ANALYSIS OF THE STABILITY OF A FLAT TEXTILE STRUCTURE

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Abstract

This paper deals with a specific method of examining the state of equilibrium of a flat textile structure. This kind of structure is modelled as an inextensible elastica loaded with its dead weight and axial force. The elastica represents, as an example, a longitudinal section of a fabric. It is assumed that the elastica rests on a flat, immovable base. We considered only those forms of deformed elastica where it's the two ends were supported by pivot bearings, and the tangent at those points lay on the immovable supporting plane. In the analysis, the shape of the deflection curve was determined for a given axial force, and it was examined whether a given position is stable or unstable. The analysis was carried out on the basis of the energetic method, by examining the potential energy of the system. The investigations can be used for simulations of fabric buckling, folding and other applications of textile mechanics.

Key words:

stability, fabrics, textile mechanics, elastica, deflection curve, states of equilibrium

1. Assumptions of model and initial equations

We first consider a flat fabric resting on a fixed, immovable base (figure 1). Under the action of compressive forces, folds arise on its surface, which remain there due to the occurrence of friction forces.

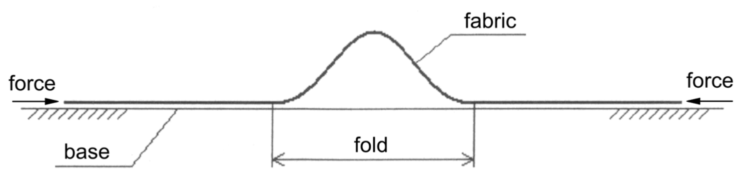

Figure 1. An example of fabric deformation in the form of a fold

Depending on the friction force quantity, those folds will remain or disappear after the action of deforming forces. In order to enable a thorough examination of the deformed fabric's stability, a substitute model was assumed which was limited to its deformed shape, i.e. to the fold. To generalise our further considerations, the term 'elastica' in place of 'fabric' is introduced hereafter.
The heavy elastica is loaded with the axial force P and continuous load q in the coordinate system, as in Figure 2. The elastica rests on a flat, fixed base and is supported on both ends by pivot bearings. It is inextensible, so it cannot change its length l under the influence of the loads acting on it.
However, it is subject to Hooke's law while being bent, and the known relation for the bending moment M is applicable to it:

$$M = EI\frac{1}{\rho}\,, \tag{1}$$

where ρ stands for the radius of curvature, and EI means the bending rigidity.

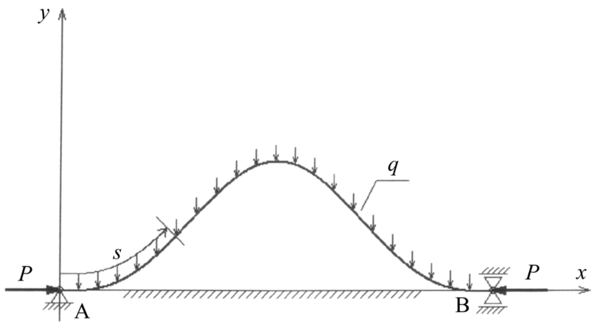

Figure 2. The load scheme of elastica in the coordinate system

No simplifications are applied to the curvature $1/\rho$, as is done with the theory of bending beams, because large deformations are involved here.
In this case, the existence of the rigid base causes the limitation of the y coordinate. It must be greater or equal to zero for each value of the arc coordinate s, which is measured along the deflection curve. The boundary conditions for this load scheme are the following:

$$y = 0\big|_{s=0} \ , \ y = 0\big|_{s=l} \ ,$$
$$M = 0\big|_{s=0} \ , \ M = 0\big|_{s=l} \ . \qquad (2)$$

The zero M moment at the points of support A and B results from the fact that apart from the fold, the elastica rests flat on the base and its curvature $1/\rho$ amounts to zero. It also results from that fact that the tangent at the points of support must be horizontal. Thus, we obtain additional boundary conditions, namely:

$$\frac{dy}{ds} = 0\bigg|_{s=0} \ , \ \frac{dy}{ds} = 0\bigg|_{s=l} \ . \qquad (3)$$

