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MATHEMATICAL PROBLEMS IN CREATING LARGE ASTRONOMICAL CATALOGS

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Abstract. The next stage after performing observations and their primary reduction is to transform the set of observations into a catalog. To this end, objects that are irrelevant to the catalog should be excluded from observations and gross errors should be discarded. To transform such a prepared data set into a high-precision catalog, we need to identify and correct systematic errors. Therefore, each object of the survey should be observed several, preferably many, times. The problem formally reduces to solving an overdetermined set of equations. However, in the case of catalogs this system of equations has a very specific form: it is extremely sparse, and its sparseness increases rapidly with the number of objects in the catalog. Such equation systems require special methods for storing data on disks and in RAM, and for the choice of the techniques for their solving. Another specific feature of such systems is their high "stiffness", which also increases with the volume of a catalog. Special stable mathematical methods should be used in order not to lose precision when solving such systems of equations. We illustrate the problem by the example of photometric star catalogs, although similar problems arise in the case of positional, radial-velocity, and parallax catalogs.

Key words: methods: data analysis – catalogs: photometric

1. INTRODUCTION

Let us now consider the general problems of creating large high-precision catalogs in the case of photometric star surveys, although practically the same challenges have to be addressed in the case of positional, radial-velocity, or parallax catalogs. To illustrate these problems, we consider three high-precision photometric catalogs: the Alma-Ata *WBVR* catalog of Sternberg Astronomical Institute (Kornilov et al. 1991, 1996), the photometric catalog of the planned "Lyra-B" space mission (Zakharov et al. 2013a,b), and the photometric star catalog of *Gaia* astrometric survey (Lindegren et al. 2008; Jordi et al. 2010).

The common feature of these three catalogs is their high precision and homogeneity of photometric measurements involved[1]. The *WBVR* catalog contains a

[1] The precision of *Gaia* multicolor photometry is not yet known. It will possibly become clear after the publication of the first release of *Gaia* catalogs in late 2016 – early 2017.

small number of objects (8500 stars), "Lyra-B" catalog will be a medium-sized (250–400 million stars), and *Gaia*, a large catalog (1 billion stars). See Table 1 for more detailed data about the above catalogs.

Not every survey can be used as a basis for a high-precision catalog. To qualify the task, the survey should meet the following conditions:

- It should be possible to extract a homogeneous component from the survey data. For example, in the case of photometric catalogs, variable stars should be excluded from the subset of objects for final reduction. Furthermore, this subset should include no erroneous or "bad" observations.
- To identify systematic effects, each object (star) has to be observed several, preferably many, times. In this case, objects are observed under the conditions with different values of parameters that determine systematic variations. Furthermore, multiple measurements make it possible to distinguish constant stars from photometric variables.
- Special measures should be taken during observations in order to reduce systematic errors and thus to allow the linearization of the equation system for computing the corresponding corrections. Otherwise the correction of systematic errors becomes practically impossible.

2. EXPERIMENT DESIGN MATRIX

Suppose that observational data contains a stable subset – the preliminary catalog – meeting the above requirements. Suppose that, in addition to these measurements, models of systematic corrections have been constructed, and the parameter values on which these corrections depend are known at the time of each measurement. In this case, transforming the preliminary catalog into the final high-precision catalog requires only one extra step – composing and solving the system of equations, which simultaneously determines precise values of stellar parameters (magnitudes) and small systematic corrections to these quantities.

The form of equations depends on the type of measurements (observations) used when conducting the survey: direct or differential.

A direct observation involves only one measured star and the quantities that determine the corresponding measurement conditions:

$$m_i = f(s_i^{(k)})\,. \tag{1}$$

Here, m_i is the magnitude of ith star; $s_i^{(k)}$, the parameters of kth measurement of ith star, and $f(s)$, the observation model.

Direct observations are typical for most of the instruments equipped with array detectors (CCD or CMOS). A typical measurement procedure can be simply described as follows: from the frame calibrated using a standard procedure[2], a fragment is selected that contains the image of one star. The background level is determined at the edges of this fragment and the excess above this level is interpreted as signal. This is the working principle of the popular SExtractor program

[2] Standard photometric calibration of a frame acquired with an array detector includes dark count subtraction, bias subtraction, and flatfielding (Tobin 1993; Manfroid 1996).

