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ON CLUSTER ROTATION IN THE GALACTIC TIDAL FIELD

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Abstract. The dynamics of a rotating star cluster moving along a circular orbit in the axisymmetrical steady Galaxy is considered. The generalized tensor virial theorem allows to estimate its rotation speed. Conditions for direct and retrograde rotation in the galactic plane are found.

Key words: stellar dynamics – methods: analytical – globular clusters: general – open clusters and associations: general

1. INTRODUCTION

The dynamics of real star clusters in the Galaxy is governed by many factors (e.g., Spitzer 1987). Following the classical study by Bok (1934), we will analyze only two of them, namely, the self-gravitation of a cluster and the tidal force of the Galaxy. Earlier, such a model was studied by many authors who considered dynamics of a protoplanetary cloud or atmospheres of comets moving along a circular or elliptic orbit (Roche 1859; Callandreau 1892, 1902; Picart 1904; Lowell 1911, 1912; Fessenkov 1951). Most of these works were critically reviewed by Rein (1936a).

Bok (1934) considered a cluster on a circular orbit in the steady axisymmetric galactic field. He found that star orbits are finite if the cluster density

$$\varrho > \varrho_* = \kappa_R^2/(\beta\pi G), \tag{1}$$

where G is the gravitational constant, $\kappa_R^2 = 4A(A-B)$, A, B being Oort's dynamic coefficients, and β depends on the cluster shape and is equal to 4/3 for spherical systems and 0.2 for disks (e.g., Chandrasekhar 1942; Ogorodnikov 1965).

Bok (1934), Rein (1936b) and Cimino (1956) discussed the Hill stability for stars of a cluster. A steady cluster can be stable only if its diameter is smaller than the minimal size of Hill's critical surface. Then the critical density of a cluster can be estimated (Nezhinsky & Ossipkov 1987; Ossipkov et al. 1997; Ossipkov 2006). It is larger by almost an order of magnitude than that according to condition (1).

Mineur (1939) found a series of triaxial homogeneous ellipsoids with an isotropic velocity distribution in equilibrium under the joint action of self-gravitation and tides. He established that such ellipsoids could exist only if the inequality (1) was fulfilled with $\beta = \beta(\kappa_R/\kappa_z)$, where $\kappa_z = C$ is the frequency of small vertical star oscillations in the Galaxy (Kuzmin's parameter). Mineur gave a table of this function. In the solar neighborhood, it is $\beta \approx 4.2$. Mayot (1945) applied the classical theory of equilibrium figures to Mineur ellipsoids. Kondratiev (2001) developed a similar theory to find the equilibrium shape of globules. The theory was generalized by Kondratiev & Trubitsina (2010) for ellipsoids with internal flows.

Van Wijk (1949) and Kuzmin (1963) applied the tensor virial theorem to study the problem and tried to generalize Mineur's analysis for non-homogeneous ellipsoidal clusters with an isotropic velocity distributions. Lee & Rood (1969) found a generalization of the virial theorem for a cluster that rotated and moved in the tidal field. Ossipkov (2006) considered an equilibrium of triaxial non-rotating clusters with anisotropic velocity distribution. Small virial oscillations of gravitating systems in the tidal galactic field were studied by him in other works (Ossipkov 1993, 2001). His analysis was based on equations of gross-dynamics for gravitating stellar systems (Ossipkov 1985, 2000, 2004). In the present paper, we generalize this theory for rotating clusters.

2. THE BOK PROBLEM

At first, we recall equations of star motion for a cluster moving with an angular velocity Ω at the distance R_0 from the galactic center. Let $\mathbf{x} = (x, y, z)$ be a rotating frame of reference, its axis x being directed to the galactic anticenter. Then:

$$\ddot{\mathbf{x}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{x}} + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times (\mathbf{R}_0 + \mathbf{x}) = \nabla(\Phi_c + \Phi_g). \tag{2}$$