Let us consider the infinitesimal elastica section presented in Figure 3.
As has already been mentioned, the elastica is inextensible, thus $dx^2 + dy^2 = ds^2$.
Therefore we obtain the following geometrical condition:

$$dy/ds < 1 \ . \qquad (4)$$

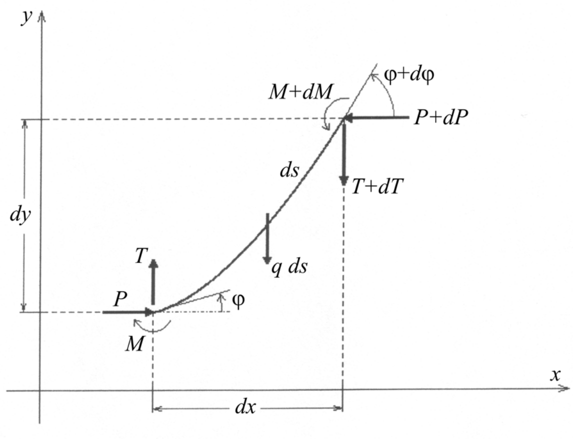

Figure 3. The infinitesimal section of elastica

Now let us write the elementary equations of equilibrium for Figure 3's section:

$$-dP = 0 \ ,$$
$$-dT - qds = 0 \ , \qquad (5)$$
$$dM - T\,dx + Pdy = 0 \ .$$

Based on the above equations of equilibrium, we obtain the principle of virtual work on the virtual displacements δx, δy, $\delta\varphi$. To do this, we multiply the equations (5) by the appropriate virtual displacements. Then, by adding the sides and integrating within the limits from 0 to l, we obtain

$$-dP\delta x - dT\delta y - qds\delta y + dM\delta\varphi - Tdx\delta\varphi + Pdy\delta\varphi = 0\ , \tag{6}$$

$$-\int_0^l \frac{dP}{ds}\delta x ds - \int_0^l \frac{dT}{ds}\delta y ds - \int_0^l T\frac{dx}{ds}\delta\varphi ds + \int_0^l P\frac{dy}{ds}\delta\varphi ds - \int_0^l q\delta y ds + \int_0^l \frac{dM}{ds}\delta\varphi ds = 0\ . \tag{7}$$

After integrating by parts we can write the following:

$$\begin{aligned} &-P\delta x\Big|_0^l + \int_0^l P\delta\left(\frac{dx}{ds}\right)ds - T\delta y\Big|_0^l + \int_0^l T\delta\left(\frac{dy}{ds}\right)ds - \int_0^l T\frac{dx}{ds}\delta\varphi ds + \\ &\quad + \int_0^l P\frac{dy}{ds}\delta\varphi ds - \int_0^l q\delta y ds + M\delta\varphi\Big|_0^l - \int_0^l M\delta\left(\frac{d\varphi}{ds}\right)ds = 0\ . \end{aligned} \tag{8}$$

Considering that

$$\frac{dx}{ds} = \cos\varphi\ , \quad \frac{dy}{ds} = \sin\varphi\ , \quad \frac{1}{\rho} = \frac{d\varphi}{ds}\ ,$$

and taking the boundary conditions and the relationship (1), the formula (8) after reduction takes the form of

$$\int_0^l q\delta y ds + P\delta x_B + \int_0^l \frac{1}{2}EI\delta\left(\frac{d\varphi}{ds}\right)^2 ds = 0\ . \tag{9}$$

Thus, in the end:

$$\delta\left[q\int_0^l y ds + Px_B + \frac{1}{2}EI\int_0^l\left(\frac{d\varphi}{ds}\right)^2 ds\right] = 0\ . \tag{10}$$

The obtained equation (9) represents the principle of virtual work, which stipulates that in the state of equilibrium, the sum of work of all actual forces (both external and internal) acting on the system for any virtual displacements is equal to zero.
The functional occurring in the square brackets in formula (10) is the total potential energy of the system (potential of external and internal forces).

$$J[y] = q\int_0^l y ds + Px_B + \frac{1}{2}EI\int_0^l\left(\frac{d\varphi}{ds}\right)^2 ds\ . \tag{11}$$

Equation (10) can be thus written in the form of

$$\delta J[y] = 0\ . \tag{12}$$

This is the necessary condition for the existence *in extremum* of the functional $J[y]$.
If the equilibrium is stable (stability), then the potential energy reaches a minimum in the balance point. In the case of maximum potential energy, however, we are dealing with an unstable state of equilibrium (labile equilibrium) [1], [2].