(Bertin & Arnouts 1996). The same principle is used to determine the photometric parameters of stars in the *Gaia* and "Lyra-B" experiments.

A differential observation involves two stars and the result of measurement has the form of the difference of their magnitudes:

$$m_i - m_j = f_d(s_{i,j}^{(k)}). \tag{2}$$

Here, i and j ($i \neq j$) are the numbers of the primary and secondary star used in the observation, respectively; m_i and m_j are the magnitudes of the primary and secondary star, respectively; $s_{i,j}^{(k)}$, the measurement parameters of the primary and secondary stars for kth measurement of the primary star, and $f_d(s)$, the model of the differential observation.

Differential observations are typical for measurements performed with diaphragm photometers. The measurements of stars for the *WBVR* catalog were performed in this way.

Direct measurements can be easily transformed into differential measurements:

$$m_i - m_j = f(s_i^{(k)}) - f(s_j^{(\ell)}) = f_d(s_{i,j}^{(k)}). \tag{3}$$

In this case, the dispersion of the random error of the magnitude difference increases[3]. However, if stars in the pair are close in terms of some parameters, then the systematic error shifts their magnitude estimates in the same direction and the systematic error of the magnitude difference may be smaller than the systematic error for each star, and this may compensate for the increase of the random error. This technique mostly determines the choice of the pairs[4].

In a frame with N stars there are $N(N-1)$ different star pairs[5], however, in this case we have to choose $N-1$ different pairs. This provides ample possibilities for the choice, and the underlying principle of selection of stars for pairs should be determined by the particularities of the survey.

The linear equation system obtained as a result of preparing the sample can be represented as follows:

$$A \times X = B\,, \tag{4}$$

where A is the rectangular design matrix with the number of columns N' equal to the number of unknowns[6] and the number of rows M equal to the number of observations ($M > N$); X, the vector of unknowns of size N', and B, the vector of right-hand sides of size M.

Equation system (4) is overdetermined ($M > N$), and the vector B of its right-hand parts includes observed quantities that are measured with some errors and therefore this system cannot be solved exactly. Usually, its solution is considered to be the set of values (vector) X of independent variables that minimizes the

[3] By a factor of 2 if $m_i = m_j$.

[4] For example, in the case of ground-based measurements, the so-called "equal-altitude" technique (Cousins 1985; Romanishin 2014) is used, where observations are planned in such a way that stars in pairs would be at equal atmospheric air masses.

[5] We consider star pairs to be ordered, i.e., the pairs (i, j) and (j, i) as different entities.

[6] The number of unknowns, N', can be greater than, or equal to, the number of measured stars, N.

residual:

$$\sum_{i=1}^{M} w_i \left\| \sum_{j=1}^{N'} A_{i,j} X_j - B_j \right\| \to \min . \tag{5}$$

Here w_i is the weight of ith equation in the equation system to be solved and $||\dots||$ is the norm. If the adopted norm has the form of the sum of squares ($||\dots|| = (\dots)^2$), then equation (5) describes the Least Square Method (LSM). Hereafter we assume by default that overdetermined equation systems (4) are solved using the LSM.

In the "zero" approximation, i.e., without the allowance for systematic errors, equation system (4) for direct measurements acquires the following form:

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ \dots & & & \\ 1 & & & \\ & 1 & & \\ & 1 & & \\ & \dots & & \\ & 1 & & \\ & \vdots & \ddots & \vdots \\ & & & 1 \\ & & & 1 \\ & & & \dots \\ & & & 1 \end{pmatrix} \times \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix} = \begin{pmatrix} f(s_1^{(1)}) \\ f(s_1^{(2)}) \\ \dots \\ f(s_1^{(k_1)}) \\ \vdots \\ f(s_N^{(1)}) \\ f(s_N^{(2)}) \\ \dots \\ f(s_N^{(k_N)}) \end{pmatrix} . \tag{6}$$

The equation system has N independent variables, and its design matrix has N columns and M rows ($M > N$). The rows of the matrix of equation system (6) are ordered by the numbers assigned to observed stars: the first k_1 rows (equations) refer to observations of the first star, they are followed by k_2 observations of the second star, ... , the last k_N rows refer to observations of the last (Nth) star; $\sum_{i=1}^{N} = M$. Each row of the design matrix contains one unity element – it is at the position corresponding to the number of the observed star – and all other elements of the row are equal to zero.