Here $\mathbf{R}_0 = (R_0, 0, 0)$, $\boldsymbol{\Omega} = (0, 0, \Omega)$, $\Phi_c(\mathbf{x}, t)$ is the (positive) potential of the cluster, $\Phi_g(R, z)$ is the (positive) galactic potential. We set

$$\Phi_g(R, z) = \Phi_0 + k_1(R - R_0) + k_2(R - R_0)^2 + k_3 z^2 \tag{3}$$

(the tidal approximation). It is evident that $k_1 = -\Omega^2 R_0$. Here $4k_2 = \kappa_R^2 - \Omega^2$, $2k_3 = \kappa_z^2$ in the solar neighborhood. The author suggested to call κ_R the 'tidal increment' (Ossipkov 2003). Maybe, it would be more correct to call it the 'centrifugal increment'. Recall that $\Omega \approx 27 \mathrm{km \cdot s^{-1} \cdot kpc^{-1}}$, $\kappa_R \approx 42\ \mathrm{km \cdot s^{-1} \cdot kpc^{-1}}$, $\kappa_z \approx 85\,\mathrm{km \cdot s^{-1} \cdot kpc^{-1}}$. Then:

$$\ddot{x} - 2\Omega v_y = \partial\Phi_c/\partial x + \kappa_R^2 x\,, \tag{4}$$

$$\ddot{y} + 2\Omega v_x = \partial\Phi_c/\partial y\,, \tag{5}$$

$$\ddot{z} = \partial\Phi_c/\partial z - \kappa_z^2 z \tag{6}$$

(e.g., Ogorodnikov 1965; Ossipkov 2007; Buliga & Ossipkov 2011; Proskurin & Ossipkov 2013).

The problem of studying these equations is known as the Bok problem (Bok 1934; Rein 1936b; Ossipkov 2007; Davydenko 2013).

$\mathbf{x}(t) \equiv 0$ is an evident equilibrium solution of Eqs (4)–(6). To study its stability, one must set

$$\Phi_c(\mathbf{x}) = \Phi_c(0) - \frac{1}{2}(a^2x^2 + b^2y^2 + c^2z^2) + \cdots$$

and analyse variational equations:

$$\ddot{x} - 2\Omega v_y = (\kappa_R^2 - a^2)x,$$

$$\begin{aligned}\ddot{y} + 2\Omega v_x &= -b^2 y,\\ \ddot{z} &= -(\kappa_z^2 + c^2) z.\end{aligned}$$

Substituting $\mathbf{x} = \mathbf{C}e^{\mathrm{i}\lambda t}$ yields a characteristic equation (Ossipkov 2007):

$$p^2 - [a^2 + b^2 + 4\Omega^2 - \kappa_R^2]\, p - b^2(a^2 - \kappa_R^2) = 0, \qquad p = \lambda^2.$$

We conclude that the necessary condition for the stability of the trivial solution of Eqs. (4)–(6) has the following form:

$$a^2 > \kappa_R^2. \tag{7}$$

The central density of the cluster is $\varrho(0) = (a^2+b^2+c^2)/4\pi G$. Hence the condition (7) can be rewritten as follows:

$$\varrho(0) > \varrho_c = \frac{\kappa_R^2}{4\pi G}\,(1 + s_y + s_z),$$

with $s_y = b^2/a^2$, $s_z = c^2/a^2$. For spherical clusters,

$$\varrho_c = \frac{3}{4\pi G}\,\kappa_R^2.$$

For homogeneous models, the critical density ϱ_c coincides with that found by Bok (1934). However, this stability analysis is not self-consistent.

3. GROSS-DYNAMIC EQUATIONS

To find self-consistent conditions for the existence of a steady cluster, we will work with gross-dynamic equations. Gross-dynamics is a branch of galactic dynamics dealing with integral parameters of stellar systems as a whole, such as the inertia moment, potential energy and so on (Kuzmin 1965). Generally, there exists a hierarchy of gross-dynamic equations (Chandrasekhar & Lee 1968; Ossipkov 1985; Ossipkov 2014). To derive it, we start with a collisionless kinetic equation for the distribution function $f(\mathbf{x}, \mathbf{v}, t)$:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}}\,\mathbf{v} + \frac{\partial f}{\partial \mathbf{v}}\,\ddot{\mathbf{x}} = 0,$$

substitute the equations of motion (4)–(6), multiply it by $\prod_{i=1}^{3} x_i^{k_i} v_i^{l_i}$ and integrate over the phase space. For any function $g(\mathbf{x}, \mathbf{v}, t)$, we denote

$$\langle g\rangle = \int g(\mathbf{x}, \mathbf{v}, t)\, f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{v}/M,$$

where $M = \int f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{v}$ is the cluster mass. We introduce the following tensors: $I_{ij} = M\langle x_i x_j\rangle$, $L_{ij} = M\langle x_i v_j\rangle$, $K_{ij} = M\langle v_i v_j\rangle$, $W_{ij} = M\big\langle x_i \partial\Phi_c/\partial x_j\big\rangle$. Here I_{ij} is the inertia tensor; K_{ij}, the kinetic energy tensor; W_{ij}, the potential energy tensor; and L_{ij} can be called the angular momentum tensor. If $\mathbf{V}(\mathbf{x}) =$

$(V_i) = (\int v_i f \,\mathrm{d}^3\mathbf{v})/\varrho$ is the streaming velocity, then $L_{ij} = \int \varrho(\mathbf{x}) x_i V_j(\mathbf{x}) \,\mathrm{d}^3\mathbf{x}$, where $\varrho(\mathbf{x})$ is the density.