2. Deflection curve

As is already known, the deflection curve of the elastica in the state of equilibrium should present the functional extremum (11), respectively the minimum for stable equilibrium and the maximum for labile equilibrium. In order to determine the functional extremum, let us take for granted the equation of the deflection curve, which fulfils the given boundary conditions. Let the deflection curve be described by the formula

$$y = A\sin\left(\frac{\pi s}{l}\right) + B\sin\left(\frac{3\pi s}{l}\right), \tag{13}$$

where *A* and *B* are coefficients which are unknown for the time being.

It can be readily noted that function (13) fulfils the boundary conditions (2).
As regards the additional boundary conditions (3) concerning the derivative dy/ds, we obtain a relationship between the *A* and *B* coefficients from them in the following form:

$$B = -\frac{1}{3}A .$$

Eventually, after transformations, the deflection curve is defined by the following equation:

$$y = \frac{4}{3}A\sin^3\left(\frac{\pi s}{l}\right). \tag{14}$$

The range of admissible values of parameter A will be presented in the next point.

3. Admissible values for the shape parameter

We have obtained the deflection curve defined by the equation (14). The shape parameter A occurring in the equation will hereafter be presented in the dimensionless form $a = A/l$, related to the length l. This parameter cannot take the full range of values. Below there is a precise definition of the interval of admissible values of a.
In the model assumptions, it was stated that the existence of a fixed base in the system imposes the condition

$$y \geq 0 .$$

Thus

$$\frac{4}{3}A\sin^3\left(\frac{\pi s}{l}\right) \geq 0 . \tag{15}$$

Since we are only discussing the interval from s=0 to s=l, then from the function curve $\sin^3(\pi s/l)$ within this interval, it follows that for condition (15) to be fulfilled, it must be $A \geq 0$, that is:

$$a \geq 0. \tag{16}$$

Apart from that, condition (4) was given in the model assumptions, due to the inextensibility of the elastica, which concerns the derivative dy/ds. Applying it now, and substituting $\xi = \pi s/l$, we obtain for $0 \leq \xi \leq \pi$

$$4a\pi \sin^2(\xi)\cos(\xi) < 1 . \tag{17}$$

To find the value a from that, we first determine the maximum value of the function within the interval $0 \leq \xi \leq \pi$:

$$f = \sin^2(\xi)\cos(\xi), \tag{18}$$

because to satisfy the inequality (17), it is sufficient to substitute function (18) with its maximum value.

On examining function (18), it can be demonstrated that in the given interval it has only one maximum, amounting to

$$f_{\max} = 2/\left(3\sqrt{3}\right).$$

When the maximum value is inserted into the inequality (17), the following is obtained:

$$8a\pi / \left(3\sqrt{3}\right) < 1 .$$

Thus, in effect,

$$a < 3\sqrt{3}/(8\pi) \cong 0.2067 \ . \tag{19}$$

Eventually, we have the following interval of admissible values of the shape parameter:

$$0 \le a < a_{\mathrm{gr}} = 3\sqrt{3}/(8\pi). \tag{20}$$

4. Potential energy of the system

Let us consider for the functional (11) the deflection function described by the relation (14). After substituting this function in equation (11), $J[y]$ becomes a function of a single variable A.

