Charge-offs, Defaults and the Financial Accelerator

TECHNICAL APPENDIX

Christopher M. Gunn *
Carleton University

Alok Johri †
McMaster University

Marc-André Letendre ‡
McMaster University

April 2022

*Department of Economics, 1125 Colonel By Dr., Ottawa, ON, Canada K1S 5B6; chris.gunn@carleton.ca

†Department of Economics, 1280 Main Street West, Hamilton, ON, Canada L8S 4M4; Tel.: +1 905-525-9140 ext.23830; fax: +1 905-521-8232; johria@mcmaster.ca

‡Department of Economics, 1280 Main Street West, Hamilton, ON, Canada L8S 4M4; Tel.: +1 905-525-9140 ext.23810; fax: +1 905-521-8232; letend@mcmaster.ca

Technical Appendix

This technical Appendix accompanies the paper with the same title.

A Additional Model Detail

A.1 Agency problem and debt-contract

The discussion in the main text regarding the financial intermediary implies that in each aggregate state in period t, the financial intermediary's budget constraint is

$$\xi_t = R_t^a A_t, \tag{A.1}$$

where ξ_t is the intermediary's return on its entire loan portfolio after idiosyncratic uncertainty has been realized, and where R_t^a and A_t are predetermined.

In the financial contract, the cut-off value $\bar{\omega}_{it}$ is defined as

$$\bar{\omega}_{it+1} R_{t+1}^k q_t K_{it+1} = R_{t+1}^l B_{it+1}. \tag{A.2}$$

If the entrepreneur's realization exceeds the threshold such that $\omega_{t+1}(i) \geq \bar{\omega}_{t+1}(i)$, the entrepreneur pays the financial intermediary the contracted amount $R_{it+1}^l B_{it+1}$, keeping the amount $\omega_{it+1} R_{t+1}^k q_t K_{it+1} - R_{it+1}^l B_{it+1}$. If $\omega_{it+1} < \bar{\omega}_{it+1}$, the entrepreneur defaults, receives nothing, and the financial intermediary receives $(1-\theta_t)\omega_{it+1} R_{t+1}^k q_t K_{it+1}$. As with R_{it}^l, $\bar{\omega}_{it}$ adjusts to reflect the aggregate ex-post realizations of the aggregate state in period t.

Given these contract details, we can write the financial intermediary's expected return on a given loan contract in a given aggregate contingency in period $t+1$ as

$$\xi_{it+1} = [1 - F_t(\bar{\omega}_{it+1})] R_{it+1}^l B_{it+1} + (1-\theta_{t+1}) \int_0^{\bar{\omega}_{it+1}} \omega R_{t+1}^k q_t K_{it+1} dF(\omega, \sigma_t^e) \tag{A.3}$$

Substituting in (A.2), we can write (A.3) in terms of the cut-off $\bar{\omega}$ as

$$\xi(\bar{\omega}_{it+1}, \theta_{t+1}) = \left[[1 - F_t(\bar{\omega}_{it+1})]\bar{\omega}_{it+1} + (1 - \theta_{t+1}) \int_0^{\bar{\omega}_{it+1}} \omega dF(\omega, \sigma_t^e) \right] R_{t+1}^k q_t K_{it+1}. \quad \text{(A.4)}$$

Defining the financial intermediary's expected share of gross returns $\Gamma(\bar{\omega})$ as

$$\Gamma_t(\bar{\omega}_{it}) = [1 - F_t(\bar{\omega}_{it})]\bar{\omega}_{it} + \int_0^{\bar{\omega}_{it}} \omega dF(\omega, \sigma_t^e), \quad \text{(A.5)}$$

and defining $G_t(\bar{\omega})$ as

$$G_t(\bar{\omega}_{it}) = \int_0^{\bar{\omega}_{it}} \omega dF(\omega, \sigma_t^e), \quad \text{(A.6)}$$

we can re-write the financial intermediary's expected return on a given loan contract in a given aggregate contingency as

$$\xi_{t+1}(\bar{\omega}_{it+1}, \theta_{t+1}) = [\Gamma_t(\bar{\omega}_{it+1}) - \theta_{t+1} G_t(\bar{\omega}_{it+1})] R_{t+1}^k q_t K_{it+1}, \quad \text{(A.7)}$$

where the terms in square brackets represent the financial intermediary's share of profits net of default costs. The requirement that the financial intermediary earn an expected return in every aggregate contingency equal to its opportunity cost of funds,

$$\xi_{t+1}(\bar{\omega}_{it+1}, \theta_{t+1}) = R_{t+1} B_{it+1} \quad \text{(A.8)}$$

