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THE TIDAL POTENTIAL OF A HOMOGENEOUS TORUS WITH AN ELLIPTICAL SECTION OF THE SLEEVE

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Abstract. In this paper the problem of the tidal potential of a homogeneous gravitating torus with an elliptical cross-section sleeve is solved. In particular, the potentials in analytical form in the vicinity of the center of the torus and its external region are found. This torus can serve as a gravitational model of the Kuiper belt.

Key words: gravitating torus, tidal potential

1. INTRODUCTION

In various problems of celestial mechanics and dynamical astronomy a knowledge of the gravitational potential of the bodies of different form and concentration of the substance is often required. The potential theory has many striking applications in dynamics of stellar systems (see an example in the paper by Kondratyev et al. 2015a).

A special class of bodies exists that form ring- and torus-shaped figures and regions often met in the Solar system and in galaxies. However, because of complexity of a problem, the potential of a homogeneous circular torus has received much attention only recently; the potential on the axis of symmetry was found in the monograph by Kondratyev (2003), and the potential of the torus in all space was given in the book by Kondratyev (2007). Subsequently, the potential of a circular torus was studied extensively in the works by Kondratyev et al. (2009, 2012), Kondratyev (2010) and Kondratyev & Trubitsyna (2010).

In cylindrical coordinates a circular torus is limited to the surface

$$(r - R_0)^2 + x_3^2 = r_0^2, \tag{1}$$

where R_0 is the radius of centerline of the torus, and r_0 is the radius of the meridional section of the torus sleeve. Such a torus is formed by rotating the circle (1) with radius r_0 about the axis Ox_3. Let us fill the shell of the torus (1) with gravitating substance (or insulator with homogeneous distribution of static electric charge). Then the spatial potential of a homogeneous torus with a circular cross section of the sleeve is given by the following equation (Kondratyev 2007):

$$\frac{\varphi_{\text{torus}}(r, x_3)}{2\sqrt{2}\rho\, r_0 R_0} =$$

$$= \int_0^{2\pi} \left\{ \left[c + 2\left(R_1^2 - \frac{r^2}{R_0^2} \right) \right] K(k) + (a-c)\, E(k) - 2\frac{(x_3 - r_0 \sin\theta)^2}{R_0^2} \varPi[n,k] \right\} \cdot$$

$$\cdot \frac{\cos\theta}{\sqrt{a-c}} d\theta\,, \quad (2)$$

where $K(k), E(k)$ and $\varPi[n,k]$ are the complete elliptical integrals of the first, second, and third kind, respectively, and

$$R_1 = 1 + \frac{r_0}{R_0}\cos\theta, \qquad a = 2\frac{r^2 + (x_3 - r_0\sin\theta)^2}{R_0^2},$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = \frac{a}{2} - R_1^2 \pm \sqrt{\left(\frac{a}{2} - R_1^2\right)^2 - 4R_1^2 \frac{(x_3 - r_0\sin\theta)^2}{R_0^2}}, \quad (3)$$

$$k = \frac{1-\tilde{k}}{1+\tilde{k}}, \qquad \tilde{k} = \sqrt{\frac{\frac{a}{2} + R_1^2 - 2R_1\frac{r}{R_0}}{\frac{a}{2} + R_1^2 + 2R_1\frac{r}{R_0}}}, \qquad n = \frac{a-b}{2\frac{r^2}{R_0^2}}.$$

Note that the same formula (2) represents the potential at any point of space (both inside and outside the torus).

However, besides the torus with a circular cross-section of the sleeve, in practice often meets the more general case of torus with an elliptical section of the sleeve. From a mathematical point of view the problem of the potential of the torus with an elliptical cross-section of the sleeve is even more difficult than in the case of torus with a circular cross-section. However, sometimes it is enough to know the potential of this torus only in the tidal approximation. This problem is solved in this paper.