This general system of equations breaks into independent overdetermined linear equation systems for individual stars. Their LSM solutions are obtained by weighted averaging of the measurements:

$$m_i^{(0)} = \frac{\sum\limits_{k=1}^{k_i} w_k f(s_i^{(k)})}{\sum\limits_{k=1}^{k_i} w_k} , \qquad i = 1 \,..\, N , \tag{7}$$

where w_k is the weight of kth measurement of ith star, and k_i is the number of observations of ith star.

In the next approximation, we take into account the systematic errors of the measurements. The vector of independent variables of equation system (6), which

consists of N unknown quantities m_i, is extended with R parameters t_i on which systematic errors depend. In this case the system of equations acquires the following form:

$$\left(\begin{array}{ccccc|cccc}
1 & & & & & c_{1,1}^{(1)} & c_{2,1}^{(1)} & \dots & c_{R,1}^{(1)} \\
1 & & & & & c_{1,1}^{(2)} & c_{2,1}^{(2)} & \dots & c_{R,1}^{(2)} \\
\dots & & & & & \dots & \dots & \dots & \dots \\
1 & & & & & c_{1,1}^{(k_1)} & c_{2,1}^{(k_1)} & \dots & c_{R,1}^{(k_1)} \\
 & 1 & & & & c_{1,2}^{(1)} & c_{2,2}^{(1)} & \dots & c_{R,2}^{(1)} \\
 & 1 & & & & c_{1,2}^{(2)} & c_{2,2}^{(2)} & \dots & c_{R,2}^{(2)} \\
 & \dots & & & & \dots & \dots & \dots & \dots \\
 & 1 & & & & c_{1,2}^{(k_2)} & c_{2,2}^{(k_2)} & \dots & c_{R,2}^{(k_2)} \\
 & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
 & & & 1 & & c_{1,N}^{(1)} & c_{2,N}^{(1)} & \dots & c_{R,N}^{(1)} \\
 & & & 1 & & c_{1,N}^{(2)} & c_{2,N}^{(2)} & \dots & c_{R,N}^{(2)} \\
 & & & \dots & & \dots & \dots & \dots & \dots \\
 & & & 1 & & c_{1,N}^{(k_N)} & c_{2,N}^{(k_N)} & \dots & c_{R,N}^{(k_N)}
\end{array}\right) \times \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \\ t_1 \\ t_2 \\ \vdots \\ t_R \end{pmatrix} = \begin{pmatrix} f(s_1^{(1)}) \\ f(s_1^{(2)}) \\ \dots \\ f(s_1^{(k_1)}) \\ \vdots \\ f(s_N^{(1)}) \\ f(s_N^{(2)}) \\ \dots \\ f(s_N^{(k_N)}) \end{pmatrix}. \tag{8}$$

The design matrix in this equation system has $N' = N + R$ columns and M rows. The left-hand part of the design matrix (left of the vertical line) coincides with the matrix of equation system (6), where each row contains only one non-zero element. The structure of the right-hand part of the design matrix depends on the adopted model of systematic errors. In the most general case, this part of the matrix has a block structure, albeit some groups of columns may be completely filled. The number of parameters associated with systematic errors in equation system (8) should not be too large and, possibly, should not be greater than $R < N^{1/4} - N^{1/2}$. The equation systems for both approximations, (6) and (8), have identical right-hand side vectors.