then serves as a restriction to define a menu of contracts over loan quantity and cut-off value for the entrepreneur. Substituting in $q_t K_{it+1} = X_{it+1} + B_{it+1}$ and (A.7) we can then write this as

$$[\Gamma_t(\bar{\omega}_{it+1}) - \theta_{t+1} G_t(\bar{\omega}_{it+1})] R_{t+1}^k q_t K_{t+1}(i) = R_{t+1}^a (q_t K_{it+1} - X_{it+1}) \quad \text{(A.9)}$$

which for a given level of net-worth X_{it+1} defines a menu of contracts relating the entrepreneur's choice of K_{it+1} to the cut-off $\bar{\omega}_{it+1}$.

A.2 Entrepreneur's contract problem

The entrepreneur's expected gross return, conditional on the ex-post realization of the aggregate state but before the resolution of idiosyncratic risk, is given by

$$V^k_{it+1} = \int_{\bar{\omega}_{it+1}}^{\infty} \omega R^k_{t+1} q_t K_{it+1} dF(\omega) - R^l_{it+1} B_{it+1}. \tag{A.10}$$

Substituting in the definitions above yields

$$V^k_{it+1} = [1 - \Gamma_t(\bar{\omega}_{it+1})] R^k_{t+1} q_t K_{it+1}, \tag{A.11}$$

where $1 - \Gamma(\bar{\omega}_{it+1})$ is the entrepreneur's expected share of gross returns.

For a given level of net-worth X_{it+1}, the entrepreneur's optimal contacting problem is then

$$max_{K_{it+1}, \bar{\omega}_{it+1}} E_t\{V^k_{it+1}\} \tag{A.12}$$

subject to the condition that the financial intermediary's expected return on the contract equal its opportunity cost of its borrowing, equation (A.9). Letting λ_{it+1} be the ex-post value of the Lagrange multiplier conditional on realization of the aggregate state, the first-order conditions are then

$$\Gamma'_t(\bar{\omega}_{it+1}) - \lambda_{t+1} [\Gamma'_t(\bar{\omega}_{it+1}) - \theta_{t+1} G'_t(\bar{\omega}_{it+1})] = 0 \tag{A.13}$$

$$E_t \left\{ [1 - \Gamma_t(\bar{\omega}_{it+1})] \frac{R^k_{t+1}}{R^a_{t+1}} + \lambda_{t+1} \left([\Gamma_t(\bar{\omega}_{it+1}) - \theta_{t+1} G_t(\bar{\omega}_{it+1})] \frac{R^k_{t+1}}{R^a_{t+1}} - 1 \right) \right\} = 0 \tag{A.14}$$

$$[\Gamma_t(\bar{\omega}_{it+1}) - \theta_{t+1} G_t(\bar{\omega}_{it+1})] R^k_{it+1} q_t K_{it+1} - R^a_{t+1} (q_t K_{it+1} - X_{it+1}) = 0 \tag{A.15}$$

where (A.13) and (A.15) hold in each contingency, but (A.14) holds only in expectation.

A.3 Household

The stand-in household's lifetime utility is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t J_t \left[\log(C_t - bC_{t-1}) - \Psi_L \frac{{N_t^h}^{1+\sigma_L}}{1+\sigma_L} \right] \tag{A.16}$$

where C_t is consumption, N_t is hours-worked, β is the subjective discount factor and J_t follows an exogenous stochastic preference process which we refer to as a *preference shock*.

The household enters into each period with real financial securities A_t which serve as deposits with the financial intermediary, and nominal bonds B_t^n, earning risk-free gross real rate of return R_t^a and risk-free gross nominal rate of return R_t^n respectively, receiving nominal wage W_t^h for supplying hours N_t^h to the labour union, and receiving a share of real profits from the capital-producers, goods-producers, financial intermediary, labour union and employment agency, denoted collectively as F_t. At the end of the period, the household chooses its consumption C_t, its holdings of financial securities A_{t+1} and nominal nominal bonds B_{t+1}^n. The household's period t budget constraint is given by

$$C_t + A_{t+1} + \frac{B_{t+1}^n}{P_t} = R_t^a A_t + R_t^n \frac{B_t^n}{P_t} + \frac{W_t^h}{P_t} N_t^h + F_t, \tag{A.17}$$

where P_t is the price of the final good in terms of the nominal unit under the control of the central bank. The household's problem is to choose sequences of C_t, N_t^h, A_{t+1} and B_{t+1}^n to maximize (A.16) subject to (A.17).