The following equations can be easily found (Ossipkov 2001, 2014):

$$\frac{d}{dt} I_{ij} = L_{ij} + L_{ji}, \tag{8}$$

and

$$\frac{d}{dt} L_{ix} = K_{ix} + W_{ix} + \kappa_R^2 I_{ix} + 2\Omega L_{iy}, \tag{9}$$

$$\frac{d}{dt} L_{iy} = K_{iy} + W_{iy} - 2\Omega L_{ix}, \tag{10}$$

$$\frac{d}{dt} L_{iz} = K_{iz} + W_{iz} - \kappa_z^2 I_{iz} \tag{11}$$

(the generalized Lagrange-Jacobi equations).

It follows from Eq. (8) that the angular momentum tensor is antisymmetrical for steady systems, $L_{ij} = -L_{ji}$. If $dL_{ij}/dt \equiv 0$, then Eqs (9)–(11) will be equations of the tensor virial theorem for a cluster in a tidal field.

4. NON-ROTATING STELLAR CLUSTER

At first, we will consider the simplest case of a non-rotating cluster (Ossipkov 2006). The streaming velocity is $\mathbf{V}(\mathbf{x}) \equiv 0$, hence then angular momentum tensor vanishes, $L_{ij} \equiv 0$. An elementary analysis of Eqs (9–11) yields the following.

Theorem 1. The inertia tensor is diagonal for a cluster in relative equilibrium without streaming motions, principal inertia axes coinciding with the x, y, z axes, i.e. $I_{xy} = I_{xz} = I_{yz} \equiv 0$.

Now we will suppose (!) that the potential energy tensor is also diagonal, $W_{ij} \equiv 0, i \neq j$. Then it follows from the tensor virial theorem that the kinetic energy tensor is also diagonal. The latter means that principal axes of the velocity ellipsoid (averaged over the cluster) coincide with principal inertia axes.

Denote

$$\tau_x^2 = I_{xx}/(-W_{xx}). \tag{12}$$

It is evident that τ_x (and τ_z defined below) are close to the crossing time.

Theorem 2 (Ossipkov 2006). An inequality

$$\kappa_R^2 \tau_x^2 \leq 1 \tag{13}$$

is a general necessary condition for the existence of a cluster in relative equlibrium without streaming motions.

For homogeneous clusters, the inequality (13) is reduced to Bok's condition (1).

5. ROTATION AROUND THE z AXIS

Now we consider a cluster in rigid rotation in the galactic plane. Then, the components of the streaming velocity are $V_x = -ny$, $V_y = nx$, $V_z \equiv 0$, n being the angular velocity. $n > 0$ means direct rotation of the cluster, and retrograde rotation will occur for $n < 0$. Then:

$$L_{xy} = nI_{xx}, \quad L_{yx} = -nI_{yy},$$

and we conclude that $I_{xx} = I_{yy} = I_\parallel$ for steady systems (as for an axisymmetric cluster). We see that

$$K_{xx} = M\sigma_x^2 + n^2 I_\parallel, \quad K_{yy} = M\sigma_y^2 + n^2 I_\parallel, \quad K_{zz} = M\sigma_z^2,$$

where σ_x^2, σ_y^2, σ_z^2 are dispersions of residual velocities, and

$$L_{xx} = -nI_{xy} \equiv 0, \quad L_{yy} = nI_{xy} \equiv 0, \quad L_{zz} \equiv 0.$$

Principal axes of the mean residual velocity ellipsoid coincide with the coordinate axes.