Equation systems for differential measurements somewhat differ from those discussed above. Thus in the "zero" approximation (without the allowance for systematic errors) the design matrix has the following form:

$$\begin{pmatrix}
+1 & \dots & \dots & \dots & -1 \\
+1 & \dots & \dots & -1 & \dots \\
\dots & \dots & \dots & \dots & \dots \\
+1 & \dots & -1 & \dots & \dots \\
 & +1 & -1 & \dots & \dots \\
 & +1 & \dots & -1 & \dots \\
\dots & \dots & \dots & \dots & \dots \\
 & +1 & \dots & \dots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\dots & -1 & \dots & \dots & +1 \\
\dots & \dots & \dots & -1 & +1 \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & -1 & \dots & +1
\end{pmatrix} \times \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix} = \begin{pmatrix} f_d(s_{1,j_{1,1}}^{(1)}) \\ f_d(s_{1,j_{1,2}}^{(2)}) \\ \dots \\ f_d(s_{1,j_{1,k_1}}^{(k_1)}) \\ \vdots \\ f_d(s_{N,j_{N,1}}^{(1)}) \\ f_d(s_{N,j_{N,2}}^{(2)}) \\ \dots \\ f_d(s_{1,j_{N,k_N}}^{(k_N)}) \end{pmatrix}. \tag{9}$$

This equation system, like (6), has N independent variables and its design matrix has N columns and M rows, however, each row of the matrix contains two nonzero elements: the "+1" element is at the position corresponding to the number of the primary star of the measured pair, and the "−1" element is at the position corresponding to the number of the secondary star. Matrix rows in equation system (9) are ordered by the numbers assigned to the primary of observed stars: the first k_1 rows (equations) refer to observations of the first star; these are followed by k_2 observations of the second star, etc. Note that the number of the secondary star may vary from one observation to another. This number is described by subscript $j_{i,k}$, where i is the number of the primary star in the observed pair and k, the running number of observation of ith primary star.

With systematic corrections taken into account, the equation system for differential measurements acquires the following form:

$$\left(\begin{array}{ccccc|ccccc}
+1 & \dots & \dots & \dots & -1 & c_{1,1}^{(1)} & c_{2,1}^{(1)} & \dots & c_{R,1}^{(1)} \\
+1 & \dots & \dots & -1 & \dots & c_{1,1}^{(2)} & c_{2,1}^{(2)} & \dots & c_{R,1}^{(2)} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
+1 & \dots & -1 & \dots & \dots & c_{1,1}^{(k_1)} & c_{2,1}^{(k_1)} & \dots & c_{R,1}^{(k_1)} \\
 & +1 & -1 & \dots & \dots & c_{1,2}^{(1)} & c_{2,2}^{(1)} & \dots & c_{R,2}^{(1)} \\
 & +1 & \dots & -1 & \dots & c_{1,2}^{(2)} & c_{2,2}^{(2)} & \dots & c_{R,2}^{(2)} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 & +1 & \dots & \dots & -1 & c_{1,2}^{(k_2)} & c_{2,2}^{(k_2)} & \dots & c_{R,2}^{(k_2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\dots & -1 & \dots & \dots & +1 & c_{1,N}^{(1)} & c_{2,N}^{(1)} & \dots & c_{R,N}^{(1)} \\
\dots & \dots & \dots & -1 & +1 & c_{1,N}^{(2)} & c_{2,N}^{(2)} & \dots & c_{R,N}^{(2)} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & -1 & \dots & +1 & c_{1,N}^{(k_N)} & c_{2,N}^{(k_N)} & \dots & c_{R,N}^{(k_N)}
\end{array}\right) \times \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \\ t_1 \\ t_2 \\ \vdots \\ t_R \end{pmatrix} = \begin{pmatrix} f_d(s_{1,j_{1,1}}^{(1)}) \\ f_d(s_{1,j_{1,2}}^{(2)}) \\ \dots \\ f_d(s_{1,j_{1,k_1}}^{(k_1)}) \\ \vdots \\ f_d(s_{N,j_{N,1}}^{(1)}) \\ f_d(s_{N,j_{N,2}}^{(2)}) \\ \dots \\ f_d(s_{1,j_{N,k_N}}^{(k_N)}) \end{pmatrix}. \tag{10}$$

In this equation system, the number of unknown variables increases to $N' = N + R$ and the design matrix is expanded with the right-hand part (right of the vertical line), which in the general case has block structure. The right-hand side vector is the same as in equation system (9).

Given that equation systems (9) and (10) contain only magnitude differences, their ranks are equal to $N - 1$ and $N' - 1$, respectively. To make this system solvable, it should be extended so as to make it definite. One of the possible ways consists in assigning an a priori magnitude to one of the stars, i.e., to complement the equation set by the following equation:

$$m_i = \text{const}\,. \tag{11}$$

As a result, the number of equations in equation systems increases to $M + 1$. The important problem is how to determine the weight w_i assigned to the extra equation.