$$J[y] = V(A).$$

To find the value of A coefficient, the Ritz method should be applied [3]. This method uses the necessary condition of existence of the $V(A)$ function extremum, that is, the following equation:

$$dV/dA = 0 \ . \tag{21}$$

The formula (11) must first be transformed and the individual integrals calculated.
For the first addend, it follows that

$$q\int_0^l y\,ds = q\int_0^l \frac{4}{3}A\sin^3\left(\frac{\pi s}{l}\right)ds = \frac{16ql}{9\pi}A \ . \tag{22}$$

In the second addend of the formula (11) there is the x_B, value which is the x coordinate of the movable end of the elastica.
This is calculated in the following way. Using the formula

$$ds^2 = dx^2 + dy^2 \ ,$$

we obtain

$$x_B = \int_0^l \sqrt{1-\left(\frac{dy}{ds}\right)^2}\,ds.$$

Since this integral cannot be calculated precisely, the approximate square root formula must be used here, leading to the result

$$x_B \cong \int_0^l \left[1-\frac{1}{2}\left(\frac{dy}{ds}\right)^2\right]ds = \int_0^l ds - \frac{1}{2}\int_0^l \left(\frac{dy}{ds}\right)^2 = l - \frac{1}{2}\int_0^l \left[\frac{4A\pi}{l}\sin^2\left(\frac{\pi s}{l}\right)\cos\left(\frac{\pi s}{l}\right)\right]^2 ds \ .$$

After transformation, we obtain

$$x_B = l - \frac{\pi^2 A^2}{2l} \ . \tag{23}$$

Thus, definitely

$$P x_B = Pl - \frac{P\pi^2 A^2}{2l} \ . \tag{24}$$

Before calculating the third addend, it is necessary to represent the curvature $d\varphi/ds$ in a somewhat different form. It is known that

$$\frac{dy}{ds} = \sin\varphi \ ,$$

$$\frac{d^2y}{ds^2} = \cos\varphi \frac{d\varphi}{ds} .$$

Thus

$$\frac{d\varphi}{ds} = \frac{d^2y/ds^2}{\sqrt{1-(dy/ds)^2}} . \tag{25}$$

For the third addend, it follows that

$$\frac{1}{2}EI\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds .$$

Now the integral has to be calculated.

$$\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds = \int_0^l \left(\frac{d^2y/ds^2}{\sqrt{1-(dy/ds)^2}}\right)^2 ds = \int_0^l \frac{\left(d^2y/ds^2\right)^2}{1-(dy/ds)^2} ds .$$

This type of integrals is discussed in publications [4] and [2], among others. As it is impossible to represent the result of the above integration in the form of elementary functions, an approximate solution should be applied. To do this, the numerator and denominator of the integral are multiplied by $1+(dy/ds)^2$.

Then, the product is

$$\int_0^l \frac{\left(d^2y/ds^2\right)^2}{1-(dy/ds)^2}\frac{1+(dy/ds)^2}{1+(dy/ds)^2} ds = \int_0^l \frac{\left(d^2y/ds^2\right)^2\left[1+(dy/ds)^2\right]}{1-(dy/ds)^4} ds . \tag{26}$$

Since, as was shown at the beginning, $dy/ds < 1$, then $(dy/ds)^4$ is much less than 1. It can be thus assumed that $1-(dy/ds)^4 \cong 1$, and thus it follows that:

$$\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds \cong \int_0^l \left(d^2y/ds^2\right)^2\left[1+(dy/ds)^2\right] ds . \tag{27}$$

A better approximation can be obtained by subsequent multiplication of the numerator and denominator of the formula (26) by $1+(dy/ds)^4$ and so on. Applying in (27) the formulae for the first and second derivatives of the y function, after integration we obtain:.

$$\frac{1}{2}EI\int_0^l \left(\frac{d\varphi}{ds}\right)^2 ds = \frac{EI\pi^4}{8l^5}\left(20l^2A^2 + 13\pi^2A^4\right). \tag{28}$$

Eventually, the formula for the total potential energy of the system takes the form of

$$V(A) = Pl - \frac{P\pi^2}{2l}A^2 + \frac{16ql}{9\pi}A + \frac{5EI\pi^4}{2l^3}A^2 + \frac{13EI\pi^6}{8l^5}A^4 . \tag{29}$$

The potential energy is also the function of the single variable A, which can be called a variable parameter of shape, and two constants connected with the external load, namely P and q.