Letting λ_t be the Lagrange multiplier associated with the household's budget constraint, the first-order conditions with respect to C_t, N_t^h, A_{t+1} and B_{t+1}^n are respectively

$$\lambda_t = \frac{J_t}{C_t - bC_{t-1}} - \beta b E_t \frac{J_{t+1}}{C_{t+1} - bC_t} \tag{A.18}$$

$$\lambda_t \frac{W_t^h}{P_t} = \Psi_L J_t {N_t^h}^{\sigma_L} \tag{A.19}$$

$$\lambda_t = \beta E_t \lambda_{t+1} R^a_{t+1} \tag{A.20}$$

$$\lambda_t = \beta E_t \lambda_{t+1} R^n_{t+1} \frac{P_t}{P_{t+1}}. \tag{A.21}$$

A.4 Final goods firm and intermediate goods firms

The final goods firm produces the final good Y_t by combining differentiated intermediate goods $y_{jt}, j \in [0,1]$, according to the technology

$$Y_t = \left[\int_0^1 y_{jt}^{\nu_{p_t}} dj \right]^{\frac{1}{\nu_{p_t}}}, \quad 0 < \nu_p \leq 1. \tag{A.22}$$

where ν_{p_t} follows an exogenous stochastic process which we refer to as a *price markup shock*. The producer acquires each j^{th} intermediate good at price P_{jt}, and sells the final good at price P_t where it may be used as a consumption or as an input into the production of investment goods. Each period the producer chooses intermediate goods y_{jt} $\forall j$ to maximize profits $P_t Y_t - \int_0^1 P_{jt} y_{jt} dj$, yielding a standard demand curve

$$y_{jt} = \left[\frac{P_{jt}}{P_t} \right]^{\frac{1}{\nu_p - 1}} Y_t, \tag{A.23}$$

for the j^{th} intermediate good, and nominal price index

$$P_t = \left[\int_0^1 P_{jt}^{\nu_p/(\nu_p-1)} dj \right]^{\frac{(\nu_p-1)}{\nu_p}}. \tag{A.24}$$

The j^{th} intermediate goods firms produces the differentiated good y_{jt} according to the technology

$$y_{jt} = z_t (\tilde{n}_{jt} \chi_t)^{\alpha} \tilde{k}_{jt}^{1-\alpha}, \tag{A.25}$$

where z_t is a stationary *technology shock* while χ_t denotes a stochastic trend in labour productivity. The growth rate of the productivity trend $g_{yt} = \chi_t / \chi_{t-1}$ follows a stationary AR(1) process. Also, $\tilde{n}_{jt}$ is total hours-worked, and $\tilde{k}_{jt}$ is physical capital services. Hours-worked is a composite of both household and entrepreneurial labour, such that

$\tilde{n}_{jt} = n_{jt}^{\Omega}(n_{jt}^{e})^{1-\Omega}$, where n_{jt} is worker labour, n_{jt}^{e} is entrepreneurial labour , and where Ω parameterizes the elasticity of the hours composite to household labour. Capital services is defined by $\tilde{k}_{jt} = u_{jt}k_{it}$, where k_{jt} is the stock of physical capital and u_{jt} is the utilization rate of that stock, chosen by the entrepreneurs.

The j^{th} firm hires n_{jt} and n_{jt}^{e} at wage rates W_t and w_t^e respectively, rents capital services $\tilde{k}_{jt}$ at rate r_t, and sells its output at price P_{jt}. Intermediate goods firms have market power, and can thus set prices subject to the demand curve (A.23). The firms face Calvo frictions in setting their prices such that each period they can re-optimize prices with probability $1-\zeta_p$. A firm that is unable to re-optimize its price in a given period re-sets it according to the indexation rule $P_{jt} = P_{jt-1}\pi_{t-1}^{\iota_p}\pi^{1-\iota_p}, \quad 0 \leq \iota_p \leq 1$, where $\pi_t = P_t/P_{t-1}$ and π is its steady state, and where $0 \leq \iota_p \leq 1$. A firm that can re-optimize its price in period t chooses its price P_{jt}^{*} to maximize

$$E_t \sum_{s=0}^{\infty} \zeta_p^s \beta^s \frac{\lambda_{t+s}P_t}{\lambda_t P_{t+1}} \left[P_{jt}^{*}(\Pi_{k=1}^{s}\pi_{t+k-1}^{\iota_w}\pi^{1-\iota_w})y_{jt+s} - P_{t+s}S(y_{jt+s})\right], \tag{A.26}$$

where $\beta^s \frac{\lambda_{t+s}P_t}{\lambda_t P_{t+1}}$ is the household owner's nominal discount factor, given the production technology (A.25) and the demand curve for y_{jt}, and where $S(y_{jt})$ is the firm's real cost function as a solution to its cost-minimization problem for a given level of output y_{jt}.