Instead of adding equation (11) to equation system (9) or (10), the m_i value can be substituted into these systems, and the resulting constants can be added to the right-hand side of the corresponding equation B_i. In this case, the number

of equations remains unchanged (M), and the number of independent variables decreases by one (down to $N-1$ or $N'-1$, respectively) and becomes equal to the rank of the design matrix.

Finally, the values of all the remaining independent variables in linear equation systems (9) and (10) can be expressed in terms of variable m_i.

3. SOLUTION OF EQUATION SYSTEMS

Equation systems (8) and (10) can be solved directly, i.e., as overdetermined linear equation systems with rectangular design matrix A, or they can be transformed into a system of normal equations where the number of equations is equal to the number of unknowns. Each method has its advantages and disadvantages.

A very good modern method to directly solve overdetermined linear equation systems with rectangular design matrices is based on the singular value decomposition (SVD) technique (Forsite et al. 1977). This method is stable, it allows determining the condition of the equation system and controlling the significance of the solutions obtained. SVD works both for square and rectangular matrices. A downside of this technique is its higher computing requirements compared to a number of other methods of linear algebra.

The idea of the SVD method consists in the following. The singular value decomposition of a real $N \times M$ matrix A is its decomposition into a product of three matrices of the following form:

$$A = U \times \Sigma \times V^{\mathrm{T}}, \tag{12}$$

where Σ is a rectangular matrix of size $N \times M$ with the upper diagonal populated by positive singular values, and U and V are unitary matrices of sizes $M \times M$ and $N \times N$ containing left- and right-singular vectors, respectively.

The condition number of matrix A is equal to the ratio of the largest ($\lambda_{\max}$) to the smallest ($\lambda_{\min}$) singular values: $\mathrm{cond}(A) = \lambda_{\max}/\lambda_{\min}$. The condition number determines how the error of input data may affect the solution of the equation system. The larger the condition number, the poorer is the condition of the equation system. Well-conditioned equation systems can be solved using practically any solution methods, whereas only stable methods (e.g., SVD) can be applied to ill-conditioned equation systems. Stable methods are usually more computer intensive.

Equation system (4) is transformed into a set of normal equations by multiplying both its parts by matrix A^{T}:

$$\begin{aligned} A^{\mathrm{T}} \times A \times X &= A^{\mathrm{T}} \times B\,, \\ G \times X &= P\,. \end{aligned} \tag{13}$$

Here, $G = A^{\mathrm{T}} \times A$ is the matrix of normal equations of size $N' \times N'$, and $P = A^{\mathrm{T}} \times B$, the vector of its right-hand parts of size N'.

Matrix G is squared, symmetric, and positive definite. Many methods have been developed for solving such systems. However, reduction of an overdetermined equation system to normal form degrades its condition: $G = A^{\mathrm{T}} \times A$ and therefore $\mathrm{cond}(G) = \mathrm{cond}^2(A)$.

Another parameter affecting the choice of the solution method for equation systems (4) and (13) is the size of the system. If the matrix of the equation system

Table 1. Number of parameters in sparse design matrices for three surveys.

Survey	Stars (columns)	Measurements (equations)	Matrix elements	Direct method 0 approx., non zero elem. (sparse factor)	Differential 0 approx., non zero elem. (sparse factor)	Next approx., non zero elem. (sparse factor)
WBVR	8000	70 000	$560 \cdot 10^6$	—	16 000 (1/35 000)	770 000 ($\sim 1/1500$)
Lyra-B	$300 \cdot 10^6$	$33 \cdot 10^9$	$9.9 \cdot 10^{18}$	—	$33 \cdot 10^9$ ($1/10^8$)	$4.3 \cdot 10^{12}$ ($1/2.3 \cdot 10^6$)
Gaia	$\sim 10^9$	$70 \cdot 10^9$	$70 \cdot 10^{18}$	$70 \cdot 10^9$ ($1/10^9$)	—	$12 \cdot 10^{12}$ ($1/5.6 \cdot 10^6$)

fits completely into the computer memory, then common computational methods can be applied. If the number of elements in the matrix exceeds the capacity of the computer's RAM, then special computing methods have to be used.

