s-(s-1)$. As in Case A, $\Delta_{\min}$ is increasing in l and $v_{\bar{\pi}_S}$ is decreasing in l . Plugging the value $l = l(s)$ in eq. [17] gives

$$\frac{v_{\bar{\pi}_S}(S)}{s} = (a-c)^2 \frac{2(2s+1)}{(2+4s+n+sn-2s^2)^2} \quad [20]$$

Using eq. [20] we have that

$$\frac{v(N)}{n} \geq \frac{v_{\bar{\pi}_S}(S)}{s} \Leftrightarrow \eta(n, s) \geq 0 \quad [21]$$

where $\eta(n, s) = (1 + s)^2 n^2 + 4(-1 - s + s^2 - s^3) + 4s(2 + 2s - 4s^2 + s^3)$. But for $s < n/2$, $\eta(n, s) \geq 0$. Hence for all $l \geq l(s)$ coalition S does not deviate.

3.2 Linear Aggregative Games

In this section we restrict attention to linear aggregative normal form games. This class was introduced by Martimort and Stole (2010) and it involves aggregative games where the payoff of player i satisfies some sort of linearity in own strategy x_i for any fixed sum of all players' strategies. For this class of games we will identify a new threshold on the minimum number of outside coalitions that support a non-empty core. This new threshold is going to be (weakly) lower than⁹ $l(s)$.

We follow Martimort and Stole (2010) and we consider a bilinear form, i.e., a mapping defined on two linear spaces which is linear in each argument separately. More formally let V and W be two linear spaces. A bilinear form is a mapping $\langle \cdot, \cdot \rangle : V \times W \rightarrow \mathbb{R}$ where for $f, \tilde{f} \in V$, $h, \tilde{h} \in W$ and a scalar λ the following hold:

$$\langle f + \tilde{f}, \cdot \rangle = \langle f, \cdot \rangle + \langle \tilde{f}, \cdot \rangle, \quad \langle \cdot, h + \tilde{h} \rangle = \langle \cdot, h \rangle + \langle \cdot, \tilde{h} \rangle \quad [22]$$

$$\langle \lambda f, h \rangle = \langle f, \lambda h \rangle = \lambda \langle f, h \rangle \quad [23]$$

A linear aggregative normal form game arises when the payoff of player i has the form

$$u_i \left(x_i, \sum_{k \in N} x_k \right) = a_i \left(\sum_{k \in N} x_k \right) + \left\langle x_i, \tilde{u}_i \left(\sum_{k \in N} x_k \right) \right\rangle \quad L1$$

where $a_i(\cdot)$ and $\tilde{u}_i(\cdot)$ are appropriately defined real-valued functions and $\langle \cdot, \cdot \rangle$ is a bilinear mapping. An example satisfying $L1$ is a market where firms compete in quantities under constant returns to scale. Assume the average cost in the market is c . Then

$$u_i \left(q_i, \sum_{k=1}^n q_k \right) = q_i \left(p \left(\sum_{k=1}^n q_k \right) - c \right) = q_i \tilde{u}_i \left(\sum_{k=1}^n q_k \right) = \left\langle q_i, \tilde{u}_i \left(\sum_{k=1}^n q_k \right) \right\rangle$$

The next result discusses the non-emptiness of the core under a linear aggregative normal form structure. This result does not require assumption $A2$ (or $A1$). It requires though to set $a_i \left(\sum_{k \in N} x_k \right) = 0$, for all $\sum_{k \in N} x_k$.

First, define the number

⁹ Recall that $l(s)$ is given by eq. [7].

$$\widehat{l}(s) = \begin{cases} 1, & \text{if } \frac{n}{2} \leq s \leq n-1 \\ \frac{n}{s} - 1, & \text{if } 1 \leq s < \frac{n}{2} \end{cases} \quad [24]$$

Given coalition S recall that Π_{-S}^l is the set of all partitions of the outsiders that have at least l coalitions.