5. Analysis of states of equilibrium

To begin analysis of the states of equilibrium, the above-mentioned condition (21) should be applied, on basis of which the value of the shape parameter A can be determined. This is the parameter on which the kind of equilibrium depends with a given load defined by P and q. Thus, we have

$$\frac{dV}{dA} = -\frac{P\pi^2}{l}A + \frac{5EI\pi^4}{l^3}A + \frac{13EI\pi^6}{2l^5}A^3 + \frac{16ql}{9\pi} = 0 \, . \tag{30}$$

From equation (30), the following relationship between the force P and parameter A is calculated:

$$P = \frac{5EI\pi^2}{l^2} + \frac{13EI\pi^4}{2l^4}A^2 + \frac{16ql^2}{9\pi^3}\frac{1}{A} \, . \tag{31}$$

To make further discussion more general, let us represent the energy V, force P and continuous load q in the dimensionless form, relating them to Euler's critical force $P_{cr} = \pi^2 EI/l^2$.
Let us make the following transformations

$$a = \frac{A}{l} \, , \qquad p = \frac{P}{P_{cr}} \, ,$$

$$w = \frac{ql}{P_{cr}} \, , \qquad v = \frac{V}{P_{cr}\, l} \, . \tag{32}$$

Thus, we obtain

$$v = p - \frac{\pi^2}{2}pa^2 + \frac{5\pi^2}{2}a^2 + \frac{13\pi^4}{8}a^4 + \frac{16w}{9\pi}a \, , \tag{33}$$

$$p = 5 + \frac{13\pi^2}{2}a^2 + \frac{16w}{9\pi^3}\frac{1}{a} \, . \tag{34}$$

Since relation (34) was obtained by use of the formula (21) expressing the necessary condition for existence of the function extremum, the points lying on the p curves thus correspond to the extremum of the function of potential energy. The location of the minimum and maximum of the energy must still be defined.

Here, the second derivative of potential energy is used as equated to zero:

$$\frac{d^2V}{dA^2} = -\frac{P\pi^2}{l} + \frac{5EI\pi^4}{l^3} + \frac{39EI\pi^6}{2l^5}A^2 = 0 \, .$$

Therefore

$$g = \frac{P}{P_{cr}} = \frac{39\pi^2}{2}a^2 + 5 \, . \tag{35}$$

The curve g described by equation (35) is a diagram of the compressing force p represented as a function of the parameter a, but corresponding only to the points for which the second derivative d^2v/da^2 (or in another way d^2V/dA^2) is equal to zero. Figure 4 presents the dependence of the force p on the dimensionless parameter a for several values w, and the drawn curve g. This is a boundary curve. Right of it, on each of the p curves, with $w>0$ there are points for which $d^2v/da^2 > 0$, which corresponds to the minimum of potential energy v, that is, to the state of stable equilibrium. Moreover, it should be noted that the curve g crosses the functions p in their minimumminimum points. The boundary value of the shape parameter a_{gr} is also marked in the diagram.

Now let us discuss in more detail the states of equilibrium for two possible cases of continuous load w ($w=0$ and $w>0$). It should be remembered that everything is considered with the condition $A>0$ or, which follows, $a>0$.

Case I ($w=0$).
Here we consider Figure 4 and the formulae of potential energy v and its derivatives. Substituting $w=0$ in the formula (33), we have