One of the ways to reduce memory requirements is to use sparse matrices. In sparse matrices most of the elements are equal to zero and only significant, non-zero elements are stored in memory. In such matrices, special addressing algorithms are used and computing methods are subject to the requirement that they should preserve the structure of sparse matrices: computations should result in few or no new non-zero elements.

Matrices of equation systems (6), (8), (9), and (10) used for creating catalogs are very sparse. Table 1 lists the parameters of the corresponding matrices for *WBVR*, "Lyra-B", and *Gaia* surveys. We note that the number of non-zero elements in design matrices (8) and (10) was computed assuming that the number of parameters is equal to $R = N^{1/4}$ and the parameter domain in matrices is completely filled.

Matrices G of the normal equation system for direct and differential measurements are also sparse. Thus, matrix G for direct measurements has the form:

$$G = \begin{array}{|ccccc|c|} \hline O_1 & & & & & \vdots \\ & O_2 & & & & \vdots \\ & & \ddots & & & \vdots \\ & & & & O_N & \vdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \tag{14}$$

Matrix G is symmetric and its upper right part of size of $N \times N$ is diagonal. The diagonal of this matrix is populated by positive numbers O_i equal to the number of observations of ith star including the fictitious observation defined by equation (11). These numbers are strictly positive, because each star included into the system should have been observed at least once. The remaining part of the matrix (of width $R \ll N$) is filled or has a block structure. (This part is absent in zero approximation.)

If G is computed using matrices A of the form (9) or (10) before formal equation (11) is added to them, the rank of the normal equation system is equal to $N - 1$, allowing the magnitude differences, but not the magnitudes themselves, to be determined. In this case all O_i are equal to the number of actually performed

observations of the corresponding star. When design matrices (9) and (10) are folded and after equation (11) is added to them, the coefficient O_i whose number appears in equation (11) increases by 1, the rank of matrix G is equal to N, and normal equation system (13) can be solved in the usual way.

In the case of differential measurements, matrix G_d has a more complicated form:

$$G_d = \left(\begin{array}{cccccccc|c}
O_1 & & \dots & -D_{1,i} & \dots & -D_{1,k} & \dots & & \vdots \\
& O_2 & -D_{2,j} & \dots & \dots & \dots & -D_{2,m} & & \vdots \\
\dots & -D_{j,2} & \ddots & & \dots & & & & \vdots \\
-D_{i,1} & \dots & & O_i & \dots & & & & \vdots \\
\dots & \dots & \dots & \dots & \ddots & \dots & \dots & -D_{n,N} & \vdots \\
-D_{k,1} & \dots & & & \dots & O_k & \dots & & \vdots \\
\dots & -D_{m,2} & & & \dots & \dots & O_m & -D_{m,N} & \vdots \\
& & & & -D_{N,n} & & -D_{N,m} & O_N & \vdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right) \tag{15}$$

The upper right-hand part of matrix G_d of size $N \times N$ has a symmetric form. Its diagonal is populated by positive numbers O_i equal to the number of observations of ith star irrespectively of whether it is a primary or secondary star in the observed pair. The number of observations includes the fictitious observation defined by equation (11). The cells outside the diagonal of this part of the matrix are populated by negative integer numbers $-D_{i,j}$ or zeros. The absolute value of $D_{i,j}$ is equal to the number of joint observations of ith and jth stars, $D_{i,j} = 0$ if there were no observations of the corresponding pair. Stars are measured in pairs and therefore the sum of elements over each column or row is equal to zero. The only exceptions are the row and column with the number appearing in the added equation (11): the sum of elements in this row and the sum of elements in this column are equal to unity. The remaining part of matrix G_d (of width $R \ll N$) is, like in the case of matrix (14), filled or has a block structure.

The reasoning about the rank of the normal equation system presented above for the case of direct observations also fully applies to the case of equations for differential measurements.