Lemma 6 Assume $L1$ holds with $a_i(\sum_{k \in N} x_k) = 0$, for all $\sum_{k \in N} x_k$. Then $\frac{v(N)}{n} \geq \frac{v_{\pi_S}(S)}{s}$, for all $\pi_S = \{S, \pi_{-S}\}$ s.t. $\pi_{-S} \in \widehat{\Pi}_{-S}^{l(s)}$.

Proof We first compute the worth of the grand coalition using the fact that player symmetry implies that $\tilde{u}_i(\sum_{k \in N} x_k) = \tilde{u}_j(\sum_{k \in N} x_k)$, for any i, j . Hence using symmetry, $L1$ and eq. [22] the objective function of the grand coalition can be written as

$$\begin{aligned} u_N(x_1, x_2, \dots, x_n) &= \sum_{i \in N} u_i \left(x_i, \sum_{k \in N} x_k \right) = \sum_{i \in N} \left\langle x_i, \tilde{u}_i \left(\sum_{k \in N} x_k \right) \right\rangle \\ &= \left\langle \sum_{i \in N} x_i, \tilde{u}_i \left(\sum_{k \in N} x_k \right) \right\rangle \end{aligned}$$

I.e., the grand coalition simply selects the value of the sum $\sum_{i \in N} x_i$.

Consider next a deviant coalition S with s members. Take a partition $\pi_{-S} = \{S_1, S_2, \dots, S_l\}$ of the outsiders and let $\pi_S = \{S, \pi_{-S}\}$. Using, again, symmetry, $L1$ and eq. [22] we can write the objective function of S as

$$\begin{aligned} u_S(x_1, x_2, \dots, x_n) &= \sum_{i \in S} u_i \left(x_i, \sum_{k \in N} x_k \right) = \sum_{i \in S} \left\langle x_i, \tilde{u}_i \left(\sum_{k \in N} x_k \right) \right\rangle \\ &= \left\langle \sum_{i \in S} x_i, \tilde{u}_i \left(\sum_{k \in N} x_k \right) \right\rangle \end{aligned}$$

Therefore S selects simply the sum $\sum_{i \in S} x_i$. For simplicity denote $\sum_{i \in S} x_i = x_S$. Notice

next that any other coalition $S_j \in \pi_{-S}$ selects a similar sum $\sum_{i \in S_j} x_j = x_{S_j}$. Therefore the

formation of the partition $\pi_S = \{S, \pi_{-S}\}$ gives rise to an $l + 1$ symmetric normal form game. Since l is the only parameter that matters for S (given condition $L1$), from now we use the notation π_{-S}^l to denote any partition of the outsiders into l coalitions; moreover we denote $\pi_{S,l} = \{S, \pi_{-S}^l\}$. Let $x_S^{\pi_{S,l}}$ be the equilibrium choice of S under $\pi_{S,l}$;

and $x_{S_j}^{\pi_{S,l}}$ the equilibrium choice of $S_j \in \pi_{-S}^l$. By symmetry $x_S^{\pi_{S,l}} = x_{S_j}^{\pi_{S,l}}$, any $S_j \in \pi_{-S}^l$. Denote by $v_{\pi_{S,l}}(S)$ the corresponding worth of S . Then

$$v_{\pi_{S,l}}(S) = \langle x_S^{\pi_{S,l}}, \tilde{u}_i((l+1)x_S^{\pi_{S,l}}) \rangle$$

We thus have

$$\begin{aligned} \frac{v(N)}{n} &= \frac{\langle \sum_{i \in N} x_i^{\pi^*}, \tilde{u}_i(\sum_{k \in N} x_k^{\pi^*}) \rangle}{n} \geq \frac{\langle (l+1)x_S^{\pi_{S,l}}, \tilde{u}_i((l+1)x_S^{\pi_{S,l}}) \rangle}{n} \\ &= \frac{(l+1)\langle x_S^{\pi_{S,l}}, \tilde{u}_i((l+1)x_S^{\pi_{S,l}}) \rangle}{n} = \frac{(l+1)v_{\pi_{S,l}}(S)}{n} \end{aligned}$$

where the inequality is due to the fact that $\sum_{k \in N} x_k^{\pi^*}$ maximizes the bilinear mapping $\langle \sum_{k \in N} x_k, \tilde{u}_i(\sum_{k \in N} x_k) \rangle$; and the second equality is due to eq. [23].