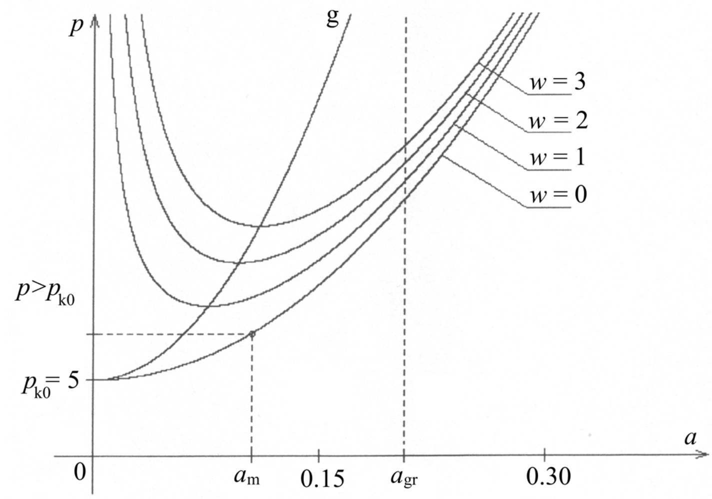

Figure 4. A diagram of the compressing force p corresponding to the extremum of potential energy

$$v = p - \frac{\pi^2}{2} pa^2 + \frac{5\pi^2}{2} a^2 + \frac{13\pi^4}{8} a^4 ,$$
$$\frac{dv}{da} = -\pi^2 pa + 5\pi^2 a + \frac{13\pi^4}{2} a^3 ,$$
$$\frac{d^2 v}{da^2} = -\pi^2 p + 5\pi^2 + \frac{39\pi^4}{2} a^2 .$$

It may be noted that the minimum value of the force p amounts to p_{k0}=5, which occurs with a=0.

- If $p<p_{k0}$, then the function v has the only extremum for a=0, and it is the minimum ($dv/da = 0$, $d^2v/da^2 > 0$ for a=0).
- If $p=p_{k0}$, then the function v also has its minimum for a=0, but it is flatter at this point (for a=0, all differential coefficients of the function v with respect to a up to the third degree inclusive are equal to zero, while $d^4v/da^4 > 0$).
- When $p>p_{k0}$, then the derivative dv/da when a>0 has already two zero points: one for a=0, the other for $a=a_m$ defined by the formula

$$a_m = (1/\pi)\sqrt{2(p-5)/13} . \quad (36)$$

It can be checked that for a=0 the derivative $d^2v/da^2 < 0$, so in this point there is the maximum of potential energy v. On the other hand, for $a=a_m$ the derivative $d^2v/da^2 > 0$, so there is the minimum of potential energy v.
To conclude, for $p \le p_{k0}$ only the rectilinear form of equilibrium exists, that is, the stable position is only for a=0.
However if $p>p_{k0}$, there are two positions of equilibrium. The first one, for a=0, is unstable, whereas the other, for a_m defined by formula (36) is the position of stable equilibrium.

Case II (w>0)
As above, here we consider the formulae of potential energy v and its derivatives.

$$v = p - \frac{\pi^2}{2} pa^2 + \frac{5\pi^2}{2} a^2 + \frac{13\pi^4}{8} a^4 + \frac{16w}{9\pi} a ,$$
$$\frac{dv}{da} = -\pi^2 pa + 5\pi^2 a + \frac{13\pi^4}{2} a^3 + \frac{16w}{9\pi} ,$$

$$\frac{d^2v}{da^2} = -\pi^2 p + 5\pi^2 + \frac{39\pi^4}{2}a^2 \ .$$

Based on Figure 4, it can be seen that the minimum value of the force p, which for further consideration will be marked as p_k, is greater than it was in the previous situation for $w=0$ ($p_k>p_{k0}=5$).
It can be also seen that the minimum occurs with $a>0$. Let this point be designated as a_k, as in Figure 5. To determine the values p_k and a_k the minimum of the function p given by the formula (34) must be found. After appropriate transformation, we obtain the following results:

$$p_k = 5 + \sqrt[3]{416w^2/(3\pi^4)} \ , \tag{37}$$

$$a_k = \sqrt[3]{16w/(117\pi^5)} \ . \tag{38}$$

Figure 5. An example of a diagram of the compressing force p for $w=3$ and potential energy v for $w=3$ and $p=8$