Matrix G (14) in zero approximation contains N nonzero elements, and its sparseness factor is equal to $1/N$. The same parameters for matrix G_d (15) can be estimated as follows: the diagonal of G_d contains N positive numbers, the average number of observations of ith star is equal to M/N, and therefore ith row and ith column should also contain another M/N non-negative elements and the total number of such elements in matrix G_d is $2M$. Hence matrix G_d contains a total of $N + 2M$ non-zero elements[7], and its sparseness factor is equal to $1/(N + 2M)$.

[7] $N+2M$ is the upper limit of the number of non-zero elements, it does not take into account the possibility of multiple observations of pairs and is achieved in the case where all non-zero elements outside the diagonal are equal to –1.

Table 2. Parameters of normal equations for sparse matrices for two surveys.

Survey	Stars (columns) N	Measurements (equations) M	Direct method 0 approx., non zero elem. (sparse factor)	Differential 0 approx., non zero elem. (sparse factor)	Next approx., non zero elem. (sparse factor)
Lyra-B	$300 \cdot 10^6$	$33 \cdot 10^9$	$300 \cdot 10^6$ $(1/3 \cdot 10^8)$	$66 \cdot 10^9$ $(1/66 \cdot 10^9)$	$96 \cdot 10^9$ $(1/9.6 \cdot 10^{10})$
Gaia	$\sim 10^9$	$70 \cdot 10^9$	10^9 $(1/10^9)$	$1.4 \cdot 10^{11}$ $(1/1.4 \cdot 10^{11})$	$2.7 \cdot 10^{11}$ $(1/2.7 \cdot 10^{11})$

In the next approximations, a large number of non-zero elements is contained in the strip of parameters. If it is fully filled then the number of such elements is approximately equal to $2NR$, or $2N^{5/4}$ in the case of $R = N^{1/4}$. Table 2 lists the numerical values of these parameters for "Lyra-B" and *Gaia* surveys.

"Lyra-B" and *Gaia* catalogs can be accommodated in the RAM of modern supercomputers if stored in the form of sparse matrices. Given that there is a version of SVD procedure for sparse matrices (Berry 1992; Berry et al. 2003; Kontoghiorghes 2005; Brand 2006; Yand et al. 2014), such equation systems can be solved directly.

4. "PHYSCIAL" REDUCTION OF THE SIZE OF THE EQUATION SYSTEM

Another method of solving the equation system for the creation of a high-precision catalog consists in setting up equation system (4) only for a part of the stars measured in the process of the survey rather than for all observed stars. The number of stars rapidly increases with magnitude, and the selection can be performed as follows: all bright stars, a small fraction of medium-brightness stars, and an even smaller fraction of faint stars. Another possible algorithm involves selecting all observations for stars brighter than a certain critical magnitude, and a fixed number of observations per unit magnitude for fainter stars. There are also other possible ways of selecting stars.

The resulting equation system is substantially smaller than the initial complete system, and it can be solved with less powerful computers in a much shorter time.

The representativeness of the resulting sample is a very important factor. It can be validated statistically – by the presence of stars with different characteristics. This fact can also be validated by comparing the solutions obtained for equations for several different samples.

Solution of the equation system for a sample of stars yields the magnitudes of the stars of the sample and the parameters of systematic errors. The parameters of errors are assumed to be the same for all stars of the survey and are used for the subsequent computation of the magnitudes and magnitude corrections for stars not included into the sample. The magnitude of ith star not included into the sample is computed using only the measurements of this star and therefore does not require much computing power.

5. CONCLUSIONS

This study analyzes the form of linear equation systems for constructing high-precision catalogs from initial data by identifying and correcting small systematic errors. We show that for surveys with the data size comparable to that of *Gaia* (Lindegren et al. 2008) or smaller, simultaneous processing of these data on

modern computers appears quite realistic. Larger surveys based on observations collected with PanSTARR (Stubbs et al. 2010) or (future) LSST (LSST Science Collaboration 2009) telescopes are too large to be completely processed.

High-precision multicolor photometric star catalogs are in great demand in a substantial part of astronomy and its applications, and the studies cited here (Gorynya et al. 1996; Shakura et al. 1998; Popov & Prokhorov 2006; Samus et al. 2009; Sil'chenko et al. 2012), which we selected based on our subjective opinion, cover only a small part of the applications of these catalogs.

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