Finally notice that $\frac{l+1}{n} \geq \frac{1}{s}$ if $l \geq \frac{n}{s} - 1 \equiv \widehat{l}(s)$. Hence for $l \geq \widehat{l}(s)$ the inequality $\frac{v(N)}{n} \geq \frac{v_{\pi_{S,l}}(S)}{s}$ holds. This completes the proof. ■

To better comprehend the above result, consider the equal split allocation of the value of the grand coalition. This allocation gives the members of S the fraction $\frac{s}{n}$ of $v(N)$. By deviating from N , the members of S obtain the fraction $\frac{1}{l+1}$ of $\langle (l+1)x_S^{\pi_{S,l}}, \tilde{u}_i((l+1)x_S^{\pi_{S,l}}) \rangle$, where $v(N) \geq \langle (l+1)x_S^{\pi_{S,l}}, \tilde{u}_i((l+1)x_S^{\pi_{S,l}}) \rangle$ as explained above. Hence for S not to deviate it suffices that $\frac{s}{n} \geq \frac{1}{l+1}$ or $l \geq \frac{n}{s} - 1$ as Lemma 6 showed.

To continue notice by eq. [24] that if $s \geq n/2$ then $\widehat{\Pi}_{-S}^{l(s)} = \Pi_{-S}$. So define

$$\widehat{\mathcal{T}}_{-S} = \begin{cases} \Pi_{-S}, & \text{if } \frac{n}{2} \leq |S| = s \leq n-1 \\ \widehat{\Pi}_{-S}^{l(s)}, & \text{if } 1 \leq |S| = s < \frac{n}{2} \end{cases} \quad [25]$$

Consider the set of games

$$\mathcal{U}^{\widehat{\mathcal{T}}} = \{(N, v_{\pi_S}(S)) : \text{for each } S \subset N, \pi_S = \{S, \pi_{-S}\} \text{ with } \pi_{-S} \in \widehat{\mathcal{T}}(S)\}$$

The discussion of this section implies the following.

Proposition 3 Assume L1 holds. Then each game in $\mathcal{U}^{\widehat{\mathcal{T}}}$ has non-empty core.

Let us finally compare the two thresholds $\widehat{l}(s)$ and $l(s)$. Note that for $s \geq n/2$ we have $\widehat{l}(s) = l(s) = 1$; and for $s < n/2$ we have $\widehat{l}(s) < l(s)$. Hence, in general, a larger set of beliefs supports non-empty core when we impose L1.

4 Conclusions

The notion of γ -core is widely used in cooperative games with externalities. There are good reasons for this. First, the scenario behind the γ -core is more plausible than the scenarios behind other concepts used in the literature, such as the α and β -cores. The latter notions require an excessive or even unrealistic degree of pessimism: the members of the deviant coalition believe that the outsiders will necessarily punish their deviation, even if this is not in the best interest of the outsiders. Clearly, there are many economic environments where this pessimism cannot be justified. On the other hand, the γ -core does not have the consistency of the recursive core: under the latter framework, the characteristic function of a coalition is defined by computing recursively the cores of all reduced games. Nonetheless, the γ -core is easier to define and compute in applications. Finally, the γ -core scenario is dictated in some cases by institutional arrangements which allow for the formation of one coalition only (we refer the reader to the relevant discussion in Currarini and Marini (2003)).

The current paper analyzed various versions of the γ -core for cooperative games generated by aggregative normal form games. The latter games are often encountered in economic applications. The contribution of the paper was two-fold. First, it derived a result on the existence of the γ -core per se. This result relies on the assumption that the payoff of a player in the underlying normal form game decreases in the aggregate value of all players' strategies. In spite of its merits, the γ -core still relies on a specific conjecture of the deviant coalition. So, in the second part of the paper we allowed a multitude of beliefs over the reactions of the outsiders and we derived bounds on the beliefs which are compatible with core non-emptiness.