- If $p<p_k$, then the function v in the interval for $a\geq 0$ has no extremum. At the point $a=0$ and for each $a>0$ the derivative $dv/da>0$, so the function v is increasing while the value a increases. From the function analysis, it follows that for $a=0$ the potential energy accepts the least value in the present interval.
- If $p=p_k$, then for $0<a<a_k$ the derivative $dv/da>0$, so the function v is increasing. At the point $a=a_k$ derivatives of the function v up to the second degree inclusive with respect to a, are equal to zero, while $d^3v/da^3>0$; this means that at this point there is the point of inflexion. For $a>a_k$ again $dv/da>0$, so in this interval the function v increases again.
- The situation of $p>p_k$ is illustrated as an example in Figure 5, together with the diagrams of the force p for $w=3$ and of the potential energy v, in the case when $w=3$ and $p=8>p_k$ (for our example $p_k=7.3399$ and $a_k=0.1103$).
  In point $a=0$, the derivative $dv/da>0$. Based on Figure 5, it can be clearly seen that while increasing the value a from point $a=0$, the function v increases up to the local maximum which is attained at $a=a_{ns}$. Then the function decreases until the local minimum occurring at $a=a_s$, at which point it increases again.

To conclude, if $p<p_k$, then there is only a rectilinear form of stable equilibrium (state of stability for $a=0$). If $p=p_k$, then the stable equilibrium also occurs for the point $a=0$, while at the point $a=a_k$ there is the critical state at which the neutral equilibrium occurs (the deflection point in the diagram of energy v).
Eventually, for $p>p_k$ there are three forms of equilibrium.

1. The rectilinear form for $a=0$ corresponds with the state of stable equilibrium.
2. The curvilinear form corresponding to the left part of the curve (for $a=a_{ns}$) is unstable (local maximum of energy v).

3. The curvilinear form corresponding to the right part of the curve (for a=a_s) is stable (local minimum of energy v).

Let us now calculate the points a_s and a_{ns}.
Considering that $a \geq 0$, let us multiply both sides of the equation (34) by a. We then obtain the following:

$$\frac{13\pi^2}{2}a^3 + (5-p)a + \frac{16w}{9\pi^3} = 0\ . \qquad (39)$$

Equation (39) is a cubic equation with respect to a.

The roots of a have to be calculated with the given p and w. According to the earlier analysis, for $p>p_k$ and under assumption that $a \geq 0$ there should be two roots, respectively of a_s and a_{ns}, while $a_{ns}<a_s$.
To solve the equation (39) Cardan's formulae will be applied.
Equation (39) can be transformed to the shape of

$$a^3 + a(10-2p)/(13\pi^2) + (32w)/(117\pi^5) = 0\ . \qquad (40)$$

The roots of equation (40) depend on the value of the expression

$$R = (1/4)\left[32w/(117\pi^5)\right]^2 + (1/27)\left[(10-2p)/(13\pi^2)\right]^3\ . \qquad (41)$$

If $p>p_k$, then $R<0$. It means that equation (40) has three real roots. They are

$$a_{1,2,3} = 2\sqrt{\frac{2p-10}{39\pi^2}}\cos\left(\frac{\lambda}{3} + k\frac{2\pi}{3}\right),\ \ (k = 0,1,2)\ . \qquad (42)$$

The angle λ occurring in formula (42) is calculated from the formula

$$\cos(\lambda) = -\sqrt{\frac{416w^2}{3\pi^4(p-5)^3}}\ . \qquad (43)$$

From the analysis of the function $\cos(\lambda)$ it follows that for p=p_k $\cos(\lambda) = -1$, that is λ=π, while for $p \to \infty$, $\cos(\lambda) \to 0$, so $\lambda \to \pi/2$. The root of a_2 of equation (42) for k=1 is always negative, and so must be rejected. There are two roots left, of which a_1 (for k=0) is greater than a_3 (for k=2).
Eventually we obtain the following results:

$$a_s = 2\sqrt{\frac{2p-10}{39\pi^2}}\cos\left(\frac{\lambda}{3}\right), \qquad (44)$$

$$a_{ns} = 2\sqrt{\frac{2p-10}{39\pi^2}}\cos\left(\frac{\lambda}{3} + \frac{4\pi}{3}\right), \qquad (45)$$

where the angle λ is defined by the formula (43).
For example, from Figure 5 we obtain for w=3 and p=8 the values a_s=0.1781, a_{ns}=0.0626.