2. STATEMENT AND SOLUTION OF THE PROBLEM

Let us consider a circular torus with an elliptical section of the sleeve. The equation of a surface of such a torus is

$$\frac{(r - R_0)^2}{a_1^2} + \frac{x_3^2}{a_3^2} = 1, \quad (4)$$

where a_1 and a_3 are semiaxes of an ellipse in the meridional section of the sleeve of the torus. It is formed by rotating an ellipse around a straight line, which is located outside the ellipse and is parallel to the axis Ox_3.

The mass of the torus with the density ρ is

$$M = 2\pi^2 a_1 a_3 R_0\, \rho. \quad (5)$$

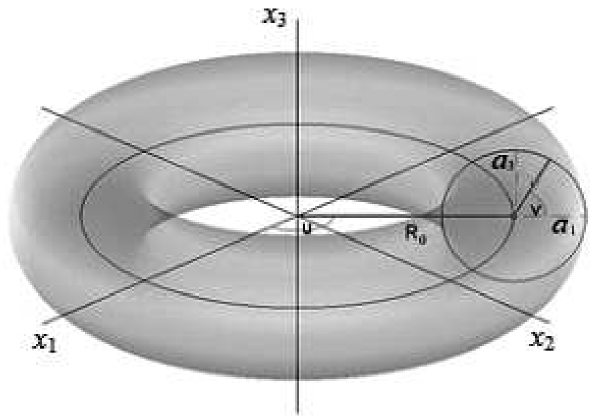

Fig. 1. Circular torus with an elliptical cross-section of the sleeve.

In a general view, the potential of the torus in the point $(x_1,\, x_2\,,\, x_3)$ is given by the integral

$$\varphi\,(x_1,\, x_2,\, x_3) = G\rho \iiint\limits_V \frac{dx_1'\, dx_2'\, dx_3'}{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}}. \tag{6}$$

The integrand in (6) is the distance between the test point and the point of integration $(x_1',\, x_2'\,,\, x_3')$,

$$D = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}, \tag{7}$$

which is represented in the form

$$D = \sqrt{x_1' + x_2' + x_2' - 2\,(x_1 x_1' + x_2 x_2' + x_3 x_3') + x_1^2 + x_2^2 + x_3^2}\,. \tag{8}$$

At this stage there are two versions of this problem.

2.1. The case of the interior sampling point

In this case, the sampling point is near the center of the torus and its coordinates x_i satisfy the inequalities

$$x_i \ll x_i'\,. \tag{9}$$

Then the inverse distance $1/D$ can be represented by a series in powers x_i; then, up to the quadratic terms, we have

$$\begin{aligned} \frac{1}{D} = {} & \frac{1}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}} + \frac{x_1 x_1' + x_2 x_2' + x_3 x_3'}{(x_1'^2 + x_2'^2 + x_3'^2)^{3/2}} + \\ & + \frac{3(x_1 x_1' + x_2 x_2' + x_3 x_3')^2 - \left(x_1^2 + x_2^2 + x_3^2\right)\left(x_1'^2 + x_2'^2 + x_3'^2\right)}{2(x_1'^2 + x_2'^2 + x_3'^2)^{5/2}}\,. \end{aligned} \tag{10}$$

It should be noted that, due to the symmetry of the torus, while integrating the coordinates of the attracting point x'_i in the expression (10) disappear all members, odd relative to (x'_1, x'_2, x'_3).

To solve this problem, we exclude in (10) the main (first from right) member. Then the desired potential is equal to

$$\varphi_t = \frac{G\rho}{2} \iiint\limits_V \frac{\left(2x_1^2 - x_2^2 - x_3^2\right) x_1'^2 + \left(2x_2^2 - x_1^2 - x_3^2\right) x_2'^2 + \left(2x_3^2 - x_1^2 - x_2^2\right) x_3'^2}{\left(x_1'^2 + x_2'^2 + x_3'^2\right)^{5/2}} \cdot$$

$$\cdot\, dx'_1\, dx'_2\, dx'_3. \quad (11)$$

As a result, we find that the inner tidal potential of the torus is expressed in the analytical form by the formula

$$\varphi = \frac{G\rho}{2} \left\{ \left(2x_1^2 - x_2^2 - x_3^2\right) \cdot A + \left(2x_2^2 - x_1^2 - x_3^2\right) \cdot B + \left(2x_3^2 - x_1^2 - x_2^2\right) \cdot C \right\}. \quad (12)$$