The biggest shortcoming of our analysis is that it focuses on symmetric players only. The extension to the non-symmetric case is not straightforward and cannot rely on the approach used in the current paper. A more fruitful approach might be to find conditions under which the Shapley-Bondareva theorem is applicable, i.e., find conditions under which the cooperative game at hand is balanced. This we leave as a future research task.

Appendix

Derivation of characteristic function [16]

In this part we provide the details of the computation of the characteristic function of coalition S in the oligopoly market example (pages 14–16).

Consider the objective function of S given by eq. [15]. The first-order condition for the optimal quantity of firm $i \in S$ is

$$\frac{\partial u_S(q_1, q_2, \dots, q_n)}{\partial q_i} = a - c - Q - 2q_i - \sum_{\substack{r \in S \\ r \neq i}} q_r = 0$$

which by symmetry reduces to

$$(1 + s)q_i = a - c - Q$$

Deriving similar conditions for all coalitions in π_{-S} and re-arranging terms gives us the system of equations

$$q_i = \frac{a - c - Q}{1 + s}, \quad i \in S \quad [26]$$

$$q_j = \frac{a - c - Q}{1 + s_j}, \quad j \in S_j \quad [27]$$

$$\vdots$$

$$q_l = \frac{a - c - Q}{1 + s_l}, \quad l \in S_l \quad [28]$$

Adding the left parts of all equations and doing the same for their right parts gives

$$Q = s \frac{a - c - Q}{1 + s} + \sum_{j=1}^l s_j \frac{a - c - Q}{1 + s_j}$$

from which we get

$$Q = (a - c) \frac{\Delta - 1}{\Delta} \quad [29]$$

where $\Delta = 1 + \frac{s}{1+s} + \sum_{j=1}^l \frac{s_j}{1+s_j}$. Using eq. [29], the quantities in eq. [26]–[28] become

$$q_i^{\pi_S} = \frac{a - c}{(1 + s)\Delta}, \quad q_j^{\pi_S} = \frac{a - c}{(1 + s_j)\Delta}, \quad \dots, \quad q_m^{\pi_S} = \frac{a - c}{(1 + s_m)\Delta}$$

Plugging the above quantities in eq. [15] gives us the function in eq. [16]. $\square$

References

- Acemoglu, D., and M. K. Jensen. 2013. "Aggregate Comparative Statics." *Games and Economic Behavior* 81:27–49.
- Amir, R., and V. E. Lambson. 2000. "On the Effects of Entry in Cournot Markets." *Review of Economic Studies* 67:235–54.

- Aumann, R. 1967. "A Survey of Cooperative Games without Side Payments." In *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, edited by M. Shubik, 3–27. Princeton: Princeton University Press.
- Chander, P. 2010. "The Core of Games with Positive Externalities." Working Paper, CORE.
- Chander, P., and H. Tulkens. 1997. "The Core of an Economy with Multilateral Environmental Externalities." *International Journal of Game Theory* 26:379–401.
- Currarini, S., and M. Marini. 2003. "A Sequential Approach to the Characteristic Function and the Core in Games with Externalities." In *Advances in Economic Design*, edited by M. Sertel and A. Kara, 233–49. Berlin: Springer Verlag.
- Helm, C. 2001. "On the Existence of a Cooperative Solution for a Cooperative Game with Externalities." *International Journal of Game Theory* 30:141–6.
- Huang, C. Y., and T. Sjöström. 2003. "Consistent Solutions for Cooperative Games with Externalities." *Games and Economic Behavior* 43:196–213.
- Koczy, L. 2007. "A Recursive Core for Partition Function Form Games." *Theory and Decision* 63:41–51.
- Lekeas, P. 2013. "Coalition Beliefs in Cournot Oligopoly TU Games." *International Game Theory Review* 15:1–21.
- Marini, M. 2013. "The Sequential Core of an Economy with Environmental Externalities." *Environmental Sciences* 1:79–82.
- Martimort, D., and L. Stole. 2010. "Aggregate Representations of Aggregate Games." Working Paper.