A.5 Employment agency and employment unions

The employment agency combines differentiated labour $n_{qt}, q \in [0,1]$, into a composite N_t according to

$$N_t = \left[\int_0^1 n_{qt}^{\nu_{wt}} dq\right]^{\frac{1}{\nu_{wt}}}, \quad 0 < \nu_w \leq 1. \tag{A.27}$$

where ν_{wt} follows an exogenous stochastic process which we refer to as a *wage markup shock*. Each period the agency acquires each q^{th} differentiated labour service at wage W_{qt} from the labour union, and sells the composite labour to the intermediate goods producers for wage W_t. The agency chooses n_{qt} $\forall q$ to maximize profits $W_tN_t - \int_0^1 W_{qt}n_{qt}dq$, yielding a demand function

$$n_{qt} = \left[\frac{W_{qt}}{W_t}\right]^{\frac{1}{\nu_w - 1}} N_t, \tag{A.28}$$

for the q^{th} labour type, and wage index

$$W_t = \left[\int_0^1 W_{qt}^{\nu_w/(\nu_w-1)} dq\right]^{\frac{(\nu_w-1)}{\nu_w}}. \tag{A.29}$$

The q^{th} labour union acquires labour N_t^h from the household at wage W_t^h, differentiates it into labour type $n_{qt}, q \in [0,1]$, and then sells it to the employment agency for wage W_{qt}. The unions have market power, and can thus choose the wage for each labour type subject to the labour demand curve (A.28). The unions face Calvo frictions in setting their wages, such that each period they can re-optimize wages with probability $1 - \zeta_w$. A union that is unable to re-optimize wages re-sets it according to the indexation rule $W_{qt} = W_{qt-1}\pi_{t-1}^{\iota_w}\pi^{1-\iota_w}, \quad 0 \leq \iota_w \leq 1$, where $\pi_t = P_t/P_{t-1}$ and π is its steady state, and where $0 \leq \iota_w \leq 1$. A union that can re-optimize its wage in period t chooses its wage W_{qt}^* to maximize

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \beta^s \frac{\lambda_{t+s} P_t}{\lambda_t P_{t+1}} \left[W_{qt}^*(\Pi_{k=0}^s \pi_{t+k-1}^{\iota_w}\pi^{1-\iota_w}) - W_{t+s}^h\right] n_{qt+s}, \tag{A.30}$$

subject to the demand curve for n_{qt}.

A.6 Capital-producer

The competitive capital-goods producer operates a technology that combines existing capital with new investment goods to create new installed capital. At the end of each period it purchases existing capital K_t^k from entrepreneurs at price $\bar{q}_t$, combining it with investment I_t to yield new capital stock K_t^{nk}, which it sells back to entrepreneurs in the same period at price q_t. The capital-producer faces investment adjustment costs in the creation of new capital, and incurs depreciation in the process, so that

$$K_t^{nk} = (1-\delta)K_t^k + I_t \left[1 - S\left(\frac{m_t I_t}{I_{t-1}}\right)\right], \tag{A.31}$$

where m_t follows an exogenous stochastic process that we refer to as a *marginal efficiency of investment (MEI) shock* and $S(x)$ is an investment adjustment cost function with the properties $S(x) = 0$, $S'(x) = 0$, and $S''(x) = s''$, where s'' is a parameter. The capital producer's period t profits are given by $\Pi_t^k = q_t K_t^{nk} - \bar{q}_t K_t^k - I_t$. Since the capital producer faces intertemporal investment adjustment costs, it faces a dynamic problem, choosing K_t^{nk}, K_t^k and I_t to maximize

$$E_0 \sum_{t=0}^{\infty} \frac{\beta^t \lambda_t}{\lambda_0} \Pi_t^k \tag{A.32}$$

subject to (A.31).