This potential is a quadratic function of the coordinates of the trial point, and the dimensionless coefficients A, B, C are

$$A = \iiint\limits_V \frac{x_1'^2}{\left(x_1'^2 + x_2'^2 + x_3'^2\right)^{5/2}} dx'_1\, dx'_2\, dx'_3;$$

$$B = \iiint\limits_V \frac{x_2'^2}{\left(x_1'^2 + x_2'^2 + x_3'^2\right)^{5/2}} dx'_1\, dx'_2\, dx'_3; \quad (13)$$

$$C = \iiint\limits_V \frac{x_3'^2}{\left(x_1'^2 + x_2'^2 + x_3'^2\right)^{5/2}} dx'_1\, dx'_2\, dx'_3\,.$$

The integration in (13) is held on the torus volume V (it is more convenient to do the integration at first on dx'_3, and then on dx'_2 and dx'_1).

The formula of the tidal potential (12) is suitable for any homogeneous body with three planes of symmetry. In the case of the torus, which holds the azimuthal symmetry $A = B$, the expression for the potential (12) is even more simplified and takes the following form:

$$\varphi\,(x) = \frac{G\rho}{2} \left(A - C\right) \left(x_1^2 + x_2^2 - 2x_3^2\right). \quad (14)$$

To calculate the coefficients of the potential we introduce polar coordinates (r, θ). Omitting strokes, we have the expressions given in (15).

$$A = B = \iiint\limits_V \frac{r^2 \cos^2\theta \cdot r\,dr\,dx_3\,d\theta}{\left(r^2 + x_3^2\right)^{5/2}} =$$

$$= 2\pi \int\limits_{R_0-a_1}^{R_0+a_1} r^3 dr \int\limits_0^{a_3\sqrt{1-\frac{(r-R_0)^2}{a_1^2}}} \frac{dx_3}{\left(r^2 + x_3^2\right)^{5/2}} = \tag{15}$$

$$= 2\pi a_3 \int\limits_{R_0-a_1}^{R_0+a_1} \frac{dr}{r} \left[\frac{\sqrt{1 - \frac{(r-R_0)^2}{a_1^2}}}{\sqrt{r^2 + a_3^2 \frac{(r-R_0)^2}{a_1^2}}} - \frac{a_3^2}{3} \frac{\left(1 - \frac{(r-R_0)^2}{a_1^2}\right)^{3/2}}{\left(r^2 + a_3^2 \frac{(r-R_0)^2}{a_1^2}\right)^{3/2}} \right].$$

Denoting here for the sake of brevity

$$x = \frac{r - R_0}{a_1}, \quad r = a_1 x + R_0, \quad dr = a_1\,dx, \tag{16}$$

we find

$$\begin{gathered} A = 2\pi a_1 a_3 \int\limits_{-1}^{1} \frac{dx}{R_0 + a_1\,x} \times \\ \times \left[\frac{\sqrt{1 - x^2}}{\sqrt{\left(R_0 + a_1 x\right)^2 + a_3^2\,x^2}} - \frac{a_3^2}{3} \frac{\left(1 - x^2\right)^{3/2}}{\left[\left(R_0 + a_1 x\right)^2 + a_3^2\,x^2\right]^{3/2}} \right]. \end{gathered} \tag{17}$$