The tensor virial equations can be written as follows:

$$M\sigma_x^2 = -W_\parallel - (\kappa_R^2 - \omega)I_\parallel, \tag{14}$$

$$M\sigma_y^2 = -W_\parallel + \omega I_\parallel, \tag{15}$$

$$M\sigma_z^2 = -W_{zz} + \kappa_z^2 I_{zz}. \tag{16}$$

Here $\omega = -(n^2 + 2\Omega n)$, and we again supposed that $W_{xx} = W_{yy} = W_\parallel$. Divide Eqs (14), (16) by Eq. (15). Then:

$$\frac{\sigma_x^2}{\sigma_y^2} = \frac{1 + (\omega - \kappa_R^2)\tau_x^2}{1 + \omega\tau_x^2}, \quad \frac{\sigma_z^2}{\sigma_y^2} = \varepsilon_{yz}\,\frac{1 + \kappa_z^2\tau_z^2}{1 + \omega\tau_x^2}. \tag{17}$$

Here $\tau_z^2 = I_{zz}/(-W_{zz})$, $\varepsilon_{yz} = (-W_\parallel)/(-W_{zz})$. The following theorem follows immediately from Eq. (17).

Theorem 3. Inequalities $1 + (\omega - \kappa_R^2)\tau_x^2 > 0$, $1 + \omega\tau_x^2 > 0$ are necessary conditions for existence of a steady cluster with rigid rotation in the xy plane.

Denote $n_{1,2} = -\Omega \pm \sqrt{\Omega^2 + \tau_x^{-2}}$. An elementary analysis of the second inequality (that is weaker) yields the following. If $n < n_2 < 0$ or $n > n_1 > 0$, then no equilibrium solution is possible; if $n \in (n_2, 0)$, then retrograde rotation is possible; if $n \in (0, n_1)$, then the rotation is direct.

Denote

$$\varrho_e = \frac{1}{\pi G\tau_x^2}, \quad \varrho_0 = \frac{\kappa_R^2}{\pi G}.$$

ϱ_e can be considered as an effective density of the cluster. An analysis of the first inequality of Theorem 3 leads to the following assertions.

Theorem 4. If $\varrho_e < \varrho_0$, then steady models rotating in the $z = 0$ plane are possible, rotation is retrograde, and

$$n \in \left[-\Omega - \sqrt{\Omega^2 - \kappa_R^2 + \tau_x^{-2}}, \quad \Omega + \sqrt{\Omega^2 - \kappa_R^2 + \tau_x^{-2}}\right].$$

Theorem 5. If $\varrho_e > \varrho_0$, that is the inequality (13) is fulfilled, then both direct and retrograde steady rotations are possible, and the angular velocity

$$n \in \left[-\Omega - \sqrt{\Omega^2 + \kappa_R^2 - \tau_x^{-2}}, \quad \Omega + \sqrt{\Omega^2 + \kappa_R^2 - \tau_x^{-2}}\right].$$

6. STEADY ROTATION AROUND THE y AND x AXES

If a cluster rotates in the xz plane, then $V_x = -nz$, $V_z = nx$, $L_{xx} = nI_{xx}$, $L_{zz} = -I_{zz}$, $I_{xx} = I_{zz}$. It can be easily found that:

$$K_{xx} - m\sigma_x^2 + n^2 I_{zz}, \quad K_{yy} = M\sigma_y^2, \quad K_{zz} = M\sigma_z^2 + n^2 I_{xx}.$$

Suppose that $W_{xx} = W_{zz}$. The tensor virial theorem yields the following.

Theorem 6. If $\omega_z \tau_z^2 < 1$, where $\omega_z = n^2 - \kappa_z^2$, then steady rotation in the xz plane is possible, and the squared angular velocity is:

$$n^2 = \frac{1}{\tau_z^2}\left[1 + \tau_z^2\kappa_z^2 + \frac{M\sigma_z^2}{-W_{zz}}\right].$$

If a cluster rotates around the x axis, $V_y = -nz$, $V_z = ny$. In this case, $I_{yy} = I_{zz}$, $K_{xx} = M\sigma_x^2$, $K_{yy} = M\sigma_y^2 + n^2 I_{zz}$, $K_{zz} = M\sigma_z^2 + n^2 I_{yy}$.

Theorem 7. If $\kappa_R^2\tau_x^2 < 1$, $n^2\tau_z^2 < 1$, then rotation in the xy plane is possible, and the squared angular velocity is:

$$n^2 = -M\,\frac{\sigma_y^2}{I_{zz}} + \frac{1}{\tau_z^2}.$$

7. CONCLUSIONS

It follows from the above analysis that the spin rotation of not very dense clusters must be retrograde. It is in accordance with the fact that most stars on direct orbits escape from clusters, especially at their periphery (e.g., Davydenko 2013). Probably, this remains valid for clusters whose orbits are not circular. We assumed that rotation was rigid, but this is not necessary, and n can be considered as an effective angular velocity.

Unfortunately, we could not find reliable observational data on internal kinematics of open clusters to check the prediction of the theory.

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