6. Discussion of the range of values of axial force and continuous load

In point 3 it was stated that a cannot take any optional value due to the specific length of the elastica. Admissible values of a belong to the interval $0 \leq a \leq a_{gr}$ where $a_{gr} = 3\sqrt{3}/(8\pi) \cong 0.2067$.

On analysing Figure 4, we see that if we increase the value w, then the stable curvilinear solutions a_s located right of the curve g will be greater than a_{gr} for the value w above a certain amount.

Since we always try to remain within the admissible limits of the value a, then let us consider what the maximum value w should be for stable curvilinear solutions a_s while still remaining within the interval.

Let the value a_k defined by the equation (38) be less than a_{gr}. By virtue of the above,

$$\sqrt[3]{16w/\left(117\pi^5\right)} < 3\sqrt{3}/(8\pi),$$

thus

$$w < 9477\sqrt{3}\,\pi^2/8192 = w_{\max} \ . \qquad (46)$$

The boundary value for w amounts approximately to $w_{\max} \cong 19.7761$.

Similarly, when looking at Figure 5 it can be seen that with the fixed w, increasing the force p above a certain value results in the value a_s being greater than the admissible one.

Since the value a_s depends not only on the value p, but also on the value λ (dependent in turn on w), thus for various w the maximum values of the axial force $p_{\max}$ will be different, and above them there are no more stable curvilinear solutions within the discussed interval of admissible values of a.

When w=0, it is sufficient for the value a_m as defined by the formula (36) to be less than a_{gr}.

Thus

$$\frac{1}{\pi}\sqrt{\frac{2(p-5)}{13}} < \frac{3\sqrt{3}}{8\pi}.$$

Therefore

$$p < 5 + \frac{351}{128} = p_{\max}^{0} \ . \qquad (47)$$

The approximate value of the maximum axial force in case of w=0 amounts to $p_{\max}^{0} \cong 7.7422$.

For the second case of $w>0$, the values of $p_{\max}$ for subsequent $w \le w_{\max}$ are calculated numerically.

To this end, with a fixed w, the value a_s was calculated from formula (44) for subsequent forces p increasing by even steps, beginning from the value p_k, until the moment that the value a_{gr} is exceeded.

Then, the calculation was repeated for the next value of w. From the values obtained, a diagram of maximum axial force $p_{\max}$ as a function of the continuous load w was drawn up.

Based on formula (37), a diagram of the critical force p_k as a function of continuous load w was also made. Both the diagrams are presented in Figure 6.

Figure 6. A diagram of the maximum force $p_{\max}$ and critical force p_k represented as function of the continuous load w

7. Conclusions

In conclusion, let us sum up the problem of the stability of the discussed elastica when $w>0$.
As follows from the above discussion, in this case the rectilinear form of equilibrium will always be stable, while we consider infinitesimal deviations from the point of balance. It can be seen clearly that with the increasing value p, the local maximum of energy occurs at a value approaching closer and closer to $a=0$, but not reaching it. This only proves the fact that for large values of p, it is easier to unbalance the system when it is in a stable, rectilinear state of equilibrium, causing some finite displacement to it. In order, however, for the system to assume a new, curvilinear form of stable equilibrium, it is necessary to pass the maximum of potential energy corresponding to the unstable form of equilibrium (Figure 5). The greater the force p is, the less displacement is needed for the system to assume a new form of equilibrium.
The force p_k as defined by formula (37) can be called the critical force, above which (except for the rectilinear form of stability) there is also a curvilinear form of stable equilibrium of the system.
Due to the assumption of inextensibility of the elastica, there arose the limitation of the value of the shape parameter a, which has to be less than a_{gr}. Thus, in turn, some limitations for the value w and axial force p followed in the determination of the curvilinear stable solutions a_s within the limits of admissible values a, which are illustrated in Figure 6.

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