Thus, the required coefficients A, B, C of the tidal potential inside the torus are the following:

$$A = B = 2\pi a_1 a_3 \int\limits_{-1}^{1} \frac{dx}{R_0 + a_1\,x} \times$$

$$\times \left[\frac{\sqrt{1 - x^2}}{\sqrt{R_0^2 + 2R_0 a_1 x + (a_1^2 + a_3^2)\,x^2}} - \frac{a_3^2}{3} \frac{\left(1 - x^2\right)^{3/2}}{[R_0^2 + 2R_0 a_1 x + (a_1^2 + a_3^2)\,x^2]^{3/2}} \right], \tag{18}$$

$$C = \iiint\limits_V \frac{x_3^2}{\left(r^2 + x_3^2\right)^{5/2}}\,dx_1 dx_2 dx_3.$$

We notice that the specified coefficients A, B, C are expressed through standard elliptic integrals of the third kind. At the same time, they can be found also by numerical method. Thus, the desired tidal potential inside the hole of the torus is given by (14).

2.2. The case of a distant outer point

If

$$x_i \gg x_i' , \tag{19}$$

then the expansion $1/D$ has the following form:

$$\frac{1}{D} \approx \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \times$$

$$\left[1 + \frac{x_1 x_1' + x_2 x_2' + x_3 x_3'}{x_1^2 + x_2^2 + x_3^2} + \frac{3(x_1 x_1' + x_2 x_2' + x_3 x_3')^2 - \left(x_1^2 + x_2^2 + x_3^2\right)\left(x_1'^2 + x_2'^2 + x_3'^2\right)}{2(x_1^2 + x_2^2 + x_3^2)^2}\right]. \tag{20}$$

As in the first case, the problem now is simplified by the fact that after integration in the expression (20) disappear all terms regarding odd (x_1', x_2', x_3') .

The tidal potential of the torus in this case is

$$\varphi_t(x) =$$

$$= \frac{G\rho}{2} \iiint\limits_V \frac{\left(2x_1'^2 - x_2'^2 - x_3'^2\right) x_1^2 + \left(2x_2'^2 - x_1'^2 - x_3'^2\right) x_2^2 + \left(2x_3'^2 - x_1'^2 - x_2'^2\right) x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} \cdot$$

$$\cdot\, dx_1'\, dx_2'\, dx_3', \tag{21}$$

or,

$$\varphi(x) = \frac{G}{2} \left\{ \frac{(2J_1 - J_2 - J_3)\, x_1^2 + (2J_2 - J_1 - J_3)\, x_2^2 + (2J_3 - J_1 - J_2)\, x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} \right\}. \tag{22}$$

In (22) we introduced the moments of inertia of the body with three planes of symmetry:

$$\begin{aligned} J_1 &= \rho \iiint\limits_V x_1'^2 dx_1'\, dx_2'\, dx_3' \,; \\ J_2 &= \rho \iiint\limits_V x_2'^2 dx_1'\, dx_2'\, dx_3' \,; \\ J_3 &= \rho \iiint\limits_V x_3'^2 dx_1'\, dx_2'\, dx_3' \,. \end{aligned} \tag{23}$$

The formula for the tidal potential (22) fits for any homogeneous body with three planes of symmetry. In the case of torus $J_1 = J_2$, the expression (22) simplifies to

$$\varphi(x) = \frac{G}{2}(J_1 - J_3) \frac{x_1^2 + x_2^2 - 2x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}. \tag{24}$$

The moments of inertia of the torus are

$$J_1 = J_2 = \frac{1}{2}M\left(R_0^2 + \frac{3}{4}a_1^2\right), \quad J_3 = \frac{1}{4}M \cdot a_3^2, \tag{25}$$

where the mass of torus is given by Eq. (5).

3. DISCUSSION

The knowledge of the potential of the torus with an elliptic cross section of the sleeve is of great practical and theoretical value. For example, such a torus can serve as a gravitational model of the Kuiper belt. Indeed, the Kuiper belt has the shape of torus with an elliptical cross section of the sleeve. With the help of the tidal potential of the Kuiper Belt, we can study the motion and evolution of asteroids and dwarf planets.

Note, also, that in the paper by Kondratyev et al. (2015b) the formula for tidal gravity in a torus of dark matter was used to study the limiting oblateness of elliptical galaxies.

The final form of the potential coefficients A, B, C will be obtained in the following work.

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