The capital producer's first-order conditions are given by

$$\bar{q}_t = q_t(1-\delta) \tag{A.33}$$

$$q_t - \frac{1}{\Upsilon_t} - q_t S\left(\frac{m_t I_t}{I_{t-1}}\right) - q_t S'\left(\frac{m_t I_t}{I_{t-1}}\right)\frac{m_t I_t}{I_{t-1}} + E_t\left\{\frac{\beta\lambda_{t+1}}{\lambda_t} q_{t+1} m_{t+1} \frac{I_{t+1}^2}{I_t^2} S'\left(\frac{m_{t+1} I_{t+1}}{I_t}\right)\right\}. \tag{A.34}$$

A.7 Monetary policy

Monetary policy takes the form of a monetary authority that sets the gross nominal interest rate R_{t+1}^n according to a rule in the form

$$\frac{R_{t+1}^n}{R^n} = \left(\frac{R_t^n}{R^n}\right)^{\rho_R} \left[\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\right]^{1-\rho_R} \eta_t, \tag{A.35}$$

where variables without subscripts are steady-state values, Π_t is the gross inflation rate, and η_t follows an exogenous stochastic process that we refer to as a *monetary policy shock*.

A.8 Equilibrium

Equilibrium in this economy is defined by contingent sequences of C_t, $c_t^e(i)\forall i$, N_t, N_{t^h}, $n_{jt}\forall j$, $u_{jt}\forall j$, $n_{jt}^e\forall j$, $P_{jt}\forall j$, $y_{jt}\forall j$, I_t, A_{t+1}, $K_{it+1}\forall i$, $u_{it}\forall i$, $B_{it+1}\forall i$, $\bar{\omega}_{it+1}\forall i$, K_t^{nk}, K_t^k, B_{t+1}^n, W_t, W_t^h, W_t^e, W_{qt}, r_t, R_{t+1}^a, $R_{it+1}^l\forall i$, R_t^k, $\bar{q}_t$, q_t, R_t^n, P_t, that satisfy the following conditions: (i) the allocations solve the household's, final goods-producer's, intermediate goods producers', financial intermediary's, entrepreneurs', capital producer's, employment agency's and employment union's problems, taking prices as given, (ii) all markets clear, (iii) the resource constraint $C_t + C_t^e + G_t + I_t + \theta_t G(\bar{\omega}_t) R_t^k q_{t-1} K_t + a(u_t)K_t = Y_t$ holds, where $\int_0^1 K_{it+1} = K_{t+1}$, $\int_0^1 B_{it+1} = B_{t+1}$, $\int_0^1 X_{it+1} = X_{t+1}$, $\int_0^1 c_{it+1}^e = C_{t+1}^e$, $\int_0^1 N_i^e = N^e = 1$ and where all entrepreneurs choose the same cut-off such that $\bar{\omega}_{it+1} = \bar{\omega}_{t+1} \quad \forall i$, and therefore $R_{it+1}^l = R_{t+1}^l \quad \forall i$.

Equilibrium in the capital goods market implies that $K_t^{nk} = K_{t+1}$ and $K_t^k = K_t$, and equilibrium in the securities market implies that $A_t = B_t$. Nominal bonds are in zero net-supply such that $B_t^n = 0$.

In equilibrium the financial intermediary's return on its entire loan portfolio just covers its opportunity cost of funds, implying that its budget constraint holds in every aggregate contingency and after idiosyncratic uncertainty is resolved as

$$\left[\Gamma_t(\bar{\omega}_{t+1}) - \theta_{t+1} G_t(\bar{\omega}_{t+1})\right] R_{t+1}^k q_t K_{t+1} = R_{t+1}^a A_{t+1}. \tag{A.36}$$

Aggregate net-worth evolves as the accumulated gross returns of surviving entrepreneurs plus their labour income. Letting V_t be aggregate gross entrepreneurial returns, we can compute it as the average gross idiosyncratic returns,

$$V_t = \left[1 - \Gamma_t(\bar{\omega}_t)\right] R_t^k q_{t-1} K_t, \tag{A.37}$$

which after making substitutions yields

$$V_t = R_t^k q_{t-1} K_t - \left[R_t^a B_t + \theta_t G_t(\bar{\omega}_t) R_t^k q_{t-1} K_t\right], \tag{A.38}$$

so that aggregate net-worth evolves as

$$X_{t+1} = \gamma V_t + w_t^e. \tag{A.39}$$

Finally, entrepreneurial consumption C_t^e is equal to the aggregated gross return of dying entrepreneurs,

$$C_t^e = (1 - \gamma) V_t. \tag{A.40}$$

For reference later in the discussion of our results, we also define the equilibrium real risk-free net interest rate as $r_t^f = \frac{1}{E_t \beta \frac{\lambda 1t+1}{\lambda_{1t}}} - 1$, the credit spread as $R_t^l - R_t^a$, and leverage as $L_t = \frac{q_t K_{t+1}}{X_{t+1}}$.